BI-ALGEBRAIC GEOMETRY AND THE ANDRÉ-OORT CONJECTURE

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Contents

1. The André-Oort conjecture

The André-Oort conjecture describes the distribution of special points on Shimura varieties. It is the analog in a Hodge-theoretic context of the Manin-Mumford conjecture (a theorem of Raynaud [Ray88]) stating that an irreducible subvariety of a complex Abelian variety containing a Zariski-dense set of torsion points is the translate of an Abelian subvariety by a torsion point. This text is a survey of the André-Oort conjecture, its context and its proof in the case of the Shimura variety $\mathcal{A}_g$ moduli space of principally polarized complex Abelian varieties of dimension g (and more generally for mixed Shimura varieties whose pure part is of Abelian type) following a strategy proposed by Pila and Zannier and obtained through the work of many authors.

1.1. **The Hodge theoretic motivation.** Let us start by explaining the algebro-geometric problem underlying the conjecture. Let $f : \mathcal{X} \longrightarrow S$ be a smooth family of algebraic varieties over a quasi-projective smooth base S. Can we describe the locus of points $s \in S$ where the fiber $\mathcal{X}_s$ (and its Cartesian powers) contain more algebraic cycles than the very general fiber (and its Cartesian powers)? We work over $\mathbb{C}$ and consider the Hodge incarnation of this problem. Let $\mathbb{V} \to S$ be an admissible variation of mixed $\mathbb{Z}$-Hodge structures on the complex quasi-projective smooth base S (cf. [PS08, Def. 14.49]). In particular $\mathbb{V}$ is a $\mathbb{Z}$-local system on S such that each fiber $\mathbb{V}_s$, $s \in S$, carries a graded-polarized mixed Hodge structure. This is an abstraction of the geometric case corresponding to $\mathbb{V} = (R^p f_* \mathbb{Z})_{\mathrm{prim}}$ (for some $p > 0$) for f as above. One wants to understand the Hodge locus $\mathrm{HL}(S, \mathbb{V}) \subset S$, namely the subset of points s in S for which exceptional Hodge classes of type $(0,0)$ do occur in some $\mathbb{V}_{\mathbb{Q},s}^{a} \otimes (\mathbb{V}_{\mathbb{Q},s}^{\vee})^{b}$, where $\mathbb{V}_{\mathbb{Q},s}^{\vee}$ denotes the $\mathbb{Q}$-Hodge structure dual to $\mathbb{V}_{\mathbb{Q},s}$.

The Tannakian formalism available for Hodge structures is particularly useful for describing $\mathrm{HL}(S, \mathbb{V})$. Recall that for every $s \in S$, the Mumford-Tate group $\mathbf{MT}_s$ of the Hodge structure $\mathbb{V}_{\mathbb{Q},s}$ is the Tannakian group of the subcategory $< \mathbb{V}_{\mathbb{Q},s}^{\otimes} >$ of the Tannakian category of pure Hodge structures tensorially generated by $\mathbb{V}_{\mathbb{Q},s}$ and $\mathbb{V}_{\mathbb{Q},s}^{\vee}$. Equivalently, the group $\mathbf{MT}_s$ is the stabiliser of the Hodge classes of type $(0,0)$ in the rational Hodge structures tensorially generated by $\mathbb{V}_{\mathbb{Q},s}$ and its dual. A point $s \in S$ is said to be Hodge generic if $\mathbf{MT}_s$ is maximal when s varies in its connected component. If S is connected, two Hodge generic points of S have the same Mumford-Tate group, called the generic Mumford-Tate group $\mathbf{MT}_{S,\mathrm{gen}}$ of $(S, \mathbb{V})$. The Hodge locus $\mathrm{HL}(S, \mathbb{V})$ is the subset of points of S which are not Hodge generic.

A fundamental result of Cattani-Deligne-Kaplan [CDK95] states that $\mathrm{HL}(S, \mathbb{V})$ is a countable union of closed irreducible algebraic subvarieties of S, each not contained in the union of the others. The irreducible components of the intersections of these algebraic subvarieties are called *special subvarieties* of $(S, \mathbb{V})$. Hodge subvarieties of dimension zero are called *special points* of $(S, \mathbb{V})$. We would like to understand the distribution of special points in S.

1.2. **The André-Oort conjecture for $\mathbb{C}^2$.** The André-Oort conjecture answers this question when S is a Shimura variety. We start with its most explicit incarnation.

The simplest Shimura variety is the classical modular curve $Y(1)$. As a complex analytic space it is the quotient $Y(1) := \mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}$, where $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ is the Poincaré upper-half plane and the group $\mathbf{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau + b}{c\tau + d}.$$

The space $Y(1)$ can also be interpreted as the set of complex elliptic curves up to isomorphism:

$$\mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H} \longrightarrow \{E/\mathbb{C}\}/\cong, \qquad \tau \mapsto [E_\tau := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})] \ .$$

As complex elliptic curves up to isomorphism are classified by their j-invariant, the quotient map $\pi : \mathcal{H} \longrightarrow Y(1)$ identifies with the holomorphic j-map $j : \mathcal{H} \longrightarrow \mathbb{C}$ given by

$$\tau \mapsto j(E_\tau) = q^{-1} + 744 + 196884q + \cdots, \quad q = e^{2\pi i\tau}.$$

Hence the quotient $Y(1) \overset{j}{\simeq} \mathbb{C}$ is the moduli space of complex elliptic curves; it is an algebraic variety, defined over $\mathbb{Q}$ (even over $\mathbb{Z}$) because we have the notion of elliptic curves over arbitrary schemes.

Ignoring the stacky issues, the universal family of elliptic curves over $Y(1) \simeq \mathbb{C}$ defines a Hodge locus in $Y(1)$, i.e. special points. For $\tau \in \mathcal{H}$, $\mathrm{End}\,(E_\tau) = \{z \in \mathbb{C} : z\cdot(\mathbb{Z}\tau+\mathbb{Z}) \subset \mathbb{Z}\tau + \mathbb{Z}\}$. Hence $\mathrm{End}\,(E_\tau) = \mathbb{Z}$ if $\dim_\mathbb{Q} \mathbb{Q}(\tau) \neq 2$ and $\mathrm{End}\,(E_\tau)$ is an order in $\mathbb{Q}(\tau)$ if $\dim_\mathbb{Q} \mathbb{Q}(\tau) = 2$, in which case E_τ is a CM-elliptic curve. It follows easily that the Mumford-Tate group at $j(\tau)$ is $\mathbf{GL}(2, \mathbb{Q})$ in the first case, while it is $\mathrm{Res}_{\mathbb{Q}(\tau)/\mathbb{Q}}\mathbf{G}_m$ in the second. Hence special points (also called CM-points) in $\mathbb{C}$ correspond to imaginary quadratic τ's in $\mathcal{H}$, in particular they are dense (even for the analytic topology) in $\mathbb{C}$.

Let us now consider $Y(1)^2 \simeq \mathbb{C}^2$ as the moduli space of pairs of elliptic curves. Once more the Hodge locus for this family can be explicitly described:
- a point $x = (x_1, x_2) \in \mathbb{C}^2$ is special if both $x_1 \in \mathbb{C}$ and $x_2 \in \mathbb{C}$ are special.
- a special curve is either a line $\{x_1\}\times\mathbb{C}$ with x_1 special, a line $\mathbb{C}\times\{x_2\}$ with x_2 special, or the image T_n in $\mathbb{C}^2$ of the modular curve $Y(n)$ parametrizing isogenies $\mathbb{Z}/n\mathbb{Z} \hookrightarrow E_1 \twoheadrightarrow E_2$ between two elliptic curves. The curve T_n is obtained from $Y(n)$ by forgetting the isogeny (an equivalent definition of T_n is given below).

Each of these special curves contains a dense set of special points. Conversely André [An89] conjectured:

Conjecture 1.1. *Let $\Sigma \subset \mathbb{C}^2$ be a set of special points, and let Z be an irreducible component of the its Zariski-closure $\overline{\Sigma}^{\mathrm{Zar}}$. Then Z is one of the following:*

(1) *a special point,*
(2) *$\{x_1\} \times \mathbb{C}$ with x_1 special,*
(3) *$\mathbb{C} \times \{x_2\}$ with x_2 special,*
(4) *the image T_n (a Hecke correspondence) of*

$$t_n : \mathcal{H} \to \mathcal{H} \times \mathcal{H} \to \mathbb{C}^2, \quad \tau \mapsto (\tau, n\tau) \mapsto (j(\tau), j(n\tau))$$

for some $n \in \mathbb{Z}_{\geq 1}$,

(5) *$\mathbb{C}^2$ itself.*

Conjecture 1.1 was proven by Edixhoven [Ed98] under the Generalized Riemann Hypothesis (GRH) and by André [An98] unconditionally.

1.3. **The Conjecture.** We turn to the general case, first in an informal way. A pure Shimura variety S (resp. a mixed Shimura variety) is a complex quasi-projective moduli space of pure polarized (resp. mixed graded-polarized) Hodge structures with additional data, such that the universal family above S defines an admissible variation of (mixed) Hodge structure $\mathbb{V}$ over S. As explained by Deligne [De79] this restricts severely the possible types of Hodge structures we can consider. The prototype of a pure Shimura variety is the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties of dimension g, the variation $\mathbb{V}$ over $\mathcal{A}_g$ is the Hodge incarnation $R^1 f_* \mathbb{Z}$ of the universal Abelian variety $f : \mathfrak{A}_g \longrightarrow \mathcal{A}_g$ over $\mathcal{A}_g$. The prototype of a mixed Shimura variety is $\mathfrak{A}_g$, the variation $\mathbb{V}$ over $\mathfrak{A}_g$ is the Hodge incarnation of the universal semi-Abelian variety over $\mathfrak{A}_g$.

As in Section 1.1 the variation $\mathbb{V}$ over S defines special subvarieties in S. A special point of $\mathcal{A}_g$, also called a CM-point, corresponds to an Abelian variety with Complex Multiplication (CM). A special point on $\mathfrak{A}_g$ is a torsion point on a CM-Abelian variety.

A crucial feature of Shimura varieties is their purely group-theoretic description: any Shimura variety S is defined thanks to a Shimura datum $(\mathbf{G}, X)$, where $\mathbf{G}$ is a connected linear algebraic group over $\mathbb{Q}$ and X is a certain homogeneous space under a subgroup of $\mathbf{G}(\mathbb{C})$. Special subvarieties of S also have a purely group theoretic description: they are precisely the images of the natural morphisms between Shimura varieties. In the next section we review this formalism for pure Shimura varieties.

Once we know that any Shimura variety S contains one special point, this group-theoretic formalism also implies that any special subvariety of S contains a dense (even for the Archimedean topology) set of special points. The André-Oort conjecture is the converse statement:

Conjecture 1.2 (André-Oort)**.** *Let Z be an irreducible subvariety of a mixed Shimura variety S. If Z contains a Zariski-dense set of special points then Z is a special subvariety of S.*

1.4. **History and results.** Motivated by transcendence questions about periods of Shimura varieties, André [An89, p.215, Problem 1] formulated Conjecture 1.2 for a curve Z contained in a pure Shimura variety. Oort [Oort94] was interested in the study of Jacobians with complex multiplication and proposed Conjecture 1.2 for $S = \mathcal{A}_g$. Hence the name of the conjecture.

Both André and Oort were aware of the analogy with the Manin-Mumford conjecture. This analogy has inspired all the strategies for proving Conjecture 1.2.

(a) The p-adic methods of Raynaud's proof [Ray88] of the Manin-Mumford conjecture inspired works on Conjecture 1.2 when S is a pure Shimura variety and for the Zariski-closure Z of a set of special points having good reduction properties at one fixed place p [Moo98,II], [Ya05].

(b) Edixhoven developed an approach to Conjecture 1.2, based on Galois techniques and intersection theory, retrospectively close in spirit to Hindry's approach to the Manin-Mumford conjecture [Hin88]. This method uses in a crucial way effective Cebotarev type results, known only under the Generalized Riemann Hypothesis (GRH). In [Ed98]

Edixhoven proves Conjecture 1.2 under GRH for S a product of two modular curves ; in [EdYa03] Edixhoven and Yafaev obtain the result under GRH for Z a curve in an arbitrary pure Shimura variety S; and in [Ed05] Edixhoven proves Conjecture 1.2 under GRH for Z an arbitrary subvariety of a product of modular curves. This approach, allied with ideas *à la Margulis-Ratner* from ergodic theory on homogeneous spaces ([CloUl05], [U07]), culminated in the following result [UY14a], [KY14] (announced in 2006 and published in 2014):

Theorem 1.3. *The André-Oort Conjecture 1.2 for pure Shimura varieties is true under the Generalized Riemann Hypothesis. It is also true unconditionally if Z is the Zariski-closure of a set of special points contained in a Hecke orbit.*

This text will say nothing about Edixhoven's approach, for which many surveys are available. We refer in particular to [Ya07] and [Panorama] and the references therein.

(c) Pila and Zannier [PiZa08] developed a method based on o-minimal geometry for proving the Manin-Mumford conjecture. Pila adapted it to obtain an unconditional proof of Conjecture 1.2 for S an arbitrary product $\mathbb{C}^n \times \mathbf{G}_m^k$ [Pil11] (as we already mentioned, André obtained an unconditional proof for S the product of two modular curves but his method using Puiseux expansion did not generalize). The combination of the work of many authors (whose contributions are detailed below) then lead to the following:

Theorem 1.4. *The André-Oort Conjecture 1.2 is true for $\mathcal{A}_g$ and more generally for any mixed Shimura variety whose pure part is of Abelian type.*

The goal of this text is to present the ideas around Conjecture 1.2 and sketch the proof of Theorem 1.4 following the Pila-Zannier strategy. Following [U14], Conjecture 1.2 for a general connected mixed Shimura variety S uniformized by $\pi : X^+ \longrightarrow S := \Gamma\backslash X^+$ follows from three main ingredients (two of which are known in full generality while the third one is known only under GRH or unconditionally for mixed Shimura varieties whose pure part is of Abelian type):

The first ingredient is the *definability in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ of the restriction of π to a semi-algebraic fundamental set $\mathcal{F}$ for the action of Γ on X^+*: see Theorem 6.2. This result is obtained by Peterzil-Starchenko [PetStar13] for $S = \mathcal{A}_g$, by Klingler-Ullmo-Yafaev [KUY16] for an arbitrary pure Shimura variety and extended by Gao [Gao16] to any mixed Shimura variety.

The second ingredient is *the Ax-Lindemann conjecture for Shimura varieties*, see Theorem 4.25, which says that the Zariski-closure $\pi(Y)$ of any algebraic subvariety Y of X^+ (in the sense of Example 4.6) should be weakly special (in the sense of Section 3.3). This is the main geometric ingredient in the Pila-Zannier strategy for solving the Manin-Mumford-André-Oort problem for Shimura varieties.

Theorem 4.25 is proven by Pila [Pil11] when S is a product $Y(1)^n \times (\mathbb{C}^*)^k$, by Ullmo-Yafaev [UY14b] for projective Shimura varieties, by Pila-Tsimerman [PT14] for $\mathcal{A}_g$, by Klingler-Ullmo-Yafaev [KUY16] for any pure Shimura variety and extended by Gao [Gao16] to any mixed Shimura variety. All these proofs use o-minimal geometry as a tool. Mok has an entirely complex-analytic approach to the Ax-Lindemann conjecture in the pure case. We refer to [Mok10], [Mok12] for partial results.

The third ingredient is *a good lower bound for the size of Galois orbits of special points of S*. This ingredient is already crucial in the Edixhoven's approach. We refer to Section 9 for the description of the expected lower bound for an arbitrary pure Shimura variety. These expected lower bounds are known under GRH for any pure Shimura variety following results of Tsimerman [Tsi12] and Ullmo-Yafaev [UY15]. They are known unconditionally only for mixed Shimura varieties whose pure part is of Abelian type. For simplicity we restrict ourselves to the case $S = \mathcal{A}_g$. Given a point $x \in \mathcal{A}_g$ let A_x be the principally polarized Abelian variety parametrized by x and d_x the absolute value of the discriminant of the center of the ring of endomorphisms of A_x. When x is special its field of definition $k(x)$ is a number field. One wants to show that there exist real positive numbers $\alpha = \alpha(g)$ and $\beta = \beta(g)$ such that for any special point $x \in \mathcal{A}_g$ one has:

$$[k(x) : \mathbb{Q}] > \alpha \cdot d_x^{\beta} \quad . \tag{1.1}$$

Tsimerman [Tsi] remarkably noticed that the inequality (1.1) follows from the Masser-Wüstholz isogeny theorem [MaWü95] and an upper bound for the Faltings height $h_F(A_x)$ of the form

$$\forall \epsilon > 0, \quad h_F(A_x) \ll_\epsilon d_x^{\epsilon} \quad . \tag{1.2}$$

Colmez [Col93] conjectured a closed formula for the Faltings height of an Abelian variety with complex multiplication, depending only on its CM-type (E, Φ). Fixing E and averaging on the 2^g possible CM-type Φ for E one obtains a simpler formula for the average of the Faltings height of Abelian varieties with CM by the ring of integers $\mathcal{O}_E$ of E. The upper-bound (1.2) follows from this Colmez conjecture on average and classical arguments from analytic number theory (see [Tsi]).

Finally a proof of Colmez conjecture on average has been obtained independently by Andreatta-Goren-Howard-Madapusi Pera [AGHM] (studying CM-points on certain orthogonal Shimura varieties) and Yuan-Zhang [YuZh] (analyzing Heegner points on certain Shimura curves).

Daw-Orr [DawOrr15] show that the Pila-Zannier method gives a new proof of Conjecture 1.2 under GRH for an arbitrary pure Shimura variety.

Gao [Gao16] extends the Pila-Zannier method in the mixed setting, showing Conjecture 1.2 under GRH for any mixed Shimura variety and Conjecture 1.2 unconditionally for mixed Shimura varieties whose pure part is of Abelian type.

1.5. **Organization of the text.** This text is organized as follows.

For the reader's convenience Section 2 gives a short introduction to the formalism of Shimura varieties using Deligne's language of Hodge theory. For simplicity we restrict ourselves to the *pure Shimura varieties*.

Section 3 describes a general format where a reasonable Manin-Mumford-André-Oort type problem can be formulated: the notion of a *special structure* on a complex algebraic variety S, which axiomatizes the properties of the collection of special subvarieties on a Shimura variety or an Abelian variety. We also notice that in all the cases we consider, special structures are related to Kähler geometry through the notion of *weakly special subvarieties*: in the case of semi-abelian varieties or pure Shimura varieties, weakly special subvarieties are exactly the totally geodesic subvarieties for the canonical Kähler metric on S. The special subvarieties of S are precisely the weakly special ones (a purely geometric notion) containing a smooth special point (an arithmetic notion).

This dichotomy between geometry and arithmetic persists in Section 4, where we develop the idea of bi-algebraic geometry. In a few words, this is the study of the transcendence properties of the uniformization morphism $\pi : \tilde{S} \longrightarrow S$ for S a connected algebraic variety whose universal cover $\tilde{S}$ can be endowed with an algebraic structure (in the precise sense of Definition 4.1). The bi-algebraic structure is non-trivial when the uniformization map π is non-algebraic. In this situation, bi-algebraic subvarieties of S are defined by a functional transcendence constraint: they are the irreducible algebraic subvarieties of S images of algebraic subvarieties of $\tilde{S}$ (in the sense of Definition 4.2). All the special structures we consider are of bi-algebraic origin (see Section 4.3), and bi-algebraic subvarieties and weakly special subvarieties coincide. Hence special subvarieties are exactly the bi-algebraic subvarieties containing a smooth special point. In the best cases, the bi-algebraic structure can be enriched over $\overline{\mathbb{Q}}$ (see Section 4.2) and the special points are exactly the arithmetic bi-algebraic points (see Definition 4.10).

The geometry of non-trivial bi-algebraic structures is governed by a natural conjecture in functional transcendence: given a connected algebraic variety S with a bi-algebraic structure, the Ax-Lindemann conjecture predicts that the Zariski-closure $\pi(Y)$ of any algebraic subvariety Y of $\tilde{S}$ should be bi-algebraic. This is the main geometric ingredient in the Pila-Zannier strategy for solving the Manin-Mumford-André-Oort problem for special structures of bi-algebraic origin.

In Section 5 we turn to the techniques at our disposal for attacking the Ax-Lindemann conjecture and the Manin-Mumford-André-Oort problem in general. Let S be an algebraic variety endowed with a bi-algebraic structure. Whether or not this bi-algebraic structure underlies a special structure on S seems to depend on the existence of a common geometric framework for S and $\tilde{S}$, more flexible than (semi-) algebraic geometry as the map $\pi : \tilde{S} \longrightarrow S$ is far from algebraic, but topologically more constraining than analytic geometry in order to explain the special structure. Such a common framework is reminiscent of Grothendieck's idea of "tame topology" [Gro84, section 5], and is described in model theoretic language as o-minimal geometry. Section 5 presents a minimal recollection of o-minimal geometry, and state a deep diophantine criterion due to Pila and Wilkie for detecting (positive dimensional) semi-algebraic subsets of $\mathbb{R}^n$ among subsets definable in an o-minimal structure: if such a subset contains polynomially many (with respect to the height) points of $\mathbb{Q}^n$ then it contains a non-trivial positive dimensional semi-algebraic subset.

The last four sections turn to the proof of the André-Oort conjecture Theorem 1.4 for pure Shimura varieties of Abelian type. Section 6 details the first ingredient, namely the o-minimal geometry of pure Shimura varieties which culminates in Theorem 6.2. Section 7 describes the general structure of the proof. Section 8 sketch the proof of the second ingredient: the hyperbolic Ax-Lindemann Theorem 4.25, known for any Shimura variety. Section 9 considers its arithmetic part: the obtention of good lower bounds for the size of Galois orbits of special points, known only for Shimura varieties of Abelian type.

This text is largely inspired by the course on the André-Oort conjecture given by E. Ullmo at IHES in Spring 2016. For other surveys on the André-Oort conjecture following the Pila-Zannier method, we refer to [Daw16] for a more elementary introduction, to [Sca12] and [Sca16] for the description of the method in the geometrically easier case of $S = \mathbb{C}^n \times \mathbf{G}_m^k$ but with an expanded treatment of the o-minimal background.

2. Pure Shimura varieties and their special subvarieties

References for this section are [De71], [De79], [Mi05]. For mixed Shimura varieties, see [Pink89].

Recall that a pure $\mathbb{Q}$-Hodge structure on a $\mathbb{Q}$-vector space V is a linear decomposition $V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$. Equivalently it is a morphism of real algebraic groups $h\colon \mathbf{S} \longrightarrow \mathbf{GL}(V_{\mathbb{R}})$, where $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}_{\mathbf{m},\mathbb{C}}$ denotes the Deligne's torus (hence $\mathbf{S}(\mathbb{R}) = \mathbb{C}^*$). The Mumford-Tate group $\mathbf{MT}(h)$ we defined in Section 1.1 is equivalently the smallest algebraic $\mathbb{Q}$-subgroup $\mathbf{H}$ of $\mathbf{GL}(V)$ such that h factors through $\mathbf{H}_{\mathbb{R}}$. It is a reductive group if V is assumed to be polarized.

A *Shimura datum* is a pair $(\mathbf{G}, X)$, with $\mathbf{G}$ a linear connected reductive group over $\mathbb{Q}$ and X a $\mathbf{G}(\mathbb{R})$-conjugacy class of a morphism of real algebraic groups $h \in \mathrm{Hom}(\mathbf{S}, \mathbf{G}_{\mathbb{R}})$, satisfying the "Deligne's conditions" [De79, 1.1.13]:

- (D1) The Hodge structure on the Lie algebra $\mathfrak{g}$ defined by $\mathrm{Ad} \circ h$ has Hodge types $(-1, 1)$, $(0, 0)$ and $(1, -1)$ only.
- (D2) The conjugation by $h(i)$ defines a Cartan involution of the group of real points $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ of the adjoint group $\mathbf{G}^{\mathrm{ad}}$: the subgroup $\{g \in \mathbf{G}^{\mathrm{ad}}(\mathbb{C}), \quad h(i)^{-1}\overline{g}h(i) = g\}$ of $\mathbf{G}^{\mathrm{ad}}(\mathbb{C})$ is compact.
- (D3) for every simple factor $\mathbf{H}$ of $\mathbf{G}$, the composition of $h : \mathbf{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$ with the projection $\mathbf{G}_{\mathbb{R}} \longrightarrow \mathbf{H}_{\mathbb{R}}$ is non-trivial.

These conditions imply, in particular, that the connected components of X are Hermitian symmetric domains. Any Hermitian symmetric domain can be obtained in this way. A morphism of Shimura data from $(\mathbf{G}_1, X_1)$ to $(\mathbf{G}_2, X_2)$ is a $\mathbb{Q}$-morphism $f : \mathbf{G}_1 \longrightarrow \mathbf{G}_2$ mapping X_1 to X_2.

Definition 2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ (where $\mathbb{A}_{\mathrm{f}}$ denotes the ring of finite adèles of $\mathbb{Q}$). The Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ is the complex analytic space $\mathbf{G}(\mathbb{Q})\backslash(X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K)$, where $\mathbf{G}(\mathbb{Q})$ acts diagonally on $X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$.*

Proposition 2.2. *Let $\mathbf{G}(\mathbb{R})_+$ be the stabilizer in $\mathbf{G}(\mathbb{R})$ of a connected component X^+ of X and $\mathbf{G}(\mathbb{Q})_+ := \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q})$. The class group $\mathcal{C} := \mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$ is finite and one has the decomposition*

$$\mathrm{Sh}_K(\mathbf{G}, X) = \coprod_{g\in\mathcal{C}} \Gamma_g\backslash X^+ \quad , \tag{2.1}$$

where Γ_g denotes the congruence arithmetic lattice $gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ of $\mathbf{G}(\mathbb{R})_+$.

Each $\Gamma_g\backslash X^+$ has finite volume for the natural (up to a non-zero multiple scalar) $\mathbf{G}(\mathbb{R})_+$-invariant measure on the Hermitian symmetric space X^+. It follows from results of Baily and Borel [BB66] that each $\Gamma_g\backslash X^+$ has a natural structure of complex quasi-projective variety, hence also $\mathrm{Sh}_K(G, X)$. Moreover the natural analytic morphism $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1) \longrightarrow \mathrm{Sh}_{K_2}(\mathbf{G}_2, X_2)$ deduced from a morphism of Shimura data $f : (\mathbf{G}_1, X_1) \longrightarrow (\mathbf{G}_2, X_2)$ mapping a compact open subgroup $K_1 \subset \mathbf{G}_1(\mathbb{A}_\mathrm{f})$ into $K_2 \subset \mathbf{G}_2(\mathbb{A}_\mathrm{f})$ is naturally algebraic.

If Γ_g has no torsion then the algebraic variety $\Gamma_g\backslash X^+$ is smooth. Usually we work with a stronger notion of *neat compact open subgroup* $K \subset \mathbf{G}(\mathbb{A}_\mathrm{f})$, in which case $\mathrm{Sh}_K(\mathbf{G}, X)$ is smooth.

The quotient $S = \Gamma_e\backslash X^+$ is called the connected Shimura variety associated to the Shimura datum $(\mathbf{G}, X)$ and the compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_\mathrm{f})$.

The projective limit $\mathrm{Sh}(\mathbf{G}, X)_\mathbb{C} = \lim_K \mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$ is a $\mathbb{C}$-scheme on which $\mathbf{G}(\mathbb{A}_\mathrm{f})$ acts continuously by multiplication on the right. The multiplication by $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$ on $\mathrm{Sh}(\mathbf{G}, X)$ induces an algebraic correspondence T_g on $\mathrm{Sh}_K(G, X)$, called a Hecke correspondence.

Let $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(V)$ be a rational representation of $\mathbf{G}$. Choose a $\mathbb{Z}$-structure $V_\mathbb{Z}$ on V such that $\rho(K) \subset \mathbf{GL}(V_{\hat{\mathbb{Z}}})$. Every point $x \in X$ defines a polarized $\mathbb{Z}$-Hodge structure

$$\rho \circ x : \mathbf{S} \xrightarrow{x} \mathbf{G}_\mathbb{R} \xrightarrow{\rho} \mathbf{GL}(V_\mathbb{R})$$

on $V_\mathbb{Z}$. These $\rho \circ x$, $x \in X$, aggregate to form a polarized variation of $\mathbb{Z}$-Hodge structure $\mathbb{V}_\rho$ on $\mathrm{Sh}_K(\mathbf{G}, X)$. The collection of special subvarieties on $\mathrm{Sh}_K(\mathbf{G}, X)$ associated with $\mathbb{V}_\rho$ is shown to be independent of the choice of the faithful representation ρ and has a purely group-theoretic description: a subvariety $V \subset \mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$ is *special* if and only if there is a Shimura datum $(\mathbf{H}, X_\mathbf{H})$, a morphism of Shimura data $f : (\mathbf{H}, X_\mathbf{H}) \longrightarrow (\mathbf{G}, X)$ and an element $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$ such that V is an irreducible component of the image of the *Hecke-Shimura morphism*, also called a *Hecke correspondence*:

$$\mathrm{Sh}(\mathbf{H}, X_\mathbf{H}) \xrightarrow{\mathrm{Sh}(f)} \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{\cdot g} \mathrm{Sh}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X) \ .$$

It can also be shown that the Shimura datum $(\mathbf{H}, X_\mathbf{H})$ can be chosen in such a way that $\mathbf{H} \subset \mathbf{G}$ is the generic Mumford-Tate group on $X_\mathbf{H}$. A *special* point is a special subvariety of dimension zero. One sees that a point $[x, gK] \in \mathrm{Sh}_K(\mathbf{G}, X)$ (where $x \in X$ and $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$) is *special* if and only if the group $\mathbf{MT}(x)$ is commutative (in which case $\mathbf{MT}(x)$ is a torus).

Given a special subvariety S of $\mathrm{Sh}_K(\mathbf{G}, X)$, the set of special points of $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$ contained in S is dense in V for the strong (and in particular for the Zariski) topology. Indeed, one shows that S contains a special point, say s. Let $\mathbf{H}$ be a reductive group defining S and let $\mathbf{H}(\mathbb{R})^+$ denote the connected component of the identity in the real Lie group $\mathbf{H}(\mathbb{R})$. The fact that $\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$ is dense in $\mathbf{H}(\mathbb{R})^+$ implies that the "$\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$-orbit" of s, which is contained in S, is dense in S. This "orbit" (sometimes referred to as the Hecke orbit of s) consists of special points.

For the precise definition of a Shimura datum (resp. a Shimura variety) of Abelian type we refer to [Mi05, section 8]. We only recall that if $(\mathbf{G}, X^+)$ is a connected Shimura datum with $\mathbf{G}$ simple then:

- if $\mathbf{G}^{\mathrm{ad}}$ is of type A, B or C then $(\mathbf{G}, X^+)$ is of Abelian type.
- if $\mathbf{G}^{\mathrm{ad}}$ is of type E_6 or E_7 then $(\mathbf{G}, X^+)$ is not of Abelian type.
- if $\mathbf{G}^{\mathrm{ad}}$ is of type D then $(\mathbf{G}, X^+)$ may or may not be of Abelian type.

3. Special structures on algebraic varieties

3.1. **Pre-special structure.** In this section we introduce a general format in which a Manin-Mumford-André-Oort type problem can be formulated: the notion of a special structure on an algebraic variety. We refer to [U16] for more details and [Zil13] for a study of special subvarieties from the point of view of model theory.

Definition 3.1. *(pre-special structure) Let S be a complex quasi-projective variety. A* pre-special structure *on S is the datum of a countable set $\Sigma(S)$ of irreducible algebraic subvarieties of S, called* special subvarieties *of S, satisfying the following properties:*

(i) *$S \in \Sigma(S)$, i.e. S is special.*
(ii) *an irreducible component of an intersection of special subvarieties of S is a special subvariety of S.*
(iii) *Let $\Sigma_i(S) \subset \Sigma(S)$ be the set of special subvarieties of S of dimension i. For any $W \in \Sigma(S)$, special points of S (i.e. elements of $\Sigma_0(S)$) are dense in W.*

It follows from property (ii) that for any irreducible algebraic subvariety Z of S, there exists a unique smallest special subvariety of S containing Z. One says that Z is Hodge generic if it not contained in any strict special subvariety of S.

The following are natural examples of complex algebraic varieties endowed with a pre-special structure:

(1) a complex semi-Abelian variety W extension of an Abelian variety A by a torus $T \simeq \mathbf{G}_m^n$, with special subvarieties the translate by a torsion point of an algebraic subgroup. Special points are torsion points.

(2) a Shimura variety S with its special subvarieties.

3.2. **Special structure.** Notice that many other examples of pre-special structures are obtained as follows. Suppose S is a quasi-projective variety over $\overline{\mathbb{Q}}$ and define an irreducible subvariety of $S_{\mathbb{C}}$ to be special if it is defined over $\overline{\mathbb{Q}}$. Such a pre-special structure is called trivial as there seems to be nothing interesting to say in general about the distribution of special points. We exclude such trivial examples by strengthening the definition of a pre-special structure as follows:

Definition 3.2. *(special structure) Let S be a complex quasi-projective variety. A* special structure *on S is the datum of a pre-special structure on each S^r, $r \in \mathbb{Z}_{\geq 1}$, such that for any $W \in \Sigma(S)$ then W^r belongs to $\Sigma(S^r)$ and satisfies:*

(a) *$\Sigma_0(W^r) = \Sigma_0(W)^r$.*
(b) *for any r-tuple $(a_1, \cdots, a_r)$ then*

$$\prod_{i=1}^{r} \Sigma_{a_i}(W) \subset \Sigma_a(W^r) \quad ,$$

where $a = \sum_{i=1}^{r} a_i$.

(c) *The set $\Sigma_{d_W}(W^2)$ contains a countable infinite set of bi-étale algebraic correspondences of W, where d_W denotes the dimension of W. Here an algebraic correspondence of W is understood to be an algebraic subvariety V of W^2 whose projections on both factors are finite and surjective; it is said to be bi-étale if moreover these projections are étale.*

As any finite product of semi-Abelian varieties is a semi-Abelian variety and a finite product of Shimura varieties is a Shimura variety, one easily checks that the pre-special structure on a semi-Abelian variety or a Shimura variety is special. The only non-trivial condition is (c): the existence of infinitely many bi-étale algebraic correspondences is provided by endomorphisms of a semi-abelian variety, and Hecke correspondences of a Shimura variety.

An abstract Manin-Mumford-André-Oort type conjecture in the format defined above is one of the following equivalent statements (1) or (2) below (the equivalence follows from the properties (ii) and (iii) of pre-special varieties):

Conjecture 3.3. *(Abstract Manin-Mumford-André-Oort) Let S be a complex quasi-projective variety endowed with a special structure.*

(1) Let Z be an irreducible algebraic subvariety of S containing a Zariski-dense set of special points. Then Z is a special subvariety of S.

(2) Let Z be an algebraic subvariety of S. The set of special subvarieties of S contained in Z and maximal for these properties is finite.

Conjecture 3.3 in the case of semi-abelian varieties is the classical Manin-Mumford conjecture, while we recover the André-Oort Conjecture 1.2 in the case of Shimura varieties.

Remark 3.4. Notice that any semi-Abelian variety can be realized as a subvariety of a mixed Shimura variety. However only the ones whose Abelian part is CM can be realized a special subvarieties of a mixed Shimura variety. Hence the André-Oort Conjecture 1.2 implies the Manin-Mumford conjecture only for such semi-Abelian varieties. In [Zil02] and [Pink05], Zilber and Pink propose a general conjecture (now called the Zilber-Pink conjecture) about atypical intersections in mixed Shimura varieties, which implies both the Manin-Mumford and the André-Oort conjecture. We refer the reader to the volume [Panorama] for an exposition of the Zilber-Pink conjecture.

3.3. **Weakly special subvarieties.** This section relates special structures and Kähler geometry.

Notice first that any semi-Abelian variety A is endowed with an essentially canonical Kähler metric coming from the flat Euclidean metric on its uniformization $\mathbb{C}^n$. Define a *weakly special* subvariety of A as an irreducible algebraic subvariety whose smooth locus is totally geodesic in A. Thus special subvarieties are weakly special, and a weakly special subvariety is special if and only if it contains a special point.

Similarly, a connected pure Shimura variety S (assumed to be smooth) inherits an essentially canonical Kähler metric from its universal cover X^+: any locally symmetric Kähler metric on the Hermitian symmetric space X^+ is invariant under Γ hence descends to $S = \Gamma\backslash X^+$. Notice that the locally symmetric Kähler metric on X^+ is unique (up to a scalar) if X^+ is irreducible as a symmetric space: it coincides with the Bergman metric of the bounded Harish-Chandra realization of X^+.

Define once more a *weakly special* subvariety of S as an irreducible algebraic subvariety whose smooth locus is totally geodesic in S. Every special subvariety of S is easily seen to be weekly special. Similarly to the case of semi-Abelian varieties, Moonen [Moo98,I] proved:

Theorem 3.5. *Let S be a pure connected Shimura variety. A weakly special subvariety of S is special if and only if it contains a special point.*

More precisely: let $(\mathbf{H}, X_{\mathbf{H}})$ be a sub-Shimura datum of the Shimura datum $(\mathbf{G}, X)$ defining S. Assume that the adjoint Shimura datum $(\mathbf{H}^{\mathrm{ad}}, X_{\mathbf{H}^{\mathrm{ad}}})$ splits as a product:

$$(\mathbf{H}^{\mathrm{ad}}, X_{\mathbf{H}^{\mathrm{ad}}}) = (\mathbf{H}_1, X_1) \times (\mathbf{H}_2, X_2) \ .$$

Let x_2 be a point of X_2 and Z the image of $X_1^+ \times x_2$ in S. Then Z is weakly special, and Z is special if and only if x_2 is a special point of X_2. Conversely any weakly special subvariety of S is obtained in this way.

When S is a general mixed Shimura variety, Pink [Pink05, def.4.1] defines the weakly special subvarieties of S in terms of mixed Shimura data. Once more the special subvarieties are exactly the weakly special ones containing one special point.

4. Bi-algebraic geometry and the Ax-Lindemann property.

4.1. **Complex bi-algebraic geometry.** Let X and S be (connected) complex algebraic varieties and suppose $\pi : X^{\mathrm{an}} \longrightarrow S^{\mathrm{an}}$ is a complex analytic, *non-algebraic*, morphism between the associated complex analytic spaces. In this situation the image $\pi(Y)$ of a generic algebraic subvariety $Y \subset X$ is usually highly transcendental and the pairs $(Y \subset X, V \subset S)$ of irreducible algebraic subvarieties such that $\pi(Y) = V$ are rare and of particular geometric significance. We are especially interested in the case where X is the universal cover $\tilde{S}$ of S. In this case however, the requirement that $\tilde{S}$ is a complex algebraic variety is too restrictive for practical purposes. We relax it as follows:

Definition 4.1. *A bi-algebraic structure on a connected complex algebraic variety S is a pair*

$$(D : \tilde{S} \longrightarrow \hat{X}, \quad h : \pi_1(S) \longrightarrow \mathrm{Aut}(\hat{X}))$$

where $\tilde{S}$ denotes the universal cover of S, $\hat{X}$ is a complex algebraic variety, $\mathrm{Aut}(\hat{X})$ *its group of algebraic automorphisms, $h : \pi_1(S) \longrightarrow \mathrm{Aut}(\hat{X})$ is a group morphism (called the holonomy representation) and D is a local biholomorphism (called the developing map).*

Definition 4.2. *Let S be a connected complex algebraic variety S endowed with a bi-algebraic structure (D, h).*

(i) *An irreducible analytic subvariety $Y \subset \tilde{S}$ is said to be an irreducible algebraic subvariety of $\tilde{S}$ if $D(Y)$ is open (for the analytic topology) in its Zariski-closure $D(Y)^{\mathrm{Zar}} \subset \hat{X}$.*

(ii) *An irreducible algebraic subvariety $Y \subset \tilde{S}$, resp. $W \subset S$, is said to be* bi-algebraic *if $\pi(Y)$ is an algebraic subvariety of S, resp. any (equivalently one) analytic irreducible component of $\pi^{-1}(W)$ is an irreducible algebraic subvariety of $\tilde{S}$.*

Example 4.3. (Tori)
The paradigm of a bi-algebraic structure is provided by the multi-exponential

$$\pi := (\exp(2\pi i \cdot), \ldots, \exp(2\pi i \cdot)) : \mathbb{C}^n \longrightarrow (\mathbb{C}^*)^n \ .$$

In this case $\tilde{S} = \hat{X} = \mathbb{C}^n$ and D is the identity morphism. One easily shows that an irreducible algebraic subvariety $Y \subset \mathbb{C}^n$ (resp. $W \subset (\mathbb{C}^*)^n$)) is bi-algebraic if and only if Y is a translate of a rational linear subspace of $\mathbb{C}^n = \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{C}$ (resp. W is a translate of a subtorus of $(\mathbb{C}^*)^n$).

For the choice of the factor $2\pi i$ in the exponential, see Section 4.2.

Example 4.4. (Abelian varieties)
Let $\pi : \mathrm{Lie}A \simeq \mathbb{C}^n \longrightarrow A$ be the uniformizing map of a complex Abelian variety A of dimension n. Once more $\tilde{S} = X = \mathbb{C}^n$ and D is the identity morphism. One checks that an irreducible algebraic subvariety $W \subset A$ is bi-algebraic if and only if W is the translate of an Abelian subvariety of A (cf. [UY11, prop. 5.1] for example).

Example 4.5. (Semi-abelian varieties)
Any semi-abelian variety admits a canonical semi-algebraic structure generalizing Example 4.3 and Example 4.4 (we leave the details to the reader).

Example 4.6. (Shimura varieties)
Let $S = \Gamma\backslash X^+$ be a connected pure Shimura variety associated to a Shimura datum $(\mathbf{G}, X)$ (with the notations of Section 2). For simplicity we assume that Γ is torsion-free, equivalently that S is smooth (the meticulous reader will easily extend Definition 4.1 and Definition 4.2 to the orbifold case). Hence $\pi : X^+ \longrightarrow S$ is the universal cover of S. Fix a faithful algebraic representation $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(V)$. As X is a $\mathbf{G}(\mathbb{R})$-conjugacy class of morphisms from $\mathbf{S}$ to $\mathbf{G}_{\mathbb{R}}$, any point $x \in X^+$ defines a morphism $\rho \circ x : \mathbf{S} \longrightarrow \mathbf{GL}(V)$, i.e. a Hodge structure V_x on V. Let $F_x^\bullet$ be the corresponding Hodge filtration on $V_{\mathbb{C}}$. The Borel embedding $D : X^+ \longrightarrow \hat{X}$ associates to a point $x \in X^+$ the filtration F_x in the complex algebraic flag variety $\hat{X}$ parametrizing filtrations of $V_{\mathbb{C}}$ of a given type. This is an open holomorphic embedding of X^+ in its dual compact space of X^+. The flag variety $\hat{X}$ is homogeneous under the algebraic action of $\mathbf{G}^{\mathrm{ad}}(\mathbb{C})$ and the open embedding D is equivariant under the natural inclusion $h : \Gamma \hookrightarrow \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ \hookrightarrow \mathbf{G}^{\mathrm{ad}}(\mathbb{C})$, hence (D, h) defines a bi-algebraic structure on S.

The identification of the bi-algebraic varieties for this bi-algebraic structure is due to Ullmo and Yafaev [UY11]:

Theorem 4.7. *Let S be a pure connected Shimura variety endowed with its canonical bi-algebraic structure. The bi-algebraic subvarieties of S are the weakly special ones.*

Let us indicate the proof of Theorem 4.7, which illustrates typical reduction steps and monodromy arguments.

Let $(\mathbf{G}, X)$ be the Shimura datum defining S (hence S is a connected component of the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$, for some compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_{\mathrm{f}})$).

Any weakly special subvariety W of S is an algebraic subvariety of S image under $\pi : X^+ \longrightarrow S = \Gamma\backslash X^+$ of a totally geodesic Hermitian subdomain $X^+_{\mathbf{H}} \subset X^+$. As $X^+_{\mathbf{H}}$ is the intersection of the algebraic subvariety $\hat{X}_{\mathbf{H}} \subset \hat{X}$ with X^+, the weakly special W is bi-algebraic.

Conversely we want to show that any bi-algebraic subvariety of S is weakly special. Let $W \subset S$ be an algebraic subvariety.

- Replacing if necessary S by its smallest special subvariety containing W, we can assume without loss of generality that W is Hodge generic in S.

- The morphism $\psi : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ from $\mathbf{G}$ to its adjoint group extends to a morphism of Shimura data $\psi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$. Let $K^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$ be a compact open subgroup containing the image of K. We thus have a morphism of Shimura varieties $\psi : \mathrm{Sh}_K(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, K^{\mathrm{ad}})$. In this situation one immediately checks that W is weakly special if and only if $\psi(W)$ is weakly special. Moreover as the connected components of X and X^{ad} coincide, W is bi-algebraic if and only if $\psi(W)$ is bi-algebraic. Hence we can assume that $\mathbf{G}$ is adjoint.

- Changing the level if necessary we can also assume without loss of generality that K is sufficiently small so that S is smooth.

Fix a faithful rational representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(V)$ and an integral structure $V_{\mathbb{Z}} \subset V$ such that $\Gamma \subset \mathbf{GL}(V_{\mathbb{Z}})$. This defines a polarized $\mathbb{Z}$-variation of Hodge structures $\mathbb{V}$ on S. Let $\rho : \pi_1(W^{\mathrm{sm}}) \longrightarrow \Gamma \subset \mathbf{GL}(V_{\mathbb{Z}})$ be the monodromy representation of the induced variation on the smooth locus W^{sm} of W and $\Gamma_W := \rho(\pi_1(W^{\mathrm{sm}}))$. Let $\widetilde{W} \subset X^+$ be an analytic irreducible component of $\pi^{-1}(W)$. Hence the group Γ_W is exactly the stabilizer of $\widetilde{W}$ in Γ.

Suppose from now on that W is bi-algebraic. Hence $\widetilde{W} \subset X^+$ is algebraic of the form $\hat{W} \cap X^+$, where $\hat{W} \subset \hat{X}$ is the Zariski-closure of W in $\hat{X}$. In particular $\hat{W}$ is stabilized by the algebraic monodromy $\mathbf{G}_1$, connected component of the Zariski-closure of Γ_W in $\mathbf{G}$. Recall the following result of Deligne (generalized by André [An92] in the mixed case):

Theorem 4.8. *Let $\mathbb{V}$ be an admissible variation of mixed Hodge structures on a smooth quasi-projective variety S with generic Mumford-Tate group $\mathbf{G}$.*

(i) *The algebraic monodromy group $\mathbf{G}_1 \subset \mathbf{G}$ is a normal subgroup of the derived group $\mathbf{G}^{\mathrm{der}}$.*
(ii) *If moreover S contains a CM-point then $\mathbf{G}_1 = \mathbf{G}^{\mathrm{der}}$.*

Applying (i) and as $\mathbf{G}$ is adjoint, we obtain a decomposition of Shimura data

$$(\mathbf{G}, X) = (\mathbf{G}_1, X_1) \times (G_2, X_2)$$

and one checks that W is the π- image of $X_1^+ \times x_2$ for a Hodge generic point $x_2 \in X_2^+$. If follows from Moonen's Theorem 3.5 that W is weakly special.

□

The construction of a natural bi-algebraic structure on a pure Shimura variety extends to mixed Shimura varieties, as well as the identification of bi-algebraic subvarieties with weakly special ones (see [Gao16]).

4.2. **$\overline{\mathbb{Q}}$-bi-algebraic geometry.** Let S be a complex algebraic variety with a bi-algebraic structure as in Section 4.1. While positive dimensional bi-algebraic subvarieties are usually rare and of geometric significance, any point of S is bi-algebraic in the sense of Definition 4.2. To obtain a more meaningful definition of bi-algebraic points we refine Definition 4.1 as follows:

Definition 4.9. *A $\overline{\mathbb{Q}}$-bi-algebraic structure on a complex algebraic variety S is a complex bi-algebraic structure $(D : \tilde{S} \longrightarrow \hat{X}, h : \pi_1(S) \longrightarrow \mathrm{Aut}(\hat{X}))$ such that:*

(1) *S is defined over $\overline{\mathbb{Q}}$.*
(2) *$\hat{X} = \hat{X}_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ and the homomorphism h takes value in $\mathrm{Aut}_{\overline{\mathbb{Q}}} X_{\overline{\mathbb{Q}}}$.*

Definition 4.10. *Let (D, h) be a $\overline{\mathbb{Q}}$-bi-algebraic structure on S. A point $s \in S(\mathbb{C})$ is said to be an* arithmetic bi-algebraic point *if $s \in S(\overline{\mathbb{Q}})$ and any (equivalently one) π-pre-image $\tilde{s} \in \tilde{S}$ satisfies $D(\tilde{s}) \in \hat{X}_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$.*

Let us emphasize that the choice of the $\overline{\mathbb{Q}}$-structure on $\hat{X}$ and the normalization of the developing map D crucially determines the existence of a large supply of arithmetic bi-algebraic points.

Example 4.11. (Tori)
If we endow $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ with their standard rational structure $\mathbb{Q}^n$ and $(\mathbb{Q}^*)^n$, the arithmetic bi-algebraic points of $(\mathbb{C}^*)^n$ for the $\overline{\mathbb{Q}}$-bi-algebraic structure defined in Example 4.3 are exactly the torsion points. Indeed, without loss of generality we can assume $n = 1$. The Gelfond-Schneider theorem [Ge60] states that if α and β are complex numbers such that $\alpha \neq 0$ and e^α, β and $e^{\alpha\beta}$ are all in $\overline{\mathbb{Q}}$ then $\beta \in \mathbb{Q}$. Applying to $\alpha = 2\pi i$, we see that $\beta \in \mathbb{C}^*$ is bi-algebraic if and only if it is of the form $\exp(2\pi i\beta)$ with $\beta \in \mathbb{Q}$, i.e. α is a torsion point.

Notice that if we had chosen for the uniformization map the usual exponential $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ rather than $\exp(2\pi i\cdot) : \mathbb{C} \longrightarrow \mathbb{C}^*$ (keeping the same rational structures $\mathbb{Q} \subset \mathbb{C}$ and $\mathbb{Q}^* \subset \mathbb{C}^*$), or if we had kept the same uniformization map but chosen the rational structure $\mathbb{Q}(1)$ of $\mathbb{C}$, the only arithmetic bi-algebraic point for $\mathbb{C}^*$ would have been 1 by the Hermite-Lindemann theorem [Ge60].

Example 4.12. (Abelian varieties with CM)
In the setting of Example 4.4, suppose from now on that A is an Abelian variety over $\overline{\mathbb{Q}}$. If we define a $\overline{\mathbb{Q}}$-bi-algebraic structure on $A_{\mathbb{C}}$ by choosing the standard $\overline{\mathbb{Q}}$-model $\mathrm{Lie}(A_{\overline{\mathbb{Q}}})$ of $\mathrm{Lie}(A_{\mathbb{C}})$, the unique bi-algebraic point of $A_{\mathbb{C}}$ is the identity (see [Lang66, thm.3 p.28]).

When A is a complex Abelian variety of dimension g with CM (hence A is in particular defined over $\overline{\mathbb{Q}}$) one can consider a better $\overline{\mathbb{Q}}$-structure on $\mathrm{Lie}(A_{\mathbb{C}})$: in this case the lattice of periods $\Gamma := \mathrm{Ker}\,\pi \subset \mathrm{Lie}(A)$ generates a $\overline{\mathbb{Q}}$-vector space $V_{\overline{\mathbb{Q}}} \subset \mathrm{Lie}(A)$ of dimension g, hence defines a $\overline{\mathbb{Q}}$-structure on $\mathrm{Lie}(A)$. In [Ma76] Masser proved:

Theorem 4.13. *(Masser) Let A be a complex Abelian variety of dimension g with CM. Let $V_{\overline{\mathbb{Q}}} \subset \mathrm{Lie}(A)$ be the $\overline{\mathbb{Q}}$-vector space generated by the lattice of periods Γ. Arithmetic bi-algebraic points for this $\overline{\mathbb{Q}}$-bi-algebraic structure on A are exactly the torsion points of A.*

Example 4.14. (Semi-abelian varieties whose Abelian part has CM) Example 4.11 and Example 4.12 can be combined to define a $\overline{\mathbb{Q}}$-bi-algebraic structure on any semi-abelian variety whose Abelian part has CM. Once more the arithmetic bi-algebraic points are the torsion points. We leave the details to the reader.

Example 4.15. (Shimura varieties)
Let $(\mathbf{G}, X)$ be a pure Shimura datum and $K \subset \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ a compact open subgroup. A fundamental result of the theory of Shimura varieties is that the complex quasi-projective variety $\mathrm{Sh}_K(\mathbf{G}, X)$ is defined over a number field $E(\mathbf{G}, X)$ (called the reflex field) depending only on the Shimura datum $(\mathbf{G}, X)$. It follows that any pure connected Shimura variety $S = \Gamma\backslash X^+$, connected component of $\mathrm{Sh}_K(\mathbf{G}, X)$, is defined over an Abelian extension of $E(\mathbf{G}, X)$.

With the notations of Section 2, the flag variety $\hat{X}$ is naturally defined over $\mathbb{Q}$ as V is. This defines a $\overline{\mathbb{Q}}$-bi-algebraic structure on S. The arithmetic bi-algebraic points of S for this $\overline{\mathbb{Q}}$-bi-algebraic structure on S are the points of $S(\overline{\mathbb{Q}})$ whose pre-images lie in $X^+ \cap \hat{X}(\overline{\mathbb{Q}})$. An easy argument given in [UY11, section 3.4] shows that special points are always arithmetic bi-algebraic points.

What about the converse? When $(\mathbf{G}, X) = (\mathbf{GL}_2, \mathcal{H}^{\pm})$ and S is the modular curve $Y(1) \simeq \mathbb{C}$, Schneider's theorem [Schn37] states that if $\tau \in \mathcal{H} \cap \overline{\mathbb{Q}}$ and $x = j(\tau) \in \overline{\mathbb{Q}}$ then τ is imaginary quadratic i.e. x is a CM-point. Hence the bi-algebraic points are exactly the special points.

Cohen [Co96] and Shiga-Wolfart [ShWo95] generalize this result to $\mathcal{A}_g$. A formal argument generalize their result to Shimura varieties of Abelian type:

Theorem 4.16. *(Cohen, Shiga, Wolfart) A point $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$ is an arithmetic bi-algebraic point if and only if it is special.*

More generally let $(\mathbf{G}, X)$ be a Shimura datum of Abelian type, $K \subset \mathbf{G}(\mathbb{A}_f)$ a compact open subgroup and S a connected component of $\mathrm{Sh}_K(\mathbf{G}, X)$ endowed with the $\overline{\mathbb{Q}}$-bi-algebraic structure defined above. A point of S is bi-algebraic if and only if it is special.

Using Example 4.14, both the definition of a natural $\overline{\mathbb{Q}}$-bi-algebraic structure and Theorem 4.16 extend to mixed Shimura varieties whose pure part is of Abelian type.

Remark 4.17. It is worth underlining that all *numerical* transcendence results used to define interesting $\overline{\mathbb{Q}}$-bi-algebraic structures are subsumed in the fundamental analytic subgroup theorem of Wüstholz [Wus89]:

Theorem 4.18. *Let $\mathbf{G}$ be a commutative algebraic group over $\overline{\mathbb{Q}}$ with Lie algebra $\mathfrak{g}$ and $\exp : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbf{G}(\mathbb{C})$ its complex exponential map. Let $\mathfrak{b} \subset \mathfrak{g}$ be a $\overline{\mathbb{Q}}$-vector subspace of positive dimension and $B := \exp(\mathfrak{b} \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$.*

Then $B \cap \mathbf{G}(\overline{\mathbb{Q}}) \neq 0$ if and only if there exists a positive dimensional $\overline{\mathbb{Q}}$-algebraic subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{H}(\mathbb{C}) \subset B$.

4.3. **Special structures and bi-algebraic structures.**

Definition 4.19. *A special structure on a complex algebraic variety S is of bi-algebraic origin if S admits a bi-algebraic structure such that the special subvarieties of S are its bi-algebraic subvarieties containing a special point. Such a bi-algebraic structure is said to underly the special structure.*

A special structure on a complex algebraic variety S is said to be of $\overline{\mathbb{Q}}$-bi-algebraic origin if it admits an underlying $\overline{\mathbb{Q}}$-bi-algebraic structure whose arithmetic bi-algebraic points are the special points.

Thus the special structures we defined on semi-abelian varieties and mixed Shimura varieties are of bi-algebraic origin. If moreover the Abelian part of the semi-Abelian variety has CM or the pure part of the mixed Shimura variety is of Abelian type, it follows from Example 4.14 and Example 4.15 that the special structure is of $\overline{\mathbb{Q}}$-bi-algebraic origin.

4.4. **The Ax-Lindemann conjecture.**

In the abstract context of bi-algebraic geometry, the Ax-Lindemann conjecture is the following functional transcendence statement:

Conjecture 4.20. *(abstract Ax-Lindemann): Let S be an irreducible algebraic variety endowed with a bi-algebraic structure. For any irreducible algebraic subvariety $Y \subset \tilde{S}$, the Zariski-closure $\overline{\pi(Y)}^{\mathrm{Zar}}$ is a bi-algebraic subvariety of S.*

Equivalently: for any algebraic subvariety $V \subset S$, any irreducible algebraic subvariety Y of X contained in $\pi^{-1}(V)$ and maximal for this property is bi-algebraic.

Example 4.21. (semi-Abelian varieties) Ax [Ax71] showed that the abstract Ax-Lindemann conjecture is true for any semi-Abelian variety endowed with the bi-algebraic structure of Example 4.5:

Theorem 4.22. *(Ax) Let A be a semi-Abelian variety and $\pi : \mathbb{C}^n \longrightarrow A$ its uniformization. For any irreducible algebraic subvariety $Y \subset \mathbb{C}^n$, the Zariski-closure $\overline{\pi(Y)}^{\mathrm{Zar}}$ of $\pi(Y)$ is the translate of an algebraic subgroup of A.*

Remark 4.23. Notice that Ax's theorem for $\pi := (\exp(2\pi i\cdot), \ldots, \exp(2\pi i\cdot)) : \mathbb{C}^n \longrightarrow (\mathbb{C}^*)^n$ is the functional analog of the classical Lindemann transcendence theorem stating that if $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$. This explain our terminology.

Example 4.24. (Shimura variety)

Theorem 4.25. *(Hyperbolic Ax-Lindemann) Let $\pi : X \longrightarrow S$ be the uniformization map of a connected mixed Shimura variety. We endow S with the bi-algebraic structure of Example 4.6.*

- (i) *For any irreducible algebraic subvariety $Y \subset X$, the Zariski-closure $\overline{\pi(Y)}^{\mathrm{Zar}}$ of $\pi(Y)$ is weakly special.*
- (ii) *Equivalently, let W be an algebraic subvariety of S. Irreducible algebraic subvarieties of X contained in $\pi^{-1}W$ and maximal for this property are precisely the irreducible components of the pre-images of maximal weakly special subvarieties of S contained in W.*

5. O-minimal geometry and the Pila-Wilkie's theorem

5.1. **O-minimal structures.** For a more detailed treatment of o-minimality we refer to [vdD98], [PW06], [PetStar10], [Pil] and [Sca16].

Definition 5.1. *A structure $\mathbf{S}$ is a collection $\mathbf{S} = (S_n)_{n\in\mathbb{N}}$, where S_n is a set of subsets of $\mathbb{R}^n$, called the* definable sets *of the structure, such that for every $n \in \mathbb{N}$:*

(1) *all algebraic subsets of $\mathbb{R}^n$ are in S_n.*
(2) *S_n is a boolean subalgebra of the power set of $\mathbb{R}^n$.*
(3) *If $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$.*
(4) *Let $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ be a linear projection. If $A \in S_{n+1}$ the $p(A) \in S_n$.*

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be definable if its graph is.

A dual point of view starts from the functions, namely considers sets definable in a first-order structure

$$\langle \mathbb{R}, +, \times, <, (f_i)_{i\in I} \rangle$$

where I is a set and the $f_i : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$, $i \in I$, are functions. A subset $Z \subset \mathbb{R}^n$ is definable if it can be defined by a formula

$$Z := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \ / \ \phi(x_1, \cdots, x_n) \text{ is true}\} \ ,$$

where ϕ is a first-order formula that can be written using only the quantifiers $\forall$ and $\exists$ applied to real variables, logical connectors, algebraic expressions written with the f_i's, $<$ and fixed parameters $\lambda_i \in \mathbb{R}$. When the set I is empty the definable subsets are the semi-algebraic sets. Semi-algebraic subsets are thus always definable.

The o-minimal axiom for a structure $\mathbf{S}$ guarantees the possibility of doing geometry using definable sets as basic blocks. In particular it excludes Cantor sets from $\mathbf{S}$:

Definition 5.2. *A structure $\mathbf{S}$ is said to be o-minimal if the definable subsets of $\mathbb{R}$ are precisely the finite unions of points and intervals (i.e. the semi-algebraic subsets of $\mathbb{R}$).*

Example 5.3. The structure $\mathbb{R}_{\text{sin}} := \langle \mathbb{R}, +, \times, <, \sin \rangle$ is not o-minimal. Indeed the infinite union of points $\pi\mathbb{Z} = \{x \in \mathbb{R}, \ \sin x = 0\}$ is a definable subset of $\mathbb{R}$ in this structure.

A deep theorem of Wilkie [Wil96] states:

Theorem 5.4. *The structure $\mathbb{R}_{\exp} := \langle \mathbb{R}, +, \times, <, \exp \rangle$ is o-minimal.*

Definition 5.5. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a restricted analytic function if it is zero outside $[0,1]^n$ and if there exists a real analytic function g on a neighbourhood of $[0,1]^n$ such that f and g are equal on $[0,1]^n$.*

One defines $\mathbb{R}_{\text{an}} := \langle \mathbb{R}, +, \times, <, \{f\} \text{ for } f \text{ restricted analytic function} \rangle$.

A theorem of Van den Dries based on Gabrielov's results [Ga68] shows:

Theorem 5.6. *The structure $\mathbb{R}_{\text{an}}$ is o-minimal.*

In diophantine geometry we will use the structure

$$\mathbb{R}_{\text{an,exp}} := \langle \mathbb{R}, +, \times, <, \exp, \{f\} \text{ for } f \text{ restricted analytic function} \rangle$$

generated by $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\exp}$. The structure generated by two o-minimal structures is not o-minimal in general, but Van den Dries and Miller [vdDM85] prove in this case:

Theorem 5.7. *The structure $\mathbb{R}_{\text{an,exp}}$ is o-minimal.*

5.2. **Pila-Wilkie's counting theorem.** Let H denote the standard multiplicative height function on $\overline{\mathbb{Q}}$. Thus if L is a number field, M_L its set of places, and $x \in L$ then

$$H(x) := \prod_{v \in M_L} \max(1, |x|_v) \ .$$

We also denote by H its extension to $\overline{\mathbb{Q}}^n$ defined by $H(x_1, \cdots, x_n) := \max_i H(x_i)$. Given a subset $Z \subset \mathbb{R}^n$, a positive integer d and a real number T we define

$$\Theta_d(Z,T) := \{(x_1, \cdots, x_n) \in Z \ / \ \max_i [\mathbb{Q}(x_i) : \mathbb{Q}] \leq d \text{ and } H(x_1, \cdots, x_n) \leq T\} \ , \tag{5.1}$$

and

$$N_d(Z,T) := |\Theta_d(Z,T)| \ . \tag{5.2}$$

Definition 5.8. *Let $Z \subset \mathbb{R}^n$. We denote by Z^{alg} the union of all positive dimensional semi-algebraic subsets of $\mathbb{R}^n$ contained in Z.*

Theorem 5.9. *(Pila-Wilkie [PW06]) Let $Z \subset \mathbb{R}^n$ be a subset definable in an o-minimal expansion of $\mathbb{R}$ (typically: $\mathbb{R}_{\mathrm{an,exp}}$). Let d be a positive integer and ε a positive real number. There exists a constant $c = c(Z, d, \varepsilon)$ such that*

$$\forall T > 0, \quad N_d(Z - Z^{\mathrm{alg}}, T) \leq c \cdot T^{\varepsilon} \quad . \tag{5.3}$$

In particular if there exists $\alpha > 0$ and $c' = c'(d, Z) > 0$ such that for any T sufficiently large we have $N_d(Z, T) \geq c' \cdot T^{\alpha}$ then Z^{alg} is non-empty.

Example 5.10. Let $Z \subset I^2$ be the intersection of a real analytic curve C defined in a neighborhood of I^2 with I^2 (where $I = [0, 1]$). Hence Z definable in $\mathbb{R}_{\mathrm{an}}$. Suppose that there exist a positive integer d, and real numbers $\alpha > 0$ and $c' = c'(d, Z) > 0$ such that for any T sufficiently large we have $N_d(Z, T) \geq c' \cdot T^{\alpha}$. Then the real analytic curve C is real algebraic.

Theorem 5.9 can be refined in two directions, which are used in the Pila-Zannier strategy. The first refinement uses the notion of semi-algebraic bloc.

Definition 5.11. *A semi-algebraic bloc W in $\mathbb{R}^n$ for an o-minimal expansion of $\mathbb{R}$ is a connected definable subset of $\mathbb{R}^n$, regular at every point, such that there exists a connected positive dimensional semi-algebraic set $B \subset \mathbb{R}^n$ containing W and which coincide with W in the neighbourhood of every point of W. In particular a semi-algebraic bloc is covered by open semi-algebraic sets.*

Example 5.12. Let $W := \{(x, y) \in \mathbb{R}^2, y < \exp(x)\}$. This is a semi-algebraic bloc of $\mathbb{R}_{\exp}$ with $B = \mathbb{R}^2$.

Theorem 5.13. *Let $Z \subset \mathbb{R}^n$ be a subset definable in an o-minimal expansion of $\mathbb{R}$. Let d be a positive integer and ε a positive real number. There exists a constant $c = c(Z, d, \varepsilon)$ such that $\Theta_d(Z, T)$ is contained in at most $c \cdot T^{\varepsilon}$ semi-algebraic blocs contained in Z.*

The second refinement deals with families.

Definition 5.14. *A definable family $Z := \{Z_b\}_{b \in B}$ of subsets of $\mathbb{R}_n$ is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m$ whose projection on the second factor is $B \subset \mathbb{R}^m$.*

In this case every fiber $Z_b \subset \mathbb{R}^n$ for $b \in B$ is definable.

Theorem 5.15. *Let $Z := \{Z_b\}_{b \in B}$ be a definable family of subsets of $\mathbb{R}^n$ in an o-minimal expansion of $\mathbb{R}$. Let ε be a positive real number. There exists a constant $c := c(\varepsilon, Z)$ and a definable family $Y := \{Y_b\}_{b \in B}$ of subsets of $\mathbb{R}^n$ such that, for every $b \in B$, one has the inclusion $Y_b \subset Z_b^{\mathrm{alg}}$ and*

$$N_d(Z_b - Y_b, T) \leq c \cdot T^{\varepsilon} \quad . \tag{5.4}$$

Remarks 5.16.
(a) The crux of this refinement is the uniformity (the constant c does not depend of $b \in B$).
(b) The use of the definable family $\{Y_b\}_{b \in B}$ is needed as the set Z^{alg} associated to a definable set Z is usually not definable. Consider for example [Sca16, Rem. 4.5] the $\mathbb{R}_{\exp}$-definable subset of $\mathbb{R}^3$ defined as

$$Z := \{(x, y, z) \in \mathbb{R}_+^3 \; ; \; z = x^y\}$$

whose algebraic part Z^{alg} is the union of triples $(x, y, z) \in Z$ such that $y \in \mathbb{Q}$.

The proof of Theorem 5.9 and its refinements relies on a reparametrization theorem generalizing a result of Gromov and Yomdin for semi-algebraic sets:

Theorem 5.17. *Let r be an integer. Let $Z \subset (0,1)^n$ be a definable set in an o-minimal expansion of $\mathbb{R}$, of dimension m. There exists a finite set $I := I(Z,r)$, uniformly bounded when Z varies in a definable family, such that*

$$Z = \bigcap_{i \in I} \phi_i((0,1)^m)$$

where $\phi_i : (0,1)^m \longrightarrow (0,1)^n$ is of class $\mathcal{C}^r$ and $|\partial_\alpha \phi_i| \leq 1$ for any multi-index α of length $|\alpha| \leq r$.

6. O-minimality and Shimura varieties

We will not pursue here how to use o-minimality in the general context of special structures of bi-algebraic origin. From now on we restrict ourselves to the context of (mixed) Shimura varieties.

6.1. **Definability of π restricted to a fundamental set.** Let $\pi : X^+ \longrightarrow S := \Gamma \backslash X^+$ be the uniformization of a connected mixed Shimura variety S. The realization $X^+ \subset \hat{X}$ defines X^+ as a real semi-algebraic subset of $\hat{X}$. Of course the map π cannot be definable in any o-minimal structure as it is periodic under the infinite group Γ. We remove this difficulty by restricting π to a fundamental set of X^+ for the action of Γ:

Definition 6.1. *A fundamental set for the action of Γ on X^+ is a connected open subset $\mathcal{F}$ of X^+ such that $\Gamma\overline{\mathcal{F}} = X^+$ and such that the set $\{\gamma \in \Gamma \ |\gamma\mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is finite.*

An essential step for using o-minimal geometry in the context of Shimura varieties is the following result:

Theorem 6.2. *There exists a semi-algebraic fundamental set $\mathcal{F}$ for the action of Γ on X^+ such that the restriction $\pi_{|\mathcal{F}} \colon \mathcal{F} \longrightarrow S$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.*

The special case of Theorem 6.2 when S is pure and compact is easy, see [UY14b, Prop.4.2]. In this case, the map $\pi_{|\mathcal{F}}$ is even definable in $\mathbb{R}_{\mathrm{an}}$. Theorem 6.2 in the case where $X = \mathbf{H}_g$ is the Siegel upper half plane of genus g was proven by Peterzil and Starchenko (see [PetStar13] and [PetStar10]): in this case they use an explicit description for π in terms of θ-functions and delicate computations with these. Notice moreover that this particular case implies Theorem 6.2 for any special subvariety S of $\mathcal{A}_g$ (see Proposition 2.5 of [U14]). On the other hand Peterzil and Starchenko's method does not generalize to general arithmetic varieties, where an explicit description of π is not available. The paper [KUY16] provides a completely geometric proof of Theorem 6.2 for any pure Shimura variety using the general theory of toroidal compactifications of arithmetic varieties (cf. [AMRT75]). Gao generalizes this result to mixed Shimura varieties in [Gao16].

Let us give the proof of Theorem 6.2 in the baby-case of $S = Y(1)$ and $\pi = j : \mathbf{H} \longrightarrow Y(1) = \mathbf{SL}(2,\mathbb{Z}) \backslash \mathbf{H} \simeq \mathbb{C}$. In this case we consider for $\mathcal{F}$ the usual semi-algebraic fundamental set:

$$\mathcal{F} := \{z = x + iy \in \mathbf{H},\ -\frac{1}{2} < x < \frac{1}{2} \text{ and } y > \frac{\sqrt{3}}{2}\}. \tag{6.1}$$

Let us consider the diagram of holomorphic maps:

$$\mathcal{F} \subset \mathbf{H} \overset{z \mapsto e^{2\pi i z}}{\longrightarrow} \Delta^* \overset{q}{\longrightarrow} S = \mathbb{C}\ ,$$

where $\Delta^* := \{z \in \mathbb{C}^*,\ |z| < \exp(-\pi\sqrt{3})\}$. We claim that this composite is definable in $\mathbb{R}_{\text{an,exp}}$. It follows from the following observations:

- $\exp(2\pi i z) = \exp(-2\pi \mathrm{Im}(z)) \cdot \exp(2\pi i \mathrm{Re}(z))$. The first factor is definable in $\mathbb{R}_{\exp}$. On the other hand $\mathrm{Re}(x)$ is bounded on $\mathcal{F}$, hence the second factor restricted to $\mathcal{F}$ is definable in $\mathbb{R}_{\text{an}}$.
- The function $q : \Delta^* \longrightarrow \mathbb{C}$ extends to $\Delta \longrightarrow \mathbb{P}^1\mathbb{C}$ hence is definable in $\mathbb{R}_{\text{an}}$.

For a general pure connected Shimura variety S associated with a Shimura datum $(\mathbf{G}, X)$, the fundamental set $\mathcal{F}$ is a semi-algebraic Siegel set, whose construction we recall now (see [Bor69] for a general reference). Without loss of generality we can assume that $\mathbf{G}$ is semi-simple of adjoint type. Let $\mathbf{P}$ be a minimal $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$ and $K_\infty \subset \mathbf{G}(\mathbb{R})$ a maximal compact subgroup such that $K_\infty \cap \mathbf{P}(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{P}(\mathbb{R})$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$ and let $\mathbf{A}$ be a maximal split torus of $\mathbf{P}$. We denote by $\mathbf{S}$ a maximal split torus of $\mathbf{GL}(V)$ containing $\rho(\mathbf{A})$, by $\mathbf{M}$ the maximal anisotropic subgroup of the connected centralizer $\mathbf{Z}(\mathbf{A})^0$ of $\mathbf{A}$ in $\mathbf{P}$ and by Δ the set of positive simple roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and $\mathbf{P}$. We denote by $A \subset \mathbf{S}(\mathbb{R})$ the real torus $\mathbf{A}(\mathbb{R})$. For any real number $t > 0$ we let

$$A_t := \{a \in A \mid a^\alpha \geq t \text{ for any } \alpha \in \Delta\}\ .$$

A Siegel set for $\mathbf{G}(\mathbb{R})$ for the data $(K_\infty, \mathbf{P}, \mathbf{A})$ is a product:

$$\Sigma'_{t,\Omega} := \Omega \cdot A_t \cdot K_\infty \subset \mathbf{G}(\mathbb{R})$$

where Ω is a compact neighborhood of e in $\mathbf{M}^0(\mathbb{R}) \cdot \mathbf{U}(\mathbb{R})$.

The image

$$\Sigma_{t,\Omega} := \Omega \cdot A_t \cdot x_0 \subset X^+$$

of $\Sigma'_{t,\Omega}$ in X^+ (where x_0 is the point of $X^+ = \mathbf{G}(\mathbb{R})/K_\infty$ fixed under K_∞) is called a Siegel set in X^+.

Theorem 6.3. *There exist a semi-algebraic Ω, a real number t and a finite subset J of $\mathbf{G}(\mathbb{Q})$ such that $\mathcal{F} := J \cdot \Sigma_{t_0,\Omega}$ is a fundamental set for the action of Γ on X^+ satisfying Theorem 6.2.*

6.2. **Heights of special points.** In Example 4.15 we define an $\overline{\mathbb{Q}}$-bi-algebraic structure on any Shimura variety S whose pure part is of Abelian type: special points are exactly the arithmetic bi-algebraic points. A crucial ingredient for applying the Pila-Wilkie's Theorem 5.9 in this context consists in showing that for any special point $x \in S$, the fiber $\pi^{-1}(x)$ consists of algebraic points of X^+ defined over extensions of uniformly bounded degree over $\mathbb{Q}$; moreover one controls the height of points of $\pi^{-1}(x) \cap \mathcal{F}$. For simplicity let us state the result for $S = \mathcal{A}_g$ (the first part is classical, the second is due to Pila and Tsimerman):

Theorem 6.4. (1) *The uniformization $\pi : \mathbf{H}_g \longrightarrow \mathcal{A}_g = \mathbf{Sp}(2g,\mathbb{Z})\backslash\mathbf{H}_g$ can be normalized in such a way that the coordinates of the inverse images by π of CM-points of $\mathcal{A}_g$ lie in algebraic extensions of uniformly bounded degree.*

(2) *One can choose the fundamental set $\mathcal{F}$ in Theorem 6.2 for the action of $\mathbf{Sp}(2g,\mathbb{Z})$ on $\mathbf{H}_g$, and positive real numbers $\alpha = \alpha(g)$ and $c = c(g)$ such that if $x \in \mathcal{A}_g$ is a CM-point parametrizing the Abelian variety A_x and if $\tilde{x} \in \mathcal{F} \cap \pi^{-1}(x)$ then*

$$H(\tilde{x}) \leq c \cdot d_x^{\alpha} \quad ,$$

where H denotes the canonical height on $M_g(\overline{\mathbb{Q}}) \cap \mathbf{H}_g \subset \overline{\mathbb{Q}}^{n^2}$ and d_x is the absolute value of the discriminant of the center of the ring of endomorphisms of A_x.

Let us write explicitly the case of $Y(1)$. Let $\tau \in \mathcal{F}$ where $\mathcal{F}$ denotes the fundamental set defined in Equation (6.1). If the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ has complex multiplication then τ satisfies a reduced equation $aX^2 + bX + c = 0$ for integers a, b and c such that $|b| \leq a \leq c$. In particular the coordinates of

$$\tau = -\frac{b}{2a} + i\frac{\sqrt{4ac-b^2}}{2a}$$

lie in extensions of degree at most 2 of $\mathbb{Q}$. Moreover $\mathrm{End}\,(E_\tau) = \mathbb{Z}[\tau]$ and the absolute value d_τ of the discriminant of $\mathrm{End}\,(E_\tau)$ is $4ac - b^2$. With our conventions on the height:

$$H(\tau) = \max(H(\frac{b}{2a}), H(\frac{\sqrt{4ac-b^2}}{2a})).$$

On the one hand $H(\frac{b}{2a}) = \max(|b|, 2|a|) = 2|a| \leq d_\tau$. On the other hand $\frac{\sqrt{4ac-b^2}}{2a}$ is a root of the integral polynomial $4a^2X^2 - d_\tau$ hence:

$$H(\frac{\sqrt{4ac-b^2}}{2a}) \leq \max(4a^2, d_\tau) \leq \frac{4}{3}d_\tau \quad ,$$

where the last inequality follows by noticing that

$$3a^2 \leq 4ac - b^2 = d_\tau$$

in view of the inequalities satisfied by (a,b,c).

Finally we obtain $H(\tau) \leq \frac{4}{3}d_\tau$.

7. Strategy of the proof of the André-Oort conjecture for $\mathcal{A}_g$

For $x \in \mathcal{A}_g$ we denote by A_x the principally polarized Abelian variety parametrized by x and d_x the absolute value of the discriminant of the center of the ring of endomorphisms of A_x. We denote by $\pi : \mathbf{H}_g \longrightarrow \mathbf{Sp}(2g,\mathbb{Z})\backslash\mathbf{H}_g = \mathcal{A}_g$ the uniformization map.

There are two main steps in the proof of Theorem 1.4:

Theorem 7.1. *Let $W \subset \mathcal{A}_g$ be an algebraic subvariety. There exists a constant $C = C(g,W)$ with the following property. Let x be a special point of $\mathcal{A}_g$ contained in W. If $d_x \geq C$ then there exists a positive dimensional special subvariety Z_x of $\mathcal{A}_g$ contained in W and containing x.*

As there exist only finitely many special points x with d_x smaller than a given constant, Theorem 7.1 implies that there are only finitely many special subvarieties of $\mathcal{A}_g$ contained in W and maximal for these properties which are points. Notice that this proves Theorem 1.4 if W is a curve.

The second step is proven in [U14]:

Theorem 7.2. *Let W be a Hodge generic subvariety of a pure connected Shimura' variety S. We assume that if $S = S_1 \times S_2$ is a product of connected Shimura varieties then W is not of the form $W = S_1 \times W_2$ for a special subvariety W' of S_2.*

Then the set of weakly special positive dimensional subvarieties contained in W is not Zariski-dense in W. In particular the set of positive dimensional special subvarieties contained in W is not Zariski dense in W.

7.1. **Proof of Theorem 7.1.** The main ingredient in the proof of Theorem 7.1 is the following result, due to Tsimerman [Tsi] and based on the results of Andreatta-Goren-Howard-Madapusi Pera [AGHM] and Yuan-Zhang [YuZh] on the Colmez conjecture:

Theorem 7.3. *There exist positive constants $c_1 = c_1(g)$ and β such that for any special point x of $\mathcal{A}_g$ one has:*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| = [\mathbb{Q}(x) : \mathbb{Q}] \geq c_1 d_x^{\beta}. \tag{7.1}$$

We will detail Theorem 7.3 in Section 9.4. For now let us show how Theorem 7.3 and o-minimal technics imply Theorem 7.1.

Let $W \subset \mathcal{A}_g$ be an algebraic subvariety defined over $\overline{\mathbb{Q}}$. Replacing W by the union of its conjugate under $\mathrm{Gal}(\overline{\mathbb{Q}})/\mathbb{Q})$ we can assume without loss of generality that W is defined over $\mathbb{Q}$.

Let $\mathcal{F} \subset \mathbf{H}_g$ be a semi-algebraic fundamental set for the action of $\mathbf{Sp}(2g, \mathbb{Z})$ on $\mathbf{H}_g$ such that $\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{A}_g$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$ (see Theorem 6.2). Hence the set $\widetilde{W_{\mathcal{F}}} := \pi^{-1}(W) \cap \mathcal{F}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Let $x \in V$ be a special point. Notice that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $d_{\sigma \cdot x} = d_x$. It follows from Theorem 6.4 that any point y in $\pi^{-1}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x) \cap \widetilde{W_{\mathcal{F}}}$ is defined in an extension of $\mathbb{Q}$ of uniformly bounded degree and satisfies

$$H(y) \leq c \cdot d_x^{\alpha} \quad . \tag{7.2}$$

It follows from Pila-Wilkie Theorem 5.9 and the inequalities (7.1) and (7.2) that if d_x is sufficiently large, there exists a semi-algebraic subset $Y \subset \widetilde{W_{\mathcal{F}}}$ of positive dimension, containing one point y in $\pi^{-1}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x)$. Let Z be an irreducible algebraic subvariety of $\mathbf{H}_g$ contained in $\pi^{-1}(W)$, containing y, and maximal for these properties. Hence Z is positive dimensional. Moreover it follows from the Ax-Lindemann Theorem 4.25 that $\pi(Z)$ is a special subvariety of $\mathcal{A}_g$ contained in V and containing a Galois conjugate $\sigma \cdot x$ of x. As V is defined over $\mathbb{Q}$ the positive dimensional special subvariety $\sigma^{-1}(\pi(Z))$ of $\mathcal{A}_g$ is contained in V and contains x.

□

7.2. **Proof of Theorem 7.2.** We follow [U16]. Let $\mathcal{E}(W)$ be the set of weakly special subvarieties contained in W. For a positive integer r we denote by $\mathcal{E}_r(W)$ the subset of $\mathcal{E}(W)$ consisting of weakly special subvarieties of real dimension r. Let d be the maximum r such that $\mathcal{E}_r(W)$ is non-empty.

It follows from the description of weakly special subvarieties that there exists a semi-simple group $\mathbf{H}$ of $\mathbf{G}_\mathbb{R}$ and $z_0 \in \mathcal{F}$ such that $\pi(\mathbf{H}(\mathbb{R})) \cdot z_0$ is a weakly special subvariety of W of dimension d. Without loss of generality we can assume that $\mathbf{H}$ has no compact simple real factor.

Let us define

$$B_\mathbf{H} := \{(t,z) \in \mathbf{G}(\mathbb{R}) \times \mathcal{F}, \quad \pi(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z) \subset V\}.$$

It follows from analytic continuation that $B_\mathbf{H}$ can also be described as:

$$B_\mathbf{H} := \{(t,z) \in \mathbf{G}(\mathbb{R}) \times \mathcal{F}, \quad \pi_{|\mathcal{F}}(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z) \subset V\}.$$

As $\pi_{|\mathcal{F}}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$ (see Theorem 6.2) it follows that $B_\mathbf{H}$ is a definable subset of $\mathbf{G}(\mathbb{R}) \times \mathcal{F}$.

Lemma 7.4. *Let $(t,z) \in B_\mathbf{H}$. Then $\pi(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z)$ is a weakly special subvariety of W.*

Proof. Let $(t,z) \in B_\mathbf{H}$. It follows from the definition of $B_\mathbf{H}$ that $t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z$ is a semi-algebraic subset of X^+ whose projection $\pi(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z)$ is contained in W. On the other hand the real dimension of $t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z$ is at least the dimension of $\mathbf{H}(\mathbb{R})^+ \cdot z$, with equality if and only if $\mathrm{Stab}_{\mathbf{G}(\mathbb{R})} \cap t\mathbf{H}(\mathbb{R})^+ t^{-1}$ is a maximal compact subgroup of $t\mathbf{H}(\mathbb{R})^+ t^{-1}$.

Let Y be an irreducible algebraic subvariety of X^+, containing $t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z$, such that $\pi(Y) \subset W$, and maximal for these properties. By the Ax-Lindemann Theorem 4.25, $\pi(Y)$ is weakly special. It follows from the definition of d that:

$$\dim(\pi(Y)) \leq d = \dim(\mathbf{H}(\mathbb{R})^+ \cdot z_0) \leq \dim(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z) \leq \dim(\pi(Y)) \ .$$

Hence $\pi(Y) = \pi(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z)$, hence $\pi(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z)$ is weakly special. $\square$

Lemma 7.5. *The set $C(\mathbf{H}, W)$ of conjugacy classes $t\mathbf{H}(\mathbb{R})^+ t^{-1}$, $t \in \mathbf{G}(\mathbb{R})$, for which there exists $z \in \mathcal{F}$ satisfying $\pi(T\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z) \subset W$, is finite.*

Proof. Consider the map $\psi : B_\mathbf{H} \longrightarrow \mathbf{G}(\mathbb{R})/N_{\mathbf{G}(\mathbb{R})}(\mathbf{H}(\mathbb{R})^+)$ deduced from the projection on the first factor. Hence $C(\mathbf{H}, W)$ is in bijection with $\psi(B_\mathbf{H})$. As $B_\mathbf{H}$ is definable and ψ is algebraic, the image $\psi(B_\mathbf{H})$ is definable. Moreover if $(t,z) \in B_\mathbf{H}$ then $\pi(t\mathbf{H}(\mathbb{R})^+ t^{-1} \cdot z)$ is weakly special by Lemma 7.4. From the description of weakly special subvarieties there exists a $\mathbb{Q}$-algebraic subgroup $\mathbf{H}_{t,\mathbb{Q}} \subset \mathbf{G}$ such that $t\mathbf{H}_\mathbb{R} t^{-1} = \mathbf{H}_{t,\mathbb{R}}^{\mathrm{nc}}$. As the set of $\mathbb{Q}$-algebraic subgroup of $\mathbf{G}$ is countable, it follows that $C(\mathbf{H}, B)$ is countable. Any countable set definable in some o-minimal structure is finite hence $C(\mathbf{H}, W)$ is finite. $\square$

Lemma 7.6. *Under the hypotheses of Theorem 7.2 the union $\bigcup_{V \in \mathcal{E}_d(W)} V$ of the weakly special subvarieties contained in W of maximal dimension d is not Zariski-dense in W.*

Proof. As $\mathbf{G}_\mathbb{R}$ has only finitely many conjugacy classes of semi-simple subgroups, there exists only finitely many (up to $\mathbf{G}(\mathbb{R})$-conjugacy) subgroups $\mathbf{H}_\mathbb{R}$ of $\mathbf{G}_\mathbb{R}$ for which there exists $z_0 \in \mathcal{F}$ with $\pi(\mathbf{H}(\mathbb{R})^+ \cdot z_0) \in \mathcal{E}_d(W)$ and such that $\mathbf{H}_\mathbb{R} = \mathbf{H}_\mathbb{R}^{\mathrm{nc}}$.

For such an $\mathbf{H}_\mathbb{R}$, there exists a semi-simple subgroup $\mathbf{H}_\mathbb{Q} \subset \mathbf{G}$ satisfying $(\mathbf{H}_\mathbb{Q})_\mathbb{R}^{\mathrm{nc}} = \mathbf{H}_\mathbb{R}$ and the number of such $\mathbf{H}_\mathbb{Q}$ is finite by Lemma 7.5.

Let $\mathbf{H}_{\mathbb{Q}}$ be such a subgroup.

If $\mathbf{H}_{\mathbb{Q}}$ is a factor of $\mathbf{G}$ then S decomposes as $S_1 \times S_2$. Any weakly special subvarieties of the form $\pi(\mathbf{H}_{\mathbb{R}})^+ \cdot z)$ with $z \in \mathcal{F}$ is of the form $S_1 \times \{x_2\}$ for some $x_2 \in S_2$. The Zariski-closure of the union of weakly special subvarieties V of the form $\pi(\mathbf{H}_{\mathbb{R}})^+ \cdot z)$ is $S_1 \times W'$, where W' denotes the Zariski-closure of the set of x_2 for which $S_1 \times \{x_2\} \subset W$. As W is not of the form $S_1 \times W'$, this union is not Zariski-dense in W.

If $\mathbf{H}_{\mathbb{Q}}$ is not normal in $\mathbf{G}$, one shows the following:

Proposition 7.7. *Suppose $\mathbf{H}_{\mathbb{Q}}$ is not normal in $\mathbf{G}$. Then the union of weakly special subvarieties of the form $\pi(\mathbf{H}_{\mathbb{R}})^+ \cdot z)$ is contained in a finite union $\bigcup_{1\leq i\leq r} V_i$ of strict special subvarieties V_i of S.*

As W is Hodge generic, the intersection $W \cap \bigcup_{1\leq i\leq r} V_i$ is not Zariski-dense in W. This finishes the proof of Lemma 7.6. □

One concludes the proof of Theorem 7.2 by induction on the dimension of the weakly special subvarieties of S contained in W. Let us indicate the argument.

Let $d_1 < d$ be the maximal dimension of a weakly special subvariety of W not contained in $\mathcal{E}_d(W)$. There exist a semi-simple subgroup $\mathbf{H}_1 = \mathbf{H}_1^{\mathrm{nc}}$ of $\mathbf{G}_{\mathbb{R}}$ and $z_1 \in \mathcal{F}$ such that $\pi(\mathbf{H}_1(\mathbb{R})^+ \cdot z_1) \subset W$ is of dimension d_1 and is not in $\mathcal{E}_d(W)$. Up to $\mathbf{G}(\mathbb{R})$-conjugacy there are only finitely any possibilities fo $\mathbf{H}_1$. The proof of Lemma 7.4 shows that if $(z,t) \in B_{\mathbf{H}_1}$ and if $\pi(t\mathbf{H}_1(\mathbb{R})^+t^{-1} \cdot z)$ is not contained in $\mathcal{E}_d(W)$ then $\pi(t\mathbf{H}_1(\mathbb{R})^+t^{-1} \cdot z)$ is weakly special contained in W. The proof of Lemma 7.5 shows that the set $C(\mathbf{H}_1, W, \mathcal{E}_d(W))$ of conjugacy classes $t\mathbf{H}_1(\mathbb{R})^+t^{-1}$, $t \in \mathbf{G}(\mathbb{R})$, such that there exists $z \in \mathcal{F}$ with $\pi(t\mathbf{H}_1(\mathbb{R})^+t^{-1} \cdot z) \subset W$ and $\pi(t\mathbf{H}_1(\mathbb{R})^+t^{-1} \cdot z)$ does not belong to $\mathcal{E}_d(W)$, is finite. As in the proof of Lemma 7.6 one concludes that the set of weakly special subvarieties of W of dimension at leat d_1 is not Zariski-dense in W.

By decreasing induction on r one concludes $\bigcup_{r\geq 0}\bigcup_{V\in\mathcal{E}_r(W)} V$ is not Zariski-dense in W.

□

8. The hyperbolic Ax-Lindemann conjecture

In this section we give indications on the proof of the hyperbolic Ax-Lindemann Theorem 4.25.

8.1. Equivalence of (i) and (ii). Let us first show that the two statements (i) and (ii) of Theorem 4.25 are indeed equivalent.

We first assume (ii). Let Y be an irreducible algebraic subvariety of X^+. Let W be the Zariski-closure of $\pi(Y)$. Let Z be an irreducible algebraic subvariety of $\pi^{-1}(W)$ containing Y, and maximal for these properties. By hypothesis, $\pi(Z)$ is weakly special, in particular $\pi(Z)$ is irreducible algebraic. As $\pi(Y) \subset \pi(Z) \subset W$, it follows that $\pi(Z) = W$, hence W is weakly special.

Conversely let us assume (i). Let W be an algebraic subvariety of S and Y an irreducible algebraic subvariety of $\pi^{-1}(W)$, maximal for these properties. By hypothesis the Zariski-closure W' of $\pi(Y)$ is weakly special. As $W' \subset W$, there exists an analytic irreducible component Y' of $\pi^{-1}(W')$ containing Y. As W' is weakly special, Y' is irreducible algebraic. By maximality of Y, one obtains $Y = Y'$ and $\pi(Y) = W'$ is weakly special.

8.2. **Stabilizers of maximal algebraic subvarieties of $\pi^{-1}(W)$.** We work with (ii). Let $W \subset S$ be an irreducible algebraic subvariety and $Y \subset \pi^{-1}W$ an irreducible algebraic subvariety of X^+, maximal for these properties. We want to show that $\pi(Y)$ is weakly special. The main intermediate step is the following:

Proposition 8.1. *There exists a connected $\mathbb{Q}$-algebraic subgroup $\mathbf{H}_Y$ of $\mathbf{G}$, of positive dimension, such that $\mathbf{H}_Y(\mathbb{R})^+ \subset \mathrm{Stab}_{\mathbf{G}(\mathbb{R})^+}(Y)$.*

Let us show how to deduce Theorem 4.25 from Proposition 8.1. The arguments are close to the ones used in the proof of Theorem 4.7.

Let $\mathbf{H}_Y$ be the largest connected $\mathbb{Q}$-algebraic subgroup of $\mathbf{G}$ such that $\mathbf{H}_Y(\mathbb{R})^+ \subset \mathrm{Stab}_{\mathbf{G}(\mathbb{R})^+}(Y)$. By Proposition 8.1 the group $\mathbf{H}_Y$ is positive dimensional.

Let $W' \subset S$ be the Zariski-closure of $\pi(Y)$. Replacing W by W' we can assume that $\pi(Y)$ is Zariski-dense in W. One also can assume that W is Hodge generic, replacing S by the smallest special subvariety of S containing W. In this situation it follows that $\pi(Y)$ is also Hodge-generic in the sense that $\pi(Y)$ is not contained in any strict special subvariety S' of S. Otherwise $\pi(Y) \subset S' \cap W \subsetneq W$ contradicting the Zariski-density of $\pi(Y)$ in W.

Lemma 8.2. *Let $\tilde{W}$ be an irreducible component of $\pi^{-1}(W)$ containing Y. Then $\mathbf{H}_Y(\mathbb{Q})$ stabilizes $\tilde{W}$.*

Proof. Let $h \in \mathbf{H}_Y(\mathbb{Q})$. As $Y \subset \tilde{W} \cap h\tilde{W}$ is irreducible algebraic there exists an irreducible component Z of $\tilde{W} \cap h\tilde{W}$ containing Y. Notice that $\pi(Z)$ is an irreducible component of $W \cap T_h(W)$ containing $\pi(Y)$. As $\pi(Y)$ is Zariski-dense in W it follows that $\pi(Z) = W$. Hence $\tilde{W} = h\tilde{W}$. □

Without loss of generality we can assume that $\mathbf{G}$ is semi-simple of adjoint type. Indeed consider the morphism of Shimura data $\psi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$. Let $K^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$ be a compact open subgroup containing the image of K. We thus have a morphism of Shimura varieties $\psi : \mathrm{Sh}_K(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, K^{\mathrm{ad}})$ and the conjectures for W and $\psi(W)$ are equivalent.

For simplicity let us first assume that $\mathbf{G}$ is $\mathbb{Q}$-simple.

We choose a Hodge-generic point $z \in W^{\mathrm{sm}}$ and a point $\tilde{z} \in \tilde{W}$ above z. Let $\rho : \pi_1(W^{\mathrm{sm}}, z) \longrightarrow \Gamma \subset \mathbf{GL}(V_{\mathbb{Z}})$ be the associated monodromy representation with image $\Gamma_W := \rho(\pi_1(W^{\mathrm{sm}}, z)) \subset \Gamma$. By Galois theory Γ_W is the subgroup of Γ stabilizing $\tilde{W}$. In particular the group Γ_W contains

$$\mathbf{H}_Y(\mathbb{Z}) := \mathbf{H}_Y(\mathbb{Q}) \cap \Gamma = \mathbf{H}_Y(\mathbb{Q}) \cap \mathbf{G}(\mathbb{Z}).$$

Deligne's Theorem 4.8 then states that the Zariski-closure $\overline{\Gamma_W}$ of Γ_W is normal in $\mathbf{G}$. As we assumed that $\mathbf{G}$ is simple, it follows that $\overline{\Gamma_V} = \mathbf{G}$.

Lemma 8.3. *The group Γ_W normalizes $\mathbf{H}_Y$.*

Proof. Let $\gamma \in \Gamma_W$. Thus $\gamma \mathbf{H}_Y \gamma^{-1} \cdot \tilde{W} = \tilde{W}$. Hence

$$Y' := \gamma \mathbf{H}_Y \gamma^{-1} \cdot Y \subset \tilde{W} \quad .$$

But Y' is semi-algebraic and contains Y. In this situation Y' is contained in an irreducible algebraic subvariety of X^+ contained in $\tilde{W}$ and maximal for these properties. By our maximality assumption on Y it follows that $Y = Y'$. Hence $\gamma \mathbf{H}_Y \gamma^{-1}$ fixes Y and it follows that $\gamma \mathbf{H}_Y \gamma^{-1} = \mathbf{H}_Y$. □

We finish the proof of Theorem 4.25 in this case by noticing that the normaliser of $\mathbf{H}_Y$ is algebraic and contains Γ_W. Hence it contains $\overline{\Gamma_W} = \mathbf{G}$. As we supposed that $\mathbf{G}$ is simple if follows that $\mathbf{G} = \mathbf{H}_Y$. Hence $\mathbf{G}$ stabilizes $\tilde{W}$ and Y. Finally $Y = \tilde{W} = X^+$ and $\pi(Y) = W = S$.

In general the adjoint group $\mathbf{G}$ is a product of simple factors. One obtains a decomposition $(\mathbf{G}, X) = (\mathbf{G}_1, X_1) \times (\mathbf{G}_2, X_2)$ with $\mathbf{G}_1$ the Zariski-closure of the monodromy Γ_W. The same kind of arguments as in the simple case then show that

$$\pi(Y) = W = \pi(X_1^+ \times \{x_2\})$$

for some point $x_2 \in X_2$.

□

8.3. **O-minimal arguments and hyperbolic geometry.** To prove Proposition 8.1 we introduce the set

$$\Theta(Y) := \{g \in \mathbf{G}(\mathbb{R}) : \dim(gY \cap \pi^{-1}V \cap \mathcal{F}) = \dim(Y)\} \ ,$$

where $\mathcal{F}$ is a fundamental set for the action of Γ on X^+ as in Theorem 6.2.

Theorem 6.2 implies that $\Theta(Y)$ is definable in $\mathbb{R}_{\text{an,exp}}$. This relies on the fact that the dimension function is a well-defined definable function in any o-minimal theory [vdD98].

The inclusion $g \cdot Y \subset \pi^{-1}(W)$ holds for any $g \in \Theta(Y)$. This follows from the inclusion $g \cdot Y \cap \mathcal{F} \subset \pi^{-1}(W)$ and analytic continuation.

Lemma 8.4.

$$\Theta(Y) \cap \Gamma = \{\gamma \in \Gamma, \quad \gamma^{-1}\mathcal{F} \cap Y \neq \emptyset\} \ .$$

Moreover for any $\gamma \in \Theta(Y) \cap \Gamma$ the translate $\gamma \cdot Y$ is irreducible algebraic in $\pi^{-1}(W)$, maximal for these properties.

Proof. The Γ-invariance of $\pi^{-1}(V)$ implies:

$$\Theta(Y) \cap \Gamma = \{\gamma \in \Gamma, \quad \dim(\gamma Y \cap \pi^{-1}V \cap \mathcal{F}) = \dim(Y)\}$$
$$\{\gamma \in \Gamma, \quad \dim(Y \cap \gamma^{-1}\mathcal{F}) = \dim(Y)\} \ .$$

As $\mathcal{F}$ is open in X^+ the conditions $\dim(Y \cap \gamma^{-1}\mathcal{F}) = \dim(Y)$ and $\gamma^{-1}\mathcal{F} \cap Y \neq \emptyset$ are the same. The first part of the lemma follows. The second part follows from the inclusion $\gamma \cdot Y \subset \pi^{-1}(V)$ obtained by analytic continuation as above and the maximality of Y among the irreducible algebraic subvarieties of X^+ contained in $\pi^{-1}(W)$. □

The heart of the proof of Proposition 8.1 is the following statement. Let H be the height function H on $\mathbf{G}(\overline{\mathbb{Q}})$ deduced from the canonical height function on $\mathrm{End}\,(V_{\overline{\mathbb{Q}}}) \simeq \overline{\mathbb{Q}}^{n^2}$ and the embedding $\mathbf{G}(\overline{\mathbb{Q}}) \subset \mathbf{GL}(V_{\overline{\mathbb{Q}}}) \subset \mathrm{End}\,(V_{\overline{\mathbb{Q}}})$. For every positive real number T we define

$$N_Y(T) := \{\gamma \in \Gamma, \quad Y \cap \gamma^{-1}\mathcal{F} \neq \emptyset, \quad H(\gamma) \leq T\} \ .$$

Theorem 8.5. *There exists a positive real number a such that for T large enough*

$$N_Y(T) \geq T^a \quad .$$

Indications on the proof of Theorem 8.5 will be given in the next section. For now let us show how it implies Proposition 8.1.

First notice that if B is a semi-algebraic bloc of $\Theta(Y)$ containing an element $\gamma \in \Theta(Y) \cap \Gamma$ then

$$B \subset \gamma \cdot \mathrm{Stab}_G(Y) \quad .$$

Indeed if U_γ is an open semi-algebraic subset of B containing γ then $U_\gamma \cdot Y$ is semi-algebraic contained in $\pi^{-1}(W)$ and contains the maximal algebraic Y of $\pi^{-1}(Y)$. Hence $U_\gamma \cdot Y = \gamma Y$. For $b \in B$ one can construct a connected semi-algebraic set $U(\gamma, b)$ of B containing γ and b. The same argument shows that

$$\gamma \cdot Y = b \cdot Y = B \cdot Y \quad .$$

Applying the bloc version Theorem 5.13 of Pila-Wilkie's counting theorem, we obtain positive real numbers b_1 and b_2 such that for T sufficiently large, there exists a bloc B in $\Theta(Y)$ such that

$$|\{\gamma \in B \cap \Gamma, H(\gamma) \leq T^{b_1}\}| \geq T^{b_2} \quad .$$

If we fix $\gamma_0 \in B \cap \Gamma$ the previous discussion shows that the subset $\gamma_0^{-1} \cdot (B \cap \Gamma) \subset \mathrm{Stab}_G(Y)$ contains at least T^{b_2} elements. It follows that $\mathrm{Stab}_G(Y) \cap \Gamma$ is infinite. Hence the algebraic subgroup of $\mathbf{G}$ generated by $\mathrm{Stab}_G(Y) \cap \Gamma$ is positive dimensional. This finishes the proof that Theorem 8.5 implies Proposition 8.1.

□

8.4. **An algebraic curve of X^+ meets many fundamental domains.** Theorem 8.5, which is the technical heart of the proof of Theorem 4.25, is a statement in hyperbolic geometry. We have to show that an irreducible algebraic subvariety Y of X^+ cuts "many" Γ-translates of $\mathcal{F}$. Hence we can assume that Y is the intersection C of an irreducible algebraic curve $\hat{C}$ of $\hat{X}$ with X^+.

The following comparison between the norm and the distance on X^+ on the one hand, the norm and the height on the other hand is a nice exercice on locally symmetric spaces:

Lemma 8.6. (i) *For any $g \in \mathbf{G}(\mathbb{R})$ the following inequality holds:*

$$\log \|g\|_\infty \leq d(g \cdot x_0, x_0) \quad . \tag{8.1}$$

(ii) *There exists a positive number B and a positive integer N such that:*

$$\forall\, \gamma \in \mathbf{G}(\mathbb{Z}), \quad \forall\, u \in \gamma\mathcal{F}, \qquad H(\gamma) \leq B \cdot \|u\|_\infty^n \quad . \tag{8.2}$$

We also have at our disposal a lower bound for the volume of complex-analytic subvariety of X^+ due to Hwang and To [HwTo02]. Let us denote by Vol_C the area on C for the restriction of the metric g_X to C. For a positive real number R we denote by $B(x_0, R)$ the geodesic ball of X^+ of center x_0 and radius R.

Theorem 8.7. *Let C be a complex analytic curve in X^+. For any point $x_0 \in C$ there exist positive constants a, b such that for any positive real number R one has :*

$$\mathrm{Vol}_C(C \cap B(x_0, R)) \geq a \exp(b \cdot R) \quad . \tag{8.3}$$

The key lemma for the proof of Theorem 8.5 is then the following upper-bound for the volume of an algebraic curve (the proof uses the geometry of toroidal compactifications):

Lemma 8.8. *There exists a constant $A > 0$ such that for any algebraic curve $C \subset X^+$ of degree d we have the bound*

$$\mathrm{Vol}_C(C \cap \mathcal{F}) \leq A \cdot d \ . \tag{8.4}$$

With all these ingredients we show Theorem 8.5 as follows. Let T be a positive real number. Let us define

$$\begin{aligned} C(T) &:= \{u \in C \text{ and } \|u\|_\infty \leq T\} \\ &= \bigcup_{\substack{\gamma\in\Gamma \\ \gamma\mathcal{F}\cap C\neq\emptyset}} \{u \in \gamma\mathcal{F} \cap C \text{ and } \|u\|_\infty \leq T\} \end{aligned} \tag{8.5}$$

It follows from the (8.2) that

$$C(T) \subset \bigcup_{\substack{\gamma\in\Gamma,\ \gamma\mathcal{F}\cap C\neq\emptyset \\ H(\gamma)\leq B\cdot T^N}} \{u \in \gamma\mathcal{F} \cap C\} \ . \tag{8.6}$$

Taking volumes:

$$\mathrm{Vol}_C(C(T)) \leq \sum_{\substack{\gamma\in\Gamma,\ \gamma\mathcal{F}\cap C\neq\emptyset \\ H(\gamma)\leq B\cdot T^N}} \mathrm{Vol}_C(\mathcal{F} \cap \gamma^{-1}C) \ , \tag{8.7}$$

hence

$$\mathrm{Vol}_C(C(T)) \leq \sum_{\substack{\gamma\in\Gamma,\ \gamma\mathcal{F}\cap C\neq\emptyset \\ H(\gamma)\leq B\cdot T^N}} \mathrm{Vol}_C(\mathcal{F} \cap \gamma^{-1}C) \ . \tag{8.8}$$

Notice that all the curves $\gamma^{-1}C$, $\gamma \in \mathbf{G}(\mathbb{Z})$, have the same degree as algebraic curves. Hence it follows from (8.4) that

$$\mathrm{Vol}_C(C(T)) \leq (A \cdot d) \cdot N_C(B \cdot T^N) \ . \tag{8.9}$$

Observe that Part (i) of Lemma 8.6 implies that $C \cap B(x_0, \log T) \subset C(T)$. Thus:

$$\mathrm{Vol}_C(C \cap B(x_0, \log T)) \subset \mathrm{Vol}_C C(T) \ . \tag{8.10}$$

Using inequality (8.9) and Theorem 8.7 it follows that

$$aT^b \leq A \cdot d \cdot N_C(B \cdot T^N) \ .$$

This finishes the proof of Theorem 8.5.

□

9. Lower bounds for Galois orbits of CM-points

9.1. **Class groups for tori and reciprocity morphisms.**

9.1.1. *Class groups for tori.* Let $\mathbf{M}$ be an algebraic torus over $\mathbb{Q}$. We denote by K_M^m the unique maximal compact subgroup of $\mathbf{M}(\mathbb{A}_{\mathrm{f}})$.

Definition 9.1. *The absolute class group of* $\mathbf{M}$ *is the finite group*

$$h_{\mathbf{M}} := \mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A}_{\mathrm{f}})/K_{\mathbf{M}}^m \quad .$$

If $K_{\mathbf{M}} \subset \mathbf{M}(\mathbb{A}_{\mathrm{f}})$ is an arbitrary compact open subgroup we define the associated relative class group as the finite group

$$h_{\mathbf{M},K_{\mathbf{M}}} := \mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A}_{\mathrm{f}})/K_{\mathbf{M}} \quad ,$$

so that $h_{\mathbf{M}} = h_{\mathbf{M},K_{\mathbf{M}}^m}$.

Notice that if F is a number field and $\mathbb{R}_F := \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ then $h_{\mathbb{R}_F}$ is equal to the classical class group h_F of the ring of integers $\mathcal{O}_F$ of F.

9.1.2. *Reciprocity morphisms.* The notations are those of Section 2. Let $x = [x,1]$ be a CM-point of S. The Mumford-Tate group $\mathbf{MT}_x$ is a $\mathbb{Q}$-torus $\mathbf{T}$ and $(\mathbf{T},\{x\})$ is a Shimura sub-datum of $(\mathbf{G},X)$. Let $K_{\mathbf{T}} := K \cap \mathbf{T}(\mathbb{A}_{\mathrm{f}})$. Then

$$\mathrm{Sh}_{K_{\mathbf{T}}}(\mathbf{T},\{x\}) = \mathbf{T}(\mathbb{Q})\backslash\left(\{x\}\times\mathbf{T}(\mathbb{A}_{\mathrm{f}})/K_{\mathbf{T}}\right) \subset \mathrm{Sh}_K(\mathbf{G},X)$$

is a zero-dimensional subvariety, of cardinality $h_{\mathbf{T},K_{\mathbf{T}}}$, defined over the reflex field $E := E(\mathbf{T},\{x\})$ of $(\mathbf{T},\{x\})$.

The theory of Complex Multiplication gives a surjective morphism, called the *reciprocity morphism*

$$r := r(\mathbf{T},\{x\}) : \mathbb{R}_E \longrightarrow \mathbf{T} \quad .$$

9.2. **Faltings height.** Let K be a number field and A_K an Abelian variety over K of dimension g. Let $p : \mathbf{A} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ be its Néron model and $\epsilon : \mathrm{Spec}(\mathcal{O}_K) \longrightarrow \mathbf{A}$ its unit section. We denote by $\omega_{\mathbf{A}_K} := \epsilon^*\Omega^g_{\mathbf{A}/\mathrm{Spec}\mathcal{O}_K}$.

Every field embedding $\sigma : K \longrightarrow \mathbb{C}$ defines a Hermitian metric on

$$\omega_{\mathbf{A}_K,\sigma} := H^0(A_\sigma(\mathbb{C}),\Omega^g_{A_\sigma(\mathbb{C})})$$

given on any section $\alpha \in H^0(A_\sigma(\mathbb{C}),\Omega^g_{A_\sigma(\mathbb{C})})$ by

$$||\sigma||_\sigma := \left|\frac{1}{(2\pi)^g}\alpha\wedge\overline{\alpha}\right|.$$

We denote $\overline{\omega_{\mathbf{A}_K}}$ the element $(\omega_{\mathbf{A}_K}, ||\cdot||_\sigma) \in \widehat{\mathrm{Pic}}(\mathcal{O}_K)$.

The Faltings height of A is defined as

$$h_F(A) := \deg_{\mathrm{Ar}}(\overline{\omega_{\mathbf{A}_K}}) \quad .$$

If A has semi-stable reduction over K the Faltings height $h_F(A)$ does not change under base change to a finite extension of K. If A has good reduction over K there exists a finite extension L of K such that $\omega_{\mathbf{A}_L} \simeq \mathcal{O}_L$. Choosing a Neron differential $\omega \in \omega_{\mathbf{A}_L}$, one then obtains

$$h_F(A) = -\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma:L\longrightarrow\mathbb{C}}\log||\omega||_\sigma \quad .$$

9.3. **Lower bounds for Galois orbits.** In 2001, motivated by applications to the André-Oort conjecture, Edixhoven formulated a conjecture that Galois orbits of CM abelian varieties of fixed dimension should grow as a uniform power of the discriminant of the center of the Endomorphism ring of the abelian varieties in question. He stated his conjecture precisely as Problem 14 in [EMO]. In private communications with the third author in 2000, Edixhoven expressed the view that his conjecture should follow easily from the Colmez conjecture on Faltings height of CM abelian varieties. Reference to Colmez' paper appears explicitly in the statement of his Problem 14 of [EMO]. It appears that before Tsimerman's announcement, the required inequality relating Faltings height to the discriminant of the centre of the Endomorphism algebra has been known to experts on transcendence theory and diophantine approximation. In his 2005 thesis Eric Villani (see [V]) explicitly works out the required inequality for abelian varieties corresponding to special points of Hilbert modular varieties as a subproduct of his main result - see 'Remarque' on page 18 of [V]. In [GR14], the authors develop all the necessary tools for the deduction and in [R], Proposition 2.10, Gaél Rémond explicitly states the deduction of the required inequality from the results of [GR14]. Until the announcements of proofs of the Colmez conjecture on average in 2014 (see [AGHM] and [YuZh]), there was very little progress on Colmez conjecture. Upon the announcement, Tsimerman immediately presented an argument for the deduction of lower bounds of Galois orbits from it, which, as explained above, has been essentially well-known to the experts. In this section we give the argument as presented by Tsimerman.

The Faltings height can be interpreted as a height on the set $\mathcal{A}_g(\overline{\mathbb{Q}})$ of algebraic points of $\mathcal{A}_g$. If $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$ parameterizes the Abelian variety A_x one define $h_F(x) = h_F(A_x)$. This function satisfies the Northcott property: given d and T positive real integers the set

$$N_{d,T}(\mathcal{A}_g) := \{x \in \mathcal{A}_g(\overline{\mathbb{Q}}), [\mathbb{Q}(x):\mathbb{Q}] \leq d \text{ and } h_F(x) \leq T\}$$

is finite. If the Falting's height h_F were uniformly bounded on CM-point of $\mathcal{A}_g$ we would directly obtain that the fields of definition of these points have a degree tending to infinity. This type of argument is used in the proof of the Manin-Mumford conjecture to obtain a lower bound or Galois-orbits of torsion points of an Abelian variety, as these are the points of canonical height zero. For $\mathcal{A}_g$ it is not true that the Faltings height is uniformly bounded but a direct consequence of the Colmez conjecture on average, (which we describe in the next section) is the following:

Theorem 9.2. *Let g be a positive integer and ϵ a positive real number. There exists a positive real number $c_1 = c_1(g,\epsilon)$ with the following property. Let E be a CM-field of degree $2g$ with discriminant d_E. Let A be a g-dimensional Abelian variety with complex multiplication by the ring of integers $\mathcal{O}_E$ of E. Then*

$$h_F(A) \leq c_1|d_E|^{\epsilon} \ .$$

The main result of Tsimerman in [Tsi] is the following corollary for the size of Galois orbits of CM-points in $\mathcal{A}_g$:

Theorem 9.3. *(Tsimerman) Let g be a positive integer. There exist positive real numbers $\alpha = \alpha(g)$ and $c_2 = c_2(g)$ with the following property. Let E be a CM-field of degree $2g$ with discriminant d_E. Let $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$ be a CM-point parameterizing an Abelian* *variety with complex multiplication by $\mathcal{O}_E$. Then*

$$[\mathbb{Q}(x):\mathbb{Q}] \geq c_2 \cdot |d_E|^{\alpha} \ .$$

The proof of Theorem 9.3 relies on a deep theorem of Masser and Wüstholz [MaWü95] (which is also the crux of an alternative proof of Mordell's conjecture):

Theorem 9.4. *(Masser-Wüstholz) Let g be a positive integer. There exist positive real numbers $\beta = \beta(g)$ and $c_3 = c_3(g)$ with the following property. Let A and B be two Abelian varieties defined over a number field k. We suppose that A and B are $\overline{\mathbb{Q}}$-isogenous. Then there exists a $\overline{\mathbb{Q}}$-isogeny from A to B of degree N with*

$$N \leq c_2 \max(h_F(A), [k:\mathbb{Q}])^{\beta} \ .$$

Tsimerman's argument is the following. Let Σ be the locus in $\mathcal{A}_g$ of Abelian varieties with complex multiplication by $\mathcal{O}_E$ and fixed CM-type Φ. For all $x, y \in \Sigma$ the Abelian varieties A_x and A_y are $\overline{\mathbb{Q}}$-isogenous. On the other hand the cardinal of Σ is the cardinal of the class group of $\mathcal{O}_E$. As E is CM the class formula gives $|\Sigma| \gg d_E^{\gamma}$ for an absolute constant $\gamma > 0$ for d_E sufficiently large.

Let us fix $x_0 \in \Sigma$. Let N be a positive integer. There exists $\delta > 0$ such that the number of $\overline{\mathbb{Q}}$-isogenies with source A_{x_0} of degree at most N is bounded above by N^{δ} for N sufficiently large. Let η be a positive real number such that $\eta < \frac{\gamma}{\delta}$. Taking $N = d_E^{\eta}$ and d_E large enough it follows that there exists $x \in \Sigma$ such that the minimal degree $d_{\min}(A_{x_0}, A_x)$ of a $\overline{\mathbb{Q}}$-isogeny from A_{x_0} to A_x satisfies

$$d_{\min}(A_{x_0}, A_x) > d_E^{\eta} \ .$$

By the Masser-Wüstholz Theorem 9.4 and the upper-bound on the Faltings height given by Theorem 9.2 it follows that:

$$d_E^{\eta} \leq c_3 \max(h_F(A_{x_0}), [k:\mathbb{Q}]^{\beta}) \leq c_3 \max(c_1 d_E^{\epsilon}, k:\mathbb{Q}]^{\beta}) \ .$$

If we choose $\epsilon < \eta$ and d_E sufficiently large one obtains

$$[k:\mathbb{Q}] \geq d_E^{\frac{\eta}{\beta}} \ .$$

□

9.4. **Colmez conjecture.** The reference for this section is [Col93]. Let A be a simple Abelian variety over $\mathbb{C}$, with complex multiplication and of dimension g. The field $E := \mathrm{End}\,(A) \otimes \mathbb{Q}$ is CM with $[E:\mathbb{Q}] = 2g$. We suppose moreover that $\mathrm{End}\, A = \mathcal{O}_E$. Let $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ be the CM type of A. Hence

$$\mathrm{Lie}(A) = \bigoplus_{\sigma \in \Phi} \mathrm{Lie}(A)_{\sigma} \ ,$$

where $\mathrm{Lie}(A)_{\sigma}$ is the subspace of $\mathrm{Lie}(A)$ on which E acts through σ.

Let K be a number field on which A is defined and has good reduction. Colmez shows that the height $h_F(A)$ depends on (E, Φ) only and conjectures a closed formula for $h_F(A)$. We will write $h_F(A) = h_F(\Phi)$ in the sequel.

Let F be a field extension of $\mathbb{Q}$. We denote by $\mathcal{G}_F$ the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and by $c \in \mathcal{G}_{\mathbb{Q}}$ the complex conjugation. Let $C(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$ be the complex vector space of locally constant complex functions on $\mathcal{G}_{\mathbb{Q}}$ and $C^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$ its subspace of central ones.

Let $\mathbb{Q}^{\mathrm{CM}} \subset \overline{\mathbb{Q}}$ be the extension of $\mathbb{Q}$ generated by CM-fields. This is a Galois extension of $\mathbb{Q}$. We denote by $\mathrm{CM}^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C}) \subset C^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$ the subspace of functions f such that $f(\sigma)$ depends only on the $\mathcal{G}_{\mathbb{Q}^{\mathrm{CM}}}$-conjugacy class of σ and such that $f(\sigma)+f(c\sigma)$ is independent of σ.

We define a Hermitian scalar product $<,>$ on $C(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$ by:

$$\forall \Theta_1, \Theta_2 \in C(\mathcal{G}_{\mathbb{Q}}, \mathbb{C}), \quad < \Theta_1, \Theta_2 >:= \frac{1}{|\mathcal{G}_{\mathbb{Q}}/\mathcal{G}_{\mathcal{F}}|} \sum_{g \in \mathcal{G}_{\mathbb{Q}}/\mathcal{G}_{\mathcal{F}}} \Theta_1(g)\overline{\Theta_2(g)} \ ,$$

where F is any finite extension of $\mathbb{Q}$ such that Θ_1 and Θ_2 depend only on conjugacy classes modulo $\mathcal{G}_F$.

The set Art of Artin characters (i.e. characters of continuous finite dimensional complex representations of $\mathcal{G}_{\mathbb{Q}}$) is an orthonormal basis of $C^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$. Given any Artin character χ, we denote by $L(\chi, s)$ its L-function. One also checks that the set of Artin characters whose L-function does not vanish at 0 form an orthonormal basis of $\mathrm{CM}^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$.

For $\Theta \in C(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$ we denote by Θ^0 its orthonormal projection

$$\Theta^0 = \sum_{\chi \in \mathrm{Art}} < \Theta, \chi > \chi$$

on $C^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$.

We also denote by $Z(\chi, s)$ the logarithmic derivative $L'(\chi, s)/L(\chi, s)$ and by $\mu_{\mathrm{Art}}(\chi)$ the logarithm $\log f_\chi$ of the Artin conductor f_χ of χ. These functions admit local decompositions

$$\mu_{\mathrm{Art}} = \sum_{p \text{ premier}\}} \mu_{\mathrm{Art},p} \log p \ ,$$

$$\forall\, \mathrm{Re}(s) > 1, \quad Z(\chi, s) = - \sum_{p \text{ premier}} Z_p(\chi, s) \log p \ .$$

For any prime p, the local factor $Z_p(\chi, s)$ lies in $\mathbb{Q}(p^{-s})$. The function $Z(\chi, s)$ admits an holomorphic extension to $\mathbb{C}$ and a functional equation.

Given a CM-type (E, Φ) we define the function $A_\Phi \in C(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$ by

$$A_\Phi(g) = \frac{|\Phi \cap g\Phi|}{[E : \mathbb{Q}]}$$

and denote by A_Φ^0 its projection on $C^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$. One checks that $A_\Phi^0 \in \mathrm{CM}^0(\mathcal{G}_{\mathbb{Q}}, \mathbb{C})$. Colmez conjecture is the following:

Conjecture 9.5. *Let A be a complex Abelian variety of CM-type (E, Φ). Then:*

$$h_F(A) = Z(A_\Phi^0) - \frac{1}{2}\mu_{\mathrm{Art}}(A_\Phi^0) = - \sum_{\chi \in \mathrm{Art}} < A_\Phi, \chi > (\frac{L'(0, \chi)}{L(0, \chi)} + \frac{1}{2}\mu_{\mathrm{Art}}(\chi)) \ .$$

Let F be the totally real subfield of E, ${}_F$ its discriminant and $d_{E/F} := N_{E/F} d_E$ the relative discriminant of E over F. Let $\chi_{E/F}$ be the associated quadratic character of F. As noticed by Colmez, Conjecture 9.5 simplifies if we average on the 2^g possible CM-types of E. It is this result which is proved by completely different method by Andreatta-Goren-Howard-Madapusi Pera [AGHM] and Yuan-Zhang [YuZh] and which implies Theorem 9.2:

Theorem 9.6. *(Colmez conjecture on average)*

$$\frac{1}{2^g}\sum_{\Phi} h_F(\Phi) = -\frac{1}{2}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} - \frac{1}{4}\log|d_{E/F}d_F| \ .$$

10. Further developments: the André-Pink conjecture.

In this section we briefly present a conjecture which is a special case of the Zilber-Pink conjecture and which is in some sense a more natural analog of the Manin-Mumord conjecture in the context of (mixed) Shimura varieties. This conjecture is now usually referred to as the André-Pink conjecture and was stated explicitly by Pink in [Pink05]. Pink also obtained a result on this conjecture under certain quite strong assumptions. We will not touch upon Pink's work in this paper and refer to Pink's (excellent) exposition in [Pink05]. The general statement of the André-Pink conjecture is as follows ([An89], Problem 3 and [Pink05], Conjecture 1.6):

Conjecture 10.1 (André-Pink)**.** *Let S be a mixed Shimura variety over $\mathbb{C}$ and $\Lambda \subset S$ be the generalised Hecke orbit of a point s of S. Let Z be a subvariety of S such that $Z \cap \Lambda$ is Zariski dense in Z. Then Z is a weakly special subvariety.*

The André-Pink conjecture is still open in general. However Martin Orr (see [Orr15]) has obtained a fairly general result, using the techniques explained in this text. The primary aim of this section is to explain Orr's result and give an idea of its proof.

Instead of looking at the Zariski closure of a set of special points, one looks at the Zariski closure of a subset of a (generalised) Hecke orbit in a (mixed) Shimura variety. The expectation is that components of this Zariski closure are weakly special. In the case of $\mathcal{A}_g$ the conjecture becomes the following.

Conjecture 10.2. *Let Λ be the isogeny class of a point $s \in \mathcal{A}_g(\mathbb{C})$. Let Z be an irreducible closed subvariety of $\mathcal{A}_g$ such that $Z \cap \Lambda$ is Zariski dense in Z. Then Z is a weakly special subvariety of $\mathcal{A}_g$.*

In the case where s is Galois-generic, Pink in [Pink05], proves that Conjecture 10.2 follows from results of Clozel, Oh and Ullmo ([CUO01]) on equidistribution of Hecke orbits. However, it would be a relatively rare occurence that s is Hodge generic. In the case where s is a special point, Conjecture 10.2 is a special case of the André-Oort conjecture, known for $\mathcal{A}_g$. In [Orr15], M. Orr proves the following.

Theorem 10.3 (Orr)**.** *Let s and Λ and Z be as in 10. Then:*

(1) *There exists a special subvariety $S \subset \mathcal{A}_g$, isomorphic to a product $S_1 \times S_2$ of Shimura varieties, such that* $\dim(S_1) > 0$ *and*

$$Z = S_1 \times Z' \subset S$$

where Z' is a closed subvariety of S_2.

(2) *If Z is a curve, then Z is weakly special.*

It is of course obvious that (2) follows from (1).

The strategy of Orr's proof is again a combination of lower bounds for Galois orbits with Pila-Wilkie theorem (the blocks version - Theorem 5.13), Ax-Lindemann (Theorem 4.25) and Ullmo's Theorem 7.2. Note that elaboration of suitable lower bounds for the Galois orbits makes essential use of the Masser-Wüstholz theorem (Theorem 9.4).

Let s be a point of $\mathcal{A}_g(\mathbb{C})$, Λ its isogeny class and Z an irreducible subvariety of $\mathcal{A}_g$ such that $Z \cap \Lambda$ is Zariski dense.

We call $\mathbf{H}_g$ the Siegel upper half space and $\pi\colon \mathbf{H}_g \longrightarrow \mathcal{A}_g$ the uniformization map. We also call $\mathcal{F}_g$ the classical Siegel fundamental domain. Let

$$\widetilde{Z} = \mathcal{F}_g \cap \pi^{-1} Z \text{ and } \widetilde{\Lambda} = \mathcal{F}_g \cap \pi^{-1}\Lambda.$$

Given a point s of $\mathcal{A}_g(\mathbb{C})$, we let A_s be the abelian variety associated to s. We define the complexity of t in Λ as the minimum degree of isogeny between A_s and A_t.

Similarly, we define the complexity of a point t in $\widetilde{\Lambda}$.

The height of a matrix in $M_n(\mathbb{Q})$ is defined as the maximum of heights of its entries.

Orr proves the following:

Proposition 10.4 ([Orr15], Proposition 3.2)**.** *Let Z be a subvariety of $\mathcal{A}_g$ and $\widetilde{s}$ a point in $\mathcal{F}_g$. Let $\epsilon > 0$. There exists an $C = c(Z, \widetilde{s}, \epsilon)$ such that for every $n \geq 1$, there is a collection of at most cn^{ϵ} semi-algebraic blocks $W_i \subset \widetilde{W}$ such that all points of $\widetilde{Z} \cap \widetilde{\Lambda}$ of complexity $\leq n$ are contained in $\bigcup_i W_i$.*

The idea of the proof is to construct a certain definable subset Y of $\mathbf{GL}_{2g}(\mathbb{R})$, show that it contains 'a lot' of points of $\mathbf{GL}_{2g}(\mathbb{Q})$ up to height n and then apply Pila-Wilkie theorem (block version) to it.

The crucial lemma is the following which is of independent interest.

Lemma 10.5 ([Orr15], Lemma 3.3)**.** *There exist constant c, k depending only on g and $\widetilde{s}$ such that: for any $\widetilde{t} \in \widetilde{Z} \cap \widetilde{\Lambda}$ of complexity n, there is a rational matrix $\gamma \in Y$ such that $\gamma\widetilde{s} = \widetilde{t}$ and the height of γ is at most cn^k.*

On the other hand, Masser-Wustholz theorem gives a polynomial (in the complexity) lower bound on the size of the Galois orbits of the points of Λ.

This implies, via Pila-Wilkie theorem and Ax-Lindemann, that positive dimensional weakly special subvarieties are dense in Z. Ullmo's Theorem 7.2 then implies the conclusion of Theorem 10.3.

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