TUBE FORMULAS FOR SELF-SIMILAR FRACTALS

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Abstract. Tube formulas (by which we mean an explicit formula for the volume of an (inner) ε-neighbourhood of a subset of a suitable metric space) have been used in many situations to study properties of the subset. For smooth submanifolds of Euclidean space, this includes Weyl's celebrated results on spectral asymptotics, and the subsequent relation between curvature and spectrum. Additionally, a tube formula contains information about the dimension and measurability of rough sets. In convex geometry, the tube formula of a convex subset of Euclidean space allows for the definition of certain curvature measures. These measures describe the curvature of sets which may be too irregular to support derivatives. In this survey paper, we describe some recent advances in the development of tube formulas for self-similar fractals, and their applications and connections to the other topics mentioned here.

1. Introduction

This survey article describes advances on the computation of tube formulas for fractal subsets of $\mathbb{R}$ and of $\mathbb{R}^d$, and relations to classical results. In particular, we show how the theory of complex dimensions can be used to calculate explicit tube formulas for a large class of self-similar fractals. We also discuss generalizations and connections to forthcoming work.

We begin by discussing results for fractal strings, that is, fractal subsets of $\mathbb{R}$. Also, we give the generalization to tube formulas of measures, and show how these results may be applied to the investigation of dimension and measurability of rough sets. In particular, we use the central notion of "complex dimensions", a concept which extends the real-valued notion of Minkowski dimension (and Hausdorff dimension, in some cases). §2 introduces these notions and describes the basic theory of fractal strings as developed more fully in [La-vF2]. §3 describes the explicit computation of a tube formula for the Koch snowflake domain, a subset of $\mathbb{R}^2$ with fractal boundary consisting of three copies of the classical self-similar Koch snowflake curve. This computation is mainly a "brute-force" calculation using elementary geometry coupled with some subtle exploitation of distributional methods. §4 contains the construction of a self-similar tiling via an iterated function system, and shows how the tiling enables one to extend the results for fractal strings to self-similar subsets of $\mathbb{R}^d$. §5 gives a brief description of some classical results from

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convex geometry and shows how these relate to the results for self-similar tilings. Finally, we conclude with several contrasting examples that illustrate the key ideas discussed.

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1.2. **Basic concepts.** We now present some of the ideas that are used throughout, and introduce basic notation.

Definition 1.1. Given $\varepsilon > 0$, the *inner ε-neighbourhood* of a set $A \subseteq \mathbb{R}^d$, $d \geq 1$, is

$$A_\varepsilon := \{x \in A : dist(x, \partial A) \leq \varepsilon\}, \tag{1.1}$$

where ∂A is the boundary of A.[1] For a given A, we are primarily interested in finding a *tube formula* for A, that is, an explicit expression for the d-dimensional Lebesgue measure of A_ε, denoted

$$V_A(\varepsilon) := \mathrm{vol}_d(A_\varepsilon).$$

Definition 1.2. A *self-similar system* is a family $\{\Phi_j\}_{j=1}^J$ (with $J \geq 2$) of contraction similitudes

$$\Phi_j(x) := \mathfrak{r}_j T_j x + a_j, \quad j = 1, \dots, J.$$

For $j = 1, \dots, J$, we have $0 < \mathfrak{r}_j < 1$, $a_j \in \mathbb{R}^d$, and $T_j \in O(d)$, the orthogonal group of rigid motions in d-dimensional Euclidean space $\mathbb{R}^d$. The number $\mathfrak{r}_j$ is the *scaling ratio* of Φ_j. For convenience, assume these ratios are indexed so that

$$1 > \mathfrak{r}_1 \geq \mathfrak{r}_2 \geq \dots \geq \mathfrak{r}_J > 0. \tag{1.2}$$

When $d = 1$, one has only $T_j = \pm 1$.

It is well known that there is a unique nonempty compact subset $F \subseteq \mathbb{R}^d$ satisfying the fixed-point equation

$$F = \Phi(F) := \bigcup_{j=1}^J \Phi_j(F). \tag{1.3}$$

This (self-similar) set F is called the *attractor* of Φ. Given a word $w = w_1 \dots w_k \in \mathcal{W}_k := \{1, 2, \dots, J\}^k$, we denote the composition of several similarity mappings by $\Phi_w := \Phi_{w_k} \circ \dots \circ \Phi_{w_2} \circ \Phi_{w_1}$.

2. Fractal strings

The essential strategy of fractal strings is to study fractal subsets of $\mathbb{R}$ by studying their complements.

Definition 2.1. A *fractal string* is any bounded open subset $L \subseteq \mathbb{R}$, that is, a countable sequence of open intervals

$$L := \{L_n\}_{n=1}^\infty. \tag{2.1}$$

[1] This is a slightly different usage of the notation A_ε, which is often used in the literature to indicate the *exterior* ε-neighbourhood; see (5.4) and Remark 5.2.

Although it is not part of the definition, the idea is that the boundary ∂L is a set one wants to study. When analyzing L in terms of characteristics which are invariant with respect to rigid motions, the position of an interval L_n within $\mathbb{R}$ is immaterial, and all pertinent data is stored in the sequence of lengths ℓ_n of L_n. For convenience, we therefore always assume that the intervals have monotonically decreasing length, i.e., they are ordered such that

$$\ell_1 \geq \ell_2 \geq \cdots > 0. \tag{2.2}$$

Indeed, this is the formulation in which the concept is used in [La-vF2], the primary reference for fractal strings.[2] For agreement with the higher-dimensional theory discussed in §4, however, we use the inradius.

Definition 2.2. The *inradius* ρ of a set $A \subseteq \mathbb{R}^d$ is

$$\rho = \rho(A) := \sup\{\varepsilon > 0 : \exists x \text{ with } B(x, \varepsilon) \subseteq A\}, \tag{2.3}$$

where $B(x, \varepsilon)$ is the ball of radius ε centered at $x \in \mathbb{R}^d$.

It is clear that if A is a bounded set, $A \subseteq A_\varepsilon$ for sufficiently large ε. Alternatively, it is apparent that for a fixed $\varepsilon > 0$, any sufficiently small set will be entirely contained within its ε-neighbourhood. The notion of *inradius* allows us to see when this phenomenon occurs; its relevance should be clear from Definition 1.1.

In accordance with (2.2), we have a monotonically decreasing sequence of inradii, so divide by the first (largest) inradius to obtain a sequence of *scales*

$$\mathcal{L} := \{r_n\}_{n=1}^{\infty}, \quad r_1 \geq r_2 \geq \cdots > 0, \quad \text{and} \quad \ell_n = 2g r_n, \tag{2.4}$$

where $g = \rho(L_1)$ is the inradius of the largest interval. Use of the term "scale" here corresponds to the implicit idea that each interval $L_n \in L$ is congruent to a copy of the largest interval which has been scaled by r_n. For a self-similar string, this mapping is explicitly given by Φ_w; see (2.5). It is a consequence of the normalization described above that one always has $r_1 = 1$. Strict positivity of the r_n avoids certain technical trivialities, and $\sum_{n=1}^{\infty} r_n < \infty$ follows from the boundedness of L.

To define a *self-similar string*, consider the self-similar system with similarity mappings $\Phi_j(x) = \mathfrak{r}_j T_j x + a_j$. The set of scaling ratios of a self-similar string will consists of the collection of all products of scaling ratios of the maps. In particular, if w is a finite word of length k on $\{1, 2, \ldots, J\}$, then the string will include

$$r_w := \mathfrak{r}_{w_1} \mathfrak{r}_{w_2} \ldots \mathfrak{r}_{w_k}, \tag{2.5}$$

the scaling ratio of $\Phi_w = \Phi_{w_k} \circ \ldots \circ \Phi_{w_2} \circ \Phi_{w_1}$. Thus, a self-similar string contains every number that arises as the scaling ratio of a composition of the similarity transformations Φ_j. (The first scale $r_1 = 1$, corresponds to a composition of 0 similarities.) The motivation for this definition is that it corresponds to the set obtained by taking the set-theoretic difference of a self-similar subset of $\mathbb{R}$ from the smallest closed interval containing it. Alternatively, the set may be constructed by selecting any bounded open interval $I \subseteq \mathbb{R}$ and examining the lengths of the intervals $\{\Phi_w(I)\}$, where w runs over all finite words. An example of a self-similar string is given by the Cantor string in Example 2.8 just below.

A *generalized fractal string* is a locally finite Borel measure on $(0, \infty)$, without mass near 0, and is denoted $\eta = \eta_{\mathcal{L}}$. Such a string may not have a geometric

[2] This notion was introduced in [LaPo1] (building on an example of [La1]) and studied extensively in [La-vF1] and [La-vF2], as well as [HeLa], [La2], [LaMa], and elsewhere.

realization. The motivation for this generalization lies in the flexibility of working in the measure-theoretic framework, and in certain specific applications. In this context, an ordinary fractal string as given in the previous definitions (self-similar or not) corresponds to a sum of Dirac masses δ_x, each located at a reciprocal of one of the numbers r_n. That is, an ordinary fractal string may be written

$$\eta_{\mathcal{L}} = \sum_{n=1}^{\infty} \delta_{1/r_n}. \tag{2.6}$$

Definition 2.3. The *scaling zeta function* $\zeta_{\mathfrak{s}}$ of a fractal string is the Mellin transform of the measure $\eta_{\mathcal{L}}$:

$$\zeta_{\mathfrak{s}}(s) := \int_0^{\infty} x^{-s}\, d\eta_{\mathcal{L}}(x), \qquad s \in \mathbb{C}. \tag{2.7}$$

In [La-vF2], this is called the geometric zeta function of $\mathcal{L}$. We have chosen the current terminology to agree with the latter sections of this paper and instead say that the *geometric zeta function* of $\zeta_{\mathcal{L}}$ is given by

$$\zeta_{\mathcal{L}}(\varepsilon, s) := \zeta_{\mathfrak{s}}(s) \frac{(2\varepsilon)^{1-s}}{s(1-s)}. \tag{2.8}$$

The function $\zeta_{\mathcal{L}}$ factors into $\zeta_{\mathfrak{s}}$ and a term which contains geometric data about open intervals, although this will not become clear until the discussion of the higher-dimensional tilings in §4. In the case when $\mathcal{L}$ is self-similar, the function $\zeta_{\mathcal{L}}(\varepsilon, \cdot) : \Omega \to \mathbb{C}$ may be meromorphically continued to all of $\mathbb{C}$. Otherwise, it is well defined in some half-plane $\{\operatorname{Re} s > a\}$ and may be continued analytically. To demarcate the domain of $\zeta_{\mathcal{L}}$ formally, we introduce the following definition.

Definition 2.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz continuous function. Then the *screen* is $S = \{f(t) + \mathbb{i}t \,:\, t \in \mathbb{R}\}$, the graph of f with the axes interchanged. Here and henceforth, $\mathbb{i} = \sqrt{-1}$. The region to the right of the screen is the *window* $W := \{z \in \mathbb{C} :\, \operatorname{Re} z \geq f(\operatorname{Im} z)\}$. We choose f so that S does not pass through any poles of $\zeta_{\mathcal{L}}$, and $\zeta_{\mathcal{L}}$ has a meromorphic extension to some neighbourhood of W.

Definition 2.5. The *(complex) scaling dimensions* of $\mathcal{L}$ are poles of $\zeta_{\mathcal{L}}$. The poles which lie in the window are called the *visible scaling dimensions*:

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W :\, \lim_{s \to \omega} |\zeta_{\mathcal{L}}(s)| = \infty\}. \tag{2.9}$$

The screen and window are useful for many purposes. In addition to demarcating the domain of $\zeta_{\mathcal{L}}$, they allow one to make precise statements about the growth of $\zeta_{\mathcal{L}}$, and they allow for certain approximation arguments that make the proof of the next theorem possible. In particular, one has the following definition.

Definition 2.6. One says that $\zeta_{\mathcal{L}}$ (or just $\mathcal{L}$) is *languid* if it satisfies certain mild growth conditions on S, and along a sequence of horizontal lines in W. For the precise statement of these conditions, see [La-vF2, Def. 5.2].

Theorem 2.7 (Tube formula for fractal strings [La-vF2, Thm. 8.1])**.** *Let $\eta_{\mathcal{L}}$ be a fractal string with geometric zeta function $\zeta_{\mathcal{L}}$ and assume that $\zeta_{\mathcal{L}}$ is languid. Then we have a tube formula*

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W)} \operatorname{res}\left(\zeta_{\mathcal{L}}(\varepsilon, s); s = \omega\right) + \{2\varepsilon \zeta_{\mathfrak{s}}(0)\} + \mathcal{R}(\varepsilon). \tag{2.10}$$

Here the term in braces is only included if $0 \in W \setminus \mathcal{D}_\eta(W)$*, and the error term is*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi\mathbb{i}} \int_S \zeta_{\mathcal{L}}(\varepsilon, s)\, ds = O(\varepsilon^{1-\sup \operatorname{Re} S}), \qquad \text{as } \varepsilon \to 0^+. \tag{2.11}$$

When the string is also self-similar, one may take $W = \mathbb{C}$ *and* $\mathcal{R}(\varepsilon) \equiv 0$*.*

If we denote the poles of the scaling zeta function separately by $\mathcal{D}_{\mathfrak{s}}$, then the result above may be rewritten

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}(W) \cup \{0\}} \operatorname{res}\left(\zeta_{\mathcal{L}}(\varepsilon, s); s = \omega\right) + \mathcal{R}(\varepsilon). \tag{2.12}$$

Formula (2.10) was originally obtained as a distribution acting on smooth functions with compact support in $(0, \infty)$. However, it has since been obtained in a pointwise fashion in [La-vF2, Thm. 8.7] under only slightly more restrictive conditions (which are always satisfied in the case of self-similar strings), so this technicality need not be emphasized.

Example 2.8. The complement of the usual Cantor set in the unit interval $[0, 1]$ consists of open intervals with lengths $\{\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \dots, \}$. This self-similar fractal string has a largest interval of length $\frac{1}{3}$ and scaling ratios 3^{-k} appearing with multiplicity 2^k. Since $g = \frac{1}{6}$ is the inradius of the largest interval, the Cantor string $\mathcal{CS}$ may be written

$$\eta_{\mathcal{CS}} = \sum_{k=0}^{\infty} 2^k \delta_{3^k}, \tag{2.13}$$

and its scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \int_0^\infty x^{-s}\, d\eta_{\mathcal{CS}}(x) = \sum_{k=0}^{\infty} \left(\frac{2}{3^s}\right)^k = \frac{1}{1 - 2 \cdot 3^{-s}}, \tag{2.14}$$

while its tube formula is

$$V_{\mathcal{C}}(\varepsilon) = \frac{1}{3\log 3} \sum_{n \in \mathbb{Z}} \left(\frac{1}{D + \mathbb{i}n\mathbf{p}} - \frac{1}{D - 1 + \mathbb{i}n\mathbf{p}}\right) \left(\frac{\varepsilon}{g}\right)^{1 - D - \mathbb{i}n\mathbf{p}} - 2\varepsilon, \tag{2.15}$$

where $D = \log_3 2$ is the Minkowski dimension, and $\mathbf{p} = 2\pi/\log 3$ is a constant called the *oscillatory period.*

Definition 2.9. The *Minkowski dimension* of $A \subseteq \mathbb{R}^d$ is

$$D = \dim_M A := \inf\{t \geq 0 : V_A(\varepsilon) = O(\varepsilon^{d-t}), \text{ as } \varepsilon \to 0^+\}, \tag{2.16}$$

and is also frequently called the box dimension; see [La1] for details. For a string, we define $\dim_M \mathcal{L} := \dim_M \partial L$.

The complex dimensions can be thought of as a generalization of Minkowski dimension because of the following result of [La2], which also appears as [La-vF2, Thm. 1.10].

Theorem 2.10. *For a string* $\mathcal{L}$*, the scaling zeta function* $\zeta_{\mathfrak{s}}$ *converges on the half-plane* $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}$ *if and only if* $\sigma \geq \dim \mathcal{L}$*.*

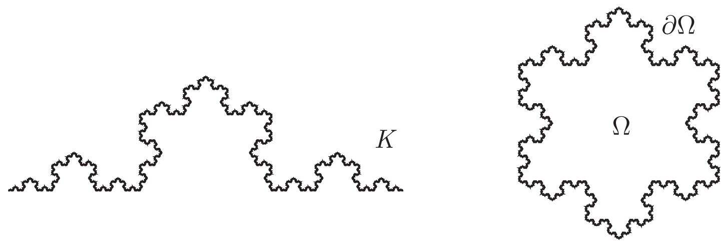

FIGURE 1. The Koch curve K and Koch snowflake domain Ω. This figure was originally published in [LaPe1] and appears with permission of JLMS.

Remark 2.11. In formula (2.14), one discovers the pleasant surprise that $\zeta_{\mathfrak{s}}$ has a nice closed form. In fact, it is shown in [La-vF2, Thm. 2.4] that all self-similar strings have a scaling zeta function of the form

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - \sum_{j=1}^{J} \mathfrak{r}_j^s}, \tag{2.17}$$

or a (finite) 'linear combination' of such terms. This remark remains true for the self-similar tilings of §4. The number-theoretic and measure-theoretic implications of this result are extensive; they are studied at length in [La-vF2, Ch. 2–3].

3. The Koch curve

The Koch curve K is the attractor of the self-similar system

$$\Phi_1(z) := \rho\overline{z} \quad \text{and} \quad \Phi_2(z) := (1-\rho)(\overline{z}-1)+1, \tag{3.1}$$

where $\rho = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$. Consequently, K is a self-similar fractal with Minkowski dimension $D := \log_3 4$; see Figure 1. In this section, we describe how to calculate the tube formula for the Koch snowflake directly.

In [LaPe1], the authors compute a tube formula for the Koch snowflake domain Ω by considering a certain sequence of curves $K_n \to K$, where convergence is with respect to Hausdorff metric. Some representative terms of this sequence are illustrated in Figure 2. The tube formula is obtained as a limit of tube formulas for V_{K_n} obtained by approximation for each K_n, as illustrated for K_2 in Figure 3. Three copies of this figure are fitted together to form the inner neighbourhood of the snowflake, as is indicated by the dashed lines at either end. This region is only an approximation of course, as one side of each rectangle should be replaced with a fractal curve. Figure 4 shows how this error is incurred and how it inherits a Cantoresque structure from the triadic character of the Koch curve. Let us refer to the dark region in Figure 4 as an *error block* and each connected component as a *trianglet*; denote a trianglet by A_k. Without taking the error blocks into account, the ε-neighbourhood of the Koch curve has approximate area

$$\widetilde{V}_{\mathrm{K}}(\varepsilon) = \varepsilon^{2-D}4^{-\{x\}}\left(\tfrac{3\sqrt{3}}{40}9^{\{x\}} + \tfrac{\sqrt{3}}{2}3^{\{x\}} + \tfrac{1}{6}\left(\tfrac{\pi}{3} - \sqrt{3}\right)\right) - \tfrac{\varepsilon^2}{3}\left(\tfrac{\pi}{3} + 2\sqrt{3}\right). \tag{3.2}$$

Here, $x := -\log_3(\varepsilon\sqrt{3})$, and we use $x = [x] + \{x\}$ to denote the decomposition of x into its integer and fractional parts. That is, $[x]$ is an integer and $0 \leq \{x\} < 1$. Furthermore, x is related to n (the index of K_n) by $n = n(\varepsilon) = [\log_3 \frac{1}{\varepsilon\sqrt{3}}] = [x]$. Thus, the level n of the approximation is determined by ε, with $n \to \infty$ as $\varepsilon \to 0$.

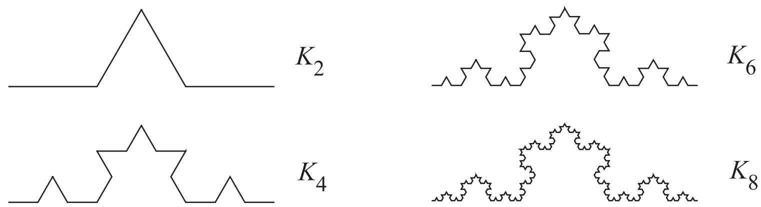

FIGURE 2. Four early stages in the geometric construction of K. This figure was originally published in [LaPe1] and appears with permission of JLMS.

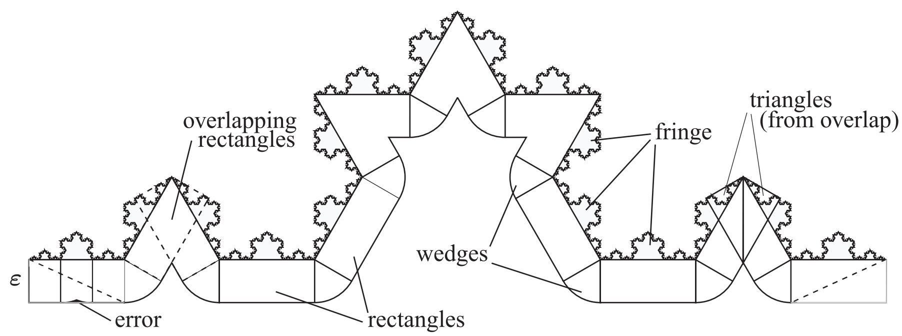

FIGURE 3. An approximation to the inner ε-neighbourhood of the Koch curve, based on the curve K_2 from Figure 2. This figure was originally published in [LaPe1] and appears with permission of JLMS.

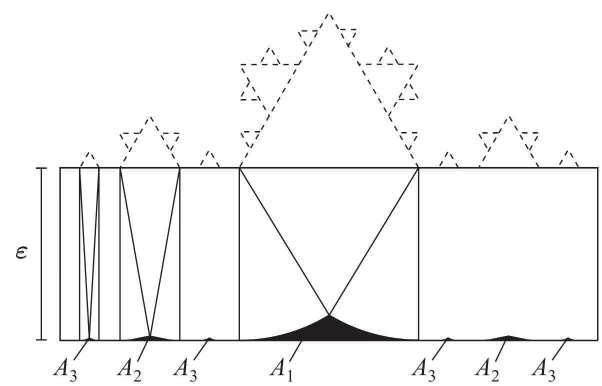

FIGURE 4. An error block for K_n. The central third of the block contains one large isosceles triangle, two wedges, and the trianglet A_1. Figure 5 contains 4 complete copies of this figure, and 12 partial copies of it. This figure was originally published in [LaPe1] and appears with permission of JLMS.

The tube formula for Ω will be obtained by summing the areas A_k of the trianglets, and multiplying by the number of error blocks occurring for a given approximation, i.e., for a particular value of ε. Unfortunately, this number is not easy to express, due to the existence of partial error blocks. See Figure 5 for a visual explanation of what is meant by *partial error blocks* and *complete error blocks*. The number of complete error blocks can be readily counted with a simple formula, but the portion of a partial error block that exists for a given value of ε is rather

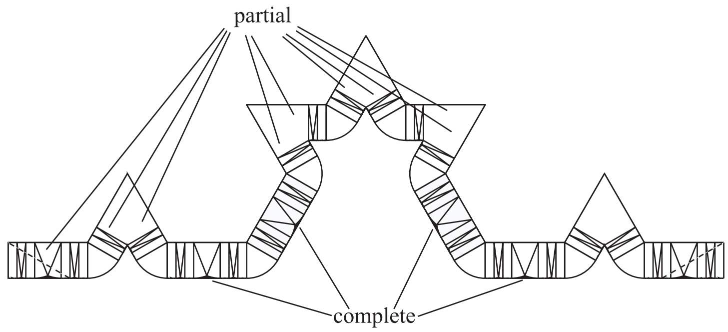

FIGURE 5. Error block formation. The ends are counted as partial (note the dotted line) because three copies of this illustration will be added together to make the entire snowflake. This figure was originally published in [LaPe1] and appears with permission of JLMS.

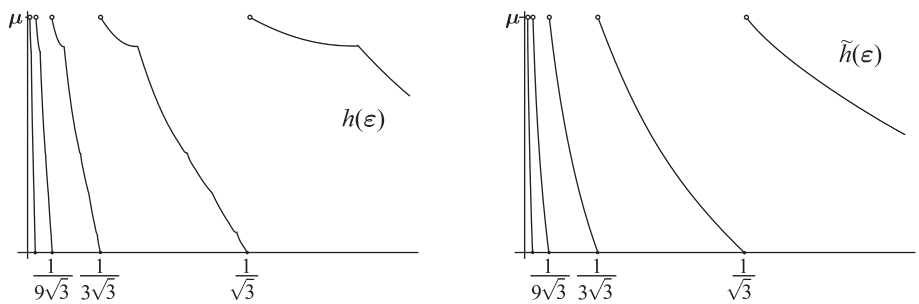

FIGURE 6. A comparison between the graph of the Cantor-like function h and the graph of a piecewise smooth approximation $\tilde{h}(\varepsilon) = \mu\cdot(-[x]-x)$, where μ is a constant and $x = -\log_3(\varepsilon\sqrt{3})$. This figure was originally published in [LaPe1] and appears with permission of JLMS.

ornery, and so we denote this quantity by $h(\varepsilon)$. We do not know $h(\varepsilon)$ explicitly, but we do know by the self-similarity of K that it has multiplicative period 3; i.e., $h(\varepsilon) = h(\frac{\varepsilon}{3})$. Once the error has been taken into account, one obtains the following theorem.

Theorem 3.1. [LaPe1, Thm. 5.1] *The area of the inner ε-neighbourhood of the Koch snowflake is given by the following tube formula:*

$$V(\varepsilon) = \sum_{n\in\mathbb{Z}} \varphi_n \varepsilon^{2-D-in\mathbf{p}} + \sum_{n\in\mathbb{Z}} \psi_n \varepsilon^{2-in\mathbf{p}}, \tag{3.3}$$

for suitable constants φ_n, ψ_n which depend only on n and are expressed in terms of the Fourier coefficients g_α of $h(\varepsilon)$.

To see the full form of (3.3), please see Remark 6.4 at the end of this paper. The result (3.3) was obtained at a time when the theory of complex dimensions was entirely restricted to fractal strings as outlined in §2 (except for the conjectures expressed in [La-vF2, Ch. 12]). Part of the motivation for proving Theorem 3.1 was to get an idea of what the complex dimensions of Ω might look like. Reasoning by analogy, one would deduce from (3.3) that the complex dimensions of Ω are obtained from the exponents appearing in (3.3); in particular, that each exponent is of the form $2-\omega$, where ω is a complex dimension of Ω. This led to the prediction that the complex dimensions of the Koch snowflake domain are

$$\mathcal{D}_{\partial\Omega} = \{D + in\mathbf{p} : n \in \mathbb{Z}\} \cup \{in\mathbf{p} : n \in \mathbb{Z}\}. \tag{3.4}$$

The reader will find in Example 6.2 and Remark 6.4 that this is not far wrong.

It is a theorem of [LlWi] that for a self-similar set, the ε-neighbourhood must be either: (i) a finite union of convex sets for every value of ε, or (ii) not a finite union of convex sets for any value of ε. We conjecture that for every self-similar set of the latter type, the tube formula involves a multiplicatively periodic function analogous to our $h(\varepsilon)$.

4. Fractal sprays and self-similar tilings

A fractal spray is the higher-dimensional counterpart of a fractal string. See [LavF2, §1.4] for a discussion; this idea also appears earlier in [LaPo2], [La2], and [La3].

Definition 4.1. Let $G \subseteq \mathbb{R}^d$ be a nonempty bounded open set, which we will call the *generator*, and let $\mathcal{L} = \{r_n\}$ be a fractal string. Then a *fractal spray* is a bounded open subset of $\mathbb{R}^d$ which is the disjoint union of open sets R_n for $n = 1, 2, \dots$, where each R_n is congruent to $r_n G$, the homothetic of G by r_n.

Thus, any fractal string can be thought of as a fractal spray on the basic shape $G = (0, 1)$, the unit interval. Every self-similar system (as in Definition 1.2) is naturally associated to a certain fractal spray called the self-similar tiling.

Definition 4.2. The *self-similar tiling* $\mathcal{T}$ corresponding to a self-similar system Φ is a fractal spray where the fractal string and generators are defined as follows. The string is the collection of all finite products of the scaling ratios $\{\mathfrak{r}_1, \dots, \mathfrak{r}_J\}$ of the self-similar system Φ. Let $C = [F]$ be the convex hull of F, and denote its interior by $\operatorname{int} C$. Recall from (1.3) that F is the self-similar set which is the attractor of Φ. Then the generators are the connected open sets G_q in the disjoint union

$$\operatorname{int} C \setminus \Phi(C) = G_1 \cup \dots \cup G_Q. \tag{4.1}$$

When there is more than one generator, it is more accurate to think of the tiling as a union of fractal sprays, one for each generator. Indeed, for purposes of computing the tube formula, it is easiest to deal with each part separately and then obtain the final result by adding the contributions from each generator. For this reason, the tube formulas below are all stated for a spray or tiling with one generator. In fact, we cannot currently exclude the possibility that there may exist examples for which $Q = \infty$. However, we have been unable to construct such an example, and for the time being we assume $Q < \infty$.

The term "self-similar tiling" is used here in a sense quite different from the one often encountered in the literature. In particular, the tiles themselves are neither self-similar nor are they all of the same size; in fact, the tiles are typically simple polyhedra. Moreover, the region being tiled is the complement of the self-similar set F within its convex hull, rather than all of $\mathbb{R}^d$; see Figure 7. In fact, it is shown in [Pe2] that when Φ satisfies the tileset condition, the collection $\{\Phi_w(G_q)\}_{w,q}$ forms an open tiling of $C \setminus F$. We now define these terms.

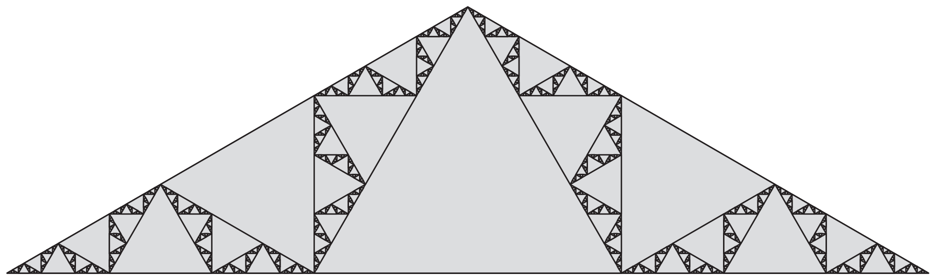

FIGURE 7. The Koch tiling $\mathcal{K}$. The generator is the largest equilateral triangle, and the system is $\{\Phi_1(z) = \xi\overline{z}, \Phi_2(z) = (1-\xi)(\overline{z}-1)+1$, for $z \in \mathbb{C}$, where $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$. This figure was originally published in [Pe2] and appears with permission of IUMJ.

Definition 4.3. An *open tiling* of $A \in \mathbb{R}^d$ is a collection of nonempty connected open sets $\{A_n\}_{n=1}^\infty$ such that

$$\text{(i) } \overline{A} = \overline{\bigcup\nolimits_{n=1}^{\infty} A_n}, \qquad \text{and} \qquad \text{(ii) } A_n \cap A_m = \varnothing \text{ for } n \neq m. \tag{4.2}$$

Figure 7 shows a tiling by open sets which are equilateral triangles in the case when F is the Koch curve.

Definition 4.4. The system Φ satisfies the *tileset condition* iff $C \nsubseteq \Phi(C)$ and

$$\operatorname{int}\Phi_j(C) \cap \operatorname{int}\Phi_\ell(C) = \varnothing, \qquad j \neq \ell. \tag{4.3}$$

The tileset condition is a separation condition which is similar to the "open set condition", but stronger. The open set condition is satisfied when there exists a "feasible open set" U which has the property that for all $j \neq \ell$, $\Phi_j(U), \Phi_\ell(U) \subseteq U$, and $\Phi_j(U) \cap \Phi_\ell(U) = \varnothing$. To see that the tileset condition implies this, let U be the the interior of $C = [F]$. For a counterexample to the converse, see [Pe2, Example 3.8].

For self-similar tilings, and more generally for fractal sprays, the scaling zeta function is defined just as it was for fractal strings. However, the geometric zeta function becomes much more complicated, due to the multifarious possibilities for the geometry of the generator. The resulting technicalities can be ignored in many cases, however, for example when the generator is sufficiently simple. This motivates the condition "Steiner-like" in the definition just below. A bounded open set in $\mathbb{R}^d$ is said to be Steiner-like if its inner parallel volume $V_G(\varepsilon)$ admits a "polynomial-like" expansion in ε of degree at most d. More precisely, we have the following definition (see §5 for an explanation of this choice of terminology).

Definition 4.5. A bounded open set $G \subseteq \mathbb{R}^d$ is *Steiner-like* iff for $0 \leq \varepsilon \leq \rho(G)$ its inner tube formula may be written

$$V_G(\varepsilon) = \sum_{k=0}^{d} \kappa_k(G,\varepsilon)\varepsilon^{d-k}, \tag{4.4}$$

where each coefficient function $\kappa_k(G,\cdot)$ is assumed to be a bounded and locally integrable function of ε for which

$$\lim_{\varepsilon \to 0^+} \kappa_k(G,\varepsilon) \tag{4.5}$$

exists, and is both positive and finite.

Definition 4.6. In the special case when G is a Steiner-like set whose terms κ_k are constant, we say the set is *diphase*. This terminology refers to the fact that its tube formula is written piecewise with only two cases: $\varepsilon \leq \rho(G)$ and $\varepsilon > \rho(G)$. If G is Steiner-like and the functions κ_k are piecewise constant on the interval $[0, \rho(G)]$, then we say G is *pluriphase*. Thus, diphase is a special case of pluriphase.

Conjecture 1. *The class of pluriphase sets includes convex sets and polyhedra.*

This result may exist in the literature, but we have been unable to find it. It is simple to show that the class of diphase sets includes balls and regular polyhedra. In particular, the examples of §6 are both diphase. There are examples of other types of diphase sets, but they resist easy description. It is easy to find convex sets which are pluriphase but not diphase; see [LaPe2, Ex. 5.7] for an example. Current research is attempting to characterize diphase and pluriphase sets in terms of other geometric properties.

In the case when G is pluriphase, the geometric zeta function is relatively simple to write down. To avoid obscuring the exposition, we give only the diphase case here (although in view of Definition 4.8 it is worth noting that the factors $\frac{1}{s-k}$ also appear in the formula for the pluriphase case).

Definition 4.7. The *geometric zeta function* of a fractal spray with a diphase generator is

$$\zeta_{\mathcal{T}}(\varepsilon, s) := \varepsilon^{d-s} \zeta_\eta(s) \sum_{k=0}^{d} \frac{g^{s-k}}{s-k} \kappa_k. \tag{4.6}$$

Definition 4.8. The set of *visible complex dimensions of a fractal spray* is

$$\mathcal{D}_{\mathcal{T}}(W) := \mathcal{D}_\eta(W) \cup \{0, 1, \ldots, d-1\}. \tag{4.7}$$

Thus, $\mathcal{D}_{\mathcal{T}}(W)$ consists of the visible scaling dimensions and the "integer dimensions" of G, and contains all the poles of $\zeta_{\mathcal{T}}$ (viewed as a function of s).

Theorem 4.9 (Tube formula for fractal sprays)**.** *Let η be a fractal spray on the Steiner-like generator G, with generating inradius $g = \rho(G) > 0$. If $\zeta_{\mathcal{T}}$ is languid, then we have a tube formula*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}(W)} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right) + \mathcal{R}(\varepsilon), \tag{4.8}$$

where the sum ranges over the set $\mathcal{D}_{\mathcal{T}}(W)$. Here, the error term $\mathcal{R}(\varepsilon)$ is

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_S \zeta_{\mathcal{T}}(\varepsilon, s)\, ds, = O(\varepsilon^{d - \sup S}), \qquad \text{as } \varepsilon \to 0^+. \tag{4.9}$$

Theorem 4.9 was first obtained distributionally in [Pe1] and appears in improved form in [LaPe2]. Since then, a pointwise proof has been obtained in [LaPeWi1]. It is important to note that $\zeta_{\mathcal{T}}$ is languid for all self-similar tilings. As we have not been able to construct a self-similar tiling which satisfies the tileset condition but fails to be Steiner-like, Theorem 4.9 applies to all known examples. *A fortiori*, it is possible to show that all self-similar tilings automatically satisfy a more stringent condition (called *strongly languid* in [La-vF2, Def 5.2]) that enables one to take $W = \mathbb{C}$ and $\mathcal{R}(\varepsilon) \equiv 0$; see [LaPe2, Thm. 8.4]. Thusly one obtains the following special case of Theorem 4.9, which for simplicity, we only state in the diphase case.

Corollary 4.10. *Let $\mathcal{T}$ be a self-similar tiling with a diphase generator G. If $\zeta_{\mathcal{T}}(s)$ has only simple poles, then*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \sum_{k=0}^{d} \operatorname{res}\left(\zeta_{\mathfrak{s}}(s); \omega\right) \varepsilon^{d-\omega} \tfrac{g^{\omega-k}}{\omega-k} \kappa_k + \sum_{k=0}^{d-1} \kappa_k \zeta_{\mathfrak{s}}(k) \varepsilon^{d-k}. \tag{4.10}$$

By comparing (4.10) with (2.10), it is easy to see how the tube formula for fractal sprays extends the results for fractal strings. Additionally, it extends classical results for convex sets, as outlined in the next section; see especially (5.7)–(5.8) and the surrounding discussion.

5. Convex geometry and the curvature measures

In order to explain the connections with convex geometry, we give a brief encapsulation of Steiner's classical result. Here, we denote the Minkowski sum of two sets in $\mathbb{R}^d$ by

$$A + B = \{x \in \mathbb{R}^d : x = a + b \text{ for } a \in A, b \in B\}.$$

Theorem 5.1. *If B^d is the d-dimensional unit ball and $A \subseteq \mathbb{R}^d$ is convex, then the d-dimensional volume of $A + \varepsilon B^d$ is given by*

$$\operatorname{vol}_d(A + \varepsilon B^d) = \sum_{k=0}^{d} \mu_k(A) \operatorname{vol}_{d\text{-}k}(B^{d-k}) \varepsilon^{d-k}, \tag{5.1}$$

where μ_k is the renormalized k-dimensional intrinsic volume *and $\operatorname{vol}_k(A)$ is the k-dimensional Lebesgue measure.*

Up to some normalizing constant, the k-dimensional intrinsic volume is the k^{th} total curvature or $(d-k)^{\text{th}}$ Quermassintegral. This valuation μ_k can be defined via integral geometry as the average measure of orthogonal projections to $(d-k)$-dimensional subspaces; see [KlRo, Ch. 7]. For now, we note that (up to a constant), there is a correspondence

μ_0	$\sim$	Euler characteristic,	μ_{d-1}	$\sim$	surface area,
μ_1	$\sim$	mean width,	μ_d	$\sim$	volume,

see [Schn2, §4.2] for more. We have chosen the term "Steiner-like" for Definition 4.5 because the intrinsic volumes satisfy the following properties:

(i) each μ_k is homogeneous of degree k, so that for any $x > 0$,

$$\mu_k(xA) = \mu_k(A)\, x^k, \text{ and} \tag{5.2}$$

(ii) each $\mu_k(A)$ is rigid motion invariant, so that for any isometry T of $\mathbb{R}^d$,

$$\mu_k(T(A)) = \mu_k(A). \tag{5.3}$$

Note that (5.1) gives the volume of the set of points which are within ε of A, including the points of A. If we denote the *exterior ε-neighbourhood* of A by

$$A_\varepsilon^{ext} := (A + \varepsilon B^d) \setminus A = \{x : d(x, A) \le \varepsilon, x \notin A\}, \tag{5.4}$$

then it is immediately clear that omitting the d^{th} term gives

$$\operatorname{vol}_d(A_\varepsilon^{ext}) = \sum_{k=0}^{d-1} C_k(A) \varepsilon^{d-k} \tag{5.5}$$

with $C_k(A) = \mu_k(A)\,\mathrm{vol}_{d\text{-}k}(B^{d-k})$. The intrinsic volumes μ_k can be localized and understood as the *curvature measures* introduced in [Fed] and described further in [Schn2, Ch. 4]. In this case, for a Borel set $\beta \subseteq \mathbb{R}^d$, one has

$$\mathrm{vol}_d\{x \in A_\varepsilon^{ext} : p(x,A) \in \beta\} = \sum_{k=0}^{d-1} C_k(A,\beta)\varepsilon^{d-k}, \tag{5.6}$$

where $p(x,A)$ is the metric projection of x to A, that is, the closest point of A to x. *In fact, the curvature measures are obtained axiomatically in [Schn2] as the coefficients of the tube formula, and it is this approach that we hope to emulate in our current work.* In other words, we believe that κ_k may also be understood as a (total) curvature, in a suitable sense, and we expect that κ_k can be localized as a curvature measure. A more rigorous formulation of these ideas is currently underway in [LaPeWi3]. Caveat: the description of κ_k given in the conditions of Definition 4.5 is intended to emphasize the resemblance between κ_k and C_k. However, κ_k may be signed (even when G is convex and $k = d-1, d$) and is more complicated in general. In contrast, the curvature measures C_k are always positive for convex bodies.

Remark 5.2 (Inner versus outer ε-neighbourhoods). The primary reason we work with the inner ε-neighbourhood instead of the exterior is that it is more intrinsic to the set; it makes the computation independent of the embedding of $\mathcal{T}$ into $\mathbb{R}^d$. At least, this should be the case, provided the 'curvature' terms κ_k of Definition 4.5 are also intrinsic. As a practical bonus, working with the inner ε-neighbourhood allows us to avoid potential issues with the intersections of the ε-neighbourhoods of different components.

In [Fed], Federer unified the tube formulas of Steiner (for convex bodies, as described in [Schn2, Ch. 4]) and of Weyl (for smooth submanifolds, as described in [BeGo], [Gr] and [We]) and extended these results to sets of *positive reach*.[3] It is worth noting that Weyl's tube formula for smooth submanifolds of $\mathbb{R}^d$ is expressed as a polynomial in ε with coefficients defined in terms of curvatures (in a classical sense) that are *intrinsic* to the submanifold [We]. See §6.6–6.9 of [BeGo] and the book [Gr]. Federer's tube formula has since been extended in various directions by a number of researchers in integral geometry and geometric measure theory, including [Schn1], [Schn2], [Zä1], [Zä2], [Fu1], [Fu2], [Sta], and most recently (and most generally) in [HuLaWe]. The books [Gr] and [Schn2] contain extensive endnotes with further information and many other references.

To emphasize the present analogy, consider that Steiner's formula (5.5) may be rewritten

$$\mathrm{vol}_d(A_\varepsilon^{ext}) = \sum_{k\in\{0,1,\ldots,d-1\}} c_k\varepsilon^{d-k}. \tag{5.7}$$

and it is clear that Corollary 4.10 may be rewritten

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega\in\mathcal{D}_{\mathfrak{s}}\cup\{0,1,\ldots,d-1\}} c_\omega\varepsilon^{d-\omega}, \tag{5.8}$$

[3]A set A has *positive reach* iff there is some $\delta > 0$ such that any point $x \in A^{\complement}$ within δ of A has a unique metric projection to A, i.e., that there is a unique point A minimizing $dist(x,A)$. Equivalently, every point q on the boundary of A lies on a sphere $\partial B(x,\delta)$ which intersects ∂A only at q, where $x \in A^{\complement}$.

where for each fixed $\omega \in \mathcal{D}_{\mathfrak{s}}$,

$$c_\omega := \operatorname{res}\left(\zeta_{\mathfrak{s}}(s);\omega\right) \sum_{k=0}^{d} \frac{g^{\omega-k}}{\omega-k}\kappa_k. \tag{5.9}$$

Note that when $\omega = k \in \{0,1,\dots,d-1\}$, one has $c_\omega = c_k = \zeta_{\mathfrak{s}}(k)\kappa_k$. The obvious similarities between the tube formulas (5.7) and (5.8) is striking. Our tube formula is a 'fractal power series' in ε, rather than just a polynomial in ε (as in Steiner's formula). Moreover, our series is summed not just over the 'integral dimensions' $\{0,1,\dots,d-1\}$, but also over the countable set $\mathcal{D}_{\mathfrak{s}}$ of scaling complex dimensions. The coefficients c_ω of the tube formula are expressed in terms of the 'curvatures' and the inradii of the generators of the tiling.

Remark 5.3. The two formulas (5.7) and (5.8) measure very different things, and so appear to be unrelated; nonetheless, they are closely linked. It can happen that the exterior ε-neighbourhood of the fractal itself is, in fact, equal to the union of the inner ε-neighbourhood of the tiling and the exterior ε-neighbourhood of its convex hull. This occurs, for example, with the Sierpinski gasket; see Remark 6.5 and Figure 10. In such cases, the tube formula for the exterior ε-neighbourhood of the fractal will be precisely the sum of the tube formula for the tiling (5.8) and Steiner's tube formula for its convex hull (5.7):

$$\operatorname{vol}_d(F_\varepsilon^{ext}) = V_{\mathcal{T}}(\varepsilon) + \operatorname{vol}_d([F]_\varepsilon^{ext}) \tag{5.10}$$

However, things do not always work out so neatly, as the example of the Koch tiling shows; see Figure 11. In the forthcoming paper [PeWi], it is shown that (5.10) holds precisely when one of the following equivalent conditions is verified:

(1) $\partial\mathcal{T} = F$.
(2) $\partial G_q \subseteq F$ for all $q = 1,\dots,Q$.
(3) $\partial(C \setminus \Phi(C)) \subseteq F$.
(4) $\partial C \subseteq F$.
(5) $F_\varepsilon^{ext} \cap C = T_\varepsilon$ for some (and, equivalently, all) $\varepsilon \geq 0$.
(6) $F_\varepsilon^{ext} \cap C^c = C_\varepsilon^{ext} \setminus C$ for some (and, equivalently, all) $\varepsilon \geq 0$.

Thus Theorem 4.9 allows one to compute explicit tube formulas for a large family of self-similar sets via (5.10).

Remark 5.4 (Comparison of $V_{\mathcal{T}}$ with the Steiner formula). In the trivial situation when the spray consists only of finitely many scaled copies of a diphase generator (so the scaling measure is supported on a finite set), the scaling zeta function will have no poles in $\mathbb{C}$, so that $\mathcal{D}_{\mathfrak{s}} = \varnothing$. Therefore, the tube formula becomes a sum over only the numbers $k = 0,1,\dots,d-1$, for which the residues simplify greatly. (For technical reasons, it turns out during the computation that the d^{th} summand always vanishes, so the sum extends only up to $d-1$.) In fact, in this case $\zeta_\eta(k) = \rho_1^k + \dots + \rho_N^k$, so each residue from (4.10) is a finite sum

$$\zeta_\eta(k)\kappa_k(\varepsilon) = \rho_1^k\kappa_k\varepsilon^{d-k} + \dots + \rho_N^k\kappa_k\varepsilon^{d-k} = \kappa_k(r_{w_1}G)\varepsilon^{d-k} + \dots + \kappa_k(r_{w_N}G)\varepsilon^{d-k}$$

where N is the number of scaled copies of the generator G, and r_{w_n} are the corresponding scaling factors. Summing over k, we obtain a tube formula for the scaled basic shape $r_{w_n}G$, for each $n = 1,\dots,N$. The pluriphase case is analogous.

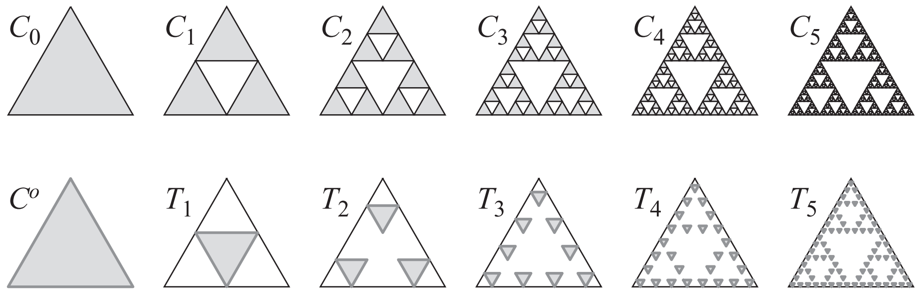

FIGURE 8. The Sierpinski gasket tiling. The first stages of the construction of the gasket are labeled C_n. The tiles $\{\Phi_w(G)\}$ are labeled T_n where $|w| = n$. This figure was originally published in [Pe2] and appears with permission of IUMJ.

6. Three illustrative examples

Example 6.1 (The Sierpinski gasket tiling)**.** The Sierpinski gasket tiling $\mathcal{SG}$ (see Figure 8) is constructed via the system

$$\Phi_1(z) := \tfrac{1}{2}z, \quad \Phi_2(z) := \tfrac{1}{2}z + \tfrac{1}{2}, \quad \Phi_3(z) := \tfrac{1}{2}z + \tfrac{1+\mathbb{i}\sqrt{3}}{4},$$

which has one common scaling ratio $\mathfrak{r} = 1/2$, with $J = 3$ and one generator G: an equilateral triangle of side length $\frac{1}{2}$ and inradius $g = \frac{1}{4\sqrt{3}}$. Thus $\mathcal{SG}$ has inradii $\rho_k = g\mathfrak{r}^k$ with multiplicity 3^k, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 3 \cdot 2^{-s}}, \tag{6.1}$$

and the scaling complex dimensions are

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_2 3,\ \mathbf{p} = \tfrac{2\pi}{\log 2}. \tag{6.2}$$

The tube formula for $\mathcal{SG}$ is readily computed to be

$$V_{\mathcal{SG}}(\varepsilon) = \tfrac{\sqrt{3}}{16 \log 2} \sum_{n \in \mathbb{Z}} \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}} \right) \left(\tfrac{\varepsilon}{g} \right)^{2-D-\mathbb{i}n\mathbf{p}} + \tfrac{3^{3/2}}{2}\varepsilon^2 - 3\varepsilon.$$

Example 6.2 (The Koch tiling)**.** The standard Koch tiling $\mathcal{K}$ (see Figure 7) is constructed via the self-similar system given in (3.1). The attractor of $\{\Phi_1, \Phi_2\}$ is the classical von Koch curve K, as in Figure 1. This system has one scaling ratio $\mathfrak{r} = |\xi| = 1/\sqrt{3}$, with $J = 2$ and one generator G: an equilateral triangle of side length $\frac{1}{3}$ and generating inradius $g = \frac{\sqrt{3}}{18}$. This tiling has inradii $\rho_k = g\mathfrak{r}^k$ with multiplicity 2^k, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 2 \cdot 3^{-s/2}}, \tag{6.3}$$

and the scaling complex dimensions are

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_3 4,\ \mathbf{p} = \tfrac{4\pi}{\log 3}. \tag{6.4}$$

By inspection, a tile with inradius $1/x$ will have tube formula

$$\gamma_G(x, \varepsilon) = \begin{cases} 3^{3/2}\left(-\varepsilon^2 + 2\varepsilon x\right), & \varepsilon \le 1/x, \\ 3^{3/2}x^2, & \varepsilon \ge 1/x. \end{cases} \tag{6.5}$$

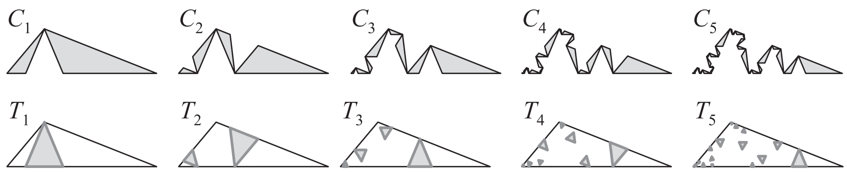

FIGURE 9. A measurable Koch tiling obtained by picking a nonlattice parameter ξ in (3.1). This figure was originally published in [Pe2] and appears with permission of IUMJ.

For fixed x, (6.5) is clearly continuous at $\varepsilon = 0^+$. Thus we have $\kappa_0 = -3^{3/2}, \kappa_1 = 2 \cdot 3^{3/2}$, and $\kappa_2 = -3^{3/2}$. Now applying (4.10), the tube formula for the Koch tiling $\mathcal{K}$ is

$$V_{\mathcal{K}}(\varepsilon) = \frac{g}{\log 3} \sum_{n \in \mathbb{Z}} \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}} \right) \left(\tfrac{\varepsilon}{g}\right)^{2-D-\mathbb{i}n\mathbf{p}} + 3^{3/2}\varepsilon^2 + \tfrac{1}{1-2\cdot 3^{-1/2}}\varepsilon, \tag{6.6}$$

where $D = \log_3 4$, $g = \frac{\sqrt{3}}{18}$ and $\mathbf{p} = \frac{4\pi}{\log 3}$ as before.

Remark 6.3 (Nonlattice Koch tilings). By replacing $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$ in (3.1) with any other complex number satisfying $|\xi|^2 + |1-\xi|^2 < 1$, one obtains a family of examples of nonlattice self-similar tilings, as illustrated in Figure 9. Computation of the tube formula does not differ significantly from the lattice case. Further discussion of nonlattice Koch tilings may be found in [Pe2]. As discussed in [LaPe2], for example, this furnishes a 1-parameter family of tilings, almost all of which are Minkowski measurable.

Remark 6.4. In §3 we discussed how to obtain a tube formula for the ε-neighbourhood of the Koch curve itself (rather than of the tiling associated with it) and this led to the prediction that the complex dimensions of the curve are

$$\mathcal{D}_{\mathcal{K}^\star} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \cup \{0 + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\},$$

where $D = \log_3 4$ and $\mathbf{p} = \frac{2\pi}{\log 3}$. The line of poles above D was expected[4], and agrees precisely with the results of §4. The meaning of the line of poles above 0 is unclear. We invite the reader to compare (6.6) with the formula for the Koch curve in (3.3). When formula (3.3) for the area of the inner ε-neighbourhood of the Koch snowflake is stated in full detail (see [LaPe1, Thm. 5.1]), it appears as follows:

$$V(\varepsilon) = G_1(\varepsilon)\varepsilon^{2-D} + G_2(\varepsilon)\varepsilon^2, \tag{6.7}$$

where G_1 and G_2 are periodic functions of multiplicative period 3, given by

$$G_1(\varepsilon) := \frac{1}{\log 3} \sum_{n \in \mathbb{Z}} \left(a_n + \sum_{\alpha \in \mathbb{Z}} b_\alpha g_{n-\alpha} \right) (-1)^n \varepsilon^{-in\mathbf{p}} \tag{6.8a}$$

$$\text{and} \quad G_2(\varepsilon) := \frac{1}{\log 3} \sum_{n \in \mathbb{Z}} \left(\sigma_n + \sum_{\alpha \in \mathbb{Z}} \tau_\alpha g_{n-\alpha} \right) (-1)^n \varepsilon^{-in\mathbf{p}}, \tag{6.8b}$$

[4]This set of complex dimensions was predicted in [La-vF1], §10.3, except for the dimensions above 0.

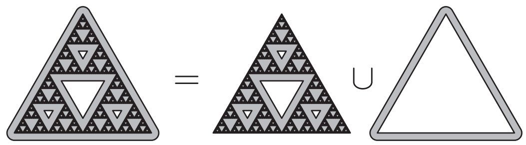

FIGURE 10. The exterior ε-neighbourhood of the Sierpinski gasket is the disjoint union of the exterior ε-neighbourhood of $C = [F]$ and the inner ε-neighbourhood of the Sierpinski gasket tiling.

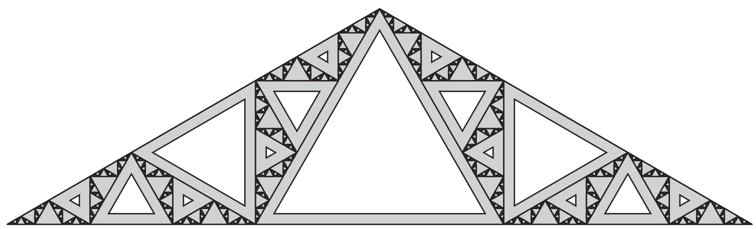

FIGURE 11. The exterior ε-neighbourhood of the Koch curve is not simply related to the exterior ε-neighbourhood of $C = [K]$ and the inner ε-neighbourhood of the Koch tiling $\mathcal{K}$.

where g_α are the Fourier coefficients of the multiplicatively periodic function $h(\varepsilon)$ discussed just before the statement of Theorem 3.1, and a_n, b_n, σ_n, and τ_n are the complex numbers given by

$$\begin{aligned} a_n &= \frac{\pi - 3^{3/2}}{2^3(D + in\mathbf{p})} + \frac{3^{3/2}}{2^3(D-1+in\mathbf{p})} - \frac{3^{5/2}}{2^5(D-2+in\mathbf{p})} + \frac{1}{2}b_n, \\ b_n &= \sum_{m=1}^{\infty} \frac{(2m)!\,(3^{2m+1}-4)}{4^{2m+1}(m!)^2(4m^2-1)(3^{2m+1}-2)(D-2m-1+in\mathbf{p})}, \\ \sigma_n &= -\log 3\left(\frac{\pi}{3} + 2\sqrt{3}\right)\delta_0^n - \tau_n, \text{ and} \\ \tau_n &= \sum_{m=1}^{\infty} \frac{(2m)!\,(3^{2m+1}-1)}{4^{2m-1}(m!)^2(4m^2-1)(3^{2m+1}-2)(-2m-1+in\mathbf{p})}. \end{aligned} \tag{6.9}$$

In the definition of σ_n we use the Kronecker delta to indicate a term that only appears in σ_0.

Remark 6.5. As mentioned in Remark 5.3, the exterior ε-neighbourhood of the Sierpinski gasket is the disjoint union of the inner ε-neighbourhood of the tiling and the exterior ε-neighbourhood of the largest triangle; see Figure 10. This means that one can immediately obtain the tube formula for the exterior ε-neighbourhood of the gasket by adding the tube formula for the tiling $\mathcal{SG}$ and Steiner's tube formula for its convex hull (5.5), the equilateral triangle C_0 (as labeled in Figure 8).

Unfortunately, this method does not apply to the Koch tiling; see Figure 11.

Example 6.6 (The pentagasket tiling)**.** The pentagasket tiling $\mathcal{P}$ is constructed via the self-similar system defined by the five maps

$$\Phi_j(x) = \tfrac{3-\sqrt{5}}{2}x + p_j, \qquad j = 1, \ldots, 5,$$

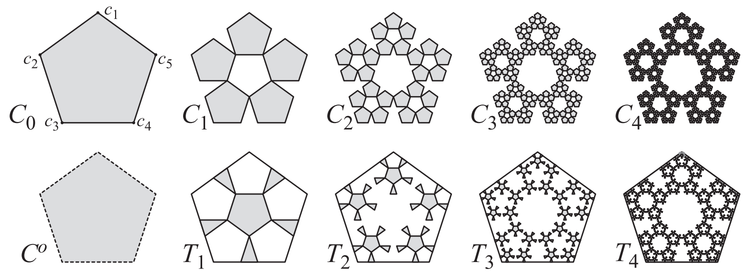

FIGURE 12. The pentagasket tiling $\mathcal{P}$. This figure was originally published in [Pe2] and appears with permission of IUMJ.

with common scaling ratio $\mathfrak{r} = \phi^{-2}$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio, and the points $\frac{p_j}{1-r} = c_j$ form the vertices of a regular pentagon of side length 1; see Figure 12.

The pentagasket $\mathcal{P}$ provides an example of multiple (noncongruent) generators G_q with $q = 1, 2, \dots, 6$. Specifically, G_1 is a regular pentagon and $G_2, \dots, G_6$ are congruent isosceles triangles, as seen in T_1 of Figure 12. This example is developed fully in [LaPe2, §9.4].

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