COHOMOLOGICAL DESCENT FOR FALTINGS' p-ADIC HODGE THEORY AND APPLICATIONS

TONGMU HE

Abstract. Faltings' approach in p-adic Hodge theory can be schematically divided into two main steps: firstly, a local reduction of the computation of the p-adic étale cohomology of a smooth variety over a p-adic local field to a Galois cohomology computation and then, the establishment of a link between the latter and differential forms. These relations are organized through Faltings ringed topos whose definition relies on the choice of an integral model of the variety, and whose good properties depend on the (logarithmic) smoothness of this model. Scholze's generalization for rigid analytic varieties has the advantage of depending only on the variety (i.e. the generic fibre). Inspired by Deligne's approach to classical Hodge theory for singular varieties, we establish a cohomological descent result for the structural sheaf of Faltings topos, which makes it possible to extend Faltings' approach to any integral model, i.e. without any smoothness assumption. An essential ingredient of our proof is a descent result of perfectoid algebras in the arc-topology due to Bhatt and Scholze. As an application of our cohomological descent, using a variant of de Jong's alteration theorem for morphisms of schemes, we generalize Faltings' main p-adic comparison theorem to any proper and finitely presented morphism of coherent schemes over an absolute integral closure of $\mathbb{Z}_p$ (without any assumption of smoothness) for torsion étale sheaves (not necessarily finite locally constant).

Contents

1. Introduction

1.1. Faltings' proof of the Hodge-Tate decomposition illustrates his approach in p-adic Hodge theory and the role of his ringed topos. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$. For a proper smooth K-scheme X, Tate conjectured that there is a canonical $\mathrm{Gal}(\overline{K}/K)$-equivariant decomposition, now called the Hodge-Tate decomposition ([Tat67, Remark, page 180]),

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widehat{\overline{K}} = \bigoplus_{0\leq q\leq n} H^q(X, \Omega^{n-q}_{X/K}) \otimes_K \widehat{\overline{K}}(q-n), \tag{1.1.1}$$

where $\widehat{\overline{K}}$ is the p-adic completion of an algebraic closure $\overline{K}$ of K, and $\widehat{\overline{K}}(q-n)$ is the $(q-n)$-th Tate twist of $\widehat{\overline{K}}$. This conjecture was settled by Faltings [Fal88, Fal02] and Tsuji [Tsu99, Tsu02] independently, and had been generalized to rigid analytic settings by Scholze [Sch13a].

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1.2. For a semi-stable $\mathcal{O}_K$-scheme X and $Y = X_{\overline{K}}$, Faltings constructed a ringed site $(\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}})$, now called the *Faltings ringed site*, whose foundation was developed by Abbes-Gros [AGT16, VI]. Faltings designed it as a bridge between the p-adic étale cohomology of Y and differential forms of X. Concretely, these links are established through the natural morphisms of sites

$$Y_{\text{ét}} \xrightarrow{\psi} \mathbf{E}^{\text{ét}}_{Y\to X} \xrightarrow{\sigma} X_{\text{ét}} \tag{1.2.1}$$

which satisfy the following properties:

(1) (Faltings' main p-adic comparison theorem, [Fal02, Thm.8, page 223], [AG20, 4.8.13]). For any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, there exists a canonical morphism

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathcal{O}_{\overline{K}} \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{1.2.2}$$

which is an almost isomorphism, that is, the cohomology groups of its cone are killed by p^r for any rational number $r > 0$.

(2) (Faltings' computation of Galois cohomology, [AG20, 6.3.8]). There exists a canonical homomorphism of $\mathcal{O}_X \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$-modules

$$\widetilde{\Omega}^q_{X/\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \longrightarrow \mathrm{R}^q\sigma_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \tag{1.2.3}$$

whose kernel and cokernel are killed by p^r for any rational number $r > \frac{2\dim(Y)+1}{p-1}$, where $\widetilde{\Omega}^q_{X/\mathcal{O}_K}$ is the module of q-th logarithmic differentials forms of X with poles in its special fibre.

Observing that $\mathbb{Z}/p^n\mathbb{Z} = \psi_*(\mathbb{Z}/p^n\mathbb{Z})$, Faltings deduced the Hodge-Tate decomposition from the degeneration and splitting of the Cartan-Leray spectral sequence for the composed functor $\mathrm{R}\Gamma(X_{\text{ét}}, -) \circ \mathrm{R}\sigma_*$, later named the Hodge-Tate spectral sequence by Scholze. Using de Jong's alteration theorem, one can deduce the Hodge-Tate decomposition for a general proper smooth K-scheme by reducing to the case where it admits a semi-stable model (cf. [Tsu02, A5]).

Recently, Abbes-Gros [AG20] generalized the Hodge-Tate spectral sequence to relative settings. Their work requires semi-stable models over $\mathcal{O}_K$. More precisely, for any projective (log-)smooth morphism between (log-)smooth log schemes over $\mathcal{O}_K$, they constructed a relative Hodge-Tate spectral sequence, which takes place in the Faltings topos of the target log scheme (cf. [AG20, 6.7.5]).

The starting point of this work is to see if the relative Hodge-Tate spectral sequence can be made free of models. A first question which has its own interest is whether we can develop p-adic Hodge theory by working over Faltings site for a general model (without any smoothness condition). Deligne [Del74] used cohomological descent of étale cohomology and Hironaka's resolution of singularities to generalize the classical Hodge theory to singular varieties. Inspired by his approach, we give a positive answer to the previous question by proving that the structural sheaf on Faltings site satisfies cohomological descent along proper hypercoverings. As an application, we generalize Faltings' main p-adic comparison theorem (which we refer to as "Faltings' comparison theorem" for short in the rest of the introduction) to general models. Other applications are expected including the extension of the relative Hodge-Tate spectral sequence to general models.

1.3. Firstly, we recall the definition of the Faltings site associated to a morphism of coherent schemes $Y \to X$ ("coherent" stands for "quasi-compact and quasi-separated") (cf. 7.7). Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the category of morphisms of coherent schemes $V \to U$ over $Y \to X$, i.e. commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{1.3.1}$$

such that U is étale over X and that V is finite étale over $Y \times_X U$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where $\{V_m \to V\}_{m\in M}$ is a finite étale covering;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where $\{U_n \to U\}_{n\in N}$ is an étale covering.

Consider the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ defined by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{1.3.2}$$

where U^V is the integral closure of U in V. It is indeed a sheaf of rings, the structural sheaf of $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. 7.6).

1.4. Recall that the cohomological descent of étale cohomology along proper hypercoverings can be generalized as follows: for a coherent S-scheme, we endow the category of coherent S-schemes $\mathbf{Sch}^{\mathrm{coh}}_{/S}$ with Voevodsky's *h-topology* which is generated by étale coverings and proper surjective morphisms of finite presentation. Then, for any torsion abelian sheaf $\mathcal{F}$ on $S_{\text{ét}}$, denoting by $a : (\mathbf{Sch}^{\mathrm{coh}}_{/S})_{\mathrm{h}} \to S_{\text{ét}}$ the natural morphism of sites, the adjunction morphism $\mathcal{F} \to \mathrm{R}a_* a^{-1}\mathcal{F}$ is an isomorphism.

This result remains true for a finer topology, the *v-topology*. A morphism of coherent schemes $T \to S$ is called a v-covering if for any morphism $\mathrm{Spec}(A) \to S$ with A a valuation ring, there exists an extension of valuation rings $A \to B$ and a lifting $\mathrm{Spec}(B) \to T$. In fact, a v-covering is a limit of h-coverings (cf. 3.6). We will describe the cohomological descent for $\overline{\mathscr{B}}$ using a new site built from the v-topology.

Definition 1.5 (cf. 3.23)**.** Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is integrally closed in S°. We define a site $\mathbf{I}_{S^\circ \to S}$ as follows:

(1) The underlying category is formed by coherent S-schemes T which are integrally closed in $S^\circ \times_S T$.
(2) The topology is generated by covering families $\{T_i \to T\}_{i\in I}$ in the v-topology.

We call $\mathbf{I}_{S^\circ \to S}$ the *v-site of S°-integrally closed coherent S-schemes*, and we call the sheaf $\mathscr{O}$ on $\mathbf{I}_{S^\circ \to S}$ associated to the presheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ the structural sheaf of $\mathbf{I}_{S^\circ \to S}$.

1.6. Let p be a prime number, $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We take $S^\circ = \mathrm{Spec}(\overline{\mathbb{Q}_p})$ and $S = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Consider a diagram of coherent schemes

$$\begin{array}{ccccc} Y & \longrightarrow & X^Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(\overline{\mathbb{Q}_p}) & \longrightarrow & \mathrm{Spec}(\overline{\mathbb{Z}_p}) & & \end{array} \tag{1.6.1}$$

where X^Y is the integral closure of X in Y and the square is Cartesian (we don't impose any condition on the regularity or finiteness of Y or X). The functor $\varepsilon^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{I}_{Y\to X^Y}$ sending $V \to U$ to U^V defines a natural morphism of ringed sites

$$\varepsilon : (\mathbf{I}_{Y\to X^Y}, \mathscr{O}) \longrightarrow (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}). \tag{1.6.2}$$

Our cohomological descent results are stated as follows:

Theorem 1.7 (Cohomological descent for Faltings sites, cf. 8.9)**.** *For any finite locally constant abelian sheaf $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{1.7.1}$$

is an almost isomorphism.

Corollary 1.8 (cf. 8.13)**.** *For any proper hypercovering $X_\bullet \to X$, if $a : \mathbf{E}^{\text{ét}}_{Y_\bullet \to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ denotes the augmentation of simplicial site where $Y_\bullet = Y \times_X X_\bullet$, then the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{1.8.1}$$

is an almost isomorphism.

The key ingredient of our proof of 1.7 is the descent of perfectoid algebras in the arc-topology (a topology finer than the v-topology) due to Bhatt-Scholze [BS19, 8.9] (cf. 5.31). The analogue in characteristic p of 1.7 is Gabber's computation of the cohomology of the structural sheaf in the h-topology (cf. 4). Theorem 1.7 allows us to descend important results for Faltings sites with nice models to Faltings sites associated to general models.

1.9. We use 1.7 to generalize Faltings' comparison theorem in the absolute case. Let A be a valuation ring extension of $\overline{\mathbb{Z}_p}$ with algebraically closed fraction field. Consider a Cartesian square of coherent schemes

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A[\frac{1}{p}]) & \longrightarrow & \mathrm{Spec}(A) \end{array} \tag{1.9.1}$$

Theorem 1.10 (Faltings' comparison theorem in the absolute case, cf. 10.18)**.** *Assume that X is proper of finite presentation over A. Then, for any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, there exists a canonical morphism*

$$\mathrm{R}\Gamma(Y_{\text{ét}},\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}} A\longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}), \tag{1.10.1}$$

which is an almost isomorphism.

We remark that the natural morphism $\psi: Y_{\text{ét}}\to \mathbf{E}^{\text{ét}}_{Y\to X}$ induces an equivalence of the categories of finite locally constant abelian sheaves on $Y_{\text{ét}}$ and $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. 10.4),

$$\mathbf{LocSys}(Y_{\text{ét}})\underset{\psi^{-1}}{\overset{\psi_*}{\rightleftarrows}}\mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X}). \tag{1.10.2}$$

As a continuation of the work of Abbes-Gros, the canonical morphism (1.10.1) (refered as *Faltings' comparison morphism*) is constructed using the acyclicity of ψ for $\mathbb{F}$, i.e. $\psi_*\mathbb{F}=\mathrm{R}\psi_*\mathbb{F}$ (so that $\mathrm{R}\Gamma(Y_{\text{ét}},\mathbb{F})=\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F})$), which is a consequence of Achinger's result on $K(\pi,1)$-schemes (cf. 10.7 and 10.9). We also propose a new way to construct Faltings' comparison morphism in the derived category of almost modules using our cohomological descent result 1.7, which avoids using the acyclicity of ψ. Indeed, there are natural morphisms of sites

$$\begin{array}{ccc}(\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a} & Y_{\text{ét}}\\ \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\psi}\\ \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X}\end{array} \tag{1.10.3}$$

and Ψ is acyclic for any torsion abelian sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, i.e. $\Psi_*(a^{-1}\mathcal{F})=\mathrm{R}\Psi_*(a^{-1}\mathcal{F})$, which allows more general coefficients and whose proof is much easier than that of ψ (cf. 3.27). We remark that this new construction won't give us a "real morphism" (1.10.1) but a canonical morphism in the derived category of almost modules (cf. 11.6).

We briefly explain the strategy for proving 1.10:

(1) Firstly, we use de Jong-Gabber-Illusie-Temkin's alteration theorem for morphisms of schemes [ILO14, X.3] to obtain a proper surjective morphism of finite presentation $X'\to X$ such that the morphism $X'\to\mathrm{Spec}(A)$ is the cofiltered limit of a system of "nice" morphisms $X'_\lambda\to T_\lambda$ of "nice" models over $\mathcal{O}_{K_\lambda}$, where K_λ is a finite extension of $\mathbb{Q}_p$ (cf. 9.11).
(2) Then, we can apply Faltings' comparison theorem in the relative case to the "nice" morphisms $X'_\lambda\to T_\lambda$ (formulated by Faltings [Fal02, Thm.6, page 266] and proved by Abbes-Gros [AG20, 5.7.4], cf. 10.14). By a limit argument, we get the comparison theorem for X'.
(3) Finally, using our cohomological descent result 1.8, we deduce the comparison theorem for X.

1.11. The site $\mathbf{I}_{Y\to X^Y}$ is also appropriate to globalize Faltings' comparison theorem. Consider a Cartesian square of coherent schemes

$$\begin{array}{ccc}Y' & \longrightarrow & X'\\ \downarrow & & \downarrow\\ Y & \longrightarrow & X\end{array} \tag{1.11.1}$$

where $Y\to X$ is Cartesian over $\mathrm{Spec}(\overline{\mathbb{Q}_p})\to\mathrm{Spec}(\overline{\mathbb{Z}_p})$. In particular, there is a natural morphism of ringed sites by the functoriality of (1.10.3),

$$f_{\mathbf{I}}:(\mathbf{I}_{Y'\to X'^{Y'}},\mathscr{O}')\longrightarrow(\mathbf{I}_{Y\to X^Y},\mathscr{O}). \tag{1.11.2}$$

Theorem 1.12 (cf. 11.11)**.** *Assume that $X'\to X$ is proper of finite presentation. Let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$ and $\mathscr{F}'=\Psi'_*a'^{-1}\mathcal{F}'$ (cf. (1.10.3)). Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow\mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}') \tag{1.12.1}$$

is an almost isomorphism.

We remark that if $\mathcal{F}' = \mathbb{Z}/p^n\mathbb{Z}$ then $\mathscr{F}' = \mathbb{Z}/p^n\mathbb{Z}$ (cf. 3.27), and that $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ is the sheafification of étale cohomologies of Y' over Y with coefficient $\mathcal{F}'$ in the v-topology (cf. 11.12). Very roughly speaking, objects of $\mathbf{I}_{Y\to X^Y}$ are "locally" the spectrums of valuation rings, and the "stalks" of (1.12.1) are Faltings' comparison morphisms (1.10.1) when $\mathcal{F}'$ is finite locally constant (cf. 11.5). Theorem 1.12 can be regarded as a scheme theoretical analogue of Scholze's comparison theorem for p-adic étale cohomology of a morphism of rigid analytic varieties [Sch13b, 3.13].

Finally, we generalize Faltings' comparison theorem in the relative case using 1.7 and 1.12.

Theorem 1.13 (Faltings' comparison theorem in the relative case, cf. 11.13 and 11.14)**.** *Assume that $Y' \to Y$ is smooth and that $X' \to X$ is proper of finite presentation. Then, for any finite locally constant abelian sheaf $\mathbb{F}'$ on $Y'_{\text{ét}}$, there exists a canonical morphism*

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{1.13.1}$$

which is an almost isomorphism, and where $f_{\text{ét}} : Y'_{\text{ét}} \to Y_{\text{ét}}$ and $f_{\mathbf{E}} : \mathbf{E}^{\text{ét}}_{Y'\to X'} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ are the natural morphisms of sites. In particular, there exists a canonical morphism

$$(\psi_*\mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{1.13.2}$$

which is an almost isomorphism, for any integer q.

1.14. The paper is structured as follows. In section 3, we establish the foundation of the site $\mathbf{I}_{S^\circ\to S}$, where proposition 3.27 discussing the cohomological properties of $\Psi : (\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \to \mathbf{I}_{S^\circ\to S}$ is the key to our new construction of Faltings' comparison morphism (cf. 11.6). Sections 4 and 5 are devoted to a detailed proof of the arc-descent for perfectoid algebras following Bhatt-Scholze [BS19, 8.9]. Since we use the language of schemes, the terminology "pre-perfectoid" is introduced for those algebras whose p-adic completions are perfectoid. In sections 6 and 7, we review the definition and some basic properties of Faltings sites and we introduce a pro-version of Faltings site to evaluate the structural sheaf on the spectrums of pre-perfectoid algebras. Then, we prove our cohomological descent result in section 8. In section 9, we review de Jong-Gabber-Illusie-Temkin's alteration theorem and apply it to schemes over a valuation ring of height 1. Section 10 is devoted to proving our generalization of Faltings' comparison theorem in the absolute case. In section 11, we give a new construction of Faltings' comparison morphism and our generalization of Faltings' comparison theorem in the relative case.

1.15. Our work suggests that the site $\mathbf{I}_{S^\circ\to S}$ could play an important role in Faltings' p-adic Hodge theory, as it allows us to work with general models (at least in the "comparison" part). A recent work of Guo [Guo19] generalized the Hodge-Tate decomposition to singular rigid analytic varieties. Thus, it seems reasonable that we could use $\mathbf{I}_{S^\circ\to S}$ to generalize the "Galois cohomology" part of Faltings' p-adic Hodge theory and relate it to Deligne-du Bois complex, and the final goal is to generalize Abbes-Gros' relative Hodge-Tate spectral sequence to morphisms of singular varieties. If one would like to be more optimistic about the site $\mathbf{I}_{S^\circ\to S}$, then it is interesting to look for an intrinsic proof of Faltings' comparison theorems on $\mathbf{I}_{S^\circ\to S}$ instead of doing alterations and cohomological descent. A further possible generalization would be over a general base $S^\circ \to S$ (not only Cartesian over $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$). On the other hand, it seems that the site $\mathbf{I}_{S^\circ\to S}$ is closely related to Bhatt-Scholze's perfectoidization in their prismatic cohomology theory [BS19, 8], and further concrete relations are waiting to be explored. The author is still studying these problems.

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2. Notation and Conventions

2.1. We fix a prime number p throughout this paper. For any monoid M, we denote by $\mathrm{Frob} : M \to M$ the map sending an element x to x^p and we call it the *Frobenius* of M. For a ring R, we denote by $R^\times$ the group of units of R. A ring R is called *absolutely integrally closed* if any monic polynomial $f \in R[T]$ has a root in R ([Sta21, 0DCK]). We remark that quotients, localizations and products of absolutely integrally closed rings are still absolutely integrally closed.

Recall that a valuation ring is a domain V such that for any element x in its fraction field, if $x \notin V$ then $x^{-1} \in V$. The family of ideals of V is totally ordered by the inclusion relation ([Bou06, VI.§1.2, Thm.1]). In particular, a radical ideal of V is a prime ideal. Moreover, any quotient of V by a prime ideal and any localization of V are still valuations rings ([Sta21, 088Y]). We remark that V is normal, and that V is absolutely integrally closed if and only if its fraction field is algebraically closed. An *extension of valuation rings* is an injective and local homomorphism of valuation rings.

2.2. Following [SGA 4_{II}, VI.1.22], a *coherent* scheme (resp. morphism of schemes) stands for a quasi-compact and quasi-separated scheme (resp. morphism of schemes). For a coherent morphism $Y \to X$ of schemes, we denote by X^Y the integral closure of X in Y ([Sta21, 0BAK]). For an X-scheme Z, we say that Z is *Y-integrally closed* if $Z = Z^{Y \times_X Z}$.

2.3. Throughout this paper, we fix two universes $\mathbb{U}$ and $\mathbb{V}$ such that the set of natural numbers $\mathbb{N}$ is an element of $\mathbb{U}$ and that $\mathbb{U}$ is an element of $\mathbb{V}$ ([SGA 4_{I}, I.0]). In most cases, we won't emphasize this set theoretical issue. Unless stated otherwise, we only consider $\mathbb{U}$-small schemes and we denote by **Sch** the category of $\mathbb{U}$-small schemes, which is a $\mathbb{V}$-small category.

2.4. Let C be a category. We denote by $\widehat{C}$ the category of presheaves of $\mathbb{V}$-small sets on C. If C is a $\mathbb{V}$-site ([SGA 4_{I}, II.3.0.2]), we denote by $\widetilde{C}$ the topos of sheaves of $\mathbb{V}$-small sets on C. We denote by $h^C : C \to \widehat{C}$, $x \mapsto h^C_x$ the Yoneda embbeding ([SGA 4_{I}, I.1.3]), and by $\widehat{C} \to \widetilde{C}$, $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{a}}$ the sheafification functor ([SGA 4_{I}, II.3.4]). Unless stated otherwise, a site in this paper stands for a site where *all finite limits are representable*.

2.5. Let $u^+ : C \to D$ be a functor of categories. We denote by $u^{\mathrm{p}} : \widehat{D} \to \widehat{C}$ the functor that associates to a presheaf $\mathcal{G}$ of $\mathbb{V}$-small sets on D the presheaf $u^{\mathrm{p}}\mathcal{G} = \mathcal{G} \circ u^+$. If C is $\mathbb{V}$-small and D is a $\mathbb{V}$-category, then u^{p} admits a left adjoint u_{p} [Sta21, 00VC] and a right adjoint ${}_{\mathrm{p}}u$ [Sta21, 00XF] (cf. [SGA 4_{I}, I.5]). So we have a sequence of adjoint functors

$$u_{\mathrm{p}},\ u^{\mathrm{p}},\ {}_{\mathrm{p}}u. \tag{2.5.1}$$

If moreover C and D are $\mathbb{V}$-sites, then we denote by $u_{\mathrm{s}}, u^{\mathrm{s}}, {}_{\mathrm{s}}u$ the functors of the topoi $\widetilde{C}$ and $\widetilde{D}$ of sheaves of $\mathbb{V}$-small sets induced by composing the sheafification functor with the functors $u_{\mathrm{p}}, u^{\mathrm{p}}, {}_{\mathrm{p}}u$ respectively. As we only consider finite complete sites, we say that the functor u^+ gives a *morphism of sites*, if u^+ is left exact and preserves covering families ([SGA 4_{I}, IV.4.9.2]). Then, we denote by

$$u = (u^{-1}, u_*) : \widetilde{D} \to \widetilde{C} \tag{2.5.2}$$

the associated morphism of topoi, where $u^{-1} = u_{\mathrm{s}}$ and $u_* = u^{\mathrm{s}} = u^{\mathrm{p}}|_{\widetilde{D}}$. We remark that the notation here, adopted by [Sta21], is slightly different with that in [SGA 4_{I}] (cf. [Sta21, 0CMZ]).

3. The v-site of Integrally Closed Schemes

Definition 3.1. Let $X \to Y$ be a quasi-compact morphism of schemes.

(1) We say that $X \to Y$ is a *v-covering*, if for any valuation ring V and any morphism $\mathrm{Spec}(V) \to Y$, there exists an extension of valuation rings $V \to W$ (2.1) and a commutative diagram (cf. [Sta21, 0ETN])

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array} \tag{3.1.1}$$

(2) Let π be an element of $\Gamma(Y, \mathcal{O}_Y)$. We say that $X \to Y$ is an *arc-covering* (resp. *π-complete arc-covering*), if for any valuation ring (resp. π-adically complete valuation ring) V of height ≤ 1 and any morphism $\mathrm{Spec}(V) \to Y$, there exists an extension of valuation rings (resp. π-adically complete valuation rings) $V \to W$ of height ≤ 1 and a commutative diagram (3.1.1) (cf. [BM20, 1.2], [CS19, 2.2.1]).

(3) We say that $X \to Y$ is an *h-covering*, if it is a v-covering and locally of finite presentation (cf. [Sta21, 0ETS]).

We note that an arc-covering is simply a 0-complete arc-covering.

Lemma 3.2. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be quasi-compact morphisms of schemes, $\pi \in \Gamma(X, \mathcal{O}_X)$, $\tau \in \{h,\ v,\ \pi\text{-complete arc}\}$.*

1. *If f is a τ-covering, then any base change of f is also a τ-covering.*
2. *If f and g are τ-coverings, then $f \circ g$ is also a τ-covering.*
3. *If $f \circ g$ is a τ-covering (and if f is locally of finite presentation when $\tau = h$), then f is also a τ-covering.*

Proof. It follows directly from the definitions. □

3.3. Let $\mathbf{Sch}^{\mathrm{coh}}$ be the category of coherent $\mathbb{U}$-small schemes, $\tau \in \{\mathrm{h}, \mathrm{v}, \mathrm{arc}\}$. We endow $\mathbf{Sch}^{\mathrm{coh}}$ with the τ-topology generated by the pretopology formed by families of morphisms $\{X_i \to X\}_{i\in I}$ with I finite such that $\coprod_{i\in I} X_i \to X$ is a τ-covering, and we denote the corresponding site by $\mathbf{Sch}^{\mathrm{coh}}_{\tau}$. It is clear that a morphism $Y \to X$ (which is locally of finite presentation if $\tau = \mathrm{h}$) is a τ-covering if and only if $\{Y \to X\}$ is a covering family in $\mathbf{Sch}^{\mathrm{coh}}_{\tau}$ by 3.2 and [SGA 4_{I}, II.1.4].

For any coherent $\mathbb{U}$-small scheme X, we endow the category $\mathbf{Sch}^{\mathrm{coh}}_{/X}$ of objects of $\mathbf{Sch}^{\mathrm{coh}}$ over X with the topology induced by the τ-topology of $\mathbf{Sch}^{\mathrm{coh}}$, i.e. the topology generated by the pretopology formed by families of X-morphisms $\{Y_i \to Y\}_{i\in I}$ with I finite such that $\coprod_{i\in I} Y_i \to Y$ is a τ-covering ([SGA 4_{I}, III.5.2]). For any sheaf $\mathcal{F}$ of $\mathbb{V}$-small abelian groups on the site $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\tau}$, we denote its q-th cohomology by $H^q_{\tau}(X, \mathcal{F})$.

Lemma 3.4. *Let $f : X \to Y$ be a quasi-compact morphism of schemes, $\pi \in \Gamma(Y, \mathcal{O}_Y)$.*

1. *If f is proper surjective or faithfully flat, then f is a v-covering.*
2. *If f is an h-covering and Y is affine, then there exists a proper surjective morphism $Y' \to Y$ of finite presentation and a finite affine open covering $Y' = \bigcup_{r=1}^{n} Y'_r$ such that $Y'_r \to Y$ factors through f for each r.*
3. *If f is an h-covering and if there exists a directed inverse system $(f_\lambda : X_\lambda \to Y_\lambda)_{\lambda\in\Lambda}$ of finitely presented morphisms of coherent schemes with affine transition morphisms $\psi_{\lambda'\lambda} : X_{\lambda'} \to X_\lambda$ and $\phi_{\lambda'\lambda} : Y_{\lambda'} \to Y_\lambda$ such that $X = \lim X_\lambda$, $Y = \lim Y_\lambda$ and that f_λ is the base change of f_{λ_0} by $\phi_{\lambda\lambda_0}$ for some index $\lambda_0 \in \Lambda$ and any $\lambda \geq \lambda_0$, then there exists an index $\lambda_1 \geq \lambda_0$ such that f_λ is an h-covering for any $\lambda \geq \lambda_1$.*
4. *If f is a v-covering, then it is a π-complete arc-covering.*
5. *Let π' be another element of $\Gamma(Y, \mathcal{O}_Y)$ which divides π. If f is a π-complete arc-covering, then it is a π'-complete arc-covering.*
6. *If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a π-complete arc-covering, then the morphism $\mathrm{Spec}(\widehat{B}) \to \mathrm{Spec}(\widehat{A})$ between the spectrums of their π-adic completions is also a π-complete arc-covering.*

Proof. (1), (2) are proved in [Sta21, 0ETK, 0ETU] respectively.

(3) To show that one can take $\lambda_1 \geq \lambda_0$ such that f_{λ_1} is an h-covering, we may assume that Y_{λ_0} is affine by replacing it by a finite affine open covering by 3.2 and (1). Thus, applying (2) to the h-covering f and using [EGA IV_3, 8.8.2, 8.10.5], there exists an index $\lambda_1 \geq \lambda_0$, a proper surjective morphism $Y'_{\lambda_1} \to Y_{\lambda_1}$ and a finite affine open covering $Y'_{\lambda_1} = \bigcup_{r=1}^{n} Y'_{r\lambda_1}$ such that the morphisms $Y'_r \to Y' \to Y$ are the base changes of the morphisms $Y'_{r\lambda_1} \to Y'_{\lambda_1} \to Y_{\lambda_1}$ by the transition morphism $Y \to Y_{\lambda_1}$, and that $Y'_{r\lambda_1} \to Y_{\lambda_1}$ factors through X_{λ_1}. This shows that f_{λ_1} is an h-covering by 3.2 and (1).

(4) With the notation in (3.1.1), if V is a π-adically complete valuation ring of height ≤ 1 with maximal ideal $\mathfrak{m}$, then since the family of prime ideals of W is totally ordered by the inclusion relation (2.1), we take the maximal prime ideal $\mathfrak{p} \subseteq W$ over $0 \subseteq V$ and the minimal prime ideal $\mathfrak{q} \subseteq W$ over $\mathfrak{m} \subseteq V$. Then, $\mathfrak{p} \subseteq \mathfrak{q}$ and $W' = (W/\mathfrak{p})_{\mathfrak{q}}$ over V is an extension of valuation rings of height ≤ 1. Since $\pi \in \mathfrak{m}$ and W' is of height ≤ 1, the π-adic completion $\widehat{W'}$ is still a valuation ring extension of V of height ≤ 1 (cf. [Bou06, VI.§5.3, Prop.5]), which proves (4).

(5) Since a π'-adically complete valuation ring V is also π-adically complete ([Sta21, 090T]), there exists a lifting $\mathrm{Spec}(W) \to X$ for any morphism $\mathrm{Spec}(V) \to Y$. After replacing W by its π'-adic completion, the conclusion follows.

(6) Let V be a π-adically complete valuation ring of height ≤ 1. Given a morphism $\widehat{A} \to V$, there exists a lifting $B \to W$ where $V \to W$ is an extension of π-adically complete valuation rings of height ≤ 1. It is clear that $B \to W$ factors through $\widehat{B}$, which proves (6). □

3.5. Let X be a coherent scheme, $\mathbf{Sch}^{\mathrm{fp}}_{/X}$ the full subcategory of $\mathbf{Sch}^{\mathrm{coh}}_{/X}$ formed by finitely presented X-schemes. We endow it with the topology generated by the pretopology formed by families of morphisms $\{Y_i \to Y\}_{i\in I}$ with I finite such that $\coprod_{i\in I} Y_i \to Y$ is an h-covering, and we denote the corresponding site by $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. It is clear that this topology coincides with the topologies induced from $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ and from $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}}$. The inclusion functors $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}} \xrightarrow{\xi^+} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\zeta^+} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ define morphisms of sites (2.5)

$$(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \xrightarrow{\zeta} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\xi} (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}. \tag{3.5.1}$$

Lemma 3.6. *Let X be a coherent scheme. Then, for any covering family $\mathfrak{U} = \{Y_i \to Y\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ with I finite,*

(i) *there exists a directed inverse system $(Y_\lambda)_{\lambda\in\Lambda}$ of finitely presented X-schemes with affine transition morphisms such that $Y = \lim Y_\lambda$, and*
(ii) *for each $i \in I$, there exists a directed inverse system $(Y_{i\lambda})_{\lambda\in\Lambda}$ of finitely presented X-schemes with affine transition morphisms over the inverse system $(Y_\lambda)_{\lambda\in\Lambda}$ such that $Y_i = \lim Y_{i\lambda}$, and*
(iii) *for each $\lambda \in \Lambda$, the family $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ is a covering in $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$.*

Proof. We take a directed set A such that for each $i \in I$, we can write Y_i as a cofiltered limit of finitely presented Y-schemes $Y_i = \lim_{\alpha\in A} Y_{i\alpha}$ with affine transition morphisms ([Sta21, 09MV]). We see that $\coprod_{i\in I} Y_{i\alpha} \to Y$ is an h-covering for each $\alpha \in A$ by 3.2.

We write Y as a cofiltered limit of finitely presented X-schemes $Y = \lim_{\beta\in B} Y_\beta$ with affine transition morphisms ([Sta21, 09MV]). By [EGA IV$_3$, 8.8.2, 8.10.5] and 3.4.(3), for each $\alpha \in A$, there exists an index $\beta_\alpha \in B$ such that the morphism $Y_{i\alpha} \to Y$ is the base change of a finitely presented morphism $Y_{i\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ by the transition morphism $Y \to Y_{\beta_\alpha}$ for each $i \in I$, and that $\coprod_{i\in I} Y_{i\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ is an h-covering. For each $\beta \geq \beta_\alpha$, let $Y_{i\alpha\beta}$ be the base change of $Y_{i\alpha\beta_\alpha}$ by $Y_\beta \to Y_{\beta_\alpha}$.

We define a category Λ^{op}, whose set of objects is $\{(\alpha,\beta) \in A \times B \mid \beta \geq \beta_\alpha\}$, and for any two objects $\lambda' = (\alpha',\beta')$, $\lambda = (\alpha,\beta)$, the set $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\lambda',\lambda)$ is

(i) the subset of $\prod_{i\in I} \mathrm{Hom}_{Y_{\beta'}}(Y_{i\alpha'\beta'}, Y_{i\alpha\beta'})$ formed by elements $f = (f_i)_{i\in I}$ such that for each $i \in I$, $f_i : Y_{i\alpha'\beta'} \to Y_{i\alpha\beta'}$ is affine and the base change of f_i by $Y \to Y_{\beta'}$ is the transition morphism $Y_{i\alpha'} \to Y_{i\alpha}$, if $\alpha' \geq \alpha$ and $\beta' \geq \beta$;
(ii) empty, if else.

The composition of morphisms $(g_i : Y_{i\alpha''\beta''} \to Y_{i\alpha'\beta''})_{i\in I}$ with $(f_i : Y_{i\alpha'\beta'} \to Y_{i\alpha\beta'})_{i\in I}$ in Λ^{op} is $(g_i \circ f'_i : Y_{i\alpha''\beta''} \to Y_{i\alpha\beta''})$, where f'_i is the base change of f_i by the transition morphism $Y_{\beta''} \to Y_{\beta'}$. We see that Λ^{op} is cofiltered by [EGA IV$_3$, 8.8.2]. Let Λ be the opposite category of Λ^{op}. For each $i \in I$ and $\lambda = (\alpha,\beta) \in \Lambda$, we set $Y_\lambda = Y_\beta$ and $Y_{i\lambda} = Y_{i\alpha\beta}$. It is clear that the natural functors $\Lambda \to A$ and $\Lambda \to B$ are cofinal ([SGA 4$_{\mathrm{I}}$, I.8.1.3]). After replacing Λ by a directed set ([Sta21, 0032]), the families $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ satisfy the required conditions. □

Lemma 3.7. *With the notation in 3.5, let $\mathcal{F}$ be a presheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, (Y_λ) a directed inverse system of finitely presented X-schemes with affine transition morphisms, $Y = \lim Y_\lambda$. Then, we have $\nu_{\mathrm{p}}\mathcal{F}(Y) = \operatorname{colim} \mathcal{F}(Y_\lambda)$, where $\nu^+ = \xi^+$ (resp. $\nu^+ = \zeta^+ \circ \xi^+$).*

Proof. Notice that the presheaf $\mathcal{F}$ is a filtered colimit of representable presheaves by [SGA 4$_{\mathrm{I}}$, I.3.4]

$$\mathcal{F} = \operatorname*{colim}_{Y'\in(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{/\mathcal{F}}} h_{Y'}. \tag{3.7.1}$$

Thus, we may assume that $\mathcal{F}$ is representable by a finitely presented X-scheme Y' since the section functor $\Gamma(Y,-)$ commutes with colimits of presheaves ([Sta21, 00VB]). Then, we have

$$\nu_{\mathrm{p}} h_{Y'}(Y) = h_{\nu^+(Y')}(Y) = \mathrm{Hom}_X(Y,Y') = \operatorname{colim} \mathrm{Hom}_X(Y_\lambda, Y') = \operatorname{colim} h_{Y'}(Y_\lambda) \tag{3.7.2}$$

where the first equality follows from [Sta21, 04D2], and the third equality follows from [EGA IV$_3$, 8.14.2]. □

Proposition 3.8. *With the notation in 3.5, let $\mathcal{F}$ be an abelian sheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, (Y_λ) a directed inverse system of finitely presented X-schemes with affine transition morphisms, $Y = \lim Y_\lambda$. Let $\tau =$ h*

and $\nu^+ = \xi^+$ *(resp.* $\tau = \mathrm{v}$ *and* $\nu^+ = \zeta^+ \circ \xi^+$*). Then, for any integer* q*, we have*

$$H^q_\tau(Y, \nu^{-1}\mathcal{F}) = \operatorname{colim} H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y_\lambda})_{\mathrm{h}}, \mathcal{F}). \tag{3.8.1}$$

In particular, the canonical morphism $\mathcal{F} \longrightarrow \mathrm{R}\nu_*\nu^{-1}\mathcal{F}$ *is an isomorphism.*

Proof. For the second assertion, the sheaf $\mathrm{R}^q\nu_*\nu^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y \mapsto H^q_\tau(Y, \nu^{-1}\mathcal{F}) = H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y})_{\mathrm{h}}, \mathcal{F})$ by the first assertion, which is $\mathcal{F}$ if $q = 0$ and vanishes otherwise.

We claim that it suffices to show that (3.8.1) holds for any injective abelian sheaf $\mathcal{F} = \mathcal{I}$ on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. Indeed, if so, then we prove by induction on q that (3.8.1) holds in general. The case where $q \leq -1$ is trivial. We set $H^q_1(\mathcal{F}) = H^q_\tau(Y, \nu^{-1}\mathcal{F})$ and $H^q_2(\mathcal{F}) = \operatorname{colim} H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y_\lambda})_{\mathrm{h}}, \mathcal{F})$. We embed an abelian sheaf $\mathcal{F}$ to an injective abelian sheaf $\mathcal{I}$. Consider the exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ and the morphism of long exact sequences

$$\begin{array}{ccccccccc} H^{q-1}_1(\mathcal{I}) & \longrightarrow & H^{q-1}_1(\mathcal{G}) & \longrightarrow & H^q_1(\mathcal{F}) & \longrightarrow & H^q_1(\mathcal{I}) & \longrightarrow & H^q_1(\mathcal{G}) \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\ H^{q-1}_2(\mathcal{I}) & \longrightarrow & H^{q-1}_2(\mathcal{G}) & \longrightarrow & H^q_2(\mathcal{F}) & \longrightarrow & H^q_2(\mathcal{I}) & \longrightarrow & H^q_2(\mathcal{G}) \end{array} \tag{3.8.2}$$

If (3.8.1) holds for any abelian sheaf $\mathcal{F}$ for degree $q-1$, then γ_1, γ_2, γ_4 are isomorphisms and thus γ_3 is injective by the 5-lemma ([Sta21, 05QA]). Thus, γ_5 is also injective since $\mathcal{F}$ is an arbitrary abelian sheaf. Then, we see that γ_3 is an isomorphism, which completes the induction procedure.

For an injective abelian sheaf $\mathcal{I}$ on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, we claim that for any covering family $\mathfrak{U} = \{(Y_i \to Y)\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_\tau$ with I finite, the augmented Čech complex associated to the presheaf $\nu_{\mathrm{p}}\mathcal{I}$

$$\nu_{\mathrm{p}}\mathcal{I}(Y) \to \prod_{i\in I} \nu_{\mathrm{p}}\mathcal{I}(Y_i) \to \prod_{i,j\in I} \nu_{\mathrm{p}}\mathcal{I}(Y_i \times_Y Y_j) \to \cdots \tag{3.8.3}$$

is exact. Admitting this claim, we see that $\nu_{\mathrm{p}}\mathcal{I}$ is indeed a sheaf, i.e. $\nu^{-1}\mathcal{I} = \nu_{\mathrm{p}}\mathcal{I}$, and the vanishing of higher Čech cohomologies implies that $H^q_\tau(Y, \nu^{-1}\mathcal{I}) = 0$ for $q > 0$ by 3.6 ([Sta21, 03F9]), which completes the proof together with 3.7. For the claim, we take the covering families $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$ constructed by 3.6. By 3.7, the sequence (3.8.3) is the filtered colimit of the augmented Čech complexes

$$\mathcal{I}(Y_\lambda) \to \prod_{i\in I} \mathcal{I}(Y_{i\lambda}) \to \prod_{i,j\in I} \mathcal{I}(Y_{i\lambda} \times_{Y_\lambda} Y_{j\lambda}) \to \cdots, \tag{3.8.4}$$

which are exact since $\mathcal{I}$ is an injective abelian sheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. □

Corollary 3.9. *Let* X *be a coherent scheme,* $\mathcal{F}$ *a torsion abelian sheaf on the site* $X_{\text{ét}}$ *formed by coherent étale* X*-schemes endowed with the étale topology,* $a : (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \to X_{\text{ét}}$ *the morphism of sites defined by the inclusion functor. Then, the canonical morphism* $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ *is an isomorphism.*

Proof. Consider the morphisms of sites defined by inclusion functors

$$(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \xrightarrow{\zeta} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\xi} (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}} \xrightarrow{\mu} X_{\text{ét}}. \tag{3.9.1}$$

Notice that the morphism $\mathcal{F} \to \mathrm{R}(\mu \circ \xi)_*(\mu \circ \xi)^{-1}\mathcal{F}$ is an isomorphism by [Sta21, 0EWG]. Hence, $\mathcal{F} \to \mathrm{R}\mu_*\mu^{-1}\mathcal{F}$ is an isomorphism by 3.8, and thus so is $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ by 3.8. □

Corollary 3.10. *Let* $f : X \to Y$ *be a proper morphism of coherent schemes,* $\mathcal{F}$ *a torsion abelian sheaf on* $X_{\text{ét}}$*. Consider the commutative diagram*

$$\begin{array}{ccc} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} & \xrightarrow{a_X} & X_{\text{ét}} \\ \downarrow{\scriptstyle f_{\mathrm{v}}} & & \downarrow{\scriptstyle f_{\text{ét}}} \\ (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a_Y} & Y_{\text{ét}} \end{array} \tag{3.10.1}$$

where f_{v} *and* $f_{\text{ét}}$ *are defined by the base change by* f*. Then, the canonical morphism*

$$a_Y^{-1}\mathrm{R}f_{\text{ét}*}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{v}*}a_X^{-1}\mathcal{F} \tag{3.10.2}$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} & \xrightarrow{\zeta_X} & (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} & \xrightarrow{b_X} & X_{\text{ét}} \\ \downarrow{\scriptstyle f_{\mathrm{v}}} & & \downarrow{\scriptstyle f_{\mathrm{h}}} & & \downarrow{\scriptstyle f_{\text{ét}}} \\ (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{\zeta_Y} & (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{h}} & \xrightarrow{b_Y} & Y_{\text{ét}} \end{array} \tag{3.10.3}$$

The canonical morphism $b_Y^{-1}\mathrm{R}f_{\text{ét}*}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{h}*}b_X^{-1}\mathcal{F}$ is an isomorphism by [Sta21, 0EWF]. It remains to show that the canonical morphism $\zeta_Y^{-1}\mathrm{R}f_{\mathrm{h}*}b_X^{-1}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{v}*}a_X^{-1}\mathcal{F}$ is an isomorphism. Let Y' be a coherent Y-scheme and we set $g : X' = Y' \times_Y X \to X$. For each integer q, $\zeta_Y^{-1}\mathrm{R}^q f_{\mathrm{h}*}b_X^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y' \mapsto H^q_{\mathrm{h}}(X', b_{X'}^{-1}g_{\text{ét}}^{-1}\mathcal{F}) = H^q_{\text{ét}}(X', g_{\text{ét}}^{-1}\mathcal{F})$ by [Sta21, 0EWH], and $\mathrm{R}^q f_{\mathrm{v}*}a_X^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y' \mapsto H^q_{\mathrm{v}}(X', a_{X'}^{-1}g_{\text{ét}}^{-1}\mathcal{F}) = H^q_{\text{ét}}(X', g_{\text{ét}}^{-1}\mathcal{F})$ by 3.9. □

Lemma 3.11. *Let A be a product of (resp. absolutely integrally closed) valuation rings* (2.1)*.*

(1) *Any finitely generated ideal of A is principal.*
(2) *Any connected component of* $\mathrm{Spec}(A)$ *with the reduced closed subscheme structure is isomorphic to the spectrum of a (resp. absolutely integrally closed) valuation ring.*

Proof. (1) is proved in [Sta21, 092T], and (2) follows from the proof of [BS17, 6.2]. □

Lemma 3.12. *Let X be a $\mathbb{U}$-small scheme, $y \rightsquigarrow x$ a specialization of points of X. Then, there exists a $\mathbb{U}$-small family $\{f_\lambda : \mathrm{Spec}(V_\lambda) \to X\}_{\lambda \in \Lambda_{y \rightsquigarrow x}}$ of morphisms of schemes such that*

(i) *the ring V_λ is a $\mathbb{U}$-small (resp. absolutely integrally closed) valuation ring, and that*
(ii) *the morphism f_λ maps the generic point and closed point of* $\mathrm{Spec}(V_\lambda)$ *to y and x respectively, and that*
(iii) *for any morphism of schemes $f : \mathrm{Spec}(V) \to X$ where V is a (resp. absolutely integrally closed) valuation ring such that f maps the generic point and closed point of V to y and x respectively, there exists an element $\lambda \in \Lambda_{y \rightsquigarrow x}$ such that f factors through f_λ and that $V_\lambda \to V$ is an extension of valuation rings.*

Proof. Let K_y be the residue field $\kappa(y)$ of y (resp. an algebraic closure of $\kappa(y)$). Let $\mathfrak{p}_y$ be the prime ideal of the local ring $\mathcal{O}_{X,x}$ corresponding to the point y, and let $\{V_\lambda\}_{\lambda \in \Lambda_{y \rightsquigarrow x}}$ be the set of all valuation rings with fraction field K_y which contain $\mathcal{O}_{X,x}/\mathfrak{p}_y$ such that the injective homomorphism $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V_\lambda$ is local. The smallness of $\Lambda_{y \rightsquigarrow x}$ and V_λ is clear, and the inclusion $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V_\lambda$ induces a morphism $f_\lambda : \mathrm{Spec}(V_\lambda) \to X$ satisfying (ii). It remains to check (iii). The morphism f induces an injective and local homomorphism $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V$. Notice that $\mathcal{O}_{X,x}/\mathfrak{p}_y \to \mathrm{Frac}(V)$ factors through K_y and that $K_y \cap V$ is a valuation ring with fraction field K_y. It is clear that $K_y \cap V \to V$ is local and injective, which shows that $K_y \cap V$ belongs to the set $\{V_\lambda\}_{\lambda \in \Lambda_{y \rightsquigarrow x}}$ constructed before. □

Lemma 3.13. *Let $f : \mathrm{Spec}(V) \to X$ be a morphism of schemes where V is a valuation ring. We denote by a and b the closed point and generic point of* $\mathrm{Spec}(V)$ *respectively. If $c \in X$ is a generalization of $f(b)$, then there exists an absolutely integrally closed valuation ring W, a prime ideal $\mathfrak{p}$ of W, and a morphism $g : \mathrm{Spec}(W) \to X$ satisfying the following conditions:*

(i) *If z, y, x denote respectively the generic point, the point $\mathfrak{p}$ and the closed point of* $\mathrm{Spec}(W)$*, then $g(z) = c$, $g(y) = f(b)$ and $g(x) = f(a)$.*
(ii) *The induced morphism* $\mathrm{Spec}(W/\mathfrak{p}) \to X$ *factors through f, and the induced morphism $V \to W/\mathfrak{p}$ is an extension of valuation rings.*

Proof. According to [EGA II, 7.1.4], there exists an absolutely integrally closed valuation ring U and a morphism $\mathrm{Spec}(U) \to X$ which maps the generic point z and the closed point y of $\mathrm{Spec}(U)$ to c and $f(b)$ respectively. After extending U, we may assume that the morphism $y \to f(b)$ factors through b ([EGA II, 7.1.2]). We denote by $\kappa(y)$ the residue field of the point y. Let V' be a valuation ring extension of V with fraction field $\kappa(y)$, and let W be the preimage of V' by the surjection $U \to \kappa(y)$. Then, the maximal ideal $\mathfrak{p} = \mathrm{Ker}(U \to \kappa(y))$ of U is a prime ideal of W, and $W/\mathfrak{p} = V'$. We claim that W is an absolutely integrally closed valuation ring such that $W_\mathfrak{p} = U$. Indeed, firstly note that the fraction fields of U and W are equal as $\mathfrak{p} \subseteq W$. Let γ be an element of $\mathrm{Frac}(W) \setminus W$. If $\gamma \in U$, then $\gamma^{-1} \in W \setminus \mathfrak{p}$ by definition since $\gamma^{-1} \in U \setminus \mathfrak{p}$ and V is a valuation ring, and then $\gamma \in W_\mathfrak{p}$. If $\gamma \notin U$, then $\gamma^{-1} \in \mathfrak{p}$ since U is a

valuation ring, and then $\gamma \notin W_{\mathfrak{p}}$. Thus, we have proved the claim, which shows that W satisfies the required conditions. □

Proposition 3.14. *Let X be a coherent $\mathbb{U}$-small scheme, X° a quasi-compact dense open subset of X. Then, there exists a $\mathbb{U}$-small product A of absolutely integrally closed $\mathbb{U}$-small valuation rings and a v-covering $\mathrm{Spec}(A) \to X$ such that $\mathrm{Spec}(A)$ is X°-integrally closed* (2.2).

Proof. After replacing X by a finite affine open covering, we may assume that $X = \mathrm{Spec}(R)$. For a specialization $y \rightsquigarrow x$ of points of X, let $\{R \to V_\lambda\}_{\lambda \in \Lambda_{y \rightsquigarrow x}}$ be the $\mathbb{U}$-small set constructed in 3.12. Let $\Lambda = \coprod_{y \in X^\circ} \Lambda_{y \rightsquigarrow x}$ where $y \rightsquigarrow x$ runs through all specializations in X such that $y \in X^\circ$. We take $A = \prod_{\lambda \in \Lambda} V_\lambda$ and $R \to A$ the natural homomorphism. As a quasi-compact open subscheme of $\mathrm{Spec}(A)$, $X^\circ \times_X \mathrm{Spec}(A)$ is the spectrum of $A[1/\pi]$ for an element $\pi = (\pi_\lambda)_{\lambda \in \Lambda} \in A$ by 3.11.(1) ([Sta21, 01PH]). Notice that $\pi_\lambda \neq 0$ for any $\lambda \in \Lambda$. We see that A is integrally closed in $A[1/\pi]$. It remains to check that $\mathrm{Spec}(A) \to X$ is a v-covering. For any morphism $f : \mathrm{Spec}(V) \to X$ where V is a valuation ring, by 3.13, there exists an absolutely integrally closed valuation ring W, a prime ideal $\mathfrak{p}$ of W and a morphism $g : \mathrm{Spec}(W) \to X$ such that g maps the generic point of W into X° and that $W/\mathfrak{p}$ is a valuation ring extension of V. By construction, there exists $\lambda \in \Lambda$ such that g factors through $\mathrm{Spec}(V_\lambda) \to X$. We see that f lifts to the composition of $\mathrm{Spec}(W/\mathfrak{p}) \to \mathrm{Spec}(V_\lambda) \to \mathrm{Spec}(A)$. □

Proposition 3.15. *Consider a commutative diagram of schemes*

$$\begin{array}{ccccc} Y' & \longrightarrow & Z' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Z & \longrightarrow & X \end{array} \tag{3.15.1}$$

where $Z' \to Z$ and $X' \to X$ are quasi-compact. Assume that $Y' \to Y \times_X X'$ is surjective, $Y \to Z$ is dominant, $Z \to X$ is separated and $Z' \to X'$ is integral. If $X' \to X$ is a v-covering, then $Z' \to Z$ is also a v-covering.

Proof. Notice that $Z' \to Z \times_X X'$ is still integral as $Z \to X$ is separated. After replacing $X' \to X$ by $Z \times_X X' \to Z$, we may assume that $Z = X$. Let $\mathrm{Spec}(V) \to Z$ be a morphism of schemes where V is a valuation ring. Since $Y \to Z$ is dominant, by 3.13, there exists a morphism $\mathrm{Spec}(W) \to Z$ where W is an absolutely integrally closed valuation ring, a prime ideal $\mathfrak{p}$ of W such that $W/\mathfrak{p}$ is a valuation ring extension of V and that the generic point ξ of $\mathrm{Spec}(W)$ is over the image of $Y \to Z$. After extending W ([Sta21, 00IA]), we may assume that there exists a lifting $\xi \to Y$ of $\xi \to Z$. The morphism $\mathrm{Spec}(W) \to Z = X$ admits a lifting $\mathrm{Spec}(W') \to X'$ where $W \to W'$ is an extension of valuation rings. We claim that after extending W', $\mathrm{Spec}(W') \to X'$ factors through Z'. Indeed, if ξ' denotes the generic point of $\mathrm{Spec}(W')$, as $Y' \to Y \times_X X'$ is surjective, after extending W', we may assume that there exists an X'-morphism $\xi' \to Y'$ which is over $\xi \to Y$. Since $\mathrm{Spec}(W')$ is integrally closed in ξ' and Z' is integral over X', the morphism $\mathrm{Spec}(W') \to X'$ factors through Z' ([Sta21, 035I]). Finally, let $\mathfrak{q} \in \mathrm{Spec}(W')$ which lies over $\mathfrak{p} \in \mathrm{Spec}(W)$, then we get a lifting $\mathrm{Spec}(W'/\mathfrak{q}) \to Z'$ of $\mathrm{Spec}(V) \to Z$, which shows that $Z' \to Z$ is a v-covering. □

3.16. Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is S°-integrally closed (2.2). For any S-scheme X, we set $X^\circ = S^\circ \times_S X$. We denote by $\mathbf{I}_{S^\circ \to S}$ the category formed by coherent S-schemes which are S°-integrally closed. Note that any S°-integrally closed coherent S-scheme X is also X°-integrally closed by definition. It is clear that the category $(\mathbf{I}_{S^\circ \to S})_{/X}$ of objects of $\mathbf{I}_{S^\circ \to S}$ over X is canonically equivalent to the category $\mathbf{I}_{X^\circ \to X}$.

Lemma 3.17 ([Sta21, 03GV])**.** *Let $Y \to X$ be a coherent morphism of schemes, $X' \to X$ a smooth morphism of schemes, $Y' = Y \times_X X'$. Then, we have $X'^{Y'} = X^Y \times_X X'$.*

Lemma 3.18. *Let $(Y_\lambda \to X_\lambda)_{\lambda \in \Lambda}$ be a directed inverse system of morphisms of coherent schemes with affine transition morphisms $Y_{\lambda'} \to Y_\lambda$ and $X_{\lambda'} \to X_\lambda$ ($\lambda' \geq \lambda$). We set $Y = \lim Y_\lambda$ and $X = \lim X_\lambda$. Then, $(X_\lambda^{Y_\lambda})_{\lambda \in \Lambda}$ is a directed inverse system of coherent schemes with affine transition morphisms and we have $X^Y = \lim X_\lambda^{Y_\lambda}$.*

Proof. We fix an index $\lambda_0 \in \Lambda$. After replacing X_{λ_0} by an affine open covering, we may assume that X_{λ_0} is affine (3.17). We write $X_\lambda = \mathrm{Spec}(A_\lambda)$ and $B_\lambda = \Gamma(Y_\lambda, \mathcal{O}_{Y_\lambda})$ for each $\lambda \geq \lambda_0$, and we set $A = \mathrm{colim}\, A_\lambda$ and $B = \mathrm{colim}\, B_\lambda$. Then, we have $X = \mathrm{Spec}(A)$ and $B = \Gamma(Y, \mathcal{O}_Y)$ ([Sta21, 009F]). Let R_λ (resp.

R) be the integral closure of A_λ in B_λ (resp. A in B). By definition, we have $X_\lambda^{Y_\lambda} = \mathrm{Spec}(R_\lambda)$ and $X^Y = \mathrm{Spec}(R)$. The conclusion follows from the fact that $R = \mathrm{colim}\, R_\lambda$. □

Lemma 3.19. *Let $S^\circ \to S$ be an open immersion of coherent schemes.*

1. *If X is an S°-integrally closed coherent S-scheme, then the open subscheme X° is scheme theoretically dense in X.*
2. *If X is an S°-integrally closed coherent S-scheme and X' is a coherent smooth X-scheme, then X' is also S°-integrally closed.*
3. *If $(X_\lambda)_{\lambda\in\Lambda}$ is a directed inverse system of S°-integrally closed coherent S-scheme with affine transition morphisms, then $X = \lim_{\lambda\in\Lambda} X_\lambda$ is also S°-integrally closed.*
4. *If $Y \to X$ is a morphism of coherent schemes over $S^\circ \to S$ such that Y is integral over X°, then the integral closure X^Y is S°-integrally closed with $(X^Y)^\circ = Y$.*

Proof. (1), (2), (3) follow from [Sta21, 035I], 3.17 and 3.18 respectively. For (4), $(X^Y)^\circ = X^\circ \times_X X^Y$ is the integral closure of X° in $X^\circ \times_X Y = Y$ by 3.17, which is Y itself. □

3.20. We take the notation in 3.16. The inclusion functor

$$\Phi^+ : \mathbf{I}_{S^\circ\to S} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/S},\ X \longmapsto X, \tag{3.20.1}$$

admits a right adjoint

$$\sigma^+ : \mathbf{Sch}^{\mathrm{coh}}_{/S} \longrightarrow \mathbf{I}_{S^\circ\to S},\ X \longmapsto \overline{X} = X^{X^\circ}. \tag{3.20.2}$$

Indeed, σ^+ is well-defined by 3.19.(4), and the adjointness follows from the functoriality of taking integral closures. We remark that $\overline{X}^\circ = X^\circ$. On the other hand, the functor

$$\Psi^+ : \mathbf{I}_{S^\circ\to S} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/S^\circ},\ X \longmapsto X^\circ, \tag{3.20.3}$$

admits a left adjoint

$$\alpha^+ : \mathbf{Sch}^{\mathrm{coh}}_{/S^\circ} \longrightarrow \mathbf{I}_{S^\circ\to S},\ Y \longmapsto Y. \tag{3.20.4}$$

Lemma 3.21. *With the notation in* 3.16*, let $\varphi : I \to \mathbf{I}_{S^\circ\to S}$ be a functor sending i to X_i. If $X = \lim X_i$ represents the limit of φ in the category of coherent S-schemes, then the integral closure $\overline{X} = X^{X^\circ}$ represents the limit of φ in $\mathbf{I}_{S^\circ\to S}$ with $\overline{X}^\circ = X^\circ$.*

Proof. It follows directly from the adjoint pair (Φ^+, σ^+) (3.20). □

It follows from 3.21 that for a diagram $X_1 \to X_0 \leftarrow X_2$ in $\mathbf{I}_{S^\circ\to S}$, the fibred product is representable by

$$X_1 \overline{\times}_{X_0} X_2 = (X_1 \times_{X_0} X_2)^{X_1^\circ \times_{X_0^\circ} X_2^\circ}. \tag{3.21.1}$$

Proposition 3.22. *With the notation in* 3.16*, let $\mathscr{C}$ be the set of families of morphisms $\{X_i \to X\}_{i\in I}$ of $\mathbf{I}_{S^\circ\to S}$ with I finite such that $\coprod_{i\in I} X_i \to X$ is a v-covering. Then, $\mathscr{C}$ forms a pretopology of $\mathbf{I}_{S^\circ\to S}$.*

Proof. Let $\{X_i \to X\}_{i\in I}$ be an element of $\mathscr{C}$. Firstly, we check that for a morphism $X' \to X$ of $\mathbf{I}_{S^\circ\to S}$, the base change $\{X'_i \to X'\}_{i\in I}$ also lies in $\mathscr{C}$, where $Z_i = X_i \times_X X'$ and $X'_i = Z_i^{Z_i^\circ}$ by 3.21. Applying 3.15 to the following diagram

$$\begin{array}{ccccc} \coprod_{i\in I} Z_i^\circ & \longrightarrow & \coprod_{i\in I} X'_i & \longrightarrow & \coprod_{i\in I} Z_i \\ \downarrow & & \downarrow & & \downarrow \\ X'^\circ & \longrightarrow & X' & \longrightarrow & X' \end{array} \tag{3.22.1}$$

we deduce that $\coprod_{i\in I} X'_i \to X'$ is also a v-covering, which shows the stability of $\mathscr{C}$ under base change.

For each $i \in I$, let $\{X_{ij} \to X_i\}_{j\in J_i}$ be an element of $\mathscr{C}$. We need to show that the composition $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ also lies in $\mathscr{C}$. This follows immediately from the stability of v-coverings under composition. We conclude that $\mathscr{C}$ forms a pretopology. □

Definition 3.23. With the notation in 3.16, we endow the category $\mathbf{I}_{S^\circ\to S}$ with the topology generated by the pretopology defined in 3.22, and we call $\mathbf{I}_{S^\circ\to S}$ the *v-site of S°-integrally closed coherent S-schemes*.

By definition, any object in $\mathbf{I}_{S^\circ\to S}$ is quasi-compact. Let $\mathscr{O}$ be the sheaf on $\mathbf{I}_{S^\circ\to S}$ associated to the presheaf $X \mapsto \Gamma(X, \mathcal{O}_X)$. We call $\mathscr{O}$ the structural sheaf of $\mathbf{I}_{S^\circ\to S}$.

Proposition 3.24. *With the notation in* 3.16*, let $f : X' \to X$ be a covering in $\mathbf{I}_{S^\circ \to S}$ such that f is separated and that the diagonal morphism $X'^\circ \to X'^\circ \times_{X^\circ} X'^\circ$ is surjective. Then, the morphism of representable sheaves $h^{\mathrm{a}}_{X'} \to h^{\mathrm{a}}_X$ is an isomorphism.*

Proof. We need to show that for any sheaf $\mathcal{F}$ on $\mathbf{I}_{S^\circ \to S}$, $\mathcal{F}(X) \to \mathcal{F}(X')$ is an isomorphism. Since the composition of $X'^\circ \to X'^\circ \times_{X^\circ} X'^\circ \to X' \overline{\times}_X X'$ factors through the closed immersion $X' \to X' \overline{\times}_X X'$ (as f is separated), the closed immersion $X' \to X' \overline{\times}_X X'$ is surjective (3.19.(1)). Consider the following sequence

$$\mathcal{F}(X) \to \mathcal{F}(X') \rightrightarrows \mathcal{F}(X' \overline{\times}_X X') \to \mathcal{F}(X'). \tag{3.24.1}$$

The right arrow is injective as $X' \to X' \overline{\times}_X X'$ is a v-covering. Thus, the middle two arrows are actually the same. Thus, the first arrow is an isomorphism by the sheaf property of $\mathcal{F}$. □

Proposition 3.25. *With the notation in* 3.16*, let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of presheaves on $\mathbf{I}_{S^\circ \to S}$. Assume that*

(i) *the morphism $\mathcal{F}_1(\mathrm{Spec}(V)) \to \mathcal{F}_2(\mathrm{Spec}(V))$ is an isomorphism for any S°-integrally closed S-scheme $\mathrm{Spec}(V)$ where V is an absolutely integrally closed valuation ring, and that*

(ii) *for any directed inverse system of S°-integrally closed affine schemes $(\mathrm{Spec}(A_\lambda))_{\lambda \in \Lambda}$ over S the natural morphism $\operatorname{colim} \mathcal{F}_i(\mathrm{Spec}(A_\lambda)) \to \mathcal{F}_i(\mathrm{Spec}(\operatorname{colim} A_\lambda))$ is an isomorphism for $i = 1, 2$* (cf. 3.19.(3))*.*

Then, the morphism of the associated sheaves $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is an isomorphism.

Proof. Let A be a product of absolutely integrally closed valuation rings such that $X = \mathrm{Spec}(A)$ is an S°-integrally closed S-scheme. Let $\mathrm{Spec}(V)$ be a connected component of $\mathrm{Spec}(A)$ with the reduced closed subscheme structure. Then, V is an absolutely integrally closed valuation ring by 3.11.(2), and $\mathrm{Spec}(V)$ is also S°-integrally closed since it has nonempty intersection with the dense open subset X° of X. Notice that each connected component of an affine scheme is the intersection of some open and closed subsets ([Sta21, 04PP]). Moreover, since A is reduced, we have $V = \operatorname{colim} A'$, where the colimit is taken over all the open and closed subschemes $X' = \mathrm{Spec}(A')$ of X which contain $\mathrm{Spec}(V)$. By assumption, we have an isomorphism

$$\operatorname{colim} \mathcal{F}_1(X') \xrightarrow{\sim} \operatorname{colim} \mathcal{F}_2(X'). \tag{3.25.1}$$

For two elements $\xi_1, \xi_1' \in \mathcal{F}_1(X)$ with $\alpha_X(\xi_1) = \alpha_X(\xi_1')$ in $\mathcal{F}_2(X)$, by (3.25.1) and a limit argument, there exists a finite disjoin union $X = \coprod_{i=1}^r X_i'$ such that the images of ξ_1 and ξ_1' in $\mathcal{F}_1(X_i')$ are the same. Therefore, $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is injective by 3.14. On the other hand, for an element $\xi_2 \in \mathcal{F}_2(X)$, by (3.25.1) and a limit argument, there exists a finite disjoin union $X = \coprod_{i=1}^r X_i'$ such that there exists an element $\xi_{1,i} \in \mathcal{F}_1(X_i')$ for each i such that the image of ξ_2 in $\mathcal{F}_2(X_i')$ is equal to $\alpha_{X_i'}(\xi_{1,i})$. Therefore, $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is surjective by 3.14. □

3.26. We take the notation in 3.16. Endowing $\mathbf{Sch}^{\mathrm{coh}}$ with the v-topology (3.3), we see that the functors σ^+ and Ψ^+ defined in 3.20 are left exact (as they have left adjoints) and continuous by 3.15 and 3.22. Therefore, they define morphisms of sites (2.5)

$$(\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \xrightarrow{\Psi} \mathbf{I}_{S^\circ \to S} \xrightarrow{\sigma} (\mathbf{Sch}^{\mathrm{coh}}_{/S})_{\mathrm{v}}. \tag{3.26.1}$$

Proposition 3.27. *With the notation in* 3.26*, let $a : (\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \to S^\circ_{\text{ét}}$ be the morphism of site defined by the inclusion functor* (3.9)*.*

(1) *For any torsion abelian sheaf $\mathcal{F}$ on $S^\circ_{\text{ét}}$, the canonical morphism $\Psi_*(a^{-1}\mathcal{F}) \to \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$ is an isomorphism.*

(2) *For any locally constant torsion abelian sheaf $\mathbb{L}$ on $\mathbf{I}_{S^\circ \to S}$, the canonical morphism $\mathbb{L} \to \mathrm{R}\Psi_* \Psi^{-1} \mathbb{L}$ is an isomorphism.*

Proof. (1) For each integer q, the sheaf $\mathrm{R}^q\Psi_*(a^{-1}\mathcal{F})$ is the sheaf associated to the presheaf $X \mapsto H^q_{\mathrm{v}}(X^\circ, a^{-1}\mathcal{F}) = H^q_{\text{ét}}(X^\circ, f_{\text{ét}}^{-1}\mathcal{F})$ by 3.9, where $f_{\text{ét}} : X^\circ_{\text{ét}} \to S^\circ_{\text{ét}}$ is the natural morphism. If X is the spectrum of a nonzero absolutely integrally closed valuation ring V, then $X^\circ = \mathrm{Spec}(V[1/\pi])$ for a nonzero element $\pi \in V$ by 3.11.(1) and 3.19.(1), which is also the spectrum of an absolutely integrally closed valuation ring (2.1). In this case, $H^q_{\text{ét}}(X^\circ, f_{\text{ét}}^{-1}\mathcal{F}) = 0$ for $q > 0$, which proves (1) by 3.25 and [SGA 4_{II}, VII.5.8].

(2) The problem is local on $\mathbf{I}_{S^\circ\to S}$. We may assume that $\mathbb{L}$ is the constant sheaf with value L. Then, $\mathrm{R}^q\Psi_*\Psi^{-1}\mathbb{L}=0$ for $q>0$ by applying (1) on the constant sheaf with value L on $S^\circ_{\text{ét}}$. For $q=0$, notice that $\mathbb{L}$ is also the sheaf associated to the presheaf $X\mapsto H^0_{\text{ét}}(X,L)$, while $\Psi_*\Psi^{-1}\mathbb{L}$ is the sheaf $X\mapsto H^0_{\text{ét}}(X^\circ,L)$ by the discussion in (1). If X is the spectrum of a nonzero absolutely integrally closed valuation ring, then so is X° and so that $H^0_{\text{ét}}(X,L)=H^0_{\text{ét}}(X^\circ,L)=L$. The conclusion follows from 3.25 and [SGA 4_{II}, VII.5.8]. □

4. The arc-Descent of Perfect Algebras

Definition 4.1. For any $\mathbb{F}_p$-algebra R, we denote by R_{perf} the filtered colimit

$$R_{\text{perf}}=\operatorname*{colim}_{\text{Frob}} R \tag{4.1.1}$$

indexed by $(\mathbb{N},\leq)$, where the transition map associated to $i\leq(i+1)$ is the Frobenius of R.

It is clear that the endo-functor of the category of $\mathbb{F}_p$-algebras, $R\mapsto R_{\text{perf}}$, commutes with colimits.

4.2. We define a presheaf $\mathcal{O}_{\text{perf}}$ on the category $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ of coherent $\mathbb{U}$-small $\mathbb{F}_p$-schemes X by

$$\mathcal{O}_{\text{perf}}(X)=\Gamma(X,\mathcal{O}_X)_{\text{perf}}. \tag{4.2.1}$$

For any morphism $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ of affine $\mathbb{F}_p$-schemes, we consider the augmented Čech complex of the presheaf $\mathcal{O}_{\text{perf}}$,

$$0\to A_{\text{perf}}\to B_{\text{perf}}\to B_{\text{perf}}\otimes_{A_{\text{perf}}}B_{\text{perf}}\to\cdots. \tag{4.2.2}$$

Lemma 4.3 ([Sta21, 0EWT])**.** *The presheaf $\mathcal{O}_{\text{perf}}$ is a sheaf on $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ with respect to the fppf topology* ([Sta21, 021L])*. Moreover, for any coherent $\mathbb{F}_p$-scheme X and any integer q,*

$$H^q_{\text{fppf}}(X,\mathcal{O}_{\text{perf}})=\operatorname*{colim}_{\text{Frob}}H^q(X,\mathcal{O}_X). \tag{4.3.1}$$

Proof. Firstly, we remark that for any integer q, the functor $H^q_{\text{fppf}}(X,-)$ commutes with filtered colimit of abelian sheaves on $(\mathbf{Sch}^{\text{coh}}_{/X})_{\text{fppf}}$ for any coherent scheme X ([Sta21, 0739]). Since the presheaf $\mathcal{O}$ sending X to $\Gamma(X,\mathcal{O}_X)$ on $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ is an fppf-sheaf, we have $H^0_{\text{fppf}}(X,\mathcal{O}^{\text{a}}_{\text{perf}})=\mathrm{colim}_{\text{Frob}}H^0_{\text{fppf}}(X,\mathcal{O})=\mathcal{O}_{\text{perf}}(X)$. Thus, $\mathcal{O}_{\text{perf}}$ is an fppf-sheaf. Moreover, $H^q_{\text{fppf}}(X,\mathcal{O}_{\text{perf}})=\mathrm{colim}_{\text{Frob}}H^q_{\text{fppf}}(X,\mathcal{O})=\mathrm{colim}_{\text{Frob}}H^q(X,\mathcal{O}_X)$ by faithfully flat descent ([Sta21, 03DW]). □

Lemma 4.4. *Let $\tau\in\{$fppf, h, v, arc$\}$. The following propositions are equivalent:*

1. *The presheaf $\mathcal{O}_{\text{perf}}$ on $\mathbf{Sch}^{\text{coh}}_{\mathbb{F}_p}$ is a τ-sheaf and $H^q_\tau(X,\mathcal{O}_{\text{perf}})=\mathrm{colim}_{\text{Frob}}H^q(X,\mathcal{O}_X)$ for any coherent $\mathbb{F}_p$-scheme X and any integer q.*
2. *For any τ-covering $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ of affine $\mathbb{F}_p$-schemes, the augmented Čech complex* (4.2.2) *is exact.*

Proof. For an affine scheme $X=\mathrm{Spec}(A)$, $H^q(X,\mathcal{O}_X)$ vanishes for $q>0$ and $H^0(X,\mathcal{O}_X)=A$. For (1) ⇒ (2), the exactness of (4.2.2) follows from the Čech-cohomology-to-cohomology spectral sequence associated to the τ-covering $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ [Sta21, 03AZ]. Therefore, (1) and (2) hold for $\tau=$ fppf by 4.3. Conversely, the exactness of (4.2.2) shows the sheaf property for any τ-covering of an affine scheme by affine schemes, which implies the fppf-sheaf $\mathcal{O}_{\text{perf}}$ is a τ-sheaf (cf. [Sta21, 0ETM]). The vanishing of higher Čech cohomologies implies that $H^q_\tau(X,\mathcal{O}_{\text{perf}})=0$ for any affine $\mathbb{F}_p$-scheme X and any $q>0$ ([Sta21, 03F9]). Therefore, for a coherent $\mathbb{F}_p$-scheme X, $H^q_\tau(X,\mathcal{O}_{\text{perf}})$ can be computed by the hyper-Čech cohomology of a hypercovering of X formed by affine open subschemes ([Sta21, 01GY]). In particular, we have $H^q_\tau(X,\mathcal{O}_{\text{perf}})=H^q_{\text{fppf}}(X,\mathcal{O}_{\text{perf}})$ for any integer q, which completes the proof by 4.3. □

Lemma 4.5 (Gabber)**.** *The augmented Čech complex* (4.2.2) *is exact for any h-covering $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ of affine $\mathbb{F}_p$-schemes.*

Proof. This is a result of Gabber, cf. [BST17, 3.3] or [Sta21, 0EWU], and 4.4. □

Lemma 4.6 ([BS17, 4.1])**.** *The augmented Čech complex* (4.2.2) *is exact for any v-covering $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ of affine $\mathbb{F}_p$-schemes.*

Proof. We write B as a filtered colimit of finitely presented A-algebras $B = \operatorname{colim} B_\lambda$. Then, $\operatorname{Spec}(B_\lambda) \to \operatorname{Spec}(A)$ is an h-covering for each λ by 3.2. Notice that $B_{\mathrm{perf}} = \operatorname{colim} B_{\lambda,\mathrm{perf}}$, then the conclusion follows from applying 4.5 on $\operatorname{Spec}(B_\lambda) \to \operatorname{Spec}(A)$ and taking colimit. □

Lemma 4.7 ([BS17, 6.3])**.** *For any valuation ring V and any prime ideal $\mathfrak{p}$ of V, the sequence*

$$0 \longrightarrow V \xrightarrow{\alpha} V/\mathfrak{p} \oplus V_{\mathfrak{p}} \xrightarrow{\beta} V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \longrightarrow 0 \tag{4.7.1}$$

is exact, where $\alpha(a) = (a, a)$ and $\beta(a, b) = a - b$. If moreover V is a perfect $\mathbb{F}_p$-algebra, then for any perfect V-algebra R, the base change of (4.7.1) *by $V \to R$,*

$$0 \longrightarrow R \longrightarrow R/\mathfrak{p}R \oplus R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \longrightarrow 0 \tag{4.7.2}$$

is exact.

Proof. The sequence (4.7.1) is exact if and only if $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$. Let $a \in \mathfrak{p}$ and $s \in V \setminus \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, $s/a \notin V$, thus $a/s \in V$ as V is a valuation ring. Moreover, we must have $a/s \in \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal. This shows the equality $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$.

The second assertion follows directly from the fact that $\operatorname{Tor}_q^A(B, C) = 0$ for any $q > 0$ and any diagram $B \leftarrow A \to C$ of perfect $\mathbb{F}_p$-algebras ([BS17, 3.16]). □

Lemma 4.8 ([BM20, 4.8])**.** *The augmented Čech complex* (4.2.2) *is exact for any arc-covering* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *of affine $\mathbb{F}_p$-schemes with A a valuation ring.*

Proof. We follow the proof of Bhatt-Mathew [BM20, 4.8]. Let $B = \operatorname{colim} B_\lambda$ be a filtered colimit of finitely presented A-algebras. Then, $\operatorname{Spec}(B_\lambda) \to \operatorname{Spec}(A)$ is also an arc-covering by 3.2. Thus, we may assume that B is a finitely presented A-algebra.

An interval $I = [\mathfrak{p}, \mathfrak{q}]$ of a valuation ring A is a pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of A. We denote by $A_I = (A/\mathfrak{p})_{\mathfrak{q}}$. The set $\mathcal{I}$ of intervals of A is partially ordered under inclusion. Let $\mathcal{P}$ be the subset consisting of intervals I such that the lemma holds for $\operatorname{Spec}(B \otimes_A A_I) \to \operatorname{Spec}(A_I)$. It suffices to show that $\mathcal{P} = \mathcal{I}$.

(1) If the valuation ring A_I is of height ≤ 1, we claim that $\operatorname{Spec}(B \otimes_A A_I) \to \operatorname{Spec}(A_I)$ is automatically a v-covering. Indeed, there is an extension of valuation rings $A_I \to V$ of height ≤ 1 which factors through $B \otimes_A A_I$. As $A_I \to V$ is faithfully flat, $\operatorname{Spec}(B \otimes_A A_I) \to \operatorname{Spec}(A_I)$ is a v-covering by 3.2 and 3.4.(1). Therefore, $I \in \mathcal{P}$ by 4.6.

(2) For any interval $J \subseteq I$, if $I \in \mathcal{P}$ then $J \in \mathcal{P}$. Indeed, applying $\otimes_{\mathbb{F}_p}(A_J)_{\mathrm{perf}}$ to the exact sequence (4.2.2) for $\operatorname{Spec}(B \otimes_A A_I) \to \operatorname{Spec}(A_I)$, we still get an exact sequence by the Tor-independence of perfect $\mathbb{F}_p$-algebras ([BS17, 3.16]).

(3) If $\mathfrak{p} \subseteq A$ is not maximal, then there exists $\mathfrak{q} \supsetneq \mathfrak{p}$ with $I = [\mathfrak{p}, \mathfrak{q}] \in \mathcal{P}$. Indeed, if there is no such I with the height of A_I no more than 1, then $\mathfrak{p} = \bigcap_{\mathfrak{q} \supsetneq \mathfrak{p}} \mathfrak{q}$, and thus,

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \operatorname*{colim}_{I=[\mathfrak{p},\mathfrak{q}],\mathfrak{q}\supsetneq\mathfrak{p}} A_I. \tag{4.8.1}$$

Since $\operatorname{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \to \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ is an h-covering as $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a field (and we have assumed that B is of finite presentation over A), there exists an interval I in the above colimit, such that $\operatorname{Spec}(B \otimes_A A_I) \to \operatorname{Spec}(A_I)$ is also an h-covering by 3.4.(3). Therefore, this I lies in $\mathcal{P}$ by 4.6.

(4) If $\mathfrak{p} \subseteq A$ is nonzero, then there exists $\mathfrak{q} \subsetneq \mathfrak{p}$ with $I = [\mathfrak{q}, \mathfrak{p}] \in \mathcal{P}$. This is similar to (3).

(5) If $I, J \in \mathcal{P}$ are overlapping, then $I \cup J \in \mathcal{P}$. Indeed, by (2) and replacing A by $A_{I\cup J}$, we may assume that $I = [0, \mathfrak{p}]$, $J = [\mathfrak{p}, \mathfrak{m}]$ with $\mathfrak{m}$ the maximal ideal. In particular, $A_I = A_{\mathfrak{p}}$, $A_J = A/\mathfrak{p}$, and $A_{I\cap J} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Since for each $R = \otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$ we have the short exact sequence (4.7.2), we get $I \cup J \in \mathcal{P}$.

In general, by Zorn's lemma, the above five properties of $\mathcal{P}$ guarantee that $\mathcal{P} = \mathcal{I}$ (cf. [BM20, 4.7]). □

Lemma 4.9 (cf. [BM20, 3.30])**.** *The augmented Čech complex* (4.2.2) *is exact for any arc-covering* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *of affine $\mathbb{F}_p$-schemes with A a product of valuation rings.*

Proof. We follow closely the proof of 3.25. Let $\operatorname{Spec}(V)$ be a connected component of $\operatorname{Spec}(A)$ with the reduced closed subscheme structure. Then, V is a valuation ring by 3.11.(2). By 4.8, the augmented Čech complex

$$0 \to V_{\mathrm{perf}} \to (B \otimes_A V)_{\mathrm{perf}} \to (B \otimes_A V)_{\mathrm{perf}} \otimes_{V_{\mathrm{perf}}} (B \otimes_A V)_{\mathrm{perf}} \to \cdots \tag{4.9.1}$$

is exact. Notice that each connected component of an affine scheme is the intersection of some open and closed subsets ([Sta21, 04PP]). Moreover, since A is reduced, we have $V = \operatorname{colim} A'$, where the colimit is taken over all the open and closed subschemes $\operatorname{Spec}(A')$ which contain $\operatorname{Spec}(V)$.

Therefore, by a limit argument, for an element $f \in \otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$ which maps to zero in $\otimes^{n+1}_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$, as $\operatorname{Spec}(A)$ is quasi-compact, we can decompose $\operatorname{Spec}(A)$ into a finite disjoint union $\coprod_{i=1}^N \operatorname{Spec}(A_i)$ such that there exists $g_i \in \otimes^{n-1}_{A_{i,\mathrm{perf}}}(B \otimes_A A_i)_{\mathrm{perf}}$ which maps to the image f_i of f in $\otimes^n_{A_{i,\mathrm{perf}}}(B \otimes_A A_i)_{\mathrm{perf}}$. Since we have

$$\otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}} = \prod_{i=1}^N \otimes^n_{A_{i,\mathrm{perf}}}(B \otimes_A A_i)_{\mathrm{perf}}, \tag{4.9.2}$$

the element $g = (g_i)_{i=1}^N$ maps to f, which shows the exactness of (4.2.2). □

Proposition 4.10 ([BS19, 8.9])**.** *Let $\tau \in \{$fppf, h, v, arc$\}$.*

(1) *The presheaf $\mathcal{O}_{\mathrm{perf}}$ is a τ-sheaf over $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$, and for any coherent $\mathbb{F}_p$-scheme X and any integer q,*

$$H^q_\tau(X, \mathcal{O}_{\mathrm{perf}}) = \operatorname*{colim}_{\mathrm{Frob}} H^q(X, \mathcal{O}_X). \tag{4.10.1}$$

(2) *For any τ-covering $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ of affine $\mathbb{F}_p$-schemes, the augmented Čech complex*

$$0 \to A_{\mathrm{perf}} \to B_{\mathrm{perf}} \to B_{\mathrm{perf}} \otimes_{A_{\mathrm{perf}}} B_{\mathrm{perf}} \to \cdots \tag{4.10.2}$$

is exact.

Proof. We follow closely the proof of Bhatt-Scholze [BS19, 8.9]. (1) and (2) are equivalent by 4.4, and they hold for $\tau \in \{$fppf, h, v$\}$ by 4.3, 4.5 and 4.6. In particular,

$$H^0_{\mathrm{v}}(\operatorname{Spec}(A), \mathcal{O}_{\mathrm{perf}}) = A_{\mathrm{perf}} \text{ and } H^q_{\mathrm{v}}(\operatorname{Spec}(A), \mathcal{O}_{\mathrm{perf}}) = 0,\ \forall q > 0. \tag{4.10.3}$$

We take a hypercovering in the v-topology $\operatorname{Spec}(A_\bullet) \to \operatorname{Spec}(A)$ such that A_n is a product of valuation rings for each degree n by 3.14 and [Sta21, 094K and 0DB1]. The associated sequence

$$0 \to A_{\mathrm{perf}} \to A_{0,\mathrm{perf}} \to A_{1,\mathrm{perf}} \to \cdots \tag{4.10.4}$$

is exact by the hyper-Čech-cohomology-to-cohomology spectral sequence [Sta21, 01GY].

Consider the double complex (A^j_i) where the i-th row $A^\bullet_i$ is the base change of (4.10.2) by $A_{\mathrm{perf}} \to A_{i,\mathrm{perf}}$, i.e. the augmented Čech complex (4.2.2) associated to $\operatorname{Spec}(B \otimes_A A_i) \to \operatorname{Spec}(A_i)$ (we set $A_{-1} = A$). On the other hand, the j-th column $A^j_\bullet$ is the associated sequence (4.10.4) to the hypercovering $\operatorname{Spec}(A_\bullet \otimes_A (\otimes^j_A B)) \to \operatorname{Spec}(\otimes^j_A B)$, which is exact by the previous discussion. Since $A^\bullet_{-1} \to \operatorname{Tot}(A^j_i)^{j\geq 0}_{i\geq 0}$ is a quasi-isomorphism ([Sta21, 0133]), for the exactness of the (-1)-row $A^\bullet_{-1}$, we only need to show the exactness of the i-th row $A^\bullet_i$ for any $i \geq 0$ but this has been proved in 4.9 thanks to our choice of the hypercovering, which completes the proof. □

5. Almost Pre-perfectoid Algebras

Definition 5.1.

(1) A *pre-perfectoid field* K is a valuation field whose valuation ring $\mathcal{O}_K$ is non-discrete, of height 1 and of residue characteristic p, and such that the Frobenius map on $\mathcal{O}_K/p\mathcal{O}_K$ is surjective.
(2) A *perfectoid field* K is a pre-perfectoid field which is complete for the topology defined by its valuation (cf. [Sch12, 3.1]).
(3) A *pseudo-uniformizer* π of a pre-perfectoid field K is a nonzero element of the maximal ideal $\mathfrak{m}_K$ of $\mathcal{O}_K$.

A *morphism of pre-perfectoid fields* $K \to L$ is a homomorphism of fields which induces an extension of valuation rings $\mathcal{O}_K \to \mathcal{O}_L$.

Lemma 5.2. *Let K be a pre-perfectoid field with a pseudo-uniformizer π. Then, the fraction field $\widehat{K}$ of the π-adic completion of $\mathcal{O}_K$ is a perfectoid field.*

Proof. The π-adic completion $\widehat{\mathcal{O}_K}$ of $\mathcal{O}_K$ is still a non-discrete valuation ring of height 1 with residue characteristic p (cf. [Bou06, VI.§5.3, Prop.5]). If $p \neq 0$, then it is canonically isomorphic to the p-adic completion of $\mathcal{O}_K$, so that there is a canonical isomorphism $\mathcal{O}_K/p\mathcal{O}_K \xrightarrow{\sim} \widehat{\mathcal{O}_K}/p\widehat{\mathcal{O}_K}$, from which we see that $\widehat{K}$ is a perfectoid field. If $p = 0$, then the Frobenius induces a surjection $\mathcal{O}_K \to \mathcal{O}_K$ if and only if $\mathcal{O}_K$ is perfect. Thus, $\widehat{\mathcal{O}_K}$ is also perfect, and we see that $\widehat{K}$ is a perfectoid field. □

5.3. Let K be a pre-perfectoid field. There is a unique (up to scalar) ordered group homomorphism $v_K : K^\times \to \mathbb{R}$ such that $v_K^{-1}(0) = \mathcal{O}_K^\times$, where the group structure on $\mathbb{R}$ is given by the addition. In particular, $\mathcal{O}_K \setminus 0 = v_K^{-1}(\mathbb{R}_{\geq 0})$ and $\mathfrak{m}_K \setminus 0 = v_K^{-1}(\mathbb{R}_{>0})$ (cf. [Bou06, VI.§4.5 Prop.7] and [Bou07, V.§2 Prop.1, Rem.2]). The non-discrete assumption on $\mathcal{O}_K$ implies that the image $v_K(K^\times) \subseteq \mathbb{R}$ is dense. We set $v_K(0) = +\infty$.

Lemma 5.4 ([Sch12, 3.2])**.** *Let K be a pre-perfectoid field. Then, for any pseudo-uniformizer π of K, there exists $\pi_n \in \mathfrak{m}_K$ for each integer $n \geq 0$ such that $\pi_0 = \pi$ and $\pi_n = u_n \cdot \pi_{n+1}^p$ for some unit $u_n \in \mathcal{O}_K^\times$, and $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$.*

Proof. If $v_K(\pi) < v_K(p)$, since the Frobenius is surjective on $\mathcal{O}_K/p$, there exists $\pi_1 \in \mathcal{O}_K$ such that $v_K(\pi - \pi_1^p) \geq v_K(p)$. Then, $v_K(\pi) = v_K(\pi_1^p)$ and thus $\pi = u \cdot \pi_1^p$ with $u \in \mathcal{O}_K^\times$. In general, since $v_K(K^\times) \subseteq \mathbb{R}$ is dense, any pseudo-uniformizer π is a finite product of pseudo-uniformizers whose valuation values are strictly less than $v_K(p)$, from which we get a p-th root π_1 of π up to a unit. Since π_1 is also a pseudo-uniformizer, we get π_n inductively. As $v_K(\pi_n)$ tends to zero when n tends to infinity, $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$. □

5.5. Let K be a pre-perfectoid field. We briefly review almost algebra over $(\mathcal{O}_K, \mathfrak{m}_K)$ for which we mainly refer to [AG20, 2.6], [AGT16, V] and [GR03]. Remark that $\mathfrak{m}_K \otimes_{\mathcal{O}_K} \mathfrak{m}_K \cong \mathfrak{m}_K^2 = \mathfrak{m}_K$ is flat over $\mathcal{O}_K$.

An $\mathcal{O}_K$-module M is called *almost zero* if $\mathfrak{m}_K M = 0$. A morphism of $\mathcal{O}_K$-modules $M \to N$ is called an *almost isomorphism* if its kernel and cokernel are almost zero. Let $\mathscr{N}$ be the full subcategory of the category $\mathcal{O}_K$-**Mod** of $\mathcal{O}_K$-modules formed by almost zero objects. It is clear that $\mathscr{N}$ is a Serre subcategory of $\mathcal{O}_K$-**Mod** ([Sta21, 02MO]). Let $\mathcal{S}$ be the set of almost isomorphisms in $\mathcal{O}_K$-**Mod**. Since $\mathscr{N}$ is a Serre subcategory, $\mathcal{S}$ is a multiplicative system, and moreover the quotient abelian category $\mathcal{O}_K$-**Mod**$/\mathscr{N}$ is representable by the localized category $\mathcal{S}^{-1}\mathcal{O}_K$-**Mod** (cf. [Sta21, 02MS]). We denote $\mathcal{S}^{-1}\mathcal{O}_K$-**Mod** by $\mathcal{O}_K^{\mathrm{al}}$-**Mod**, whose objects are called *almost $\mathcal{O}_K$-modules* or simply *$\mathcal{O}_K^{\mathrm{al}}$-modules* (cf. [AG20, 2.6.2]). We denote by

$$\alpha^* : \mathcal{O}_K\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod},\ M \longmapsto M^{\mathrm{al}} \tag{5.5.1}$$

the localization functor. It induces an $\mathcal{O}_K$-linear structure on $\mathcal{O}_K^{\mathrm{al}}$-**Mod**. For any two $\mathcal{O}_K$-modules M and N, we have a natural $\mathcal{O}_K$-linear isomorphism ([AG20, 2.6.7.1])

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(M^{\mathrm{al}}, N^{\mathrm{al}}) = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K \otimes_{\mathcal{O}_K} M, N). \tag{5.5.2}$$

The localization functor α^* admits a right adjoint

$$\alpha_* : \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K\text{-}\mathbf{Mod},\ M \longmapsto M_* = \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(\mathcal{O}_K^{\mathrm{al}}, M), \tag{5.5.3}$$

and a left adjoint

$$\alpha_! : \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K\text{-}\mathbf{Mod},\ M \longmapsto M_! = \mathfrak{m}_K \otimes_{\mathcal{O}_K} M_*. \tag{5.5.4}$$

Moreover, the natural morphisms

$$(M_*)^{\mathrm{al}} \xrightarrow{\sim} M,\ M \xrightarrow{\sim} (M_!)^{\mathrm{al}} \tag{5.5.5}$$

are isomorphisms for any $\mathcal{O}_K^{\mathrm{al}}$-module M (cf. [AG20, 2.6.8]). In particular, for any functor $\varphi : I \to \mathcal{O}_K^{\mathrm{al}}$-**Mod** sending i to M_i, the colimit and limit of φ are representable by

$$\mathrm{colim}\, M_i = (\mathrm{colim}\, M_{i*})^{\mathrm{al}},\ \lim M_i = (\lim M_{i*})^{\mathrm{al}}. \tag{5.5.6}$$

The tensor product in $\mathcal{O}_K$-**Mod** induces a tensor product in $\mathcal{O}_K^{\mathrm{al}}$-**Mod** by

$$M^{\mathrm{al}} \otimes_{\mathcal{O}_K^{\mathrm{al}}} N^{\mathrm{al}} = (M \otimes_{\mathcal{O}_K} N)^{\mathrm{al}} \tag{5.5.7}$$

making $\mathcal{O}_K^{\mathrm{al}}$-**Mod** an abelian tensor category ([AG20, 2.6.4]). We denote by $\mathcal{O}_K^{\mathrm{al}}$-**Alg** the category of commutative unitary monoids in $\mathcal{O}_K^{\mathrm{al}}$-**Mod** induced by the tensor structure, whose objects are called *almost $\mathcal{O}_K$-algebras* or simply *$\mathcal{O}_K^{\mathrm{al}}$-algebras* (cf. [AG20, 2.6.11]). Notice that R^{al} (resp. R_*) admits a

canonical algebra structure for any $\mathcal{O}_K$-algebra (resp. $\mathcal{O}_K^{\mathrm{al}}$-algebra) R. Moreover, α^* and α_* induce adjoint functors between $\mathcal{O}_K$-**Alg** and $\mathcal{O}_K^{\mathrm{al}}$-**Alg** ([AG20, 2.6.12]). Combining with (5.5.5) and (5.5.6), we see that for any functor $\varphi : I \to \mathcal{O}_K^{\mathrm{al}}$-**Alg** sending i to R_i, the colimit and limit of φ are representable by (cf. [GR03, 2.2.16])

$$\operatorname{colim} R_i = (\operatorname{colim} R_{i*})^{\mathrm{al}},\ \lim R_i = (\lim R_{i*})^{\mathrm{al}}. \tag{5.5.8}$$

In particular, for any diagram $B \leftarrow A \to C$ of $\mathcal{O}_K^{\mathrm{al}}$-algebras, we denote its colimit by

$$B \otimes_A C = (B_* \otimes_{A_*} C_*)^{\mathrm{al}}, \tag{5.5.9}$$

which is clearly compatible with the tensor products of modules. We remark that α^* commutes with arbitrary colimits (resp. limits), since it has a right adjoint α_* (resp. since the forgetful functor $\mathcal{O}_K^{\mathrm{al}}$-**Alg** $\to \mathcal{O}_K^{\mathrm{al}}$-**Mod** and the localization functor $\alpha^* : \mathcal{O}_K$-**Mod** $\to \mathcal{O}_K^{\mathrm{al}}$-**Mod** commute with arbitrary limits).

5.6. For an element a of $\mathcal{O}_K$, we denote by $(\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-**Mod** the full subcategory of $\mathcal{O}_K^{\mathrm{al}}$-**Mod** formed by the objects on which the morphism induced by multiplication by a is zero. Notice that for an $(\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-module M, M_* is an $\mathcal{O}_K/a\mathcal{O}_K$-module. Thus, the localization functor α^* induces an essentially surjective exact functor $(\mathcal{O}_K/a\mathcal{O}_K)$-**Mod** $\to (\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-**Mod**, which identifies the latter with the quotient abelian category $(\mathcal{O}_K/a\mathcal{O}_K)$-**Mod**$/\mathscr{N} \cap (\mathcal{O}_K/a\mathcal{O}_K)$-**Mod**.

Let π be a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit. The Frobenius on $\mathcal{O}_K/\pi\mathcal{O}_K$ induces an isomorphism $\mathcal{O}_K/\pi_1\mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K$. The Frobenius on $(\mathcal{O}_K/\pi)$-algebras and the localization functor α^* induce a natural transformation from the base change functor $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** $\to$ $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg**, $R \mapsto (\mathcal{O}_K/\pi) \otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R$ to the identity functor.

$$\begin{array}{ccccc} (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Alg} & \longrightarrow & (\mathcal{O}_K/\pi_1)^{\mathrm{al}}\text{-}\mathbf{Alg} & \xrightarrow{\sim} & (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Alg} \\ & \searrow^{\mathrm{id}} & \Downarrow \mathrm{Frob} & \nearrow & \end{array} \tag{5.6.1}$$

For an $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra R, we usually identify the $(\mathcal{O}_K/\pi_1)^{\mathrm{al}}$-algebra $R/\pi_1 R$ with the $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra $(\mathcal{O}_K/\pi)\otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R$, and we denote by $R/\pi_1 R \to R$ the natural morphism $(\mathcal{O}_K/\pi)\otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R \to R$ induced by the Frobenius (cf. [GR03, 3.5.6]). Moreover, the natural transformations induced by Frobenius for $(\mathcal{O}_K/\pi)$-**Alg** and $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** are also compatible with the functor α_*. Indeed, it follows from the fact that for any $(\mathcal{O}_K/\pi)$-algebra R, the composition of

$$(\mathcal{O}_K/\pi) \otimes_{(\mathcal{O}_K/\pi)} \mathrm{Hom}(\mathfrak{m}_K, R) \longrightarrow \mathrm{Hom}(\mathfrak{m}_K, (\mathcal{O}_K/\pi) \otimes_{(\mathcal{O}_K/\pi)} R) \xrightarrow{\mathrm{Hom}(\mathfrak{m}_K, \mathrm{Frob})} \mathrm{Hom}(\mathfrak{m}_K, R) \tag{5.6.2}$$

is the relative Frobenius on $(R^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, R)$.

5.7. Let C be a site. A presheaf $\mathcal{F}$ of $\mathcal{O}_K$-modules on C is called *almost zero* if $\mathcal{F}(U)$ is almost zero for any object U of C. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_K$-modules on C is called an *almost isomorphism* if $\mathcal{F}(U) \to \mathcal{G}(U)$ is an almost isomorphism for any object U of C (cf. [AG20, 2.6.23]). Let $\mathscr{N}$ be the full subcategory of the category $\mathcal{O}_K$-$\mathbf{Mod}_C$ of sheaves of $\mathcal{O}_K$-modules on C formed by almost zero objects. Similarly, $\mathscr{N}$ is a Serre subcategory of $\mathcal{O}_K$-$\mathbf{Mod}_C$. Let $\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ be the full subcategory of the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ formed by complexes with almost zero cohomologies. It is a strictly full saturated triangulated subcategory ([Sta21, 06UQ]). We also say that the objects of $\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ are *almost zero*. Let $\mathcal{S}$ be the set of arrows in $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ which induce almost isomorphisms on cohomologies. We also call the elements of $\mathcal{S}$ *almost isomorphisms*. Then, $\mathcal{S}$ is a saturated multiplicative system ([Sta21, 05RG]), and moreover the quotient triangulated category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)/\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ is representable by the localized triangulated category $\mathcal{S}^{-1}\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ ([Sta21, 05RI]). The natural functor

$$\mathcal{S}^{-1}\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C) \longrightarrow \mathbf{D}(\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}_C) \tag{5.7.1}$$

is an equivalence by [Sta21, 06XM] and (5.5.5) (cf. [GR03, 2.4.9]).

Lemma 5.8. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, M a flat $\mathcal{O}_K$-module. We fix a system of p^n-th roots $(\pi_n)_{n\geq 0}$ of π up to units* (5.4)*, then the map*

$$\bigcap_{n\geq 0} \pi_n^{-1} M \to (M^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, M),\ a \mapsto (x \mapsto xa) \tag{5.8.1}$$

where $\pi_n^{-1}M \subseteq M[1/\pi]$, is an isomorphism of $\mathcal{O}_K$-modules. Moreover, for an extension of valuation rings $\mathcal{O}_K \to R$ of height 1, we have $R = \bigcap_{n\geq 0} \pi_n^{-1}R$ and the above isomorphism coincides with the unit map $R \to (R^{\mathrm{al}})_*$.

Proof. Since $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$, any $\mathcal{O}_K$-linear morphism $f : \mathfrak{m}_K \to M$ is determined by its values $f(\pi_n) \in M$. Notice that $(\pi/\pi_n)\cdot f(\pi_n) = f(\pi)$ and M is π-torsion free, so that f must be given by the multiplication by an element $a = f(\pi)/\pi \in M[1/\pi]$. It is clear that such a multiplication sends $\mathfrak{m}_K$ to M if and only if $a \in \bigcap_{n\geq 0} \pi_n^{-1}M$, which shows the first assertion. If $\mathcal{O}_K \to R$ is an extension of valuation rings of height 1, then we directly deduce from the valuation map $v : R[1/\pi] \setminus 0 \to \mathbb{R}$ (5.3) the equality $R = \bigcap_{n\geq 0} \pi_n^{-1}R$. □

Lemma 5.9. *Let K be a pre-perfectoid field, R an $\mathcal{O}_K$-algebra, $\mathcal{O}_K \to V$ an extension of valuation rings of height 1. Then, the canonical map*

$$\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V) \longrightarrow \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}}(R^{\mathrm{al}}, V^{\mathrm{al}}) \tag{5.9.1}$$

is bijective.

Proof. There are natural maps

$$\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V) \to \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}}(R^{\mathrm{al}}, V^{\mathrm{al}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, (V^{\mathrm{al}})_*) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V), \tag{5.9.2}$$

where the middle isomorphism is given by adjunction and the last isomorphism is induced by the inverse of the unit map $V \to (V^{\mathrm{al}})_*$ by 5.8. The composition is the identity map, which completes the proof. □

Definition 5.10. Let K be a pre-perfectoid field. We say that an $\mathcal{O}_K^{\mathrm{al}}$-module M (resp. an $\mathcal{O}_K$-module M) is *flat* (resp. *almost flat*) if the functor $\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \to \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}$ given by tensoring with M is exact (resp. M^{al} is flat).

Lemma 5.11. *Let K be a pre-perfectoid field with a pseudo-uniformizer π. Then, an $\mathcal{O}_K^{\mathrm{al}}$-module M is flat if and only if M_* is π-torsion free. In particular, an $\mathcal{O}_K$-module N is almost flat if and only if the submodule of π-torsion elements of N is almost zero.*

Proof. First of all, for any $\mathcal{O}_K^{\mathrm{al}}$-modules L_1 and L_2, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(M \otimes_{\mathcal{O}_K^{\mathrm{al}}} L_1, L_2) = \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(L_1, \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(M_*, L_{2*})^{\mathrm{al}}) \tag{5.11.1}$$

by (5.5.2), (5.5.5) and (5.5.7). Therefore, the functor defined by tensoring with M admits a right adjoint, and thus it is right exact. Consider the sequence

$$0 \longrightarrow \mathcal{O}_K^{\mathrm{al}} \xrightarrow{\cdot\pi} \mathcal{O}_K^{\mathrm{al}} \longrightarrow (\mathcal{O}_K/\pi\mathcal{O}_K)^{\mathrm{al}} \longrightarrow 0, \tag{5.11.2}$$

which is exact since the localization functor α^* is exact. If M is flat, tensoring the above sequence with M and applying α_*, we deduce that M_* is π-torsion free since α_* is left exact (as a right adjoint to α^*). Conversely, if M_* is π-torsion free, then it is flat over $\mathcal{O}_K$. For any injective morphism $L_1 \to L_2$ of $\mathcal{O}_K^{\mathrm{al}}$-modules, $L_{1*} \to L_{2*}$ is also injective, and it remains injective after tensoring with M_*. Therefore, $L_1 \to L_2$ also remains injective after tensoring with M since α^* is exact. This shows that M is flat.

The second assertion follows from the almost isomorphism $N \to (N^{\mathrm{al}})_*$ and the fact that $(N^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, N)$ has no nonzero almost zero element. □

Lemma 5.12. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, M a flat $\mathcal{O}_K^{\mathrm{al}}$-module, x an element of $\mathcal{O}_K$. Then, the canonical morphism $M_*/xM_* \to (M/xM)_*$ is injective, and for any $\epsilon \in \mathfrak{m}_K$, the image of $\varphi_\epsilon : (M/\epsilon xM)_* \to (M/xM)_*$ is M_*/xM_*. In particular, the canonical morphism*

$$\varprojlim_n M_*/\pi^n M_* \longrightarrow (\varprojlim_n M/\pi^n M)_* \tag{5.12.1}$$

is an isomorphism of $\mathcal{O}_K$-modules.

Proof. We follow the proof of [Sch12, 5.3]. Applying the left exact functor α_* to the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0, \tag{5.12.2}$$

we see that $M_*/xM_* \to (M/xM)_*$ is injective.

To show that the image of φ_ϵ is M_*/xM_*, it suffices to show that φ_ϵ factors through M_*/xM_*. We identify $(M/xM)_*$ with $\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, M_*/xM_*)$ by (5.5.5) and (5.5.2) so that M_*/xM_* identifies with the subset consisting of the $\mathcal{O}_K$-morphisms $\mathfrak{m}_K \to M_*/xM_*$ sending y to ya for some element

$a \in M_*/xM_*$. For an $\mathcal{O}_K$-morphism $f : \mathfrak{m}_K \to M_*/\epsilon xM_*$, let b be an element of M_* which lifts $f(\epsilon)$. Notice that M_* is π-torsion free by 5.11. With notation in 5.8, we have $b \equiv (\epsilon/\pi_n) \cdot f(\pi_n) \mod \epsilon xM_*$ for n big enough so that the element $b/\epsilon \in M_*[1/\pi]$ lies in $\bigcap_{n\geq 0} \pi_n^{-1} M_* = M_*$. Moreover, $\pi_n \cdot (b/\epsilon) \equiv f(\pi_n) \mod xM_*$ for n big enough. As $\varphi_\epsilon(f)$ is determined by its values on π_n for n big enough, it follows that $\varphi_\epsilon(f) = a$, where a is the image of b/ϵ in M_*/xM_*.

Finally, the previous result implies that the inverse system $((M/\pi^n M)_*)_{n\geq 1}$ is Mittag-Leffler so that the "in particular" part follows immediately from the fact that α_* commutes with arbitrary limits (as a right adjoint to α^*) ([Sta21, 0596]). □

Definition 5.13. Let K be a pre-perfectoid field. For any $\mathcal{O}_K$-algebra R, we define a perfect ring $R^\flat$ as the projective limit

$$R^\flat = \varprojlim_{\text{Frob}} R/pR \tag{5.13.1}$$

indexed by $(\mathbb{N}, \leq)$, where transition map associated to $i \leq (i+1)$ is the Frobenius on R/pR. We call $R^\flat$ the *tilt* of R.

Lemma 5.14 ([Sch12, 3.4])**.** *Let K be a perfectoid field with a pseudo-uniformizer π dividing p.*

(1) *The projection induces an isomorphism of multiplicative monoids*

$$\varprojlim_{\text{Frob}} \mathcal{O}_K \longrightarrow \varprojlim_{\text{Frob}} \mathcal{O}_K/\pi\mathcal{O}_K. \tag{5.14.1}$$

In particular, the right hand side is canonically isomorphic to $(\mathcal{O}_K)^\flat$ as a ring.

(2) *We denote by*

$$\sharp : (\mathcal{O}_K)^\flat \longrightarrow \mathcal{O}_K,\ x \mapsto x^\sharp, \tag{5.14.2}$$

the composition of the inverse of (5.14.1) *and the projection onto the first component. Then $v_K \circ \sharp : (\mathcal{O}_K)^\flat \setminus 0 \to \mathbb{R}_{\geq 0}$ defines a valuation of height 1 on $(\mathcal{O}_K)^\flat$.*

(3) *The fraction field $K^\flat$ of $(\mathcal{O}_K)^\flat$ is a perfectoid field of characteristic p and the element*

$$\pi^\flat = (\cdots, \pi_1^{1/p^2}, \pi_1^{1/p}, \pi_1, 0) \in (\mathcal{O}_K)^\flat \tag{5.14.3}$$

is a pseudo-uniformizer of $K^\flat$, where $\pi = u \cdot \pi_1^p$ with $\pi_1 \in \mathfrak{m}_K$ and $u \in \mathcal{O}_K^\times$.

(4) *We have $\mathcal{O}_{K^\flat} = (\mathcal{O}_K)^\flat$, and there is a canonical isomorphism*

$$\mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat} \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K \tag{5.14.4}$$

induced by (1) *and the projection onto the first component.*

5.15. We see that the tilt defines a functor $\mathcal{O}_K$-**Alg** $\to \mathcal{O}_{K^\flat}$-**Alg**, $R \mapsto R^\flat$, which preserves almost zero objects and almost isomorphisms. For an $\mathcal{O}_K^{\mathrm{al}}$-algebra R, we set $R^\flat = ((R_*)^\flat)^{\mathrm{al}}$ and call it the *tilt* of R, which induces a functor $\mathcal{O}_K^{\mathrm{al}}$-**Alg** $\to \mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Alg**, $R \mapsto R^\flat$. Note that the tilt functor commutes with the localization functor α^* up to a canonical isomorphism, and commutes with the functor α_* up to a canonical almost isomorphism.

Definition 5.16 ([Sch12, 5.1])**.** Let K be a perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit.

(1) A *perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra* is an $\mathcal{O}_K^{\mathrm{al}}$-algebra R such that
   (i) R is flat over $\mathcal{O}_K^{\mathrm{al}}$;
   (ii) the Frobenius of $R/\pi R$ induces an isomorphism $R/\pi_1 R \to R/\pi R$ of $\mathcal{O}_K^{\mathrm{al}}$-algebras (5.6);
   (iii) the canonical morphism $R \to \varprojlim_n R/\pi^n R$ is an isomorphism in $\mathcal{O}_K^{\mathrm{al}}$-**Alg**.
   We denote by $\mathcal{O}_K^{\mathrm{al}}$-**Perf** the full subcategory of $\mathcal{O}_K^{\mathrm{al}}$-**Alg** formed by perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras.

(2) A *perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra* is a flat $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra R such that the Frobenius map induces an isomorphism $R/\pi_1 R \xrightarrow{\sim} R$. We denote by $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Perf** the full subcategory of $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** formed by perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebras.

Lemma 5.17. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit. Then, for an $\mathcal{O}_K$-algebra R, the following conditions are equivalent:*

(1) *The almost algebra $\widehat{R}^{\mathrm{al}}$ associated to the π-adic completion $\widehat{R}$ of R is a perfectoid $\mathcal{O}_{\widehat{K}}^{\mathrm{al}}$-algebra.*

(2) *The $\mathcal{O}_{\widehat{K}}$-module $\widehat{R}$ is almost flat, and the Frobenius of $R/\pi R$ induces an almost isomorphism $R/\pi_1 R \to R/\pi R$.*

Proof. We have seen that $\widehat{K}$ is a perfectoid field in 5.2 and π is obviously a pseudo-uniformizer of $\widehat{K}$. Since the localization functor $\alpha^* : \mathcal{O}_K\text{-}\mathbf{Alg} \to \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}$ commutes with arbitrary limits and colimits (5.5), we have a canonical isomorphism $\widehat{R}^{\mathrm{al}} \xrightarrow{\sim} \varprojlim_n \widehat{R}^{\mathrm{al}}/\pi^n\widehat{R}^{\mathrm{al}}$. Thus, the third condition in 5.16.(1) holds for $\widehat{R}^{\mathrm{al}}$. Since there are canonical isomorphisms

$$R/\pi_1 R \xrightarrow{\sim} \widehat{R}/\pi_1\widehat{R},\ R/\pi R \xrightarrow{\sim} \widehat{R}/\pi\widehat{R}, \tag{5.17.1}$$

the conditions (1) and (2) are clearly equivalent. □

Definition 5.18. Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit. We say that an $\mathcal{O}_K$-algebra is *almost pre-perfectoid* if it satisfies the equivalent conditions in 5.17.

We remark that in 5.18, if a morphism of $\mathcal{O}_K$-algebras $R \to R'$ induces an almost isomorphism $R/\pi^n R \to R'/\pi^n R'$ for each $n \geq 1$, then the morphism of the π-adic completions $\widehat{R} \to \widehat{R'}$ is an almost isomorphism since α^* commutes with limits. In particular, R is almost pre-perfectoid if and only if R' is almost pre-perfectoid.

Lemma 5.19. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra. If R is almost flat (resp. flat) over $\mathcal{O}_K$, then the π-adic completion $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$.*

Proof. For any integer $n > 0$, there is a canonical isomorphism

$$R/\pi^n R \xrightarrow{\sim} \widehat{R}/\pi^n\widehat{R}. \tag{5.19.1}$$

Let $x \in \widehat{R}$ be a π-torsion element. Since any π-torsion element of R is almost zero (resp. zero) by 5.11, for any $\epsilon \in \mathfrak{m}_K$ (resp. $\epsilon = 1$), the image of ϵx in $\widehat{R}/\pi^n\widehat{R}$ lies in $\pi^{n-1}\widehat{R}/\pi^n\widehat{R}$. Therefore, $\epsilon x \in \bigcap_{n>0} \pi^{n-1}\widehat{R} = 0$, which amounts to say that $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$. □

Lemma 5.20. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit, R a flat $\mathcal{O}_K$-algebra. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an injection $R/\pi_1 R \to R/\pi R$.*
(2) *For any $x \in R[1/\pi]$, if $x^p \in R$, then $x \in R$.*

Proof. We follow the proof of [Sch12, 5.7]. Assume first that $R/\pi_1 R \to R/\pi R$ is injective. Let $x \in R[1/\pi]$ with $x^p \in R$, k the minimal natural number such that $y = \pi_1^k x \in R$. If $k \geq 1$, then $y^p = \pi_1^{pk} x^p \in \pi R$. Therefore, $y \in \pi_1 R$ by the injectivity of the Frobenius. However, as R is π-torsion free, we have $y' = y/\pi_1 = \pi_1^{k-1} x \in R$ which contradicts the minimality of k.

Conversely, for any $x \in R$ with $x^p \in \pi R$, we have $(x/\pi_1)^p \in R$. Thus, $x/\pi_1 \in R$ by assumption, i.e. $x \in \pi_1 R$, which implies the injectivity of the Frobenius. □

Lemma 5.21. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit, R an $\mathcal{O}_K$-algebra which is almost flat. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an almost injection (resp. almost isomorphism) $R/\pi_1 R \to R/\pi R$.*
(2) *The Frobenius induces an injection (resp. isomorphism) $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$.*

Proof. We follow the proof of [Sch12, 5.6]. Notice that the Frobenius is compatible with the functors α^* and α_* (5.6). (2) ⇒ (1) follows from the almost isomorphism $R \to (R^{\mathrm{al}})_*$. The "injection" part of (1) ⇒ (2) follows from the inclusions (5.12)

$$(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \subseteq ((R/\pi_1 R)^{\mathrm{al}})_*,\ (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_* \subseteq ((R/\pi R)^{\mathrm{al}})_*. \tag{5.21.1}$$

For the "isomorphism" part of (1) ⇒ (2), notice that $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$ is almost surjective. Let π_2 be a p-th root of π_1 up to a unit (5.4). Then, for an element x of $(R^{\mathrm{al}})_*$, there exist elements y and x' of $(R^{\mathrm{al}})_*$ such that $\pi_2^p x = y^p + \pi_2^{p^2} x'$. Thus, $x = y'^p + \pi_2^{p^2-p} x'$ where $y' = y/\pi_2 \in (R^{\mathrm{al}})_*[1/\pi]$ (as $(R^{\mathrm{al}})_*$ is flat over $\mathcal{O}_K$ by 5.11). In fact, y' lies in $(R^{\mathrm{al}})_*$ by 5.20 and the "injection" part of (1) ⇒ (2). By applying this process to x', there exist elements y'' and x'' of $(R^{\mathrm{al}})_*$ such that $x' = y''^p + \pi_2^{p^2-p} x''$. In conclusion, we have $x = y'^p + \pi_2^{p^2-p}(y''^p + \pi_2^{p^2-p} x'') \equiv (y' + \pi_2^{p-1} y'')^p \mod \pi(R^{\mathrm{al}})_*$, which shows the surjectivity of $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$. □

Lemma 5.22. *Let K be a pre-perfectoid field, R an almost flat $\mathcal{O}_K$-algebra, π, π' pseudo-uniformizers dividing p with p-th roots π_1, π_1' respectively up to units. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an almost injection (resp. almost surjection) $R/\pi_1 R \to R/\pi R$.*
(2) *The Frobenius induces an almost injection (resp. almost surjection) $R/\pi_1' R \to R/\pi' R$.*

In particular, the definitions 5.16.(1) *and* 5.18 *do not depend on the choice of the pseudo-uniformizer.*

Proof. Notice that $(R^{\mathrm{al}})_*$ is flat over $\mathcal{O}_K$ by 5.11. The "injection" part follows from 5.20 and 5.21. For the "surjection" part, we assume that $R/\pi_1 R \to R/\pi R$ is almost surjective. Let $\epsilon \in \mathfrak{m}_K$. We can take a pseudo-uniformizer $\widetilde{\pi}$ of K dividing p with $\widetilde{\pi}_1^p = \widetilde{\pi}$ and $v_K(\pi)/3 < v_K(\widetilde{\pi}) < v_K(\pi)/2$. For any $x \in R$, by the almost surjectivity, we have $\epsilon x = y^p + \widetilde{\pi}^2 z$ for some $y, z \in R$. We also have $\widetilde{\pi} z = v^p + \pi w$ for some $v, w \in R$, then $\epsilon x = y^p + \widetilde{\pi} v^p + \widetilde{\pi}\pi w$. Since $y^p + \widetilde{\pi} v^p \equiv (y + \widetilde{\pi}_1 v)^p \mod pR$, $R'/\pi_1' R \to R/\pi' R$ is almost surjective for any pseudo-uniformizer π' dividing p with $v_K(\pi') < 4v_K(\pi)/3$. By induction, we see that $R'/\pi_1' R \to R/\pi' R$ is almost surjective in general. □

Proposition 5.23. *Let K be a pre-perfectoid field of characteristic p with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra, $\widehat{R}$ the π-adic completion of R. Then, R is almost pre-perfectoid if and only if $(\widehat{R}^{\mathrm{al}})_*$ is perfect.*

Proof. Note that $\mathcal{O}_K$ is perfect by definition. If R is almost pre-perfectoid, then $\widehat{R}$ is almost flat so that $(\widehat{R}^{\mathrm{al}})_*$ is π-adically complete by taking $M = \widehat{R}^{\mathrm{al}}$ in 5.12. Moreover, the Frobenius induces an isomorphism $(\widehat{R}^{\mathrm{al}})_*/\pi^n(\widehat{R}^{\mathrm{al}})_* \to (\widehat{R}^{\mathrm{al}})_*/\pi^{pn}(\widehat{R}^{\mathrm{al}})_*$ for any integer $n \geq 1$ by 5.21 and 5.22, which implies that $(\widehat{R}^{\mathrm{al}})_*$ is perfect. Conversely, assume that $(\widehat{R}^{\mathrm{al}})_*$ is perfect. For any π-torsion element $f \in (\widehat{R}^{\mathrm{al}})_*$, we have $\pi^{1/p^n} f = 0$ for any integer $n \geq 0$, which shows that $\widehat{R}$ is almost flat by 5.11. Moreover, it is clear that the Frobenius induces an isomorphism $(\widehat{R}^{\mathrm{al}})_*/\pi(\widehat{R}^{\mathrm{al}})_* \to (\widehat{R}^{\mathrm{al}})_*/\pi^p(\widehat{R}^{\mathrm{al}})_*$, which shows that R is almost pre-perfectoid by 5.21 and 5.22. □

Proposition 5.24. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost flat, R' the integral closure of R in $R[1/\pi]$. If the Frobenius induces an almost injection $R/\pi_1 R \to R/\pi R$, then $R \to R'$ is an almost isomorphism.*

Proof. Since $R \to (R^{\mathrm{al}})_*$ is an almost isomorphism, we may replace R by $(R^{\mathrm{al}})_*$ so that we may assume that $R = (R^{\mathrm{al}})_*$, $R \subseteq R[1/\pi]$ by 5.11 and for any $x \in R[1/\pi]$ such that $x^p \in R$, then $x \in R$ by 5.20 and 5.21. It suffices to show that R is integrally closed in $R[1/\pi]$. Suppose that $x \in R[1/\pi]$ is integral over R. There is an integer $N > 0$ such that x^r is an R-linear combination of $1, x, \ldots, x^N$ for any $r > 0$. Therefore, there exists an integer $k > 0$ such that $\pi^k x^r \in R$ for any $r > 0$. Taking $r = p^n$, we get $x \in \bigcap_{n \geq 0} \pi_n^{-1} R = (R^{\mathrm{al}})_* = R$ by 5.8, which completes our proof. □

Lemma 5.25. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost pre-perfectoid. Consider the natural morphisms*

$$\begin{array}{ccc} R & \xrightarrow{f} & R[\frac{1}{\pi}] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \widehat{R} & \xrightarrow{f'} & \widehat{R}[\frac{1}{\pi}] \end{array} \tag{5.25.1}$$

Then $f(R) \to g'^{-1}(f'(\widehat{R}))$ is an almost isomorphism.

Proof. We need to show that $f(R) \to g'^{-1}(f'(\widehat{R}))$ is almost surjective. Let $f(a)/\pi^n \in g'^{-1}(f'(\widehat{R}))$ where $a \in R$. Hence, $g'(f(a)) = \pi^n f'(b)$ for some $b \in \widehat{R}$. Notice that f' is almost injective since $\widehat{R}^{\mathrm{al}}$ is perfectoid. Therefore, $\epsilon \cdot g(a) = \epsilon \pi^n b$ for any $\epsilon \in \mathfrak{m}_K$. Since g induces an isomorphism

$$R/\pi^n R \xrightarrow{\sim} \widehat{R}/\pi^n \widehat{R}, \tag{5.25.2}$$

there exists $c \in R$ such that $\epsilon a = \pi^n c$, which implies that $\epsilon(f(a)/\pi^n) = f(c) \in f(R)$. This completes the proof. □

Proposition 5.26. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, R' the integral closure of R in $R[1/\pi]$. Then, the morphism of π-adic completions $\widehat{R} \to \widehat{R'}$ is an almost isomorphism. In particular, R' is also almost pre-perfectoid.*

Proof. We consider the following commutative diagram

$$\begin{array}{ccccc} & & \overset{f}{\frown} & & \\ R & \xrightarrow{h} & R' & \longrightarrow & R[\frac{1}{\pi}] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} \\ \widehat{R} & \xrightarrow{h'} & R'' & \longrightarrow & \widehat{R}[\frac{1}{\pi}] \\ & & \underset{f'}{\smile} & & \end{array} \tag{5.26.1}$$

where R'' is the integral closure of $\widehat{R}$ in $\widehat{R}[1/\pi]$. We claim that $R'/\pi^n R' \to R''/\pi^n R''$ is almost injective. Let $a \in R' \subseteq R[1/\pi]$ such that $g''(a) = \pi^n b$ for some $b \in R''$. Since h' is an almost isomorphism by 5.24, for any $\epsilon \in \mathfrak{m}_K$, there exists $c \in \widehat{R}$ such that $\epsilon \cdot g''(a) = \pi^n h'(c)$. Thus, $\epsilon a/\pi^n \in g'^{-1}(f'(\widehat{R}))$, hence $\epsilon^2 a/\pi^n \in f(R)$ by 5.25, and thus $\epsilon^2 a \in \pi^n R'$, which proves the claim. Now we consider

$$R/\pi^n R \longrightarrow R'/\pi^n R' \longrightarrow R''/\pi^n R''. \tag{5.26.2}$$

Its composition is an almost isomorphism since h' is an almost isomorphism. Since the second map is almost injective, the first map $R/\pi^n R \to R'/\pi^n R'$ is an almost isomorphism, which completes the proof. □

Theorem 5.27 (Tilting correspondence [Sch12, 5.2, 5.21])**.** *Let K be a perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit.*

1. *The functor $\mathcal{O}_K^{\mathrm{al}}$-**Perf** $\to (\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Perf**, $R \mapsto R/\pi R$, is an equivalence of categories.*
2. *The functor $\mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Perf** $\to (\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-**Perf**, $R \mapsto R/\pi^\flat R$ is an equivalence of categories, and the functor $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-**Perf** $\to \mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Perf**, $R \mapsto R^\flat$ is a quasi-inverse.*
3. *Let R be a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra with tilt $R^\flat$. Then, R is isomorphic to $\mathcal{O}_L^{\mathrm{al}}$ for some perfectoid field L over K if and only if $R^\flat$ is isomorphic to $\mathcal{O}_{L'}^{\mathrm{al}}$ for some perfectoid field L' over $K^\flat$.*

In conclusion, we have natural equivalences

$$\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Perf} \xrightarrow{\sim} (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Perf} \xrightarrow{\sim} (\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}\text{-}\mathbf{Perf} \xleftarrow{\sim} \mathcal{O}_{K^\flat}^{\mathrm{al}}\text{-}\mathbf{Perf}, \tag{5.27.1}$$

where the middle equivalence is given by the isomorphism (5.14.4) $\mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat} \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K$. We remark that the natural isomorphisms of the equivalence in (2) are defined as follows: for a perfectoid $\mathcal{O}_{K^\flat}^{\mathrm{al}}$-algebra R, the natural isomorphism $R \xrightarrow{\sim} (R/\pi^\flat R)^\flat$ is induced by the morphism $R_* \to (R_*/\pi^\flat R_*)^\flat$ sending x to $(\cdots, x^{1/p^2}, x^{1/p}, x)$ (notice that R_* is perfect by 5.23); for a perfectoid $(\mathcal{O}_{K^\flat}/\pi^\flat)$-algebra R, the natural isomorphism $R^\flat/\pi^\flat R^\flat \xrightarrow{\sim} R$ is induced by the projection on the first component $(R_*)^\flat \to R_*$ (cf. [Sch12, 5.17]). Consequently, for a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra R, the morphism

$$R^\flat/\pi^\flat R^\flat \longrightarrow R/\pi R \tag{5.27.2}$$

induced by the projection on the first component is an isomorphism.

Proposition 5.28. *Let K be a perfectoid field with a pseudo-uniformizer π of K dividing p, $B \leftarrow A \to C$ a diagram of perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras. Then, the π-adically completed tensor product $B\widehat{\otimes}_A C$ is also perfectoid.*

Proof. We follow closely the proof of [Sch12, 6.18]. Firstly, we claim that $(B \otimes_A C)/\pi$ is flat over $(\mathcal{O}_K/\pi)^{\mathrm{al}}$. Since $(B \otimes_A C)/\pi = (B^\flat \otimes_{A^\flat} C^\flat)/\pi^\flat$, it suffices to show the flatness of $B^\flat \otimes_{A^\flat} C^\flat$ over $\mathcal{O}_{K^\flat}^{\mathrm{al}}$, which amounts to say that the submodule of $\pi^\flat$-torsion elements of $(B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$ is almost zero as $B^\flat \otimes_{A^\flat} C^\flat = ((B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat)^{\mathrm{al}}$. If $f \in (B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$ is a $\pi^\flat$-torsion element, then by perfectness of $(B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$, we have $(\pi^\flat)^{1/p^n} f = 0$ for any $n > 0$, which proves the claim.

Thus, $(B \otimes_A C)/\pi$ is a perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra. It corresponds to a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra D by 5.27 and it induces a morphism $B\widehat{\otimes}_A C \to D$ by universal property of π-adically completed tensor product. We use dévissage to show that $(B \otimes_A C)/\pi^n \to D/\pi^n$ is an isomorphism for any integer $n > 0$.

By induction,

$$\begin{array}{ccccccc} (B\otimes_A C)/\pi^n & \xrightarrow{\cdot\pi} & (B\otimes_A C)/\pi^{n+1} & \longrightarrow & (B\otimes_A C)/\pi & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow D/\pi^n & \xrightarrow{\cdot\pi} & D/\pi^{n+1} & \longrightarrow & D/\pi & \longrightarrow & 0 \end{array} \tag{5.28.1}$$

the vertical arrows on the left and right are isomorphisms. By snake's lemma in the abelian category $\mathcal{O}_K^{\mathrm{al}}$-**Mod** ([Sta21, 010H]), we know that the vertical arrow in the middle is also an isomorphism. In conclusion, $B\widehat{\otimes}_A C \to D$ is an isomorphism, which completes the proof. □

Corollary 5.29. *Let K be a pre-perfectoid field, $B \leftarrow A \to C$ a diagram of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. Then, the tensor product $B\otimes_A C$ is also almost pre-perfectoid.*

Proof. Since α^* commutes with arbitrary limits and colimits (5.5), we have $(B\widehat{\otimes}_A C)^{\mathrm{al}} = \widehat{B}^{\mathrm{al}}\widehat{\otimes}_{\widehat{A}^{\mathrm{al}}}\widehat{C}^{\mathrm{al}}$, which is perfectoid by 5.28. □

Lemma 5.30. *Let K be a perfectoid field, $\mathcal{O}_K \to V$ an extension of valuation rings of height 1. Then, there exists an extension of perfectoid fields $K \to L$ and an extension of valuation rings $V \to \mathcal{O}_L$ over $\mathcal{O}_K$.*

Proof. Let π be a pseudo-uniformizer of K, E the fraction field of V, $\overline{E}$ an algebraic closure of E, $\overline{V}$ the integral closure of V in $\overline{E}$. Let $\mathfrak{m}$ be a maximal ideal of $\overline{V}$. It lies over the unique maximal ideal of V as $V \to \overline{V}$ is integral. Setting $W = \overline{V}_{\mathfrak{m}}$, according to [Bou06, VI.§8.6, Prop.6], $V \to W$ is an extension of valuation rings of height 1. Since W is integrally closed in the algebraically closed fraction field $\overline{E}$, the Frobenius is surjective on W/pW. Thus, the fraction field of W is a pre-perfectoid field over K. Passing to completion, we get an extension of perfectoid fields $K \to L$ by 5.2. □

Proposition 5.31 ([BS19, 8.9])**.** *Let K be a pre-perfectoid field with a pseudo-uniformizer π dividing p, $R \to R'$ a homomorphism of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. If $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a π-complete arc-covering, then for any integer $n \geq 1$, the augmented Čech complex*

$$0 \to R/\pi^n \to R'/\pi^n \to (R'\otimes_R R')/\pi^n \to \cdots \tag{5.31.1}$$

is almost exact.

Proof. We follow Bhatt-Scholze's proof [BS19, 8.9]. After replacing $\mathcal{O}_K$, R, R' by their π-adic completions, we may assume that K is a perfectoid field and that R^{al} and R'^{al} are perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras such that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a π-complete arc-covering by 3.4.(6). Since the localization functor α^* commutes with arbitrary limits and colimits (5.5), $(\widehat{\otimes}^k_R R')^{\mathrm{al}} = \widehat{\otimes}^k_{R^{\mathrm{al}}} R'^{\mathrm{al}}$ is still a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra by 5.28 for any $k \geq 0$. In particular, $\widehat{\otimes}^k_R R'$ is almost flat over $\mathcal{O}_K$. Then, by dévissage, it suffices to show the almost exactness of the augmented Čech complex when $n = 1$, i.e. the almost exactness of

$$0 \to R^\flat/\pi^\flat \to R'^\flat/\pi^\flat \to (R'^\flat \otimes_{R^\flat} R'^\flat)/\pi^\flat \to \cdots . \tag{5.31.2}$$

We claim that the natural morphism $X = \mathrm{Spec}(R'^\flat) \coprod \mathrm{Spec}(R^\flat[1/\pi^\flat]) \to Y = \mathrm{Spec}(R^\flat)$ is an arc-covering. Since $\mathrm{Spec}(R'/\pi) \to \mathrm{Spec}(R/\pi)$ is an arc-covering, $X \to Y$ is surjective. Therefore, we only need to consider the test map $\mathrm{Spec}(V) \to Y$ where V is a valuation ring of height 1. There are three cases:

1. If $\pi^\flat$ is invertible in V, then we get a natural lifting $R^\flat[1/\pi^\flat] \to V$.
2. If $\pi^\flat$ is zero in V, then we have $R/\pi = R^\flat/\pi^\flat \to V$, and there is a lifting $R'/\pi = R'^\flat/\pi^\flat \to W$.
3. Otherwise, $\mathcal{O}_{K^\flat} \to V$ is an extension of valuation rings. After replacing V by an extension (5.30), we may assume that $V[1/\pi^\flat]$ is a perfectoid field over $K^\flat$ with valuation ring V. By tilting correspondence 5.27, it corresponds to a perfectoid field over K with valuation ring $V^\sharp$, together with an $\mathcal{O}_K$-morphism $R \to V^\sharp$ by 5.9. Since $R \to R'$ gives a π-complete arc-covering, there is an extension $V^\sharp \to W$ of valuation rings of height 1 and a lifting $R' \to W$. After replacing W by an extension (5.30), we may assume that $W[1/\pi]$ is a perfectoid field over K with valuation ring W. By tilting correspondence 5.27 and 5.9, we get a lifting $R'^\flat \to W^\flat$ of $R^\flat \to V$.

Now we apply 4.10 to the arc-covering $X \to Y$ of perfect affine $\mathbb{F}_p$-schemes. We get an exact augmented Čech complex

$$0 \to R^\flat \to R'^\flat \times R^\flat[\frac{1}{\pi^\flat}] \to (R'^\flat \times R^\flat[\frac{1}{\pi^\flat}]) \otimes_{R^\flat} (R'^\flat \times R^\flat[\frac{1}{\pi^\flat}]) \to \cdots . \tag{5.31.3}$$

Since each term is a perfect $\mathbb{F}_p$-algebra, the submodule of $\pi^\flat$-torsion elements is almost zero, in other words, each term is almost flat over $\mathcal{O}_{K^\flat}$. Modulo $\pi^\flat$, we get the almost exactness of (5.31.2), which completes the proof. □

6. Brief Review on Covanishing Fibred Sites

We give a brief review on covanishing fibred sites, which are developed by Abbes and Gros [AGT16, VI]. We remark that [AGT16, VI] does not require the sites to admit finite limits (2.4).

6.1. A *fibred site* E/C is a fibred category $\pi : E \to C$ whose fibres are sites such that for a cleavage and for every morphism $f : \beta \to \alpha$ in C, the inverse image functor $f^+ : E_\alpha \to E_\beta$ gives a morphism of sites (so that the same holds for any cleavage) (cf. [SGA 4_{II}, VI.7.2]).

Let x be an object of E over $\alpha \in \mathrm{Ob}(C)$. We denote by

$$\iota_\alpha^+ : E_\alpha \to E \tag{6.1.1}$$

the inclusion functor of the fibre category E_α over α into the whole category E. A *vertical covering* of x is the image by ι_α^+ of a covering family $\{x_m \to x\}_{m\in M}$ in E_α. We call the topology generated by all vertical coverings the *total topology* on E (cf. [SGA 4_{II}, VI.7.4.2]).

Assume further that C is a site. A *Cartesian covering* of x is a family $\{x_n \to x\}_{n\in N}$ of morphisms of E such that there exists a covering family $\{\alpha_n \to \alpha\}_{n\in N}$ in C with x_n isomorphic to the pullback of x by $\alpha_n \to \alpha$ for each n.

Definition 6.2 ([AGT16, VI.5.3])**.** A *covanishing fibred site* is a fibred site E/C where C is a site. We associate to E the *covanishing topology* which is generated by all vertical coverings and Cartesian coverings. We simply call a covering family for the covanishing topology a *covanishing covering*.

Definition 6.3. Let E/C be a covanishing fibred site. We call a composition of a Cartesian covering followed by vertical coverings a *standard covanishing covering*. More precisely, a standard covanishing covering is a family of morphisms of E

$$\{x_{nm} \to x\}_{n\in N, m\in M_n} \tag{6.3.1}$$

such that there is a Cartesian covering $\{x_n \to x\}_{n\in N}$ and for each $n \in N$ a vertical covering $\{x_{nm} \to x_n\}_{m\in M_n}$.

Proposition 6.4 ([AGT16, VI.5.9])**.** *Let E/C be a covanishing fibred site. Assume that in each fibre any object is quasi-compact, then a family of morphisms $\{x_i \to x\}_{i\in I}$ of E is a covanishing covering if and only if it can be refined by a standard covanishing covering.*

6.5. Let E/C be a fibred category. Fixing a cleavage of E/C, to give a presheaf $\mathcal{F}$ on E is equivalent to give a presheaf $\mathcal{F}_\alpha$ on each fibre category E_α and transition morphisms $\mathcal{F}_\alpha \to f^{\mathrm{p}}\mathcal{F}_\beta$ for each morphism $f : \beta \to \alpha$ in C satisfying a cocycle relation (cf. [SGA 4_{II}, VI.7.4.7]). Thus, we simply denote a presheaf $\mathcal{F}$ on E by

$$\mathcal{F} = \{\alpha \mapsto \mathcal{F}_\alpha\}_{\alpha\in \mathrm{Ob}(C)}, \tag{6.5.1}$$

where $\mathcal{F}_\alpha = \iota_\alpha^{\mathrm{p}}\mathcal{F}$ is the restriction of $\mathcal{F}$ on the fibre category E_α. If E/C is a fibred site, then $\mathcal{F}$ is a sheaf with respect to the total topology on E if and only if $\mathcal{F}_\alpha$ is a sheaf on E_α for each α ([SGA 4_{II}, VI.7.4.7]). Moreover, we have the following description of a covanishing sheaf.

Proposition 6.6 ([AGT16, VI.5.10])**.** *Let E/C be a covanishing fibred site. Then, a presheaf $\mathcal{F}$ on E is a sheaf if and only if the following conditions hold:*

(v) *The presheaf $\mathcal{F}_\alpha = \iota_\alpha^{\mathrm{p}}\mathcal{F}$ on E_α is a sheaf for any $\alpha \in \mathrm{Ob}(C)$.*

(c) *For any covering family $\{f_i : \alpha_i \to \alpha\}_{i\in I}$ of C, if we set $\alpha_{ij} = \alpha_i \times_\alpha \alpha_j$ and $f_{ij} : \alpha_{ij} \to \alpha$, then the sequence of sheaves on E_α,*

$$\mathcal{F}_\alpha \to \prod_{i\in I} f_{i*}\mathcal{F}_{\alpha_i} \rightrightarrows \prod_{i,j\in I} f_{ij*}\mathcal{F}_{\alpha_{ij}}, \tag{6.6.1}$$

is exact.

7. Faltings Fibred Sites

7.1. Let $Y \to X$ be a morphism of $\mathbb{U}$-small coherent schemes, and let $\mathbf{E}_{Y\to X}$ be the category of morphisms $V \to U$ of $\mathbb{U}$-small coherent schemes over the morphism $Y \to X$, namely, the category of commutative diagrams of coherent schemes

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{7.1.1}$$

Given a functor $I \to \mathbf{E}_{Y\to X}$ sending i to $(V_i \to U_i)$, if $\lim V_i$ and $\lim U_i$ are representable in the category of coherent schemes, then $\lim(V_i \to U_i)$ is representable by $(\lim V_i \to \lim U_i)$. We say that a morphism $(V' \to U') \to (V \to U)$ of objects of $\mathbf{E}_{Y\to X}$ is *Cartesian* if $V' \to V \times_U U'$ is an isomorphism. It is clear that the Cartesian morphisms in $\mathbf{E}_{Y\to X}$ are stable under base change.

Consider the functor

$$\sigma^+ : \mathbf{E}_{Y\to X} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/X},\ (V \to U) \longmapsto U. \tag{7.1.2}$$

The fibre category over U is canonically equivalent to the category $\mathbf{Sch}^{\mathrm{coh}}_{/U_Y}$ of coherent U_Y-schemes, where $U_Y = Y \times_X U$. The base change by $U' \to U$ gives an inverse image functor $\mathbf{Sch}^{\mathrm{coh}}_{/U_Y} \to \mathbf{Sch}^{\mathrm{coh}}_{/U'_Y}$, which endows $\mathbf{E}_{Y\to X}/\mathbf{Sch}^{\mathrm{coh}}_{/X}$ with a structure of fibred category. We define a presheaf on $\mathbf{E}_{Y\to X}$ by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{7.1.3}$$

where U^V is the integral closure of U in V.

Definition 7.2. Let $Y \to X$ be a morphism of coherent schemes. A morphism $(V' \to U') \to (V \to U)$ in $\mathbf{E}_{Y\to X}$ is called *étale*, if $U' \to U$ is étale and $V' \to V \times_U U'$ is finite étale.

Lemma 7.3. *Let $Y \to X$ be a morphism of coherent schemes, $(V'' \to U'') \xrightarrow{g} (V' \to U') \xrightarrow{f} (V \to U)$ morphisms in $\mathbf{E}_{Y\to X}$.*

1. *If f is étale, then any base change of f is also étale.*
2. *If f and g are étale, then $f \circ g$ is also étale.*
3. *If f and $f \circ g$ are étale, then g is also étale.*

Proof. It follows directly from the definitions. □

7.4. Let $Y \to X$ be a morphism of coherent schemes. We still denote by $X_{\text{ét}}$ (resp. $X_{\text{fét}}$) the site formed by coherent étale (resp. finite étale) X-schemes endowed with étale topology. Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the full subcategory of $\mathbf{E}_{Y\to X}$ formed by $(V \to U)$ étale over the final object $(Y \to X)$. It is clear that $\mathbf{E}^{\text{ét}}_{Y\to X}$ is stable under finite limits in $\mathbf{E}_{Y\to X}$. Then, the functor (7.1.2) induces a functor

$$\sigma^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow X_{\text{ét}},\ (V \to U) \longmapsto U, \tag{7.4.1}$$

which endows $\mathbf{E}^{\text{ét}}_{Y\to X}/X_{\text{ét}}$ with a structure of fibred sites, whose fibre over U is the finite étale site $U_{Y,\text{fét}}$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the associated covanishing topology, that is, the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where $\{V_m \to V\}_{m\in M}$ is a finite étale covering;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where $\{U_n \to U\}_{n\in N}$ is an étale covering.

It is clear that any object of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is quasi-compact by 6.4. We still denote by $\overline{\mathscr{B}}$ the restriction of the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}_{Y\to X}$ to $\mathbf{E}^{\text{ét}}_{Y\to X}$ if there is no ambiguity.

Lemma 7.5. *Let $Y \to X$ be a morphism of coherent schemes. Then, the presheaf on $\mathbf{Sch}^{\mathrm{coh}}_{/Y}$ sending Y' to $\Gamma(X^{Y'}, \mathcal{O}_{X^{Y'}})$ is a sheaf with respect to the fpqc topology* ([Sta21, 022A]).

Proof. We may regard $\mathcal{O}_{X^{Y'}}$ as a quasi-coherent $\mathcal{O}_X$-algebra over X. It suffices to show that for a finite family of morphisms $\{Y_i \to Y\}_{i\in I}$ with $Y' = \coprod_{i\in I} Y_i$ faithfully flat over Y, the sequence of quasi-coherent $\mathcal{O}_X$-algebras

$$0 \longrightarrow \mathcal{O}_{X^Y} \longrightarrow \mathcal{O}_{X^{Y'}} \rightrightarrows \mathcal{O}_{X^{Y'\times_Y Y'}} \tag{7.5.1}$$

is exact. Thus, we may assume that $X = \mathrm{Spec}(R)$ is affine. We set $A_0 = \Gamma(Y, \mathcal{O}_Y)$, $A_1 = \Gamma(Y', \mathcal{O}_{Y'})$, $A_2 = \Gamma(Y' \times_Y Y', \mathcal{O}_{Y' \times_Y Y'})$, $R_0 = \Gamma(X^Y, \mathcal{O}_{X^Y})$, $R_1 = \Gamma(X^{Y'}, \mathcal{O}_{X^{Y'}})$, $R_2 = \Gamma(X^{Y' \times_Y Y'}, \mathcal{O}_{X^{Y' \times_Y Y'}})$. Notice that R_i is the integral closure of R in A_i for $i = 0, 1, 2$ ([Sta21, 035F]). Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_0 & \longrightarrow & R_1 & \rightrightarrows & R_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \rightrightarrows & A_2 \end{array} \tag{7.5.2}$$

We see that the vertical arrows are injective and the second row is exact by faithfully flat descent. Notice that $R_0 = A_0 \cap R_1$, since they are both the integral closure of R in A_0 as $A_0 \subseteq A_1$. Thus, the first row is also exact, which completes the proof. □

Proposition 7.6. *Let $Y \to X$ be a morphism of coherent schemes. Then, the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y \to X}$ is a sheaf.*

Proof. It follows directly from 6.6, whose first condition holds by 7.5, and whose second condition holds by 3.17 (cf. [AGT16, III.8.16]). □

Definition 7.7 ([Fal02, page 214], [AGT16, VI.10.1])**.** We call $\mathbf{E}^{\text{ét}}_{Y \to X}/X_{\text{ét}}$ the *Faltings fibred site* of the morphism of coherent schemes $Y \to X$, and call $\overline{\mathscr{B}}$ the structural sheaf of $\mathbf{E}^{\text{ét}}_{Y \to X}$.

It is clear that the localization $(\mathbf{E}^{\text{ét}}_{Y \to X})_{/(V \to U)}$ of $\mathbf{E}^{\text{ét}}_{Y \to X}$ at an object $(V \to U)$ is canonically equivalent to the Faltings fibred site $\mathbf{E}^{\text{ét}}_{V \to U}$ of the morphism $V \to U$ by 6.4 (cf. [AGT16, VI.10.14]).

Lemma 7.8. *Let X be the spectrum of an absolutely integrally closed valuation ring, Y a quasi-compact open subscheme of X. Then, for any presheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y \to X}$, we have $\mathcal{F}^{\mathrm{a}}(Y \to X) = \mathcal{F}(Y \to X)$. In particular, the associated topos of $\mathbf{E}^{\text{ét}}_{Y \to X}$ is local* ([SGA 4_{II}, VI.8.4.6])*.*

Proof. Notice that Y is also the spectrum of an absolutely integrally closed valuation ring by 3.11.(1) and that absolutely integrally closed valuation rings are strictly Henselian. Thus, any covering of $Y \to X$ in $\mathbf{E}^{\text{ét}}_{Y \to X}$ can be refined by the identity covering by 6.4. We see that $\mathcal{F}^{\mathrm{a}}(Y \to X) = \mathcal{F}(Y \to X)$ for any presheaf $\mathcal{F}$. For the last assertion, it suffices to show that the section functor $\Gamma(Y \to X, -)$ commutes with colimits of sheaves. For a colimit of sheaves $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$, let $\mathcal{G}$ be the colimit of presheaves $\mathcal{G} = \operatorname{colim} \mathcal{F}_i$. Then, we have $\mathcal{F} = \mathcal{G}^{\mathrm{a}}$ and $\Gamma(Y \to X, \mathcal{F}) = \Gamma(Y \to X, \mathcal{G}) = \operatorname{colim} \Gamma(Y \to X, \mathcal{F}_i)$. □

7.9. Let $(Y_\lambda \to X_\lambda)_{\lambda \in \Lambda}$ be a $\mathbb{U}$-small directed inverse system of morphisms of $\mathbb{U}$-small coherent schemes with affine transition morphisms $Y_{\lambda'} \to Y_\lambda$ and $X_{\lambda'} \to X_\lambda$ ($\lambda' \geq \lambda$). We set $(Y \to X) = \lim_{\lambda \in \Lambda}(Y_\lambda \to X_\lambda)$. We regard the directed set Λ as a filtered category and regard the inverse system $(Y_\lambda \to X_\lambda)_{\lambda \in \Lambda}$ as a functor $\varphi : \Lambda^{\mathrm{op}} \to \mathbf{E}$ from the opposite category of Λ to the category of morphisms of $\mathbb{U}$-small coherent schemes. Consider the fibred category $\mathbf{E}^{\text{ét}}_{\varphi} \to \Lambda^{\mathrm{op}}$ defined by φ whose fibre category over λ is $\mathbf{E}^{\text{ét}}_{Y_\lambda \to X_\lambda}$ and whose inverse image functor $\varphi^+_{\lambda'\lambda} : \mathbf{E}^{\text{ét}}_{Y_\lambda \to X_\lambda} \to \mathbf{E}^{\text{ét}}_{Y'_\lambda \to X'_\lambda}$ associated to a morphism $\lambda' \to \lambda$ in Λ^{op} is given by the base change by the transition morphism $(Y_{\lambda'} \to X_{\lambda'}) \to (Y_\lambda \to X_\lambda)$ (cf. [AGT16, VI.11.2]). Let $\varphi^+_\lambda : \mathbf{E}^{\text{ét}}_{Y_\lambda \to X_\lambda} \to \mathbf{E}^{\text{ét}}_{Y \to X}$ be the functor defined by the base change by the transition morphism $(Y \to X) \to (Y_\lambda \to X_\lambda)$.

Recall that the filtered colimit of categories $(\mathbf{E}^{\text{ét}}_{Y_\lambda \to X_\lambda})_{\lambda \in \Lambda}$ is representable by the category $\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}$ whose objects are those of $\mathbf{E}^{\text{ét}}_{\varphi}$ and whose morphisms are given by ([SGA 4_{II}, VI 6.3, 6.5])

$$\mathrm{Hom}_{\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}}((V \to U), (V' \to U')) = \operatorname*{colim}_{\substack{(V'' \to U'') \to (V \to U) \\ \text{Cartesian}}} \mathrm{Hom}_{\mathbf{E}^{\text{ét}}_{\varphi}}((V'' \to U''), (V' \to U')), \tag{7.9.1}$$

where the colimit is taken over the opposite category of the cofiltered category of Cartesian morphisms with target $V \to U$ of the fibred category $\mathbf{E}^{\text{ét}}_{\varphi}$ over Λ^{op} (distinguish with the Cartesian morphisms defined in 7.1). We see that the functors φ^+_λ induces an equivalence of categories by [EGA IV_3, 8.8.2, 8.10.5] and [EGA IV_4, 17.7.8]

$$\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi} \xrightarrow{\sim} \mathbf{E}^{\text{ét}}_{Y \to X}. \tag{7.9.2}$$

Recall that the cofiltered limit of sites $(\mathbf{E}^{\text{ét}}_{Y_\lambda \to X_\lambda})_{\lambda \in \Lambda}$ is representable by $\underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}$ endowed with the coarsest topology such that the natural functors $\mathbf{E}^{\text{ét}}_{Y_\lambda \to X_\lambda} \to \underrightarrow{\mathbf{E}}^{\text{ét}}_{\varphi}$ are continuous ([SGA 4_{II}, VI.8.2.3]).

Lemma 7.10. *With the notation in* 7.9*, for any covering family* $\mathfrak{U}=\{f_k:(V_k\to U_k)\to(V\to U)\}_{k\in K}$ *in* $\mathbf{E}^{\text{ét}}_{Y\to X}$ *with* K *finite, there exists an index* $\lambda_0\in\Lambda$ *and a covering family* $\mathfrak{U}_{\lambda_0}=\{f_{k\lambda_0}:(V_{k\lambda_0}\to U_{k\lambda_0})\to(V_{\lambda_0}\to U_{\lambda_0})\}_{k\in K}$ *in* $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$ *such that* f_k *is the base change of* $f_{k\lambda_0}$ *by the transition morphism* $(Y\to X)\to(Y_{\lambda_0}\to X_{\lambda_0})$*.*

Proof. There is a standard covanishing covering $\mathfrak{U}'=\{g_{nm}:(V'_{nm}\to U'_n)\to(V\to U)\}_{n\in N,m\in M_n}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ with N, M_n finite, which refines $\mathfrak{U}$ by 6.4. The equivalence (7.9.2) implies that there exists an index $\lambda_1\in\Lambda$ and families of morphisms $\mathfrak{U}'_{\lambda_1}=\{g_{nm\lambda_1}:(V'_{nm\lambda_1}\to U'_{n\lambda_1})\to(V_{\lambda_1}\to U_{\lambda_1})\}_{n\in N,m\in M_n}$ (resp. $\mathfrak{U}_{\lambda_1}=\{f_{k\lambda_1}:(V_{k\lambda_1}\to U_{k\lambda_1})\to(V_{\lambda_1}\to U_{\lambda_1})\}_{k\in K}$) in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_1}\to X_{\lambda_1}}$ such that g_{nm} (resp. f_k) is the base change of $g_{nm\lambda_1}$ (resp. $f_{k\lambda_1}$) by the transition morphism $(Y\to X)\to(Y_{\lambda_1}\to X_{\lambda_1})$ and that $\mathfrak{U}'_{\lambda_1}$ refines $\mathfrak{U}_{\lambda_1}$. For each $\lambda\geq\lambda_1$, let $g_{nm\lambda}:(V'_{nm\lambda}\to U'_{n\lambda})\to(V_\lambda\to U_\lambda)$ (resp. $f_{k\lambda}:(V_{k\lambda}\to U_{k\lambda})\to(V_\lambda\to U_\lambda)$) be the base change of $g_{nm\lambda_1}$ (resp. $f_{k\lambda_1}$) by the transition morphism $(Y_\lambda\to X_\lambda)\to(Y_{\lambda_1}\to X_{\lambda_1})$. Since the morphisms $\coprod_{n\in N}U'_n\to U$ and $\coprod_{m\in M_n}V'_{nm}\to V\times_U U'_n$ are surjective, there exists an index $\lambda_0\geq\lambda_1$ such that the morphisms $\coprod_{n\in N}U'_{n\lambda_0}\to U_{\lambda_0}$ and $\coprod_{m\in M_n}V'_{nm\lambda_0}\to V_{\lambda_0}\times_{U_{\lambda_0}}U'_{n\lambda_0}$ are also surjective by [EGA IV$_3$, 8.10.5], i.e. $\mathfrak{U}'_{\lambda_0}=\{g_{nm\lambda_0}\}_{n\in N,m\in M_n}$ is a standard covanishing covering in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$. Thus, $\mathfrak{U}_{\lambda_0}=\{f_{k\lambda_0}\}_{k\in K}$ is a covering family in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$. □

Proposition 7.11 ([AGT16, VI.11])**.** *With the notation in* 7.9*,* $\mathbf{E}^{\text{ét}}_{Y\to X}$ *represents the limit of sites* $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$*, and* $\overline{\mathscr{B}}=\operatorname{colim}_{\lambda\in\Lambda}\varphi_\lambda^{-1}\overline{\mathscr{B}}$*.*

Proof. (1) is proved in [AGT16, VI.11.3]. It also follows directly from the discussion in 7.9 and 7.10. For (2), notice that $\operatorname{colim}_{\lambda\in\Lambda}\varphi_\lambda^{-1}\overline{\mathscr{B}}=(\operatorname{colim}_{\lambda\in\Lambda}\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})^{\mathrm{a}}$ ([Sta21, 00WI]). It suffices to show that $\overline{\mathscr{B}}(V\to U)=\operatorname{colim}_{\lambda\in\Lambda}(\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})(V\to U)$ for each object $V\to U$ of $\mathbf{E}^{\text{ét}}_{Y\to X}$. It follows from the equivalence (7.9.2) that there exists an index $\lambda_0\in\Lambda$ and an object $V_{\lambda_0}\to U_{\lambda_0}$ of $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$ such that $V\to U$ is the base change of $V_{\lambda_0}\to U_{\lambda_0}$ by the transition morphism. For each $\lambda\geq\lambda_0$, let $V_\lambda\to U_\lambda$ be the base change of $V_{\lambda_0}\to U_{\lambda_0}$ by the transition morphism $(Y_\lambda\to X_\lambda)\to(Y_{\lambda_0}\to X_{\lambda_0})$. Then, we have $\operatorname{colim}_{\lambda\in\Lambda}(\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})(V\to U)=\operatorname{colim}_{\lambda\in\Lambda}\overline{\mathscr{B}}(V_\lambda\to U_\lambda)$ by [SGA 4$_{\mathrm{II}}$, VI 8.5.2, 8.5.7]. The conclusion follows from $\overline{\mathscr{B}}(V\to U)=\operatorname{colim}_{\lambda\in\Lambda}\overline{\mathscr{B}}(V_\lambda\to U_\lambda)$ by 3.18. □

Definition 7.12. A morphism $X\to S$ of coherent schemes is called *pro-étale* (resp. *pro-finite étale*), if there is a directed inverse system of étale (resp. finite étale) S-schemes $(X_\lambda)_{\lambda\in\Lambda}$ with affine transition morphisms such that there is an isomorphism of S-schemes $X\cong\lim_{\lambda\in\Lambda}X_\lambda$. We call such an inverse system $(X_\lambda)_{\lambda\in\Lambda}$ a *pro-étale presentation* (resp. *pro-finite étale presentation*) of X over S.

Lemma 7.13. *Let* $X\xrightarrow{g}Y\xrightarrow{f}S$ *be morphisms of coherent schemes.*

1. *If* f *is pro-étale (resp. pro-finite étale), then* f *is flat (resp. flat and integral).*
2. *Any base change of a pro-étale (resp. pro-finite étale) morphism is pro-étale (resp. pro-finite étale).*
3. *If* f *and* g *are pro-étale (resp. pro-finite étale), then so is* $f\circ g$*.*
4. *If* f *and* $f\circ g$ *are pro-étale (resp. pro-finite étale), then so is* g*.*
5. *If* f *is pro-étale with a pro-étale presentation* $Y=\lim Y_\beta$*, and if* g *is étale (resp. finite étale), then there is an index* β_0 *and an étale (resp. finite étale)* S*-morphism* $g_{\beta_0}:X_{\beta_0}\to Y_{\beta_0}$ *such that* g *is the base change of* g_{β_0} *by* $Y\to Y_{\beta_0}$*.*
6. *Let* Z *and* Z' *be coherent schemes pro-étale over* S *with pro-étale presentations* $Z=\lim Z_\alpha$*,* $Z'=\lim Z'_\beta$*, then*

$$\operatorname{Hom}_S(Z,Z')=\lim_\beta\operatorname*{colim}_\alpha\operatorname{Hom}_S(Z_\alpha,Z'_\beta). \tag{7.13.1}$$

Proof. (1) and (2) follow directly from the definition.

(3) We follow closely the proof of 3.6. Let $X=\lim X_\alpha$ and $Y=\lim Y_\beta$ be pro-étale (resp. pro-finite étale) presentations over Y and over S respectively. As Y_β are coherent, for each α, there is an index β_α and an étale (resp. finite étale) Y_{β_α}-scheme $X_{\alpha\beta_\alpha}$ such that $X_\alpha\to Y$ is the base change of $X_{\alpha\beta_\alpha}\to Y_{\beta_\alpha}$ ([EGA IV$_3$, 8.8.2, 8.10.5], [EGA IV$_4$, 17.7.8]). For each $\beta\geq\beta_\alpha$, let $X_{\alpha\beta}\to Y_\beta$ be the base change of $X_{\alpha\beta_\alpha}\to Y_{\beta_\alpha}$ by $Y_\beta\to Y_{\beta_\alpha}$. Then, we have $X=\lim_{\alpha,\beta\geq\beta_\alpha}X_{\alpha\beta}$ by [EGA IV$_3$, 8.8.2] (cf. 3.6), which is pro-finite étale over S. For (5), one can take $X=X_\alpha$.

(6) We have

$$\mathrm{Hom}_S(Z,Z') = \lim_\beta \mathrm{Hom}_S(Z,Z'_\beta) = \lim_\beta \operatorname*{colim}_\alpha \mathrm{Hom}_S(Z_\alpha, Z'_\beta) \tag{7.13.2}$$

where the first equality follows from the universal property of limits of schemes, and the second follows from the fact that $Z'_\beta \to S$ is locally of finite presentation ([EGA IV$_3$, 8.14.2]). For (4), we take $Z = X$ and $Z' = Y$. Then, for each index β, we have an S-morphism $X_\alpha \to Y_\beta$ for α big enough, which is also étale (resp. finite étale). Then, $X = \lim_\alpha X_\alpha = \lim_{\alpha,\beta} X_\alpha \times_{Y_\beta} Y$ is pro-étale (resp. pro-finite étale) over Y. □

Remark 7.14. A pro-étale (resp. pro-finite étale) morphism of $\mathbb{U}$-small coherent schemes $X \to S$ admits a $\mathbb{U}$-small pro-étale (resp. pro-finite étale) presentation. Indeed, let $X = \lim_{\lambda\in\Lambda} X_\lambda$ be a presentation of $X \to S$. We may regard Λ as a filtered category with an initial object 0. Consider the category $\mathscr{C} = {}_{X\backslash}X_{0,\text{ét,aff}}$ (resp. $\mathscr{C} = {}_{X\backslash}X_{0,\text{fét}}$) of affine (resp. finite) étale X_0-schemes which are under X. Notice that $\mathscr{C}$ is essentially $\mathbb{U}$-small and that the natual functor $\Lambda \to \mathscr{C}^{\mathrm{op}}$ is cofinal by 7.13.(6) ([SGA 4$_\mathrm{I}$, I.8.1.3]). Therefore, after replacing $\mathscr{C}^{\mathrm{op}}$ by a $\mathbb{U}$-small directed set Λ', we obtain a $\mathbb{U}$-small presentation $X = \lim_{X'\in\Lambda'} X'$ ([SGA 4$_\mathrm{I}$, I.8.1.6]).

Definition 7.15. For any $\mathbb{U}$-small coherent scheme X, we endow the category of $\mathbb{U}$-small coherent pro-étale (resp. pro-finite étale) X-schemes with the topology generated by the pretopology formed by families of morphisms

$$\{f_i : U_i \to U\}_{i\in I} \tag{7.15.1}$$

such that I is finite and that $U = \bigcup f_i(U_i)$. This defines a site $X_{\text{proét}}$ (resp. $X_{\text{profét}}$), called the *pro-étale site* (resp. *pro-finite étale site*) of X.

It is clear that the localization $X_{\text{proét}/U}$ (resp. $X_{\text{profét}/U}$) of $X_{\text{proét}}$ (resp. $X_{\text{profét}}$) at an object U is canonically equivalent to the pro-étale (resp. pro-finite étale) site $U_{\text{proét}}$ (resp. $U_{\text{profét}}$) of U. By definition, any object in $X_{\text{proét}}$ (resp. $X_{\text{profét}}$) is quasi-compact.

7.16. We compare our definitions of pro-étale site and pro-finite étale site with some other definitions existing in the literature. But we don't use the comparison result in this paper.

Let X be a $\mathbb{U}$-small Noetherian scheme. Consider the category of pro-objects pro-$X_{\text{fét}}$ of $X_{\text{fét}}$, i.e. the category whose objects are functors $F : \mathcal{A} \to X_{\text{fét}}$ with $\mathcal{A}$ a $\mathbb{U}$-small cofiltered category and whose morphisms are given by $\mathrm{Hom}(F,G) = \lim_{\beta\in\mathcal{B}} \operatorname{colim}_{\alpha\in\mathcal{A}} \mathrm{Hom}(F(\alpha), G(\beta))$ for any $F : \mathcal{A} \to X_{\text{fét}}$ and $G : \mathcal{B} \to X_{\text{fét}}$ ([Sch13a, 3.2]). We may simply denote such a functor F by $(X_\alpha)_{\alpha\in A}$. Remark that $\lim_{\alpha\in\mathcal{A}} X_\alpha$ exists which is pro-finite étale over X. Consider the functor

$$\text{pro-}X_{\text{fét}} \longrightarrow X_{\text{profét}},\ (X_\alpha)_{\alpha\in A} \mapsto \lim_{\alpha\in\mathcal{A}} X_\alpha, \tag{7.16.1}$$

which is well-defined and fully faithful by 7.13.(6) and essentially surjective by 7.14. Thus, according to [Sch13a, 3.3] and its corrigendum [Sch16], Scholze's pro-finite étale site $X^{\mathrm{S}}_{\text{profét}}$ has the underlying category $X_{\text{profét}}$ and its topology is generated by the families of morphisms

$$\{U_i \xrightarrow{f_i} U' \xrightarrow{f} U\}_{i\in I} \tag{7.16.2}$$

where I is finite and $\coprod_{i\in I} U_i \to U'$ is finite étale surjective, and there exists a well-ordered directed set Λ with a least index 0 and a pro-finite étale presentation $(U'_\lambda)_{\lambda\in\Lambda}$ of f such that $U'_0 = U$ and that for each $\lambda \in \Lambda$ the natural morphism $U'_\lambda \to \lim_{\mu<\lambda} U'_\mu$ is finite étale surjective (cf. [Ker16, 5.5], 7.13 and [EGA IV$_3$, 8.10.5.(vi)]). It is clear that the topology of our pro-finite étale site $X_{\text{profét}}$ is finer than that of $X^{\mathrm{S}}_{\text{profét}}$. We remark that if X is connected, then $X^{\mathrm{S}}_{\text{profét}}$ gives a site-theoretic interpretation of the continuous group cohomology of the fundamental group of X ([Sch13a, 3.7]). For simplicity, we don't consider $X^{\mathrm{S}}_{\text{profét}}$ in the rest of the paper, but we can replace $X_{\text{profét}}$ by $X^{\mathrm{S}}_{\text{profét}}$ for most of the statements in this paper (cf. [Ker16, 6]).

7.17. Let X be a $\mathbb{U}$-small scheme. Bhatt-Scholze's pro-étale site $X^{\mathrm{BS}}_{\text{proét}}$ has the underlying category of $\mathbb{U}$-small weakly étale X-schemes and a family of morphisms $\{f_i : Y_i \to Y\}_{i\in I}$ in $X^{\mathrm{BS}}_{\text{proét}}$ is a covering if and only if for any affine open subscheme U of Y, there exists a map $a : \{1,\dots,n\} \to I$ and affine open subschemes U_j of $Y_{a(j)}$ $(j = 1,\dots,n)$ such that $U = \bigcup_{j=1}^n f_{a(j)}(U_j)$ ([BS15, 4.1.1], cf. [Sta21, 0989]). Remark that a pro-étale morphism of coherent schemes is weakly étale by [BS15, 2.3.3.1]. Thus, for a coherent scheme X, $X_{\text{proét}}$ is a full subcategory of $X^{\mathrm{BS}}_{\text{proét}}$.

Lemma 7.18. *Let X be a coherent scheme. The full subcategory $X_{\mathrm{proét}}$ of $X^{\mathrm{BS}}_{\mathrm{proét}}$ is a topologically generating family, and the induced topology on $X_{\mathrm{proét}}$ coincides the topology defined in* 7.15*. In particular, the topoi of sheaves of $\mathbb{V}$-small sets associated to the two sites are naturally equivalent.*

Proof. For a weakly étale X-scheme Y, we show that it can be covered by pro-étale X-schemes. After replacing X by a finite affine open covering and replacing Y by an affine open covering, we may assume that X and Y are affine. Then, the result follows from the fact that for any weakly étale morphism of rings $A \to B$ there exists a faithfully flat ind-étale morphism $B \to C$ such that $A \to C$ is ind-étale by [BS15, 2.3.4] (cf. [BS15, 4.1.3]). Thus, $X_{\mathrm{proét}}$ is a topologically generating family of $X^{\mathrm{BS}}_{\mathrm{proét}}$. A family of morphisms $\{f_i : Y_i \to Y\}_{i\in I}$ in $X_{\mathrm{proét}}$ is a covering with respect to the induced topology if and only if for any affine open subscheme U of Y, there exists a map $a : \{1, \ldots, n\} \to I$ and affine open subschemes U_j of $Y_{a(j)}$ $(j = 1, \ldots, n)$ such that $U = \bigcup_{j=1}^{n} f_{a(j)}(U_j)$ ([SGA 4_{I}, III.3.3]). Notice that Y_i and Y are coherent, thus $\{f_i\}_{i\in I}$ is a covering if and only if there exists a finite subset $I_0 \subseteq I$ such that $Y = \bigcup_{i\in I_0} f_i(Y_i)$, which shows that the induced topology on $X_{\mathrm{proét}}$ coincides the topology defined in 7.15. Finally, the "in particular" part follows from [SGA 4_{I}, III.4.1]. □

Definition 7.19. Let $Y \to X$ be a morphism of coherent schemes. A morphism $(V' \to U') \to (V \to U)$ in $\mathbf{E}_{Y\to X}$ is called *pro-étale* if $U' \to U$ is pro-étale and $V' \to V \times_U U'$ is pro-finite étale. A *pro-étale presentation* of such a morphism is a directed inverse system $(V_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ étale over $V \to U$ with affine transition morphisms $U_{\lambda'} \to U_\lambda$ and $V_{\lambda'} \to V_\lambda$ $(\lambda' \geq \lambda)$ such that $(V' \to U') = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$.

Lemma 7.20. *Let $Y \to X$ be a morphism of coherent schemes, $(V'' \to U'') \xrightarrow{g} (V' \to U') \xrightarrow{f} (V \to U)$ morphisms in $\mathbf{E}_{Y\to X}$.*

1. *If f is pro-étale, then it admits a pro-étale presentation.*
2. *If f is pro-étale, then any base change of f is also pro-étale.*
3. *If f and g are pro-étale, then $f \circ g$ is also pro-étale.*
4. *If f and $f \circ g$ are pro-étale, then g is also pro-étale.*

Proof. It follows directly from 7.13 and its arguments. □

Remark 7.21. Similar to 7.14, a pro-étale morphism in $\mathbf{E}_{Y\to X}$ admits a $\mathbb{U}$-small presentation.

7.22. Let $Y \to X$ be a morphism of coherent schemes, $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ the full subcategory of $\mathbf{E}_{Y\to X}$ formed by objects which are pro-étale over the final object $Y \to X$. It is clear that $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ is stable under finite limits in $\mathbf{E}_{Y\to X}$. Then, the functor (7.1.2) induces a functor

$$\sigma^+ : \mathbf{E}^{\mathrm{proét}}_{Y\to X} \longrightarrow X_{\mathrm{proét}},\ (V \to U) \longmapsto U, \tag{7.22.1}$$

which endows $\mathbf{E}^{\mathrm{proét}}_{Y\to X}/X_{\mathrm{proét}}$ with a structure of fibred sites, whose fibre over U is the pro-finite étale site $U_{Y,\mathrm{profét}}$. We endow $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ with the associated covanishing topology. It is clear that any object in $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ is quasi-compact by 6.4. We still denote by $\overline{\mathscr{B}}$ the restriction of the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}_{Y\to X}$ to $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ if there is no ambiguity. We will show in 7.30 that $\overline{\mathscr{B}}$ is a sheaf on $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$.

Definition 7.23. We call $\mathbf{E}^{\mathrm{proét}}_{Y\to X}/X_{\mathrm{proét}}$ the *pro-étale Faltings fibred site* of the morphism of coherent schemes $Y \to X$, and call $\overline{\mathscr{B}}$ the structural sheaf of $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$.

It is clear that the localization $(\mathbf{E}^{\mathrm{proét}}_{Y\to X})_{/(V\to U)}$ of $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ at an object $V \to U$ is canonically equivalent to the pro-étale Faltings fibred site $\mathbf{E}^{\mathrm{proét}}_{V\to U}$ of the morphism $V \to U$ by 6.4.

Remark 7.24. The categories $X_{\mathrm{proét}}$, $X_{\mathrm{profét}}$ and $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ are essentially $\mathbb{V}$-small categories.

Lemma 7.25. *Let $Y \to X$ be a morphism of coherent schemes. Then, the inclusion functor*

$$\nu^+ : \mathbf{E}^{\mathrm{ét}}_{Y\to X} \longrightarrow \mathbf{E}^{\mathrm{proét}}_{Y\to X},\ (V \to U) \longmapsto (V \to U) \tag{7.25.1}$$

defines a morphism of sites $\nu : \mathbf{E}^{\mathrm{proét}}_{Y\to X} \to \mathbf{E}^{\mathrm{ét}}_{Y\to X}$ (2.5)*.*

Proof. It is clear that ν^+ commutes with finite limits and sends a standard covanishing covering in $\mathbf{E}^{\mathrm{ét}}_{Y\to X}$ to a standard covanishing covering in $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ (6.3). Therefore, ν^+ is continuous by 6.4 and defines a morphism of sites. □

Lemma 7.26. *Let $Y \to X$ be a morphism of coherent schemes. Then, the topology on $\mathbf{E}^{\text{ét}}_{Y\to X}$ is the topology induced from $\mathbf{E}^{\text{proét}}_{Y\to X}$.*

Proof. After 7.25, it suffices to show that for a family of morphisms $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$, if $\nu^+(\mathfrak{U})$ is a covering in $\mathbf{E}^{\text{proét}}_{Y\to X}$, then $\mathfrak{U}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$. We may assume that K is finite. There is a standard covanishing covering $\mathfrak{U}' = \{(V'_{nm} \to U'_n) \to (V \to U)\}_{n\in N, m\in M_n}$ in $\mathbf{E}^{\text{proét}}_{Y\to X}$ with N, M_n finite, which refines $\nu^+(\mathfrak{U})$ by 6.4. We take a directed set Ξ such that for each $n \in N$, we can take a pro-étale presentation $U'_n = \lim_{\xi\in\Xi} U'_{n\xi}$ over U, and we take a directed set Σ such that for each $n \in N$ and $m \in M_n$, we can take a pro-finite étale presentation $V'_{nm} = \lim_{\sigma\in\Sigma} V'_{nm\sigma}$ over $V \times_U U'_n$. By 7.13 (5), for each $\sigma \in \Sigma$, there exists an index $\xi_\sigma \in \Xi$ and a finite étale morphism $V'_{nm\sigma\xi_\sigma} \to V \times_U U'_{n\xi_\sigma}$ for each n and m, whose base change by $U'_n \to U'_{n\xi_\sigma}$ is $V'_{nm\sigma} \to V \times_U U'_n$. Let $V'_{nm\sigma\xi} \to V \times_U U'_{n\xi}$ be the base change of $V'_{nm\sigma\xi_\sigma} \to V \times_U U'_{n\xi_\sigma}$ by the transition morphism $U'_{n\xi} \to U'_{n\xi_\sigma}$ for each $\xi \geq \xi_\sigma$. Since $\coprod_{m\in M_n} V'_{nm\sigma} \to V \times_U U'_n$ is surjective, after enlarging ξ_σ, we may assume that $\coprod_{m\in M_n} V'_{nm\sigma\xi} \to V \times_U U'_{n\xi}$ is also surjective for $\xi \geq \xi_\sigma$ by [EGA IV$_3$, 8.10.5.(vi)]. It is clear that $\coprod_{n\in N} U'_{n\xi} \to U$ is surjective for each $\xi \in \Xi$. Therefore, $\mathfrak{U}'_{\sigma\xi} = \{(V'_{nm\sigma\xi} \to U'_{n\xi}) \to (V \to U)\}_{n\in N, m\in M_n}$ is a standard covanishing covering in $\mathbf{E}^{\text{ét}}_{V\to U}$ for each $\sigma \in \Sigma$ and $\xi \geq \xi_\sigma$. Notice that for each $n \in N$ and $m \in M_n$, there exists $k \in K$ such that the morphism $(V'_{nm\sigma\xi} \to U'_{n\xi}) \to (V \to U)$ factors through $(V_k \to U_k)$ for σ, ξ big enough by 7.13 (6), which shows that $\mathfrak{U}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$. □

Lemma 7.27. *Let $Y \to X$ be a morphism of coherent schemes, $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ a covering in $\mathbf{E}^{\text{proét}}_{Y\to X}$ with K finite. Then, there exist pro-étale presentations $(V \to U) = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$, $(V_k \to U_k) = \lim_{\lambda\in\Lambda}(V_{k\lambda} \to U_{k\lambda})$ over $Y \to X$ and compatible étale morphisms $(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)$ such that the family $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$.*

Proof. We follow closely the proof of 3.6. We take a directed set A such that for each $k \in K$ we can take a pro-étale presentation $(V_k \to U_k) = \lim_{\alpha\in A}(V_{k\alpha} \to U_{k\alpha})$ over $(V \to U)$. Then, $\mathfrak{U}_\alpha = \{(f_{k\alpha} : V_{k\alpha} \to U_{k\alpha}) \to (V \to U)\}_{k\in K}$ is a covering family in $\mathbf{E}^{\text{ét}}_{V\to U}$ for each $\alpha \in A$ by 7.26.

Let $(V \to U) = \lim_{\beta\in B}(V_\beta \to U_\beta)$ be a pro-étale presentation over $Y \to X$. For each $\alpha \in A$, there exists an index $\beta_\alpha \in B$ and a covering family $\mathfrak{U}_{\alpha\beta_\alpha} = \{f_{k\alpha\beta_\alpha} : (V_{k\alpha\beta_\alpha} \to U_{k\alpha\beta_\alpha}) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})\}_{k\in K}$ such that $f_{k\alpha}$ is the base change of $f_{k\alpha\beta_\alpha}$ by the transition morphism $(V \to U) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})$ (7.10). For each $\beta \geq \beta_\alpha$, let $f_{k\alpha\beta} : (V_{k\alpha\beta} \to U_{k\alpha\beta}) \to (V_\beta \to U_\beta)$ be the base change of $f_{k\alpha\beta_\alpha}$ by the transition morphism $(V_\beta \to U_\beta) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})$. We take $\Lambda = \{(\alpha, \beta) \in A \times B \mid \beta \geq \beta_\alpha\}$, $(V_\lambda \to U_\lambda) = (V_\beta \to U_\beta)$ and $(V_{k\lambda} \to U_{k\lambda}) = (V_{k\alpha\beta} \to U_{k\alpha\beta})$ for each $\lambda = (\alpha, \beta) \in \Lambda$. Then, the families $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ meet the requirements in the lemma (cf. 3.6). □

Lemma 7.28. *Let $Y \to X$ be a morphism of coherent schemes, $\mathcal{F}$ a presheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$, $V \to U$ an object of $\mathbf{E}^{\text{proét}}_{Y\to X}$ with a pro-étale presentation $(V \to U) = \lim(V_\lambda \to U_\lambda)$. Then, we have $\nu_{\text{p}}\mathcal{F}(V \to U) = \operatorname{colim} \mathcal{F}(V_\lambda \to U_\lambda)$, where $\nu^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{E}^{\text{proét}}_{Y\to X}$ is the inclusion functor.*

Proof. Notice that the presheaf $\mathcal{F}$ is a filtered colimit of representable presheaves by [SGA 4$_{\text{I}}$, I.3.4]

$$\mathcal{F} = \operatorname*{colim}_{(V'\to U')\in(\mathbf{E}^{\text{ét}}_{Y\to X})_{/\mathcal{F}}} h^{\text{ét}}_{V'\to U'}. \tag{7.28.1}$$

Thus, we may assume that $\mathcal{F}$ is representable by $V' \to U'$ since the section functor $\Gamma(V \to U, -)$ commutes with colimits of presheaves ([Sta21, 00VB]). Then, we have

$$\begin{aligned} \nu_{\text{p}} h^{\text{ét}}_{V'\to U'}(V \to U) =& h^{\text{proét}}_{V'\to U'}(V \to U) \\ =& \operatorname{Hom}_{\mathbf{E}^{\text{proét}}_{Y\to X}}((V \to U), (V' \to U')) \\ =& \operatorname{colim} \operatorname{Hom}_{\mathbf{E}^{\text{ét}}_{Y\to X}}((V_\lambda \to U_\lambda), (V' \to U')) \\ =& \operatorname{colim} h^{\text{ét}}_{V'\to U'}(V_\lambda \to U_\lambda) \end{aligned} \tag{7.28.2}$$

where the first equality follows from [Sta21, 04D2], and the third equality follows from [EGA IV$_3$, 8.14.2] since U' and V' are locally of finite presentation over X and $Y \times_X U'$ respectively. □

Proposition 7.29. *Let $Y \to X$ be a morphism of coherent schemes, $\mathcal{F}$ an abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$, $V \to U$ an object of $\mathbf{E}^{\text{proét}}_{Y\to X}$ with a pro-étale presentation $(V \to U) = \lim(V_\lambda \to U_\lambda)$. Then, for any*

integer q, we have

$$H^q(\mathbf{E}^{\text{pro\'et}}_{V\to U},\nu^{-1}\mathcal{F})=\operatorname{colim} H^q(\mathbf{E}^{\text{\'et}}_{V_\lambda\to U_\lambda},\mathcal{F}), \tag{7.29.1}$$

where $\nu:\mathbf{E}^{\text{pro\'et}}_{Y\to X}\to\mathbf{E}^{\text{\'et}}_{Y\to X}$ is the morphism of sites defined by the inclusion functor (7.25)*. In particular, the canonical morphism $\mathcal{F}\longrightarrow \mathrm{R}\nu_*\nu^{-1}\mathcal{F}$ is an isomorphism.*

Proof. We follow closely the proof of 3.8. For the second assertion, since $\mathrm{R}^q\nu_*\nu^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $(V\to U)\mapsto H^q(\mathbf{E}^{\text{pro\'et}}_{V\to U},\nu^{-1}\mathcal{F})=H^q(\mathbf{E}^{\text{\'et}}_{V\to U},\mathcal{F})$ by the first assertion, which is $\mathcal{F}$ if $q=0$ and vanishes otherwise.

For the first assertion, we may assume that $\mathcal{F}=\mathcal{I}$ is an abelian injective sheaf on $\mathbf{E}^{\text{\'et}}_{Y\to X}$ (cf. 3.8). We claim that for any covering in $\mathbf{E}^{\text{pro\'et}}_{Y\to X}$, $\mathfrak{U}=\{(V_k\to U_k)\to(V\to U)\}_{k\in K}$ with K finite, the augmented Čech complex associated to the presheaf $\nu_{\mathrm{p}}\mathcal{I}$,

$$0\to\nu_{\mathrm{p}}\mathcal{I}(V\to U)\to\prod_k\nu_{\mathrm{p}}\mathcal{I}(V_k\to U_k)\to\prod_{k,k'}\nu_{\mathrm{p}}\mathcal{I}(V_k\times_V V_{k'}\to U_k\times_U U_{k'})\to\cdots \tag{7.29.2}$$

is exact. Admitting this claim, we see that $\nu_{\mathrm{p}}\mathcal{I}$ is indeed a sheaf, i.e. $\nu^{-1}\mathcal{I}=\nu_{\mathrm{p}}\mathcal{I}$, and the vanishing of higher Čech cohomologies implies that $H^q(\mathbf{E}^{\text{pro\'et}}_{V\to U},\nu^{-1}\mathcal{I})=0$ for any $q>0$, which completes the proof together with 7.28. For the claim, let $(V\to U)=\lim_{\lambda\in\Lambda}(V_\lambda\to U_\lambda)$ and $(V_k\to U_k)=\lim_{\lambda\in\Lambda}(V_{k\lambda}\to U_{k\lambda})$ be the pro-étale presentations constructed in 7.27. The family $\mathfrak{U}_\lambda=\{(V_{k\lambda}\to U_{k\lambda})\to(V_\lambda\to U_\lambda)\}_{k\in K}$ is a covering in $\mathbf{E}^{\text{\'et}}_{Y\to X}$. By 7.28, the sequence (7.29.2) is the filtered colimit of the augmented Čech complexes

$$0\to\mathcal{I}(V_\lambda\to U_\lambda)\to\prod_k\mathcal{I}(V_{k\lambda}\to U_{k\lambda})\to\prod_{k,k'}\mathcal{I}(V_{k\lambda}\times_{V_\lambda}V_{k'\lambda}\to U_{k\lambda}\times_{U_\lambda}U_{k'\lambda})\to\cdots \tag{7.29.3}$$

which are exact since $\mathcal{I}$ is an injective abelian sheaf on $\mathbf{E}^{\text{\'et}}_{Y\to X}$. □

Corollary 7.30. *With the notation in* 7.29*, the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{pro\'et}}_{Y\to X}$ is a sheaf, and the canonical morphisms $\nu^{-1}\overline{\mathscr{B}}\to\overline{\mathscr{B}}$ and $\overline{\mathscr{B}}\to\mathrm{R}\nu_*\overline{\mathscr{B}}$ are isomorphisms. If moreover p is invertible on Y, then for each integer $n>0$, the canonical morphisms $\nu^{-1}(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})\to\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}$ and $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}\to\mathrm{R}\nu_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$ are isomorphisms.*

Proof. For any pro-étale presentation $(V\to U)=\lim(V_\lambda\to U_\lambda)$, we have $\nu^{-1}\overline{\mathscr{B}}(V\to U)=\operatorname{colim}\overline{\mathscr{B}}(V_\lambda\to U_\lambda)=\overline{\mathscr{B}}(V\to U)$ by 7.28 and 3.18. This verifies that $\overline{\mathscr{B}}$ is a sheaf on $\mathbf{E}^{\text{pro\'et}}_{Y\to X}$ and that $\nu^{-1}\overline{\mathscr{B}}\to\overline{\mathscr{B}}$ is an isomorphism. The second isomorphism follows from the first and 7.29. For the last assertion, notice that the multiplication by p^n is injective on $\overline{\mathscr{B}}$, so that the conclusion follows from the exact sequence

$$0\longrightarrow\overline{\mathscr{B}}\xrightarrow{\cdot p^n}\overline{\mathscr{B}}\longrightarrow\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}\longrightarrow 0. \tag{7.30.1}$$

□

8. Cohomological Descent of the Structural Sheaves

Lemma 8.1. *Let $Y\to X$ be a morphism of coherent schemes such that $Y\to X^Y$ is an open immersion. Then, the functor*

$$\epsilon^+:\mathbf{E}^{\text{pro\'et}}_{Y\to X}\longrightarrow\mathbf{I}_{Y\to X^Y},\ (V\to U)\longmapsto U^V, \tag{8.1.1}$$

is well-defined, left exact and continuous. Moreover, we have $Y\times_{X^Y}U^V=V$.

Proof. Since $U'=X^Y\times_X U$ is integral over U, we have $U^V=U'^V$. Applying 3.19.(4) to $V\to U'$ over $Y\to X^Y$, we see that the X^Y-scheme U^V is Y-integrally closed with $Y\times_{X^Y}U^V=V$, and thus the functor ϵ^+ is well-defined. Let $(V_1\to U_1)\to(V_0\to U_0)\leftarrow(V_2\to U_2)$ be a diagram in $\mathbf{E}^{\text{pro\'et}}_{Y\to X}$. By 3.21, $U_1^{V_1}\overline{\times}_{U_0^{V_0}}U_2^{V_2}=(U_1^{V_1}\times_{U_0^{V_0}}U_2^{V_2})^{V_1\times_{V_0}V_2}=(U_1\times_{U_0}U_2)^{V_1\times_{V_0}V_2}$ which shows the left exactness of ϵ^+. For the continuity, notice that any covering in $\mathbf{E}^{\text{pro\'et}}_{Y\to X}$ can be refined by a standard covanishing covering

(6.4). For a Cartesian covering family $\mathfrak{U} = \{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$ with N finite, we apply 3.15 to the commutative diagram

$$\begin{array}{ccccc} \coprod_{n\in N} V \times_U U_n & \longrightarrow & \coprod_{n\in N} U_n^{V\times_U U_n} & \longrightarrow & \coprod_{n\in N} U_n \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U^V & \longrightarrow & U \end{array} \tag{8.1.2}$$

then we see that $\epsilon^+(\mathfrak{U})$ is a covering family in $\mathbf{I}_{Y\to X^Y}$. For a vertical covering family $\mathfrak{U} = \{(V_m \to U) \to (V \to U)\}_{m\in M}$ with M finite, we apply 3.15 to the commutative diagram

$$\begin{array}{ccccc} \coprod_{m\in M} V_m & \longrightarrow & \coprod_{m\in M} U^{V_m} & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U^V & \longrightarrow & U \end{array} \tag{8.1.3}$$

then we see that $\epsilon^+(\mathfrak{U})$ is also a covering family in $\mathbf{I}_{Y\to X^Y}$. □

8.2. Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. Then, there are morphisms of sites

$$\epsilon : \mathbf{I}_{Y\to X^Y} \longrightarrow \mathbf{E}^{\text{proét}}_{Y\to X}, \tag{8.2.1}$$

$$\varepsilon : \mathbf{I}_{Y\to X^Y} \longrightarrow \mathbf{E}^{\text{ét}}_{Y\to X} \tag{8.2.2}$$

defined by (8.1.1) and the composition of (8.1.1) with (7.25.1) respectively. We temporarily denote by $\mathscr{O}^{\text{pre}}$ the presheaf on $\mathbf{I}_{Y\to X^Y}$ sending W to $\Gamma(W, \mathcal{O}_W)$ (thus $\mathscr{O} = (\mathscr{O}^{\text{pre}})^{\text{a}}$). Notice that we have $\overline{\mathscr{B}} = \epsilon^{\text{p}}\mathscr{O}^{\text{pre}}$ (resp. $\overline{\mathscr{B}} = \varepsilon^{\text{p}}\mathscr{O}^{\text{pre}}$). The canonical morphism $\epsilon^{\text{p}}\mathscr{O}^{\text{pre}} \to \epsilon^{\text{p}}\mathscr{O}$ (resp. $\varepsilon^{\text{p}}\mathscr{O}^{\text{pre}} \to \varepsilon^{\text{p}}\mathscr{O}$) induces a canonical morphism $\overline{\mathscr{B}} \to \epsilon_*\mathscr{O}$ (resp. $\overline{\mathscr{B}} \to \varepsilon_*\mathscr{O}$).

8.3. Let K be a pre-perfectoid field (5.1) of mixed characteristic $(0, p)$, $\eta = \text{Spec}(K)$, $S = \text{Spec}(\mathcal{O}_K)$, $Y \to X$ a morphism of coherent schemes such that X^Y is an S-scheme with generic fibre $(X^Y)_\eta = Y$. In particular, X^Y is an object of $\mathbf{I}_{\eta\to S}$.

Lemma 8.4. *For any ring R, there is an R-algebra R_∞ satisfying the following conditions:*

(i) *The scheme* $\text{Spec}(R_\infty[1/p])$ *is pro-finite étale and faithfully flat over* $\text{Spec}(R[1/p])$.
(ii) *The R-algebra R_∞ is the integral closure of R in $R_\infty[1/p]$.*
(iii) *Any unit t of R_∞ admits a p-th root $t^{1/p}$ in R_∞.*

Moreover, if p lies in the Jacobson radical $J(R)$ of R, and if there is a p^2-th root $p_2 \in R$ of p up to a unit, and we write $p_1 = p_2^p$, then we may require further that

(iv) *the Frobenius of R_∞/pR_∞ induces an isomorphism $R_\infty/p_1R_\infty \to R_\infty/pR_\infty$.*

Proof. Setting $B_0 = R[1/p]$, we construct inductively a ring B_{n+1} ind-finite étale over B_n and we denote by R_n the integral closure of R in B_n. For $n \geq 0$, we set

$$B_{n+1} = \operatorname*{colim}_{T\subseteq R_n^\times} \bigotimes_{B_n}^{t\in T} B_n[X]/(X^p - t) \tag{8.4.1}$$

where the colimit runs through all finite subsets T of the subset $R_n^\times$ of units of R_n and the transition maps are given by the inclusion relation of these finite subsets T. Notice that $B_n[X]/(X^p - t)$ is finite étale and faithfully flat over B_n, thus B_{n+1} is ind-finite étale and faithfully flat over B_n. Now we take $B_\infty = \text{colim}_n B_n$. The integral closure R_∞ of R in B_∞ is equal to $\text{colim}_n R_n$ by 3.18. We claim that R_∞ satisfies the first three conditions. Firstly, we see inductively that $B_n = R_n[1/p]$ $(0 \leq n \leq \infty)$ by 3.17. Thus, (i), (ii) follow immediately. For (iii), notice that we have $R_\infty^\times = \text{colim}_n R_n^\times$. For an unit $t \in R_\infty^\times$, we suppose that it is the image of $t_n \in R_n^\times$. By construction, there exists an element $x_{n+1} \in R_{n+1}$ such that $x_{n+1}^p = t_n$. Thus, t admits a p-th root in R_∞.

For (iv), the injectivity follows from the fact that R_∞ is integrally closed in $R_\infty[1/p]$ (cf. 5.20). For the surjectivity, let $a \in R_\infty$. Firstly, since R_∞ is integral over R, p also lies in the Jacobson radical $J(R_\infty)$ of R_∞. Thus, $1 + p_1a \in R_\infty^\times$, and then by (iii) there is $b \in R_\infty$ such that $b^p = 1 + p_1a$. We write $(b-1)^p = p_1a'$ for some $a' \in a + p_1R_\infty$. Thus, $1 + a' - a \in R_\infty^\times$, and then by (iii) there is $c \in R_\infty$ such that $c^p = 1 + a' - a$. On the other hand, since R_∞ is integrally closed in $R_\infty[1/p]$, we have $x = (b-1)/p_2 \in R_\infty$. Now we have $(x - c + 1)^p \equiv x^p - c^p + 1 \equiv a \pmod{pR_\infty}$, which completes the proof. □

Remark 8.5. In 8.4, it follows from the construction that $\mathrm{Spec}(R_\infty[1/p]) \to \mathrm{Spec}(R[1/p])$ is a covering in $\mathrm{Spec}(R[1/p])^{\mathrm{S}}_{\mathrm{pro\acute{e}t}}$ (7.16).

Proposition 8.6. *With the notation in* 8.3*, for any object $V \to U$ in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$, there exists a covering $\{(V_i \to U_i) \to (V \to U)\}_{i\in I}$ with I finite such that for each $i \in I$, $U_i^{V_i}$ is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid* (5.18)*.*

Proof. After replacing U by an affine open covering, we may assume that $U = \mathrm{Spec}(A)$. Consider the category $\mathscr{C}$ of étale A-algebras B such that $A/pA \to B/pB$ is an isomorphism, and the colimit $A^{\mathrm{h}} = \mathrm{colim}\, B$ over $\mathscr{C}$. In fact, $\mathscr{C}$ is filtered and $(A^{\mathrm{h}}, pA^{\mathrm{h}})$ is the Henselization of the pair (A, pA) (cf. [Sta21, 0A02]). It is clear that $\mathrm{Spec}(A^{\mathrm{h}}) \coprod \mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A)$ is a covering in $U_{\mathrm{pro\acute{e}t}}$. So we reduce to the situation where $p \in J(A)$ or $p \in A^\times$. The latter case is trivial, since the p-adic completion of $R = \Gamma(U^V, \mathcal{O}_{U^V})$ is zero if p is invertible in A. Therefore, we may assume that $p \in J(A)$ in the following.

Since $R = \Gamma(U^V, \mathcal{O}_{U^V})$ is integral over A, we also have $p \in J(R)$. Applying 8.4 to the $\mathcal{O}_K$-algebra R, we obtain a covering $V_\infty = \mathrm{Spec}(R_\infty[1/p]) \to V = \mathrm{Spec}(R[1/p])$ in $V_{\mathrm{pro\acute{e}t}}$ such that $R_\infty = \Gamma(U^{V_\infty}, \mathcal{O}_{U^{V_\infty}})$ is an $\mathcal{O}_K$-algebra which is almost pre-perfectoid by 5.4 and 5.19. □

Proposition 8.7. *With the notation in* 8.3*, if W is an object of $\mathbf{I}_{\eta\to S}$ such that W is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, then for any integer $n > 0$, the canonical morphism $\Gamma(W, \mathcal{O}_W)/p^n\Gamma(W, \mathcal{O}_W) \to \mathrm{R}\Gamma(\mathbf{I}_{W_\eta\to W}, \mathscr{O}/p^n\mathscr{O})$ is an almost isomorphism* (5.7)*.*

Proof. Let $\mathscr{C}$ be the full-subcategory of $\mathbf{I}_{\eta\to S}$ formed by the spectrums of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. It is stable under fibred product by 5.29, 5.26 and 3.21, and it forms a topologically generating family for the site $\mathbf{I}_{\eta\to S}$ by 8.1 and 8.6. It suffices to show that for any covering in $\mathbf{I}_{\eta\to S}$, $\mathfrak{U} = \{W_k \to W\}_{k\in K}$ consisting of objects of $\mathscr{C}$ with K finite, the augmented Čech complex associated to the presheaf $W \mapsto \Gamma(W, \mathcal{O}_W)/p^n\Gamma(W, \mathcal{O}_W)$ on $\mathbf{I}_{\eta\to S}$ (whose associated sheaf is just $\mathscr{O}/p^n\mathscr{O}$),

$$0 \to \Gamma(W, \mathcal{O}_W)/p^n \to \prod_k \Gamma(W_k, \mathcal{O}_{W_k})/p^n \to \prod_{k,k'} \Gamma(W_k \overline{\times}_W W_{k'}, \mathcal{O}_{W_k \overline{\times}_W W_{k'}})/p^n \to \cdots \tag{8.7.1}$$

is almost exact. Indeed, the almost exactness shows firstly that $\Gamma(W, \mathcal{O}_W)/p^n \to H^0(\mathbf{I}_{W_\eta\to W}, \mathscr{O}/p^n\mathscr{O})$ is an almost isomorphism (cf. [Sta21, 00W1]), so that the augmented Čech complex associated to the sheaf $\mathscr{O}/p^n\mathscr{O}$ is also almost exact. Then, the conclusion follows from the almost vanishing of the higher Čech cohomologies of $\mathscr{O}/p^n\mathscr{O}$ by [Sta21, 03F9].

We set $R = \Gamma(W, \mathcal{O}_W)$ and $R' = \prod_{k\in K} \Gamma(W_k, \mathcal{O}_{W_k})$. They are almost pre-perfectoid, and $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a v-covering by definition. Thus, the almost exactness of (8.7.1) follows from 5.29, 5.26 and 5.31. □

Theorem 8.8. *With the notation in* 8.3*, let $\epsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ be the morphism of sites defined in* 8.2*. Then, for any integer $n > 0$, the canonical morphism $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \mathrm{R}\epsilon_*(\mathscr{O}/p^n\mathscr{O})$ is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_{\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}})$* (5.7)*.*

Proof. Since $\mathrm{R}^q\epsilon_*(\mathscr{O}/p^n\mathscr{O})$ is the sheaf associated to the presheaf $(V \to U) \mapsto H^q(\mathbf{I}_{V\to U^V}, \mathscr{O}/p^n\mathscr{O})$ and any object in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ can be covered by those objects whose image under ϵ^+ are the spectrums of $\mathcal{O}_K$-algebras which are almost pre-perfectoid by 8.6, the conclusion follows from 8.7. □

Corollary 8.9. *With the notation in* 8.3*, let $\varepsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ be the morphism of sites defined in* 8.2*. Then, for any finite locally constant abelian sheaf $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, the canonical morphism $\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O})$ is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$* (5.7)*.*

Proof. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$, thus we may assume that $\mathbb{L}$ is the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$. Then, the conclusion follows from 8.8 and 7.30. □

Remark 8.10. In 8.9, if $\mathbb{L}$ is a bounded complex of abelian sheaves on $\mathbf{E}^{\text{ét}}_{Y\to X}$ with finite locally constant cohomology sheaves, then the canonical morphism $\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O})$ is also an almost isomorphism. Indeed, after replacing $\mathbb{L}$ by $\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}_p$, we may assume that $\mathbb{L}$ is a complex of $\mathbb{Z}/p^n\mathbb{Z}$-modules for some integer n ([Sta21, 0DD7]). Then, there exists a covering family $\{(Y_i \to X_i) \to (Y \to X)\}_{i\in I}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ such that the restriction of $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y_i\to X_i}$ is represented by a bounded complex of finite locally constant $\mathbb{Z}/p^n\mathbb{Z}$-modules ([Sta21, 094G]). Then, the conclusion follows directly from 8.9.

Corollary 8.11. *With the notation in* 8.3*, let $Y \to X_i$ $(i = 1, 2)$ be a morphism of coherent schemes such that X_i^Y is an S-scheme with generic fibre $(X_i^Y)_\eta = Y$, $\mathbb{L}$ a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X_2}$. If there is a morphism $f : X_1 \to X_2$ under Y such that the natural morphism $g : X_1^Y \to X_2^Y$ is a separated v-covering and that $g^{-1}(Y) = Y$, and if we denote by $u : \mathbf{E}^{\text{ét}}_{Y\to X_1} \to \mathbf{E}^{\text{ét}}_{Y\to X_2}$ the corresponding morphism of sites, then the natural morphism $\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}u_*(u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$ is an almost isomorphism.*

Proof. The morphism u is defined by the functor $u^+ : \mathbf{E}^{\text{ét}}_{Y\to X_2} \to \mathbf{E}^{\text{ét}}_{Y\to X_1}$ sending $(V \to U_2)$ to $(V \to U_1) = (V \to X_1 \times_{X_2} U_2)$. We set $V_0 = Y \times_{X_1} U_1 = Y \times_{X_2} U_2$. According to 3.17, $U_1^{V_0} \to U_2^{V_0}$ is the base change of $X_1^Y \to X_2^Y$ by $U_2 \to X_2$, and thus it is a separated v-covering. Notice that V_0 is an open subscheme in both $U_1^{V_0}$ and $U_2^{V_0}$, and moreover $V_0 = V_0 \times_{U_2^{V_0}} U_1^{V_0}$. Applying 3.15 to the commutative diagram

$$\begin{array}{ccccc} V & \longrightarrow & U_1^V & \longrightarrow & U_1^{V_0} \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U_2^V & \longrightarrow & U_2^{V_0} \end{array} \tag{8.11.1}$$

it follows that $U_1^V \to U_2^V$ is also a separated v-covering. Let $\varepsilon_i : \mathbf{I}_{Y\to X_i^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X_i}$ $(i = 1, 2)$ be the morphisms of sites defined in 8.2. The sheaf $\mathrm{R}^q u_*(u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$ is associated to the presheaf $(V \to U_2) \mapsto H^q(\mathbf{E}^{\text{ét}}_{V\to U_1}, u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$. We have

$$\begin{aligned} H^q(\mathbf{E}^{\text{ét}}_{V\to U_1}, u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) &\to H^q(\mathbf{I}_{V\to U_1^V}, \varepsilon_1^{-1}u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \\ &= H^q(\mathbf{I}_{V\to U_2^V}, \varepsilon_2^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \leftarrow H^q(\mathbf{E}^{\text{ét}}_{V\to U_2}, \mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \end{aligned} \tag{8.11.2}$$

where the equality follows from the fact that the morphism of representable sheaves associated to $U_1^V \to U_2^V$ on $\mathbf{I}_{\eta\to S}$ is an isomorphism by 3.24, and where the two arrows are almost isomorphisms by 8.9, which completes the proof. □

8.12. Let Δ be the category formed by finite ordered sets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ with non-decreasing maps ([Sta21, 0164]). For a functor from its opposite category Δ^{op} to the category $\mathbf{E}$ of morphisms of coherent schemes sending $[n]$ to $Y_n \to X_n$, we simply denote it by $Y_\bullet \to X_\bullet$. Then, we obtain a fibred site $\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}$ over Δ^{op} whose fibre category over $[n]$ is $\mathbf{E}^{\text{ét}}_{Y_n\to X_n}$ and the inverse image functor $f^+ : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y_m\to X_m}$ associated to a morphism $f : [m] \to [n]$ in Δ^{op} is induced the base change by the morphism $(Y_m \to X_m) \to (Y_n \to X_n)$ associated to f. We endow $\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}$ with the total topology (6.1) and call it the *simplicial Faltings site* associated to $Y_\bullet \to X_\bullet$ ([Sta21, 09WE.(A)]). The sheaf $\overline{\mathscr{B}}$ on each $\mathbf{E}^{\text{ét}}_{Y_n\to X_n}$ induces a sheaf $\overline{\mathscr{B}}_\bullet = \{[n] \mapsto \overline{\mathscr{B}}\}$ on $\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}$ with the notation in 6.5.

For an augmentation $(Y_\bullet \to X_\bullet) \to (Y \to X)$ in $\mathbf{E}$ ([Sta21, 018F]), we obtain an augmentation of simplicial site $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ ([Sta21, 0D6Z.(A)]). We denote by $a_n : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the natural morphism induced by $(Y_n \to X_n) \to (Y \to X)$. Notice that for any sheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, we have $a^{-1}\mathcal{F} = \{[n] \mapsto a_n^{-1}\mathcal{F}\}$ with the notation in 6.5 ([Sta21, 0D70]).

Corollary 8.13. *With the notation in* 8.3*, let $\mathbb{L}$ a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$, $X_\bullet \to X$ an augmentation of simplicial coherent scheme. If we set $Y_\bullet = Y \times_X X_\bullet$ and denote by $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the augmentation of simplicial site, assuming that $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\eta\to S}$, then the canonical morphism $\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet)$ is an almost isomorphism.*

Proof. Let $b : \mathbf{I}_{Y_\bullet\to X_\bullet^{Y_\bullet}} \to \mathbf{I}_{Y\to X^Y}$ be the augmentation of simplicial site associated to the augmentation of simplicial object $X_\bullet^{Y_\bullet} \to X^Y$ of $\mathbf{I}_{\eta\to S}$ ([Sta21, 09X8]). The functorial morphism of sites $\varepsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ defined in 8.2 induces a commutative diagram of topoi ([Sta21, 0D99])

$$\begin{array}{ccc} \mathbf{I}^\sim_{Y_\bullet\to X_\bullet^{Y_\bullet}} & \xrightarrow{\varepsilon_\bullet} & \mathbf{E}^{\text{ét}\sim}_{Y_\bullet\to X_\bullet} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} \\ \mathbf{I}^\sim_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}\sim}_{Y\to X} \end{array} \tag{8.13.1}$$

We denote by $a_n : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ and $b_n : \mathbf{I}_{Y_n\to X_n^{Y_n}} \to \mathbf{I}_{Y\to X^Y}$ the natural morphisms of sites. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{R}a_*(a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet) & \xleftarrow{\alpha_1} & \mathbb{L}\otimes\overline{\mathscr{B}} & \xrightarrow{\alpha_2} & \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L}\otimes\mathscr{O}) \\ \downarrow{\scriptstyle\alpha_3} & & & & \downarrow{\scriptstyle\alpha_4} \\ \mathrm{R}a_*\mathrm{R}\varepsilon_{\bullet *}\varepsilon_\bullet^{-1}(a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet) & \xrightarrow{\alpha_5} & \mathrm{R}c_*(c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet) & = & \mathrm{R}\varepsilon_*\mathrm{R}b_*b^{-1}(\varepsilon^{-1}\mathbb{L}\otimes\mathscr{O}) \end{array} \tag{8.13.2}$$

where $c = a\circ\varepsilon_\bullet = \varepsilon\circ b$, and α_2 (resp. α_5) is induced by the canonical morphism $\varepsilon^{-1}\overline{\mathscr{B}}\to\mathscr{O}$ (resp. $\varepsilon_\bullet^{-1}\overline{\mathscr{B}}_\bullet\to\mathscr{O}_\bullet$), and other arrows are the canonical morphisms.

Notice that α_2 is an almost isomorphism by 8.9, and that α_4 is an isomorphism by [Sta21, 0D8N] as $X_\bullet^{Y_\bullet}\to X^Y$ is a hypercovering in $\mathbf{I}_{\eta\to S}$. It remains to show that $\alpha_5\circ\alpha_3$ is an almost isomorphism. With the notation in 6.5, we have

$$a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet = \{[n]\mapsto a_n^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}\} \text{ and} \tag{8.13.3}$$

$$c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet = \{[n]\mapsto \varepsilon_n^{-1}a_n^{-1}\mathbb{L}\otimes\mathscr{O}\}. \tag{8.13.4}$$

Moreover, by [Sta21, 0D97] we have

$$\mathrm{R}^q\varepsilon_{\bullet *}(c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet) = \{[n]\mapsto \mathrm{R}^q\varepsilon_{n*}(\varepsilon_n^{-1}a_n^{-1}\mathbb{L}\otimes\mathscr{O})\} \tag{8.13.5}$$

for each integer q. Therefore, $a^{-1}\mathbb{L}\otimes\overline{\mathscr{B}}_\bullet\to\mathrm{R}\varepsilon_{\bullet *}(c^{-1}\mathbb{L}\otimes\mathscr{O}_\bullet)$ is an almost isomorphism by 8.9 and so is $\alpha_5\circ\alpha_3$. □

9. Complements on Logarithmic Geometry

We briefly recall some notions and facts of logarithmic geometry which will be used in the rest of the paper. We refer to [Kat89, Kat94, GR04, Ogu18] for a systematic development of logarithmic geometry, and to [AGT16, II.5] for a brief summary of the theory.

9.1. We only consider logarithmic structures in étale topology. More precisely, let X be a scheme, $X_{\text{ét}}$ the étale site of X, $\mathcal{O}_{X_{\text{ét}}}$ the structure sheaf on $X_{\text{ét}}$, $\mathcal{O}^\times_{X_{\text{ét}}}$ the subsheaf of units of $\mathcal{O}_{X_{\text{ét}}}$. A logarithmic struture on X is a homomorphism of sheaves of monoids $\alpha : \mathscr{M}\to\mathcal{O}_{X_{\text{ét}}}$ on $X_{\text{ét}}$ which induces an isomorphism $\alpha^{-1}(\mathcal{O}^\times_{X_{\text{ét}}})\xrightarrow{\sim}\mathcal{O}^\times_{X_{\text{ét}}}$. We denote by $(X,\mathscr{M})$ the associated logarithmic scheme (cf. [AGT16, II.5.11]).

9.2. Let $(X,\mathscr{M})$ be a coherent log scheme (cf. [AGT16, II.5.15]). Then, there is a maximal open subscheme X^{tr} of X on which $\mathscr{M}$ is trivial, and moreover it is functorial in $(X,\mathscr{M})$ ([Ogu18, III.1.2.8]). Let $(X,\mathscr{M})\to(S,\mathscr{L})\leftarrow(Y,\mathscr{N})$ be a diagram of fine and saturated log schemes (cf. [AGT16, II.5.15]). Then, the fibred product is representable in the category of fine and saturated log schemes by $(Z,\mathscr{P}) = (X,\mathscr{M})\times^{\text{fs}}_{(S,\mathscr{L})}(Y,\mathscr{N})$. We remark that $Z^{\text{tr}} = X^{\text{tr}}\times_{S^{\text{tr}}}Y^{\text{tr}}$, that $Z\to X\times_S Y$ is finite, and that $Z^{\text{tr}}\to Z$ is Cartesian over $X^{\text{tr}}\times_{S^{\text{tr}}}Y^{\text{tr}}\to X\times_S Y$ ([Ogu18, III 2.1.2, 2.1.6]). Moreover, if $X^{\text{tr}} = X$, then $Z = X\times_S Y$ ([Ogu18, III.2.1.3]).

9.3. For an open immersion $j : Y\to X$, we denote by $j_{\text{ét}} : Y_{\text{ét}}\to X_{\text{ét}}$ the morphism of their étale sites defined by the base change by j. Let $\mathscr{M}_{Y\to X}$ be the preimage of $j_{\text{ét}*}\mathcal{O}^\times_{Y_{\text{ét}}}$ under the natural map $\mathcal{O}_{X_{\text{ét}}}\to j_{\text{ét}*}\mathcal{O}_{Y_{\text{ét}}}$, and we endow X with the logarithmic structure $\mathscr{M}_{Y\to X}\to\mathcal{O}_{X_{\text{ét}}}$, which is called the compactifying log structure associated to the open immersion j ([Ogu18, III.1.6.1]). Sometimes we write $\mathscr{M}_{Y\to X}$ as $\mathscr{M}_X$ if Y is clear in the context.

9.4. Let $(X,\mathscr{M})$ be a fine and saturated log scheme which is regular ([Kat94, 2.1], [Niz06, 2.3]). Then, X is locally Noetherian and normal, and X^{tr} is regular and dense in X ([Kat94, 4.1]). Moreover, there is a natural isomorphism $\mathscr{M}\xrightarrow{\sim}\mathscr{M}_{X^{\text{tr}}\to X}$ ([Kat94, 11.6], [Niz06, 2.6]). We remark that if X is a regular scheme with a strict normal crossings divisor D, then $(X,\mathscr{M}_{X\setminus D\to X})$ is fine, saturated and regular ([Ogu18, III.1.11.9]).

Let $f : (X,\mathscr{M})\to(S,\mathscr{L})$ be a smooth (resp. saturated) morphism of fine and saturated log schemes (cf. [AGT16, II 5.25, 5.18]). Then, f remains smooth (resp. saturated) under the base change in the category of fine and saturated log schemes ([Ogu18, IV.3.1.2, IV.3.1.11], resp. [Ogu18, III.2.5.3]). We remark that if f is smooth, then $f^{\text{tr}} : X^{\text{tr}}\to S^{\text{tr}}$ is a smooth morphism of schemes. If moreover $(S,\mathscr{L})$ is regular, then $(X,\mathscr{M})$ is also regular ([Ogu18, IV.3.5.3]). We also remark that if f is saturated, then for any fibred product in the category of fine and saturated log schemes $(Z,\mathscr{P}) = (X,\mathscr{M})\times^{\text{fs}}_{(S,\mathscr{L})}(Y,\mathscr{N})$, we have $Z = X\times_S Y$ ([Tsu19, II.2.13]).

9.5. Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, k the residue field of $\mathcal{O}_K$, π a uniformizer of $\mathcal{O}_K$. We set $\eta = \mathrm{Spec}(K)$, $S = \mathrm{Spec}(\mathcal{O}_K)$ and $s = \mathrm{Spec}(k)$. Then, $(S, \mathscr{M}_{\eta\to S})$ is fine, saturated and regular, since $\mathbb{N} \to \Gamma(S, \mathscr{M}_{\eta\to S})$ sending 1 to π forms a chart of $(S, \mathscr{M}_{\eta\to S})$ (cf. [AGT16, II.5.13, II.6.1]). Recall that an open immersion $Y \to X$ of quasi-compact and separated schemes over $\eta \to S$ is strictly semi-stable ([dJ96, 6.3]) if and only if the following conditions are satisfied ([dJ96, 6.4], [EGA IV$_4$, 17.5.3]):

(i) For each point x of the generic fibre X_η, there is an open neighborhood $U \subseteq X_\eta$ of x and a smooth K-morphism

$$f : U \longrightarrow \mathrm{Spec}(K[s_1, \ldots, s_m]) \tag{9.5.1}$$

such that f maps x to the point associated to the maximal ideal $(s_1, \ldots, s_m)$ and that $U \setminus Y$ is the inverse image of the closed subset defined by $s_1 \cdots s_m = 0$.

(ii) For each point x of the special fibre X_s, there is an open neighborhood $U \subseteq X$ of x and a smooth $\mathcal{O}_K$-morphism

$$f : U \longrightarrow \mathrm{Spec}(\mathcal{O}_K[t_1, \ldots, t_n, s_1, \ldots, s_m]/(\pi - t_1 \cdots t_n)) \tag{9.5.2}$$

such that f maps x to the point associated to the maximal ideal $(t_1, \ldots, t_n, s_1, \ldots, s_m)$ and that $U \setminus Y$ is the inverse image of the closed subset defined by $t_1 \cdots t_n \cdot s_1 \cdots s_m = 0$.

In this case, $(X, \mathscr{M}_{Y\to X})$ is fine, saturated and regular which is smooth and saturated over $(S, \mathscr{M}_{\eta\to S})$, since locally on X there exists a chart for the morphism $(X, \mathscr{M}_{Y\to X}) \to (S, \mathscr{M}_{\eta\to S})$ subordinate to the morphism $\mathbb{N} \to \mathbb{N}^n \oplus \mathbb{N}^m$ sending 1 to $(1, \ldots, 1, 0, \ldots, 0)$ such that the induced morphism $X \to S \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{\mathbb{N}^n\oplus\mathbb{N}^m}$ is smooth (cf. [Ogu18, IV.3.1.18]).

9.6. Recall that a morphism of schemes $f : X \to S$ is called *generically finite* if there exists a dense open subscheme U of S such that $f^{-1}(U) \to U$ is finite. We remark that for a morphism $f : X \to S$ of finite type between Noetherian schemes which maps generic points to generic points, f is generically finite if and only if the residue field of any generic point η of X is a finite field extension of the residue field of $f(\eta)$ ([ILO14, II.1.1.7]).

9.7. Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, L an algebraically closed valuation field of height 1 extension of K with valuation ring $\mathcal{O}_L$, $\overline{K}$ the algebraic closure of K in L.

Consider the category $\mathscr{C}$ of open immersions between integral affine schemes $U \to T$ over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ under $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ such that T is of finite type over $\mathcal{O}_K$ and that $\mathrm{Spec}(L) \to U$ is dominant. Let $\mathscr{C}_{\mathrm{car}}$ be the full subcategory of $\mathscr{C}$ formed by those objects $U \to T$ Cartesian over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$.

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) \\ \downarrow & & \downarrow \\ U = \mathrm{Spec}(B) & \longrightarrow & T = \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_K) \end{array} \tag{9.7.1}$$

We note that the objects of $\mathscr{C}$ are of the form $(U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A))$ where A (resp. B) is a finitely generated $\mathcal{O}_K$-subalgebra of $\mathcal{O}_L$ (resp. K-subalgebra of L) with $A \subseteq B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open immersion.

Lemma 9.8. *With the notation in* 9.7*, we have:*

(1) *The category $\mathscr{C}$ is cofiltered, and the subcategory $\mathscr{C}_{\mathrm{car}}$ is initial in $\mathscr{C}$.*

(2) *The morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ represents the cofiltered limit of morphisms $U \to T$ indexed by $\mathscr{C}$ in the category of morphisms of schemes* (cf. 7.1)*.*

(3) *There exists a directed inverse system $(U_\lambda \to T_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathscr{C}_{\mathrm{car}}$ over a directed inverse system $(\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}_{K_\lambda}))_{\lambda\in\Lambda}$ of objects of $\mathscr{C}_{\mathrm{car}}$ such that K_λ is a finite field extension of K in L, that $\overline{K} = \bigcup_{\lambda\in\Lambda} K_\lambda$, that $U_\lambda \to T_\lambda$ is strictly semi-stable over $\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}_{K_\lambda})$* (9.5)*, and that $(U_\lambda \to T_\lambda)_{\lambda\in\Lambda}$ forms an initial full subcategory of $\mathscr{C}_{\mathrm{car}}$.*

Proof. (1) For a diagram $(U_1 \to T_1) \to (U_0 \to T_0) \leftarrow (U_2 \to T_2)$ in $\mathscr{C}$, let T be the scheme theoretic image of $\mathrm{Spec}(L) \to T_1 \times_{T_0} T_2$ and let U be the intersection of $U_1 \times_{U_0} U_2$ with T. It is clear that T is of finite type over $\mathcal{O}_K$ as $\mathcal{O}_K$ is Noetherian, that U and T are integral and affine, that $\mathrm{Spec}(L) \to U$ is dominant,

and that $\mathrm{Spec}(L) \to T$ factors through $\mathrm{Spec}(\mathcal{O}_L)$. Thus, $U \to T$ is an object of $\mathscr{C}$, which shows that $\mathscr{C}$ is cofiltered. For an object $(U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A))$ of $\mathscr{C}$, we write $\mathcal{O}_L$ as a filtered union of finitely generated A-subalgebras A_i. Let π be a uniformizer of K. Notice that $L = \mathcal{O}_L[1/\pi] = \operatorname{colim} A_i[1/\pi]$ and that $\mathrm{Hom}_{K\text{-}\mathbf{Alg}}(B, L) = \operatorname{colim} \mathrm{Hom}_{K\text{-}\mathbf{Alg}}(B, A_i[1/\pi])$ by [EGA IV$_3$, 8.14.2.2]. Thus, there exists an index i such that $\mathrm{Spec}(A_i[1/\pi]) \to \mathrm{Spec}(A_i)$ is an object of $\mathscr{C}_{\mathrm{car}}$ over $U \to T$.

(2) It follows immediately from the arguments above.

(3) Consider the category $\mathscr{D}$ of morphisms of $\mathscr{C}_{\mathrm{car}}$,

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{K'}) \end{array} \tag{9.8.1}$$

such that K' is a finite field extension of K. Similarly, this category is also cofiltered with limit of diagrams of schemes $(\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)) \to (\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\mathcal{O}_{\overline{K}}))$. It suffices to show that the full subcategory of $\mathscr{D}$ formed by strictly semi-stable objects is initial. For any object $U \to T$ of $\mathscr{C}_{\mathrm{car}}$, by de Jong's alteration theorem [dJ96, 6.5], there exists a proper surjective and generically finite morphism $T' \to T$ of integral schemes such that $U' = U \times_T T' \to T'$ is strictly semi-stable over $\mathrm{Spec}(K') \to \mathrm{Spec}(\mathcal{O}_{K'})$ for a finite field extension $K \to K'$. Since L is algebraically closed, the dominant morphism $\mathrm{Spec}(L) \to U$ lifts to a dominant morphism $\mathrm{Spec}(L) \to U'$ (9.6), which further extends to a lifting $\mathrm{Spec}(\mathcal{O}_L) \to T'$ of $\mathrm{Spec}(\mathcal{O}_L) \to T$ by the valuative criterion. After replacing T' by an affine open neighborhood of the image of the closed point of $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a strictly semi-stable object of $\mathscr{D}$ over $(U \to T) \to (\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K))$, which completes the proof. □

Theorem 9.9 ([ILO14, X 3.5, 3.7])**.** *Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, $(Y \to X) \to (U \to T)$ a morphism of dominant open immersions over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ between irreducible $\mathcal{O}_K$-schemes of finite type such that $X \to T$ is proper surjective. Then, there exists a commutative diagram of dominant open immersions between irreducible $\mathcal{O}_K$-schemes of finite type*

$$\begin{array}{ccc} (Y' \to X') & \xrightarrow{(\beta^\circ, \beta)} & (Y \to X) \\ {\scriptstyle (f'^\circ, f')}\downarrow & & \downarrow{\scriptstyle (f^\circ, f)} \\ (U' \to T') & \xrightarrow[(\alpha^\circ, \alpha)]{} & (U \to T) \end{array} \tag{9.9.1}$$

satisfying the following conditions:

(i) *We have $Y' = \beta^{-1}(Y) \cap f'^{-1}(U')$, i.e. $Y' \to X'$ is Cartesian over $U' \times_U Y \to T' \times_T X$* (cf. 7.1).

(ii) *The morphism $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ induced by (f'°, f') is a smooth and saturated morphism of fine, saturated and regular log schemes.*

(iii) *The morphisms α and β are proper surjective and generically finite, and f' is projective surjective.*

Proof. We may assume that T is nonempty. Recall that $\mathrm{Spec}(\mathcal{O}_K)$ is universally $\mathbb{Q}$-resolvable ([ILO14, X.3.3]) by de Jong's alteration theorem [dJ96, 6.5]. Thus, T is also universally $\mathbb{Q}$-resolvable by [ILO14, X 3.5, 3.5.2] so that we can apply [ILO14, X.3.5] to the proper surjective morphism f and the nowhere dense closed subset $X \setminus Y$. Then, we obtain a commutative diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow[\alpha]{} & T \end{array} \tag{9.9.2}$$

and dense open subsets $U' \subseteq T'$, $Y' = \beta^{-1}(Y) \cap f'^{-1}(U') \subseteq X'$ such that $(X', \mathscr{M}_{Y' \to X'})$ and $(T', \mathscr{M}_{U' \to T'})$ are fine, saturated and regular, that $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ is smooth, that α, β are proper surjective and generically finite morphisms which map generic points to generic points, and that f' is projective (since f is proper, cf. [ILO14, X 3.1.6, 3.1.7]). Since X (resp. T) is irreducible and X' (resp. T') is a disjoint union of normal integral schemes (9.4), after firstly replacing X' by an irreducible component and then replacing T' by the irreducible component under X', we may assume that X' and T' are irreducible. Then, $Y' \to U'$ is dominant (so that f' is projective surjective), since it is smooth

and Y' is nonempty ([EGA IV$_2$, 2.3.4]). We claim that α maps U' into U. Indeed, if there exists a point $u \in U'$ with $\alpha(u) \notin U$, then $f'^{-1}(u) \cap Y' = \emptyset$. However, endowing u with the trivial log structure, the log scheme $(u, \mathcal{O}_{u_{\text{ét}}}^{\times})$ is fine, saturated and regular, and the fibred product in the category of fine and saturated log schemes

$$(u, \mathcal{O}_{u_{\text{ét}}}^{\times}) \times^{\text{fs}}_{(T', \mathscr{M}_{U' \to T'})} (X', \mathscr{M}_{Y' \to X'}) \tag{9.9.3}$$

is regular with underlying scheme $f'^{-1}(u)$ (9.4, 9.2). Thus, $f'^{-1}(u) \cap Y'$ is dense in $f'^{-1}(u)$, which contradicts the assumption that $f'^{-1}(u) \cap Y' = \emptyset$ since f' is surjective. Thus, we obtain a diagram (9.9.1) satisfying all the conditions except the saturatedness of $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$.

To make $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ saturated, we apply [ILO14, X.3.7] to the morphism (f'°, f'). We obtain a Cartesian morphism $(\gamma^{\circ}, \gamma) : (U'' \to T'') \to (U' \to T')$ of dominant open immersions such that $(T'', \mathscr{M}_{U'' \to T''})$ is a fine, saturated and regular log scheme, that γ is a proper surjective and generically finite morphism which maps generic points of T'' to the generic point of T', and that the fibred product in the category of fine and saturated log schemes

$$(T'', \mathscr{M}_{U'' \to T''}) \times^{\text{fs}}_{(T', \mathscr{M}_{U' \to T'})} (X', \mathscr{M}_{Y' \to X'}) \tag{9.9.4}$$

is saturated over $(T'', \mathscr{M}_{U'' \to T''})$. The fibred product (9.9.4) is still smooth over $(T'', \mathscr{M}_{U'' \to T''})$, and thus it is regular (9.4). Let X'' be the underlying scheme of it and let $Y'' = (X'')^{\text{tr}}$. Then, the fibred product (9.9.4) is isomorphic to $(X'', \mathscr{M}_{Y'' \to X''})$ (9.4). Thus, we obtain a commutative diagram of dominant open immersions of schemes

$$\begin{array}{ccc} (Y'' \to X'') & \xrightarrow{(\delta^{\circ}, \delta)} & (Y' \to X') \\ {\scriptstyle (f''^{\circ}, f'')}\downarrow & & \downarrow{\scriptstyle (f'^{\circ}, f')} \\ (U'' \to T'') & \xrightarrow[(\gamma^{\circ}, \gamma)]{} & (U' \to T') \end{array} \tag{9.9.5}$$

Notice that $Y'' = U'' \times_{U'} Y'$ and $X'' \to T'' \times_{T'} X'$ is finite, and that $Y'' \to X''$ is Cartesian over $U'' \times_{U'} Y' \to T'' \times_{T'} X'$ (9.2). Thus, we see that $Y'' \to X''$ is Cartesian over $U'' \times_U Y \to T'' \times_T X$ and that f'' is projective. Since T' (resp. X') is irreducible and T'' (resp. X'') is a disjoint union of normal integral schemes (9.4), after firstly replacing T'' by an irreducible component and then replacing X'' by an irreducible component on which the restriction of δ° is dominant, we may assume that T'' and X'' are irreducible. In particular, δ is generically finite and so is $\beta \circ \delta$ (9.6), and again $Y'' \to U''$ is dominant so that f'' is projective surjective. □

Lemma 9.10. *Let X be a scheme of finite type over a valuation ring A of height 1. Then, the underlying topological space of X is Noetherian.*

Proof. Let η and s be the generic point and closed point of $\mathrm{Spec}(A)$ respectively. Then, the generic fibre X_η and the special fibre X_s are both Noetherian. As a union of X_η and X_s, the underlying topological space of X is also Noetherian ([Sta21, 0053]). □

Proposition 9.11. *With the notation in 9.7 and 9.8, let $Y \to X$ be a quasi-compact dominant open immersion over $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ such that $X \to \mathrm{Spec}(\mathcal{O}_L)$ is proper of finite presentation. Then, there exists a proper surjective $\mathcal{O}_L$-morphism of finite presentation $X' \to X$, an index $\lambda_1 \in \Lambda$, and a directed inverse system of open immersions $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$ over $(U_\lambda \to T_\lambda)_{\lambda \geq \lambda_1}$ satisfying the following conditions for each $\lambda \geq \lambda_1$:*

(i) *We have $Y' = Y \times_X X' = \lim_{\lambda \geq \lambda_1} Y'_\lambda$ and $X' = \lim_{\lambda \geq \lambda_1} X'_\lambda$.*
(ii) *The log scheme $(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda})$ is fine, saturated and regular.*
(iii) *The morphism $(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda}) \to (T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda})$ is smooth and saturated, and $X'_\lambda \to T_\lambda$ is projective.*
(iv) *If moreover $Y = \mathrm{Spec}(L) \times_{\mathrm{Spec}(\mathcal{O}_L)} X$, then we can require that $Y'_\lambda = U_\lambda \times_{T_\lambda} X'_\lambda$.*

Proof. We follow closely the proof of [ALPT19, 5.2.19]. Since the underlying topological space of X is Noetherian by 9.10, each irreducible component Z of X admits a closed subscheme structure such that $Z \to X$ is of finite presentation ([Sta21, 01PH]). After replacing X by the disjoint union of its irreducible components, we may assume that X is irreducible. Then, the generic fibre of $X \to \mathrm{Spec}(\mathcal{O}_L)$ is also irreducible as an open subset of X. Using [EGA IV$_3$, 8.8.2, 8.10.5], there exists an index $\lambda_0 \in \Lambda$, a proper T_{λ_0}-scheme X_{λ_0}, and an open subscheme Y_{λ_0} of $U_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$, such that $X = \mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} X_{\lambda_0}$ and

that $Y = \mathrm{Spec}(L) \times_{U_{\lambda_0}} Y_{\lambda_0}$. Let η denote the generic point of X, η_{λ_0} the image of η under the morphism $X \to X_{\lambda_0}$, Z_{λ_0} the scheme theoretic closure of η_{λ_0} in X_{λ_0}. Notice that $\mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} Z_{\lambda_0} \to X$ is a surjective finitely presented closed immersion. After replacing X by $\mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} Z_{\lambda_0}$ and replacing X_{λ_0} by Z_{λ_0}, we may assume that $X \to X_{\lambda_0}$ is a dominant morphism of irreducible schemes. Since T_{λ_0} is irreducible and L is algebraically closed, the generic fibre of $f : X_{\lambda_0} \to T_{\lambda_0}$ is geometrically irreducible. In particular, if ξ_{λ_0} (resp. η_{λ_0}) denotes the generic point of T_{λ_0} (resp. X_{λ_0}), then $\eta = \mathrm{Spec}(L) \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ ([EGA IV$_2$, 4.5.9]). In the situation of (iv), we can moreover assume that $Y_{\lambda_0} = U_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$.

By 9.9, there exists a commutative diagram of dominant open immersions of irreducible schemes,

$$\begin{array}{ccc} (Y'_{\lambda_0} \to X'_{\lambda_0}) & \xrightarrow{(\beta^\circ,\beta)} & (Y_{\lambda_0} \to X_{\lambda_0}) \\ \downarrow{\scriptstyle (f'^\circ,f')} & & \downarrow{\scriptstyle (f^\circ,f)} \\ (U'_{\lambda_0} \to T'_{\lambda_0}) & \xrightarrow[(\alpha^\circ,\alpha)]{} & (U_{\lambda_0} \to T_{\lambda_0}) \end{array} \tag{9.11.1}$$

where $Y'_{\lambda_0} \to X'_{\lambda_0}$ is Cartesian over $U'_{\lambda_0} \times_{U_{\lambda_0}} Y_{\lambda_0} \to T'_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$, and where $(X'_{\lambda_0}, \mathscr{M}_{Y'_{\lambda_0} \to X'_{\lambda_0}}) \to (T'_{\lambda_0}, \mathscr{M}_{U'_{\lambda_0} \to T'_{\lambda_0}})$ is a smooth and saturated morphism of fine, saturated and regular log schemes, and where α and β are proper surjective and generically finite, and where f' is projective surjective. We take a dominant morphism $\gamma^\circ : \mathrm{Spec}(L) \to U'_{\lambda_0}$ which lifts $\mathrm{Spec}(L) \to U_{\lambda_0}$ since L is algebraically closed and α is generically finite, the morphism $\mathrm{Spec}(\mathcal{O}_L) \to T_{\lambda_0}$ lifts to $\gamma : \mathrm{Spec}(\mathcal{O}_L) \to T'_{\lambda_0}$ by the valuative criterion. We set $Y' = \mathrm{Spec}(L) \times_{U'_{\lambda_0}} Y'_{\lambda_0}$ and $X' = \mathrm{Spec}(\mathcal{O}_L) \times_{T'_{\lambda_0}} X'_{\lambda_0}$. It is clear that $Y' \to X'$ is Cartesian over $Y \to X$ by base change. Let ξ'_{λ_0} (resp. η'_{λ_0}) be the generic points of T'_{λ_0} (resp. X'_{λ_0}). Since the generic fibre of f is geometrically irreducible, $\xi'_{\lambda_0} \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ is a single point and η'_{λ_0} maps to it ([EGA IV$_2$, 4.5.9]). Since $\mathrm{Spec}(L) \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ is the generic point of X, we see that $X' \to X$ is proper surjective and of finite presentation. It remains to construct $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$.

After replacing T'_{λ_0} by an affine open neighborhood of the image of the closed point of $\mathrm{Spec}(\mathcal{O}_L)$, lemma 9.8 implies that there exists an index $\lambda_1 \geq \lambda_0$ such that the transition morphism $(U_{\lambda_1} \to T_{\lambda_1}) \to (U_{\lambda_0} \to T_{\lambda_0})$ factors through $(U'_{\lambda_0} \to T'_{\lambda_0})$. For each index $\lambda \geq \lambda_1$, consider the fibred product in the category of fine and saturated log schemes

$$(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda}) = (T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda}) \times^{\mathrm{fs}}_{(T'_{\lambda_0}, \mathscr{M}_{U'_{\lambda_0} \to T'_{\lambda_0}})} (X'_{\lambda_0}, \mathscr{M}_{Y'_{\lambda_0} \to X'_{\lambda_0}}), \tag{9.11.2}$$

which is a fine, saturated and regular log scheme smooth and saturated over $(T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda})$ (9.2, 9.4). Moreover, we have $Y'_\lambda = U_\lambda \times_{U'_{\lambda_0}} Y'_{\lambda_0}$, $X'_\lambda = T_\lambda \times_{T'_{\lambda_0}} X'_{\lambda_0}$, and in the situation of (iv), $Y'_\lambda = U_\lambda \times_{T_\lambda} X'_\lambda$ by base change. Therefore, $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$ meets our requirements. □

10. Faltings' Main p-adic Comparison Theorem: the Absolute Case

10.1. Let $Y \to X$ be a morphism of coherent schemes. Consider the functors

$$\psi^+ : \mathbf{E}^{\text{ét}}_{Y \to X} \longrightarrow Y_{\text{ét}},\ (V \to U) \longmapsto V, \tag{10.1.1}$$

$$\beta^+ : Y_{\text{fét}} \longrightarrow \mathbf{E}^{\text{ét}}_{Y \to X},\ V \longmapsto (V \to X). \tag{10.1.2}$$

They are left exact and continuous (cf. [AGT16, VI 10.6, 10.7]). Then, we obtain morphisms of sites

$$\begin{array}{ccc} Y_{\text{ét}} & & \\ \downarrow{\scriptstyle \psi} & \searrow^{\rho} & \\ \mathbf{E}^{\text{ét}}_{Y \to X} & \xrightarrow[\beta]{} & Y_{\text{fét}} \end{array} \tag{10.1.3}$$

where $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ is defined by the inclusion functor.

Lemma 10.2. *Let Y be a coherent scheme, V a finite étale Y-scheme. Then, there exists a finite étale surjective morphism $Y' \to Y$ such that $Y' \times_Y V$ is isomorphic to a finite disjoint union of Y'.*

Proof. If Y is connected, let $\overline{y}$ be a geometric point of Y, $\pi_1(Y, \overline{y})$ the fundamental group of Y with base point $\overline{y}$. Then, $Y_{\text{fét}}$ is equivalent to the category of finite $\pi_1(Y, \overline{y})$-sets so that the lemma holds ([Sta21, 0BND]).

In general, for any connected component Z of Y, let $(Y_\lambda)_{\lambda\in\Lambda_Z}$ be the directed inverse system of all open and closed subschemes of Y which contain Z and whose transition morphisms are inclusions. Notice that $\lim_{\lambda\in\Lambda_Z} Y_\lambda$ is a closed subscheme of Y with underlying topological space Z by [Sta21, 04PL] and [EGA IV$_3$, 8.2.9]. We endow Z with the closed subscheme structure of $\lim_{\lambda\in\Lambda_Z} Y_\lambda$. The first paragraph shows that there exists a finite étale surjective morphism $Z'\to Z$ such that $Z'\times_Y V=\coprod_{i=1}^r Z'$. Using [EGA IV$_3$, 8.8.2, 8.10.5] and [EGA IV$_4$, 17.7.8], there exists an index $\lambda_0\in\Lambda_Z$, a finite étale surjective morphism $Y'_{\lambda_0}\to Y_{\lambda_0}$ and an isomorphism $Y'_{\lambda_0}\times_Y V=\coprod_{i=1}^r Y'_{\lambda_0}$. Notice that Y'_{λ_0} is also finite étale over Y. Since Z is an arbitrary connected component of Y, the conclusion follows from the quasi-compactness of Y. □

Lemma 10.3. *Let Y be a coherent scheme, $\rho: Y_{\text{ét}}\to Y_{\text{fét}}$ the morphism of sites defined by the inclusion functor. Then, the functor $\rho^{-1}: \widetilde{Y_{\text{fét}}}\to\widetilde{Y_{\text{ét}}}$ of the associated topoi induces an equivalence $\rho^{-1}:\mathbf{LocSys}(Y_{\text{fét}})\to\mathbf{LocSys}(Y_{\text{ét}})$ between the categories of finite locally constant abelian sheaves with quasi-inverse ρ_*.*

Proof. Since any finite locally constant sheaf on $Y_{\text{ét}}$ (resp. $Y_{\text{fét}}$) is representable by a finite étale Y-scheme by faithfully flat descent (cf. [Sta21, 03RV]), the Yoneda embeddings induce a commutative diagram

$$\begin{array}{ccccc}\mathbf{LocSys}(Y_{\text{fét}}) & \longrightarrow & Y_{\text{fét}} & \xrightarrow{h^{\text{fét}}} & \widetilde{Y_{\text{fét}}}\\ \downarrow{\scriptstyle\rho^{-1}} & & \downarrow & & \downarrow\\ \mathbf{LocSys}(Y_{\text{ét}}) & \longrightarrow & Y_{\text{ét}} & \xrightarrow{h^{\text{ét}}} & \widetilde{Y_{\text{ét}}}\end{array}\tag{10.3.1}$$

where the horizontal arrows are fully faithful. In particular, ρ^{-1} is fully faithful. For a finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, let V be a finite étale Y-scheme representing $\mathbb{F}$ and let $h_V^{\text{ét}}$ (resp. $h_V^{\text{fét}}$) be the representable sheaf of V on $Y_{\text{ét}}$ (resp. $Y_{\text{fét}}$). We have $\mathbb{F}=h_V^{\text{ét}}=\rho^{-1}h_V^{\text{fét}}$ ([Sta21, 04D3]). By 10.2, $h_V^{\text{fét}}$ is finite locally constant. It is clear that the adjunction morphism $h_V^{\text{fét}}\to\rho_*h_V^{\text{ét}}$ is an isomorphism, which shows that $h_V^{\text{fét}}$ is an abelian sheaf. Thus, ρ^{-1} is essentially surjective. Moreover, the argument also shows that ρ_* induces a functor $\rho_*:\mathbf{LocSys}(Y_{\text{ét}})\to\mathbf{LocSys}(Y_{\text{fét}})$ which is a quasi-inverse of ρ^{-1}. □

Proposition 10.4. *With the notation in* 10.1*, the functors between the categories of finite locally constant abelian sheaves*

$$\mathbf{LocSys}(Y_{\text{fét}})\xrightarrow{\beta^{-1}}\mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X})\xrightarrow{\psi^{-1}}\mathbf{LocSys}(Y_{\text{ét}})\tag{10.4.1}$$

are equivalences with quasi-inverses β_* and ψ_* respectively.

Proof. Notice that for any finite locally constant abelian sheaf $\mathbb{G}$ on $Y_{\text{fét}}$, the canonical morphism $\beta^{-1}\mathbb{G}\to\psi_*\rho^{-1}\mathbb{G}$, which is induced by the adjunction $\mathrm{id}\to\psi_*\psi^{-1}$ and by the identity $\psi^{-1}\beta^{-1}=\rho^{-1}$, is an isomorphism by 10.3 and the proof of [AGT16, VI.6.3.(iii)]. For a finite locally constant abelian sheaf $\mathbb{F}$ over $Y_{\text{ét}}$, we write $\mathbb{F}=\rho^{-1}\mathbb{G}$ by 10.3. Then, $\mathbb{F}=\psi^{-1}\beta^{-1}\mathbb{G}\xrightarrow{\sim}\psi^{-1}\psi_*\rho^{-1}\mathbb{G}=\psi^{-1}\psi_*\mathbb{F}$, whose inverse is the adjunction map $\psi^{-1}\psi_*\mathbb{F}\to\mathbb{F}$ since the composition of $\psi^{-1}(\beta^{-1}\mathbb{G})\to\psi^{-1}(\psi_*\psi^{-1})(\beta^{-1}\mathbb{G})=(\psi^{-1}\psi_*)\psi^{-1}(\beta^{-1}\mathbb{G})\to\psi^{-1}(\beta^{-1}\mathbb{G})$ is the identity. On the other hand, for a finite locally constant abelian sheaf $\mathbb{L}$ over $\mathbf{E}^{\text{ét}}_{Y\to X}$, we claim that $\mathbb{L}\to\psi_*\psi^{-1}\mathbb{L}$ is an isomorphism. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$. Thus, we may assume that $\mathbb{L}$ is the constant sheaf with value L where L is a finite abelian group. Let $\underline{L}$ be the constant sheaf with value L on $Y_{\text{fét}}$. Then, $\mathbb{L}=\beta^{-1}\underline{L}$, and the isomorphism $\mathbb{L}=\beta^{-1}\underline{L}\xrightarrow{\sim}\psi_*\rho^{-1}\underline{L}=\psi_*\psi^{-1}\mathbb{L}$ coincides with the adjunction map $\mathbb{L}\to\psi_*\psi^{-1}\mathbb{L}$. Therefore, $\psi^{-1}:\mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X})\to\mathbf{LocSys}(Y_{\text{ét}})$ is an equivalence with quasi-inverse ψ_*. The conclusion follows from 10.3. □

10.5. Let $f:(Y'\to X')\to(Y\to X)$ be a morphism of morphisms between coherent schemes over $\mathrm{Spec}(\mathbb{Q}_p)\to\mathrm{Spec}(\mathbb{Z}_p)$. The base change by f induces a commutative diagram of sites

$$\begin{array}{ccc}Y'_{\text{ét}} & \xrightarrow{\psi'} & \mathbf{E}^{\text{ét}}_{Y'\to X'}\\ \downarrow{\scriptstyle f_{\text{ét}}} & & \downarrow{\scriptstyle f_{\mathbf{E}}}\\ Y_{\text{ét}} & \xrightarrow{\psi} & \mathbf{E}^{\text{ét}}_{Y\to X}\end{array}\tag{10.5.1}$$

Let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. Remark that the sheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ is flat over $\mathbb{Z}$. Consider the natural morphisms in the derived category $\mathbf{D}(\overline{\mathscr{B}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$,

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\xleftarrow{\alpha_1}(\mathrm{R}f_{\mathbf{E}*}\psi'_*\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\xrightarrow{\alpha_2}\mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'),\tag{10.5.2}$$

where α_1 is induced by the canonical morphism $\psi'_*\mathbb{F}'\to\mathrm{R}\psi'_*\mathbb{F}'$, and α_2 is the canonical morphism.

10.6. We keep the notation in 10.5 and assume that X is the spectrum of an absolutely integrally closed valuation ring A and that Y is a quasi-compact open subscheme of X. Applying the functor $\mathrm{R}\Gamma(Y\to X,-)$ on (10.5.2), we obtain the natural morphisms in the derived category $\mathbf{D}(A\text{-}\mathbf{Mod})$ by 7.8,

$$\mathrm{R}\Gamma(Y'_{\text{ét}},\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A\xleftarrow{\alpha_1}\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A\xrightarrow{\alpha_2}\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}').\tag{10.6.1}$$

Definition 10.7 ([AG20, 4.8.13, 5.7.4])**.** With the notation in 10.5 (resp. 10.6), if α_1 is an isomorphism (for instance, if the canonical morphism $\psi'_*\mathbb{F}'\to\mathrm{R}\psi'_*\mathbb{F}'$ is an isomorphism), then we call the canonical morphism

$$\alpha_2\circ\alpha_1^{-1}:(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')\tag{10.7.1}$$

$$(\text{resp. }\alpha_2\circ\alpha_1^{-1}:\mathrm{R}\Gamma(Y'_{\text{ét}},\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A\longrightarrow\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'))\tag{10.7.2}$$

the *relative* (resp. *absolute*) *Faltings' comparison morphism* associated to $f:(Y'\to X')\to(Y\to X)$ and $\mathbb{F}'$. In this case, we say that *Faltings' comparison morphisms exist.*

Theorem 10.8 ([Ach17, Cor.6.9], cf. [AG20, 4.4.2])**.** *Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K. Let X be an $\mathcal{O}_K$-scheme of finite type, $\mathbb{F}$ a finite locally constant abelian sheaf on $X_{\overline{K},\text{ét}}$, $\psi:X_{\overline{K},\text{ét}}\to\mathbf{E}^{\text{ét}}_{X_{\overline{K}}\to X}$ the morphism of sites defined in* 10.1*. Then, the canonical morphism $\psi_*\mathbb{F}\to\mathrm{R}\psi_*\mathbb{F}$ is an isomorphism.*

Corollary 10.9. *Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K. Let X be a coherent $\mathcal{O}_{\overline{K}}$-scheme, $Y=\mathrm{Spec}(\overline{K})\times_{\mathrm{Spec}(\mathcal{O}_{\overline{K}})}X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$, $\psi:Y_{\text{ét}}\to\mathbf{E}^{\text{ét}}_{Y\to X}$ the morphism of sites defined in* 10.1*. Then, the canonical morphism $\psi_*\mathbb{F}\to\mathrm{R}\psi_*\mathbb{F}$ is an isomorphism.*

We emphasize that we don't need any finiteness condition of X over $\mathcal{O}_{\overline{K}}$ in 10.9. In fact, one can replace $\mathcal{O}_{\overline{K}}$ by $\overline{\mathbb{Z}_p}$ without loss of generality, where $\overline{\mathbb{Z}_p}$ is the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$. We keep working over $\mathcal{O}_{\overline{K}}$ only for the continuation of our usage of notation.

Proof of 10.9. We take a directed inverse system $(X_\lambda\to\mathrm{Spec}(\mathcal{O}_{K_\lambda}))_{\lambda\in\Lambda}$ of morphisms of finite type of schemes by Noetherian approximation, such that K_λ is a finite field extension of K and $\overline{K}=\bigcup_{\lambda\in\Lambda}K_\lambda$, and that the transition morphisms $X_{\lambda'}\to X_\lambda$ are affine and $X=\lim_{\lambda\in\Lambda}X_\lambda$ (cf. [Sta21, 09MV]). For each $\lambda\in\Lambda$, we set $Y_\lambda=\mathrm{Spec}(\overline{K})\times_{\mathrm{Spec}(\mathcal{O}_{K_\lambda})}X_\lambda$. Notice that $Y=\lim Y_\lambda$. Then, there exists an index $\lambda_0\in\Lambda$ and a finite locally constant abelian sheaf $\mathbb{F}_{\lambda_0}$ on $Y_{\lambda_0,\text{ét}}$ such that $\mathbb{F}$ is the pullback of $\mathbb{F}_{\lambda_0}$ by $Y_{\text{ét}}\to Y_{\lambda_0,\text{ét}}$ (cf. [Sta21, 09YU]). Let $\mathbb{F}_\lambda$ be the pullback of $\mathbb{F}_{\lambda_0}$ by $Y_{\lambda,\text{ét}}\to Y_{\lambda_0,\text{ét}}$ for each $\lambda\geq\lambda_0$. Notice that $\mathcal{O}_{K_\lambda}$ also satisfies the conditions in 10.8. Let $\psi_\lambda:Y_{\lambda,\text{ét}}\to\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ be the morphism of sites defined in 10.1, $\varphi_\lambda:\mathbf{E}^{\text{ét}}_{Y\to X}\to\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ the morphism of sites defined by the transition morphism. Then, we have $\mathrm{R}^q\psi_{\lambda*}\mathbb{F}_\lambda=0$ for each integer $q>0$ by 10.8, and moreover

$$\mathrm{R}^q\psi_*\mathbb{F}=\operatorname*{colim}_{\lambda\geq\lambda_0}\varphi_\lambda^{-1}\mathrm{R}^q\psi_{\lambda*}\mathbb{F}_\lambda=0\tag{10.9.1}$$

by 7.11, [SGA 4_{II}, VII.5.6] and [SGA 4_{II}, VI.8.7.3] whose conditions are satisfied because each object in each concerned site is quasi-compact. □

Lemma 10.10. *With the notation in* 10.5*, let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, the canonical morphism $f_{\mathbf{E}}^{-1}\psi_*\mathbb{F}\to\psi'_*f_{\text{ét}}^{-1}\mathbb{F}$ is an isomorphism.*

Proof. The base change morphism $f_{\mathbf{E}}^{-1}\psi_*\mathbb{F}\to\psi'_*f_{\text{ét}}^{-1}\mathbb{F}$ is the composition of the adjunction morphisms ([SGA 4_{III}, XVII.2.1.3])

$$f_{\mathbf{E}}^{-1}\psi_*\mathbb{F}\to\psi'_*\psi'^{-1}(f_{\mathbf{E}}^{-1}\psi_*\mathbb{F})=\psi'_*f_{\text{ét}}^{-1}(\psi^{-1}\psi_*\mathbb{F})\to\psi'_*f_{\text{ét}}^{-1}\mathbb{F}\tag{10.10.1}$$

which are both isomorphisms by 10.4. □

10.11. Let K be a complete discrete valuation field of characteristic 0 with valuation ring $\mathcal{O}_K$ whose residue field k is algebraically closed (a technical condition required by [AG20, 4.1.3, 5.1.3]) of characteristic $p>0$, $\overline{K}$ an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\eta=\mathrm{Spec}(K)$, $\overline{\eta}=\mathrm{Spec}(\overline{K})$, $S=\mathrm{Spec}(\mathcal{O}_K)$, $\overline{S}=\mathrm{Spec}(\mathcal{O}_{\overline{K}})$, $s=\mathrm{Spec}(k)$. Remark that $\overline{K}$ is a pre-perfectoid field with valuation ring $\mathcal{O}_{\overline{K}}$ so we are also in the situation 8.3.

10.12. With the notation in 10.11, let X be an S-scheme, Y an open subscheme of the generic fibre X_η. We simply denote by $\mathscr{M}_X$ the compactifying log structure $\mathscr{M}_{X_\eta\to X}$ (9.3). Following [AGT16, III.4.7], we say that $Y\to X$ is *adequate* over $\eta\to S$ if the following conditions are satisfied:

(i) X is of finite type over S;
(ii) Any point of the special fibre X_s admits an étale neighborhood U such that $U_\eta\to\eta$ is smooth and that $U_\eta\setminus Y$ is the support of a strict normal crossings divisor on U_η;
(iii) $(X,\mathscr{M}_{Y\to X})$ is a fine log scheme and the structure morphism $(X,\mathscr{M}_{Y\to X})\to(S,\mathscr{M}_S)$ is smooth and saturated.

In this case, $(X,\mathscr{M}_{Y\to X})\to(S,\mathscr{M}_S)$ is adequate in the sense of [AGT16, III.4.7]. We remark that for any adequate $(S,\mathscr{M}_S)$-log scheme $(X,\mathscr{M})$, $X^{\mathrm{tr}}\to X$ is adequate over $\eta\to S$ and $(X,\mathscr{M})=(X,\mathscr{M}_{X^{\mathrm{tr}}\to X})$ (cf. 9.4, 9.5). Note that if $Y\to X$ is strictly semi-stable over $\eta\to S$ then it is adequate (cf. 9.5).

10.13. We recall the statement of Faltings' main p-adic comparison theorem following Abbes-Gros [AG20]. We take the notation in 10.11. Firstly, recall that for any adequate open immersion of schemes $X^\circ\to X$ over $\eta\to S$ and any finite locally constant abelian sheaf $\mathbb{F}$ on $X^\circ_{\overline{\eta},\text{ét}}$, the canonical morphism $\psi_*\mathbb{F}\to\mathrm{R}\psi_*\mathbb{F}$ is an isomorphism, where $\psi: X^\circ_{\overline{\eta},\text{ét}}\to\mathbf{E}^{\text{ét}}_{X^\circ_{\overline{\eta}}\to X}$ is the morphism of sites defined in 10.1 ([AG20, 4.4.2]).

Let $(X'^{\triangleright}\to X')\to(X^\circ\to X)$ be a morphism of adequate open immersions of schemes over $\eta\to S$ such that $X'\to X$ is projective and that the induced morphism $(X',\mathscr{M}_{X'^{\triangleright}\to X'})\to(X,\mathscr{M}_{X^\circ\to X})$ is smooth and saturated. Let $Y'=\overline{\eta}\times_\eta X'^{\triangleright}$, $Y=\overline{\eta}\times_\eta X^\circ$, $f:(Y'\to X')\to(Y\to X)$ the natural morphism, $\mathbb{F}'$ a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. By the first paragraph, we have the relative Faltings' comparison morphism associated to f and $\mathbb{F}'$ (10.7.1),

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'). \tag{10.13.1}$$

Remark that under our assumption, the sheaf $\mathrm{R}^qf_{\text{ét}*}\mathbb{F}'$ on $Y_{\text{ét}}$ is finite locally constant for each integer q ([AG20, 5.7.2]).

Theorem 10.14 ([Fal02, Thm.6, page 266], [AG20, 5.7.4])**.** *With the notation in* 10.13*, the relative Faltings' comparison morphism associated to f and $\mathbb{F}'$ is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_{\overline{K}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$* (5.7)*, and it induces an almost isomorphism*

$$(\psi_*\mathrm{R}^qf_{\text{ét}*}\mathbb{F}')\otimes_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}^qf_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \tag{10.14.1}$$

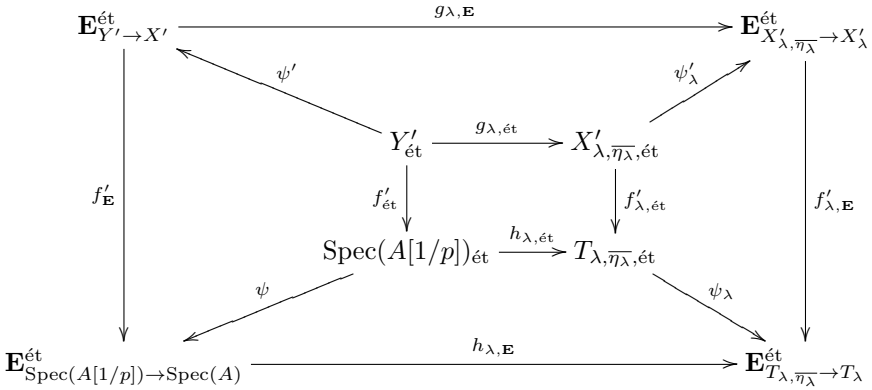
of $\mathcal{O}_{\overline{K}}$-modules for each integer q.

Proposition 10.15. *With the notation in* 10.11*, let A be an absolutely integrally closed valuation ring of height* 1 *extension of $\mathcal{O}_{\overline{K}}$, X a proper A-scheme of finite presentation, $Y=\mathrm{Spec}(A[1/p])\times_{\mathrm{Spec}(A)}X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, there exists a proper surjective morphism $X'\to X$ of finite presentation such that the relative and absolute Faltings' comparison morphisms associated to $f':(Y'\to X')\to(\mathrm{Spec}(A[1/p])\to\mathrm{Spec}(A))$ and $\mathbb{F}'$ (which exist by* 10.9*) are almost isomorphisms, where $Y'=Y\times_X X'$ and $\mathbb{F}'$ is the pullback of $\mathbb{F}$ on $Y'_{\text{ét}}$.*

Proof. Since the underlying topological space of X is Noetherian by 9.10, each irreducible component Z of X admits a closed subscheme structure such that $Z\to X$ is of finite presentation ([Sta21, 01PH]). After replacing X by the disjoint union of its irreducible components, we may assume that X is irreducible. If Y is empty, then we take $X'=X$ and thus the relative (resp. absolute) Faltings' comparison morphism associated to f' and $\mathbb{F}'$ is an isomorphism between zero objects. If Y is not empty, then we are in the situation of 9.11.(iv) by taking $\mathcal{O}_L=A$. With the notation in 9.11, we check that the morphism $X'\to X$ meets our requirements. We set $\eta_\lambda=\mathrm{Spec}(K_\lambda)$, $S_\lambda=\mathrm{Spec}(\mathcal{O}_{K_\lambda})$, $T_{\lambda,\overline{\eta_\lambda}}=\overline{\eta}\times_{\eta_\lambda}U_\lambda$, $X'_{\lambda,\overline{\eta_\lambda}}=\overline{\eta}\times_{\eta_\lambda}Y'_\lambda$, and denote by $f'_\lambda:(X'_{\lambda,\overline{\eta_\lambda}}\to X'_\lambda)\to(T_{\lambda,\overline{\eta_\lambda}}\to T_\lambda)$ the natural morphism. We obtain a commutative

diagram

(10.15.1)



Firstly notice that the site $Y'_{\text{ét}}$ (resp. $\text{Spec}(A[1/p])_{\text{ét}}$) is the limit of the sites $X'_{\lambda,\overline{\eta_\lambda},\text{ét}}$ (resp. $T_{\lambda,\overline{\eta_\lambda},\text{ét}}$) and the site $\mathbf{E}^{\text{ét}}_{Y'\to X'}$ (resp. $\mathbf{E}^{\text{ét}}_{\text{Spec}(A[1/p])\to\text{Spec}(A)}$) is the limit of the sites $\mathbf{E}^{\text{ét}}_{X'_{\lambda,\overline{\eta_\lambda}}\to X'_\lambda}$ (resp. $\mathbf{E}^{\text{ét}}_{T_\lambda,\overline{\eta_\lambda}\to T_\lambda}$) ([SGA 4_{II}, VII.5.6] and 7.11). There exists an index $\lambda_0\in\Lambda$ and a finite locally constant abelian sheaf $\mathbb{F}'_{\lambda_0}$ on $X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ such that $\mathbb{F}'$ is the pullback of $\mathbb{F}'_{\lambda_0}$ by $Y'_{\text{ét}}\to X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ (cf. [Sta21, 09YU]). Let $\mathbb{F}'_\lambda$ be the pullback of $\mathbb{F}'_{\lambda_0}$ by $X'_{\lambda,\overline{\eta_\lambda},\text{ét}}\to X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ for each $\lambda\geq\lambda_0$. We also have $\overline{\mathscr{B}}'=\operatorname{colim} g_{\lambda,\mathbf{E}}^{-1}\overline{\mathscr{B}}'$ (resp. $\overline{\mathscr{B}}=\operatorname{colim} h_{\lambda,\mathbf{E}}^{-1}\overline{\mathscr{B}}$) by 7.11. According to [SGA 4_{II}, VI.8.7.3], whose conditions are satisfied because each object in each concerned site is quasi-compact, there are canonical isomorphisms for each integer q,

$$(\mathrm{R}^q(\psi\circ f'_{\text{ét}})_*\mathbb{F}')\otimes_{\mathbb{Z}}\overline{\mathscr{B}}\xrightarrow{\sim}\operatorname{colim} h_{\lambda,\mathbf{E}}^{-1}((\mathrm{R}^q(\psi_\lambda\circ f'_{\lambda,\text{ét}})_*\mathbb{F}'_\lambda)\otimes_{\mathbb{Z}}\overline{\mathscr{B}}), \tag{10.15.2}$$

$$\mathrm{R}^q f'_{\mathbf{E}*}(\psi'^{-1}_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')\xrightarrow{\sim}\operatorname{colim} h_{\lambda,\mathbf{E}}^{-1}\mathrm{R}^q f'_{\lambda,\mathbf{E}*}(\psi'^{-1}_{\lambda*}\mathbb{F}'_\lambda\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'). \tag{10.15.3}$$

On the other hand, $(X'_\lambda,\mathscr{M}_{X'_\lambda})\to(T_\lambda,\mathscr{M}_{T_\lambda})$ is a smooth and saturated morphism of adequate $(S_\lambda,\mathscr{M}_{S_\lambda})$-log schemes with $X'_\lambda\to T_\lambda$ projective for each $\lambda\in\Lambda$ by construction. Thus, we are in the situation of 10.14, which implies that the relative Faltings' comparison morphism associated to f'_λ and $\mathbb{F}'_\lambda$,

$$(\mathrm{R}^q(\psi_\lambda\circ f'_{\lambda,\text{ét}})_*\mathbb{F}'_\lambda)\otimes_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}^q f'_{\lambda,\mathbf{E}*}(\psi'^{-1}_{\lambda*}\mathbb{F}'_\lambda\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \tag{10.15.4}$$

is an almost isomorphism for each $\lambda\geq\lambda_0$. Combining with (10.15.2) and (10.15.3), we see that the relative Faltings' comparison morphism associated to f' and $\mathbb{F}'$,

$$\mathrm{R}\psi_*(\mathrm{R}f'_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}f'_{\mathbf{E}*}(\psi'^{-1}_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'), \tag{10.15.5}$$

is an almost isomorphism (and thus so is the absolute one). □

Corollary 10.16. *With the notation in* 10.15*, there exists a proper hypercovering $X_\bullet\to X$ of coherent schemes* ([Sta21, 0DHI]) *such that for each degree n, the relative and absolute Faltings' comparison morphisms associated to $f_n:(Y_n\to X_n)\to(\text{Spec}(A[1/p])\to\text{Spec}(A))$ and $\mathbb{F}_n$ (which exist by* 10.9*) are almost isomorphisms, where $Y_n=Y\times_X X_n$ and $\mathbb{F}_n$ is the pullback of $\mathbb{F}$ by the natural morphism $Y_{n,\text{ét}}\to Y_{\text{ét}}$. In particular, $Y_\bullet\to Y$ is a proper hypercovering and $X_\bullet^{Y_\bullet}\to X^Y$ is a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$.*

Proof. Let $\mathscr{C}$ be the category of proper A-schemes of finite presentation endowed with the pretopology formed by families of morphisms $\{f_i:X_i\to X\}_{i\in I}$ with I finite and $X=\bigcup_{i\in I}f_i(X_i)$. Consider the functor $u^+:\mathscr{C}\to\mathbf{I}_{\text{Spec}(A[1/p])\to\text{Spec}(A)}$ sending X to X^Y where $Y=\text{Spec}(A[1/p])\times_{\text{Spec}(A)}X$. It is well-defined by 3.19.(4) and commutes with fibred products by 3.21 and continuous by 3.15. Lemma 10.15 allows us to take a hypercovering $X_\bullet\to X$ in $\mathscr{C}$ meeting our requirement by [Sta21, 094K and 0DB1]. We see that $Y_\bullet\to Y$ is a proper hypercovering and that $X_\bullet^{Y_\bullet}\to X^Y$ is a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$ by the properties of u^+ ([Sta21, 0DAY]). □

Lemma 10.17. *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, A a $\overline{\mathbb{Z}_p}$-algebra which is an absolutely integrally closed valuation ring, X a proper A-scheme of finite presentation, $Y=\text{Spec}(A[1/p])\times_{\text{Spec}(A)}X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Let $A'=((A/\cap_{n>0}p^nA)_{\sqrt{pA}})^\wedge$ (p-adic completion), $X'=X_{A'}$, $Y'=Y_{A'}$, $\mathbb{F}'$ the pullback of $\mathbb{F}$ on $Y'_{\text{ét}}$. Then, the following statements are equivalent:*

(1) *The absolute Faltings' comparison morphism associated to* $f : (Y \to X) \to (\mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A))$ *and* $\mathbb{F}$ *(which exists by* 10.9*) is an almost isomorphism.*
(2) *The absolute Faltings' comparison morphism associated to* $f' : (Y' \to X') \to (\mathrm{Spec}(A'[1/p]) \to \mathrm{Spec}(A'))$ *and* $\mathbb{F}'$ *(which exists by* 10.9*) is an almost isomorphism.*

Proof. If p is zero (resp. invertible) in A, then the absolute Faltings' comparison morphisms are both isomorphisms between zero objects, since Y and Y' are empty (resp. the abelian sheaves $\mathbb{F}$ and $\mathbb{F}'$ are zero after inverting p). Thus, we may assume that p is a nonzero element of the maximal ideal of A. Notice that $\cap_{n>0} p^n A$ is the maximal prime ideal of A not containing p and that $\sqrt{pA}$ is the minimal prime ideal of A containing p (2.1). Thus, $(A/\cap_{n>0} p^n A)_{\sqrt{pA}}$ is an absolutely integrally closed valuation ring of height 1 extension of $\overline{\mathbb{Z}_p}$ (2.1) and thus so is its p-adic completion A'.

We denote by $u : (Y' \to X') \to (Y \to X)$ the natural morphism. We have $\mathbb{F}' = u_{\text{ét}}^{-1}\mathbb{F}$. The natural morphisms in (10.6.1) induce a commutative diagram

$$\begin{array}{ccccc} \mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} \\ \mathrm{R}\Gamma(Y'_{\text{ét}}, \mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A' & \xleftarrow{\alpha'_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A' & \xrightarrow{\alpha'_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \end{array} \tag{10.17.1}$$

where γ_1 is induced by the canonical morphism $\mathbb{F} \to \mathrm{R}u_{\text{ét}*}u_{\text{ét}}^{-1}\mathbb{F}$, and γ_2 (resp. γ_3) is induced by the composition of $\psi_*\mathbb{F} \to \mathrm{R}u_{\mathbf{E}*}u_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \mathrm{R}u_{\mathbf{E}*}\psi'_*u_{\text{ét}}^{-1}\mathbb{F}$ (resp. and by the canonical morphism $\overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}\overline{\mathscr{B}}'$). Since α_1 and α'_1 are isomorphisms by 10.9, it suffices to show that γ_1 and γ_3 are almost isomorphisms.

Since $A/\cap_{n>0} p^n A \to (A/\cap_{n>0} p^n A)_{\sqrt{pA}}$ is injective whose cokernel is killed by $\sqrt{pA}$ (4.7), the morphism $A \to A'$ induces an almost isomorphism $A/p^nA \to A'/p^nA'$ for each n. Then, for any torsion abelian group M, the natural morphism $M \otimes_{\mathbb{Z}} A \to M \otimes_{\mathbb{Z}} A'$ is an almost isomorphism. Therefore, γ_1 is an almost isomorphism by the proper base change theorem over the strictly Henselian local ring $A[1/p]$ ([SGA 4_{III}, XII 5.5, 5.4]). For γ_3, it suffices to show that the canonical morphism $\psi_*\mathbb{F} \otimes \overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes \overline{\mathscr{B}}')$ is an almost isomorphism. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$, thus we may assume that $\psi_*\mathbb{F}$ is the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$ by 10.4. Then, $\psi'_*\mathbb{F}'$ is also the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$ by 10.10. Let $V \to U$ be an object of $\mathbf{E}^{\text{proét}}_{Y\to X}$ such that $U^V = \mathrm{Spec}(R)$ is the spectrum of an $\overline{\mathbb{Z}_p}$-algebra R which is almost pre-perfectoid. Since the almost isomorphisms $R/p^n \to (R \otimes_A A')/p^n$ $(n \geq 1)$ induces an almost isomorphism of the p-adic completions $\widehat{R} \to R\widehat{\otimes}_A A'$, the $\overline{\mathbb{Z}_p}$-algebra $R \otimes_A A'$ is still almost pre-perfectoid (5.18). The pullback of $V \to U$ in $\mathbf{E}^{\text{proét}}_{Y'\to X'}$ is the object $V_{A'} \to U_{A'}$ and $U_{A'}^{V_{A'}}$ is the spectrum of the integral closure R' of $R \otimes_A A'$ in $R \otimes_A A'[1/p]$. Since $R \otimes_A A'$ is almost pre-perfectoid, R' is also almost pre-perfectoid and the morphism $(R \otimes_A A')/p^n \to R'/p^n$ is an almost isomorphism by 5.26. Therefore, the morphism $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}(\overline{\mathscr{B}}'/p^n\overline{\mathscr{B}}')$ is an almost isomorphism by 7.30, 8.7 and 8.8. □

Theorem 10.18. *Let* $\overline{\mathbb{Z}_p}$ *be the integral closure of* $\mathbb{Z}_p$ *in an algebraic closure of* $\mathbb{Q}_p$*,* A *a* $\overline{\mathbb{Z}_p}$*-algebra which is an absolutely integrally closed valuation ring,* X *a proper* A*-scheme of finite presentation,* $Y = \mathrm{Spec}(A[1/p]) \times_{\mathrm{Spec}(A)} X$*,* $\mathbb{F}$ *a finite locally constant abelian sheaf on* $Y_{\text{ét}}$*. Then, the absolute Faltings' comparison morphism associated to* $f : (Y \to X) \to (\mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A))$ *and* $\mathbb{F}$ (10.7.2) *(which exists by* 10.9*),*

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{10.18.1}$$

is an almost isomorphism in $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod})$ (5.7).

Proof. Let K be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. By 10.17, we may assume that A is a valuation ring of height 1 extension of $\mathcal{O}_{\overline{K}}$. Let $X_\bullet \to X$ be the proper hypercovering of coherent schemes constructed in 10.16. For each degree n the canonical morphisms (10.7.2)

$$\mathrm{R}\Gamma(Y_{n,\text{ét}}, \mathbb{F}_n) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longleftarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_n\to X_n}, \psi_{n*}\mathbb{F}_n) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_n\to X_n}, \psi_{n*}\mathbb{F}_n \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \tag{10.18.2}$$

are an isomorphism and an almost isomorphism, where $\mathbb{F}_n$ is the pullback of $\mathbb{F}$ by the natural morphism $Y_{n,\text{ét}} \to Y_{\text{ét}}$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{R}\Gamma(Y_{\bullet,\text{ét}}, \mathbb{F}_\bullet) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_{1\bullet}} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}, \psi_{\bullet *}\mathbb{F}_\bullet) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_{2\bullet}} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}, \psi_{\bullet *}\mathbb{F}_\bullet \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \end{array} \tag{10.18.3}$$

where $\mathbb{F}_\bullet = \{[n] \mapsto \mathbb{F}_n\}$ with the notation in 6.5. By the functorial spectral sequence of simplicial sites ([Sta21, 09WJ]), we deduce from (10.18.2) that $\alpha_{1\bullet}$ is an isomorphism and $\alpha_{2\bullet}$ is an almost isomorphism. Since α_1 is an isomorphism by 10.9, it remains to show that the left vertical arrow is an isomorphism and the right vertical arrow is an almost isomorphism.

We denote by $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the augmentation of simplicial site and by $a_n : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the natural morphism of sites. Notice that $a^{-1}\psi_*\mathbb{F} = \{[n] \mapsto a_n^{-1}\psi_*\mathbb{F}\} = \{[n] \mapsto \psi_{n*}\mathbb{F}_n\} = \psi_{\bullet *}\mathbb{F}_\bullet$ by 10.10 ([Sta21, 0D70]). Since $X^{Y_\bullet}_\bullet \to X^Y$ forms a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$, the right vertical arrow is an almost isomorphism by 10.4 and 8.13. Finally, the left vertical arrow is an isomorphism by the cohomological descent for étale cohomology [Sta21, 0DHL]. □

11. Faltings' Main p-adic Comparison Theorem: the Relative Case for More General Coefficients

11.1. Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. We obtain from 3.26, 8.2 and 10.1 a commutative diagram of sites

$$\begin{array}{ccccc} (\mathbf{Sch}^{\text{coh}}_{/Y})_{\text{v}} & \xrightarrow{a} & Y_{\text{ét}} & & \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} & \searrow^{\rho} & \\ \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} & \xrightarrow{\beta} & Y_{\text{fét}} \end{array} \tag{11.1.1}$$

where $a : (\mathbf{Sch}^{\text{coh}}_{/Y})_{\text{v}} \to Y_{\text{ét}}$ and $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ are defined by the inclusion functors.

Lemma 11.2. *With the notation in* 11.1*, for any finite locally constant abelian sheaf* $\mathbb{F}$ *on* $Y_{\text{ét}}$*, the canonical morphism* $\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_* a^{-1}\mathbb{F}$ *is an isomorphism.*

Proof. The base change morphism $\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_* a^{-1}\mathbb{F}$ is the composition of the adjunction morphisms ([SGA 4_{III}, XVII.2.1.3])

$$\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_*\Psi^{-1}(\varepsilon^{-1}\psi_*\mathbb{F}) = \Psi_* a^{-1}(\psi^{-1}\psi_*\mathbb{F}) \to \Psi_* a^{-1}\mathbb{F} \tag{11.2.1}$$

which are both isomorphisms by 3.27.(2) and 10.4. □

11.3. We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of the p-adic number field $\mathbb{Q}_p$ and we denote by $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. We set $\eta = \mathrm{Spec}(\mathbb{Q}_p)$, $\overline{\eta} = \mathrm{Spec}(\overline{\mathbb{Q}_p})$, $S = \mathrm{Spec}(\mathbb{Z}_p)$, $\overline{S} = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Remark that $\overline{\mathbb{Q}_p}$ is a pre-perfectoid field with valuation ring $\overline{\mathbb{Z}_p}$ so we are also in the situation 8.3. Let $f : (Y' \to X') \to (Y \to X)$ be a Cartesian morphism of morphisms of coherent schemes with a Cartesian morphism $(Y \to X^Y) \to (\overline{\eta} \to \overline{S})$ (then, $Y' \to X'^{Y'}$ is Cartesian over $\overline{\eta} \to \overline{S}$ by 3.19.(4)). Thus, X^Y and $X'^{Y'}$ are objects of $\mathbf{I}_{\overline{\eta}\to\overline{S}}$. Consider the following commutative diagram of sites associated to f.

$$\begin{array}{ccccccc} & & & \overset{\psi'}{\frown} & & & \\ Y'_{\text{ét}} & \xleftarrow{a'} & (\mathbf{Sch}^{\text{coh}}_{/Y'})_{\text{v}} & \xrightarrow{\Psi'} & \mathbf{I}_{Y'\to X'^{Y'}} & \xrightarrow{\varepsilon'} & \mathbf{E}^{\text{ét}}_{Y'\to X'} \\ \downarrow{\scriptstyle f_{\text{ét}}} & & \downarrow{\scriptstyle f_{\text{v}}} & & \downarrow{\scriptstyle f_{\mathbf{I}}} & & \downarrow{\scriptstyle f_{\mathbf{E}}} \\ Y_{\text{ét}} & \xleftarrow{a} & (\mathbf{Sch}^{\text{coh}}_{/Y})_{\text{v}} & \xrightarrow{\Psi} & \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} \\ & & & \underset{\psi}{\smile} & & & \end{array} \tag{11.3.1}$$

11.4. Following 11.3, let $g : (\widetilde{Y} \to \widetilde{X}) \to (Y \to X)$ be a morphisms of coherent schemes such that $\widetilde{Y} \to \widetilde{X}^{\widetilde{Y}}$ is also Cartesian over $\overline{\eta} \to \overline{S}$. We denote by $g' : (\widetilde{Y'} \to \widetilde{X'}) \to (Y' \to X')$ the base change of g by f, and denote by $\widetilde{f} : (\widetilde{Y'} \to \widetilde{X'}) \to (\widetilde{Y} \to \widetilde{X})$ the natural morphism which is Cartesian by base change. Thus, $\widetilde{X}^{\widetilde{Y}}$ and $\widetilde{X'}^{\widetilde{Y'}}$ are also objects of $\mathbf{I}_{\overline{\eta} \to \overline{S}}$. We write the diagram (11.3.1) associated $\widetilde{f}$ equipping all labels with tildes.

Lemma 11.5. *With the notation in* 11.3 *and* 11.4*, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$. Let $\widetilde{X}$ be an object of $\mathbf{I}_{Y \to X^Y}$, $\widetilde{Y} = \overline{\eta} \times_{\overline{S}} \widetilde{X}$, $\widetilde{\mathbb{F}'} = g'^{-1}_{\text{ét}}\mathbb{F}'$, q an integer.*

1. *The sheaf $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ on $\mathbf{I}_{Y \to X^Y}$ is canonically isomorphic to the sheaf associated to the presheaf $\widetilde{X} \mapsto H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathbb{F}'})$.*
2. *The sheaf $\mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ on $\mathbf{I}_{Y \to X^Y}$ is canonically almost isomorphic to the sheaf associated to the presheaf $\widetilde{X} \mapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}, \widetilde{\psi'}_* \widetilde{\mathbb{F}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')$.*
3. *The canonical morphism $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} \to (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ is compatible with the canonical morphisms $H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathbb{F}'}) \otimes_{\mathbb{Z}} R \xleftarrow{\alpha_1} H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}, \widetilde{\psi'}_* \widetilde{\mathbb{F}'}) \otimes_{\mathbb{Z}} R \xrightarrow{\alpha_2} H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}, \widetilde{\psi'}_* \widetilde{\mathbb{F}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')$, where $R = \overline{\mathscr{B}}(\widetilde{Y} \to \widetilde{X})$* (cf.10.6.1).

Proof. Let $\widetilde{\mathscr{F}'}$ be the restriction of $\mathscr{F}'$ on $\mathbf{I}_{\widetilde{Y'} \to \widetilde{X'}^{\widetilde{Y'}}}$. We have $\widetilde{\mathscr{F}'} = \widetilde{\Psi'}_* \widetilde{a'}^{-1} \widetilde{\mathbb{F}'}$. We set $\widetilde{\mathbb{L}'} = \widetilde{\psi'}_* \widetilde{\mathbb{F}'}$ which is a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}$ by 10.4. Notice that the canonical morphisms $\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'} \to \widetilde{\mathbb{F}'}$ and $\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \to \widetilde{\mathscr{F}'}$ are isomorphisms by 10.4 and 11.2 respectively.

(1) It follows from the canonical isomorphisms

$$H^q(I_{\widetilde{Y'} \to \widetilde{X'}^{\widetilde{Y'}}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \xrightarrow{\gamma_1} H^q_{\text{v}}(\widetilde{Y'}, \widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) = H^q_{\text{v}}(\widetilde{Y'}, \widetilde{a'}^{-1}\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}) \xleftarrow{\gamma_2} H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}), \tag{11.5.1}$$

where γ_1 is induced by the canonical isomorphism $\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \xrightarrow{\sim} \mathrm{R}\widetilde{\Psi'}_*\widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}$ (3.27.(2)), and γ_2 is induced by the canonical isomorphism $\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'} \to \mathrm{R}\widetilde{a'}_*\widetilde{a'}^{-1}\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}$ (3.9).

(2) It follows from the canonical almost isomorphism

$$\gamma_3 : H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}, \widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \longrightarrow H^q(I_{\widetilde{Y'} \to \widetilde{X'}^{\widetilde{Y'}}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}') \tag{11.5.2}$$

which is induced by the canonical almost isomorphism $\widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}' \to \mathrm{R}\widetilde{\varepsilon'}_*(\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}')$ (8.9).

(3) Consider the following diagram

$$\begin{array}{ccccc}
H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}) \otimes R & \xleftarrow{\alpha_1} & H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}, \widetilde{\mathbb{L}'}) \otimes R & \xrightarrow{\alpha_2} & H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'} \to \widetilde{X'}}, \widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \\
\downarrow{\scriptstyle \gamma_2 \otimes \mathrm{id}_R} & & \downarrow & & \downarrow{\scriptstyle \gamma_3} \\
H^q_{\text{v}}(\widetilde{Y'}, \widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \otimes R & \xleftarrow[\gamma_1 \otimes \mathrm{id}_R]{\sim} & H^q(I_{\widetilde{Y'} \to \widetilde{X'}^{\widetilde{Y'}}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \otimes R & \longrightarrow & H^q(I_{\widetilde{Y'} \to \widetilde{X'}^{\widetilde{Y'}}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}')
\end{array} \tag{11.5.3}$$

where the unlabelled vertical arrow is induced by the canonical morphism $\widetilde{\mathbb{L}'} \to \mathrm{R}\widetilde{\varepsilon'}_*\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}$, and the unlabelled horizontal arrow is the canonical morphism which induces $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} \to \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ on $\mathbf{I}_{Y \to X^Y}$ by sheafification. It is clear that the diagram (11.5.3) is commutative, which completes the proof. □

11.6. We remark that 11.5 gives a new definition of the relative (resp. absolute) Faltings' comparison morphism without using 10.9. Following 11.3, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$. We set $\mathbb{L}' = \psi'_*\mathbb{F}'$, which is a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y' \to X'}$ by 10.4. Remark that the canonical morphisms $\psi'^{-1}\mathbb{L}' \to \mathbb{F}'$ and $\varepsilon'^{-1}\mathbb{L}' \to \mathscr{F}'$ are isomorphisms by 10.4 and 11.2 respectively. We also remark that $\overline{\mathscr{B}}$, $\mathscr{O}$ are flat over $\mathbb{Z}$. The canonical morphisms in the derived category $\mathbf{D}(\overline{\mathscr{B}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y \to X}})$ (cf. 10.5.2),

$$(\mathrm{R}\psi_* \mathrm{R}f_{\text{ét}*}\psi'^{-1}\mathbb{L}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \xleftarrow{\alpha_1} (\mathrm{R}f_{\mathbf{E}*}\mathbb{L}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \xrightarrow{\alpha_2} \mathrm{R}f_{\mathbf{E}*}(\mathbb{L}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{11.6.1}$$

fit into the following commutative diagram

$$\begin{array}{ccccc} \mathrm{R}\psi_*(\mathrm{R}f_{\text{ét}*}\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xleftarrow{\alpha_1} & (\mathrm{R}f_{\mathbf{E}*}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xrightarrow{\alpha_2} & \mathrm{R}f_{\mathbf{E}*}(\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \\ \downarrow{\scriptstyle\alpha_3}\wr & & \downarrow & & \downarrow{\scriptstyle\alpha_4} \\ \mathrm{R}\psi_*(\mathrm{R}a_*\mathrm{R}f_{\mathrm{v}*}\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xleftarrow[\alpha_5]{\sim} & \mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xrightarrow[\alpha_6]{} & \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}') \end{array} \tag{11.6.2}$$

(1) The morphism α_3 is induced by the canonical isomorphism $\psi'^{-1}\mathbb{L}'\to \mathrm{R}a'_*a'^{-1}(\psi'^{-1}\mathbb{L}')$ by 3.9, and thus α_3 is an isomorphism.
(2) The morphism α_5 is induced by the canonical isomorphism $\varepsilon'^{-1}\mathbb{L}'\to \mathrm{R}\Psi'_*\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}'$ by 3.27, and thus α_5 is an isomorphism.
(3) The unlabelled arrow is induced by the canonical morphism $\mathbb{L}\to \mathrm{R}\varepsilon'_*\varepsilon'^{-1}\mathbb{L}$.
(4) The morphism α_4 is induced by the canonical almost isomorphism $\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'\to \mathrm{R}\varepsilon'_*(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}')$ by 8.9, and thus α_4 is an almost isomorphism.
(5) The morphism α_6 is the composition of

$$\mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}) \tag{11.6.3}$$

$$\text{with } \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O})\longrightarrow \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}'). \tag{11.6.4}$$

In conclusion, the arrows α_3, α_5, α_6 and α_4 induce an arrow

$$\alpha_4^{-1}\circ\alpha_6\circ\alpha_5^{-1}\circ\alpha_3 : \mathrm{R}\psi_*(\mathrm{R}f_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \tag{11.6.5}$$

in the derived category of almost $\overline{\mathbb{Z}_p}$-modules on $\mathbf{E}^{\text{ét}}_{Y\to X}$ (5.7). Remark that we don't assume that α_1 is an isomorphism here. We also call (11.6.5) the *relative Faltings' comparison morphism*. Indeed, if α_1 is an isomorphism, then the relative Faltings' comparison morphism (10.7.1) induces (11.6.5) in $\mathbf{D}(\overline{\mathbb{Z}_p}^{\mathrm{al}}\text{-}\mathbf{Mod})$ due to the commutativity of the diagram (11.6.2).

If X is the spectrum of an absolutely integrally closed valuation ring A and if $Y=\overline{\eta}\times_{\overline{S}}X$, then applying the functor $\mathrm{R}\Gamma(Y\to X,-)$ on (11.6.2) we obtain the natural morphisms in the derived category $\mathbf{D}(A\text{-}\mathbf{Mod})$ by 7.8,

(11.6.6)

$$\begin{array}{ccccc} \mathrm{R}\Gamma(Y'_{\text{ét}},\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \\ \downarrow{\scriptstyle\alpha_3}\wr & & \downarrow & & \downarrow{\scriptstyle\alpha_4} \\ \mathrm{R}\Gamma((\mathbf{Sch}^{\mathrm{coh}}_{/Y'})_{\mathrm{v}},\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xleftarrow[\alpha_5]{\sim} & \mathrm{R}\Gamma(\mathbf{I}_{Y'\to X'^{Y'}},\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xrightarrow[\alpha_6]{} & \mathrm{R}\Gamma(\mathbf{I}_{Y'\to X'^{Y'}},\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}') \end{array}$$

The arrows α_3, α_5, α_6 and α_4 induce an arrow

$$\alpha_4^{-1}\circ\alpha_6\circ\alpha_5^{-1}\circ\alpha_3 : \mathrm{R}\Gamma(Y'_{\text{ét}},\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A\longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \tag{11.6.7}$$

in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}^{\mathrm{al}}\text{-}\mathbf{Mod})$ of almost $\overline{\mathbb{Z}_p}$-modules (5.7). We also call (11.6.7) the *absolute Faltings' comparison morphism*.

Lemma 11.7. *With the notation in* 11.3*, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}'=\Psi'_*a'^{-1}\mathbb{F}'$. Assume that $X'\to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}') \tag{11.7.1}$$

is an almost isomorphism.

Proof. Following 11.5, consider the following presheaves on $\mathbf{I}_{Y\to X^Y}$ for each integer q:

$$\mathcal{H}^q_1:\widetilde{X}\longmapsto H^q_{\text{ét}}(\widetilde{Y'},\widetilde{\mathbb{F}'})\otimes_{\mathbb{Z}}\overline{\mathscr{B}}(\widetilde{Y}\to\widetilde{X}), \tag{11.7.2}$$

$$\mathcal{H}^q_2:\widetilde{X}\longmapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}},\widetilde{\psi'}_*\widetilde{\mathbb{F}'})\otimes_{\mathbb{Z}}\overline{\mathscr{B}}(\widetilde{Y}\to\widetilde{X}), \tag{11.7.3}$$

$$\mathcal{H}^q_3:\widetilde{X}\longmapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}},\widetilde{\psi'}_*\widetilde{\mathbb{F}'}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'), \tag{11.7.4}$$

They satisfy the limit-preserving condition 3.25.(ii) by 7.11, [SGA 4_{II}, VII.5.6] and [SGA 4_{II}, VI 8.5.9, 8.7.3]. Moreover, if $\widetilde{X} = \mathrm{Spec}(A)$ where A is an absolutely integrally closed valuation ring with p nonzero in A, then the canonical morphisms

$$\mathcal{H}^q_1(\mathrm{Spec}(A)) \leftarrow \mathcal{H}^q_2(\mathrm{Spec}(A)) \rightarrow \mathcal{H}^q_3(\mathrm{Spec}(A)) \tag{11.7.5}$$

are an isomorphism and an almost isomorphism by 10.18. Thus, the canonical morphisms $\mathcal{H}^q_1 \leftarrow \mathcal{H}^q_2 \rightarrow \mathcal{H}^q_3$ induce an isomorphism and an almost isomorphism of their sheafifications by 3.25. The conclusion follows from 11.5. □

Lemma 11.8. *Let $Y \rightarrow X$ be an open immersion of coherent schemes, $Y' \rightarrow Y$ a finite morphism of finite presentation. Then, there exists a finite morphism $X' \rightarrow X$ of finite presentation whose base change by $Y \rightarrow X$ is $Y' \rightarrow Y$.*

Proof. Firstly, assume that X is Noetherian. We have $Y' = Y \times_X X^Y$ by 3.19.(4). We write $X^Y = \mathrm{Spec}_X(\mathcal{A})$ where $\mathcal{A}$ is an integral quasi-coherent $\mathcal{O}_X$-algebra on X, and we write $\mathcal{A}$ as a filtered colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_\alpha$ ([Sta21, 0817]). Let $\mathcal{B}_\alpha$ be the restriction of $\mathcal{A}_\alpha$ to Y. Then, $\mathcal{B} = \mathrm{colim}\, \mathcal{B}_\alpha$ is a filtered colimit of finite quasi-coherent $\mathcal{O}_Y$-algebras with injective transition morphisms. Since $Y' = \mathrm{Spec}_Y(\mathcal{B})$ is finite over Y, there exists an index α_0 such that $Y' = \mathrm{Spec}_Y(\mathcal{B}_{\alpha_0})$. Therefore, $X' = \mathrm{Spec}_X(\mathcal{A}_{\alpha_0})$ meets our requirements.

In general, we write X as a cofiltered limit of coherent schemes of finite type over $\mathbb{Z}$ with affine transition morphisms $X = \lim_{\lambda\in\Lambda} X_\lambda$ ([Sta21, 01ZA]). Since $Y \rightarrow X$ is an open immersion of finite presentation, using [EGA IV_3, 8.8.2, 8.10.5] there exists an index $\lambda_0 \in \Lambda$, an open immersion $Y_{\lambda_0} \rightarrow X_{\lambda_0}$ and a finite morphism $Y'_{\lambda_0} \rightarrow Y_{\lambda_0}$ such that the base change of the morphisms $Y'_{\lambda_0} \rightarrow Y_{\lambda_0} \rightarrow X_{\lambda_0}$ by $X \rightarrow X_{\lambda_0}$ are the morphisms $Y' \rightarrow Y \rightarrow X$. By the first paragraph, there exists a finite morphism $X'_{\lambda_0} \rightarrow X_{\lambda_0}$ of finite presentation such that $Y'_{\lambda_0} = Y_{\lambda_0} \times_{X_{\lambda_0}} X'_{\lambda_0}$. We see that the base change $X' \rightarrow X$ of $X'_{\lambda_0} \rightarrow X_{\lambda_0}$ by $X \rightarrow X_{\lambda_0}$ meets our requirements. □

Lemma 11.9. *With the notation in* 11.3*, let $g : Y'' \rightarrow Y'$ be a finite morphism of finite presentation, $\mathbb{F}''$ a finite locally constant abelian sheaf on $Y''_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}(g_{\text{ét}*}\mathbb{F}'')$. Assume that $X' \rightarrow X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{11.9.1}$$

is an almost isomorphism.

Proof. There exists a Cartesian morphism $g : (Y'' \rightarrow X'') \rightarrow (Y' \rightarrow X^Y \times_X X')$ of open immersions of coherent schemes such that $X'' \rightarrow X^Y \times_X X'$ is finite and of finite presentation by 11.8. Consider the diagram (11.3.1) associated to g:

$$\begin{array}{ccccc} Y''_{\text{ét}} & \xleftarrow{a''} & (\mathbf{Sch}^{\mathrm{coh}}_{/Y''})_{\mathrm{v}} & \xrightarrow{\Psi''} & \mathbf{I}_{Y''\rightarrow X''Y''} \\ \downarrow{\scriptstyle g_{\text{ét}}} & & \downarrow{\scriptstyle g_{\mathrm{v}}} & & \downarrow{\scriptstyle g_{\mathbf{I}}} \\ Y'_{\text{ét}} & \xleftarrow{a'} & (\mathbf{Sch}^{\mathrm{coh}}_{/Y'})_{\mathrm{v}} & \xrightarrow{\Psi'} & \mathbf{I}_{Y'\rightarrow X'Y'} \end{array} \tag{11.9.2}$$

We set $\mathscr{G}'' = \Psi''_* a''^{-1}\mathbb{F}''$. The base change morphism $a'^{-1} g_{\text{ét}*} \rightarrow g_{\mathrm{v}*} a''^{-1}$ induces a canonical isomorphism $\mathscr{F}' \xrightarrow{\sim} g_{\mathbf{I}*}\mathscr{G}''$ by 3.10. Moreover, the canonical morphism $g_{\mathbf{I}*}\mathscr{G}'' \rightarrow \mathrm{R}g_{\mathbf{I}*}\mathscr{G}''$ is an isomorphism by 11.5.(1) and 3.25, since $g : Y'' \rightarrow Y'$ is finite ([SGA 4_{II}, VIII.5.6]). By applying 11.7 to g and $\mathbb{F}''$, the canonical morphism

$$(\mathrm{R}g_{\mathbf{I}*}\mathscr{G}'') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}' \longrightarrow \mathrm{R}g_{\mathbf{I}*}(\mathscr{G}'' \otimes_{\mathbb{Z}} \mathscr{O}'') \tag{11.9.3}$$

is an almost isomorphism. Let h be the composition of $(Y'' \rightarrow X'') \rightarrow (Y' \rightarrow X^Y \times_X X') \rightarrow (Y \rightarrow X^Y)$. Note that $X'' \rightarrow X^Y$ is also proper of finite presentation. By applying 11.7 to h and $\mathbb{F}''$, the canonical morphism

$$(\mathrm{R}h_{\mathbf{I}*}\mathscr{G}'') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}h_{\mathbf{I}*}(\mathscr{G}'' \otimes_{\mathbb{Z}} \mathscr{O}'') \tag{11.9.4}$$

is an almost isomorphism. It is clear that $h_{\mathbf{I}} = f_{\mathbf{I}} \circ g_{\mathbf{I}}$. The conclusion follows from the canonical isomorphism $\mathscr{F}' \rightarrow \mathrm{R}g_{\mathbf{I}*}\mathscr{G}''$ and the canonical almost isomorphisms (11.9.3) and (11.9.4). □

Lemma 11.10. *With the notation in* 11.3*, let $\mathcal{F}'$ be a constructible abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$. Assume that $X' \to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{11.10.1}$$

is an almost isomorphism.

Proof. We prove by induction on an integer q that the canonical morphism $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}')\otimes_{\mathbb{Z}}\mathscr{O} \to \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}} \mathscr{O}')$ is an almost isomorphism. It holds trivially for each $q \leq -1$. Notice that there exists a finite morphism $g : Y'' \to Y'$ of finite presentation, a finite locally constant abelian sheaf $\mathbb{F}''$ on $Y''_{\text{ét}}$ and an injective morphism $\mathcal{F}' \to g_{\text{ét}*}\mathbb{F}''$ by [Sta21, 09Z7] (cf. [SGA 4_{III}, IX.2.14]). Let $\mathcal{G}'$ be the quotient of $\mathcal{F}' \to g_{\text{ét}*}\mathbb{F}''$, which is also a constructible abelian sheaf on $Y'_{\text{ét}}$ since $g_{\text{ét}*}\mathbb{F}''$ is so ([Sta21, 095R, 03RZ]). The exact sequence $0 \to \mathcal{F}' \to g_{\text{ét}*}\mathbb{F}'' \to \mathcal{G}' \to 0$ induces an exact sequence by 3.27.(1),

$$0 \longrightarrow \Psi'_* a'^{-1}\mathcal{F}' \longrightarrow \Psi'_* a'^{-1}(g_{\text{ét}*}\mathbb{F}'') \longrightarrow \Psi'_* a'^{-1}\mathcal{G}' \longrightarrow 0. \tag{11.10.2}$$

We set $\mathscr{H}' = \Psi'_* a'^{-1}(g_{\text{ét}*}\mathbb{F}'')$ and $\mathscr{G}' = \Psi'_* a'^{-1}\mathcal{G}'$. Then, we obtain a morphism of long exact sequences

(11.10.3)

$$\begin{array}{ccccccccc}
(\mathrm{R}^{q-1} f_{\mathbf{I}*}\mathscr{H}')\otimes\mathscr{O} & \longrightarrow & (\mathrm{R}^{q-1} f_{\mathbf{I}*}\mathscr{G}')\otimes\mathscr{O} & \longrightarrow & (\mathrm{R}^{q} f_{\mathbf{I}*}\mathscr{F}')\otimes\mathscr{O} & \longrightarrow & (\mathrm{R}^{q} f_{\mathbf{I}*}\mathscr{H}')\otimes\mathscr{O} & \longrightarrow & (\mathrm{R}^{q} f_{\mathbf{I}*}\mathscr{G}')\otimes\mathscr{O} \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\
\mathrm{R}^{q-1} f_{\mathbf{I}*}(\mathscr{H}'\otimes\mathscr{O}') & \longrightarrow & \mathrm{R}^{q-1} f_{\mathbf{I}*}(\mathscr{G}'\otimes\mathscr{O}') & \longrightarrow & \mathrm{R}^{q} f_{\mathbf{I}*}(\mathscr{F}'\otimes\mathscr{O}') & \longrightarrow & \mathrm{R}^{q} f_{\mathbf{I}*}(\mathscr{H}'\otimes\mathscr{O}') & \longrightarrow & \mathrm{R}^{q} f_{\mathbf{I}*}(\mathscr{G}'\otimes\mathscr{O}')
\end{array}$$

Notice that γ_1 and γ_2 are almost isomorphisms by induction, and that γ_4 is an almost isomorphism by 11.9. Thus, applying the 5-lemma ([Sta21, 05QA]) in the abelian category of almost $\overline{\mathbb{Z}_p}$-modules over $\mathbf{I}_{Y\to X^Y}$, we see that γ_3 is almost injective. Since $\mathcal{F}'$ is an arbitrary constructible abelian sheaf, the morphism γ_5 is also almost injective. Thus, γ_3 is an almost isomorphism. □

Theorem 11.11. *With the notation in* 11.3*, let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$. Assume that $X' \to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{11.11.1}$$

is an almost isomorphism in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{I}_{Y\to X^Y}})$ (5.7)*.*

Proof. We write $\mathcal{F}'$ as a filtered colimit of constructible abelian sheaves $\mathcal{F}' = \operatorname{colim}_{\lambda\in\Lambda}\mathcal{F}'_\lambda$ ([Sta21, 03SA], cf. [SGA 4_{III}, IX.2.7.2]). We set $\mathscr{F}'_\lambda = \Psi'_* a'^{-1}\mathcal{F}_\lambda$. We have $\mathscr{F}' = \operatorname{colim}_{\lambda\in\Lambda}\mathscr{F}'_\lambda$ by [SGA 4_{II}, VI.5.1] whose conditions are satisfied since each object in each concerned site is quasi-compact. Moreover, for each integer q, we have

$$(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} = \operatorname*{colim}_{\lambda\in\Lambda}(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'_\lambda) \otimes_{\mathbb{Z}} \mathscr{O}, \tag{11.11.2}$$

$$\mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') = \operatorname*{colim}_{\lambda\in\Lambda} \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'_\lambda \otimes_{\mathbb{Z}} \mathscr{O}'). \tag{11.11.3}$$

The conclusion follows from 11.10. □

Lemma 11.12. *With the notation in* 11.3 *and* 11.4*, let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$, $\mathcal{H} = \mathrm{R}f_{\text{ét}*}\mathcal{F}'$, and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$, $\mathscr{H} = \mathrm{R}\Psi_* a^{-1}\mathcal{H}$. Let $\widetilde{X}$ be an object of $\mathbf{I}_{Y\to X^Y}$, $\widetilde{Y} = \overline{\eta}\times_{\overline{S}}\widetilde{X}$, $\widetilde{\mathcal{F}'} = g'^{-1}_{\text{ét}}\mathcal{F}'$.*

1. *The sheaf $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ is canonically isomorphic to the presheaf $\widetilde{X} \mapsto H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathcal{F}'})$ for each integer q.*
2. *If $Y' \to Y$ is proper, then there exists a canonical isomorphism $\mathscr{H} \xrightarrow{\sim} \mathrm{R}f_{\mathbf{I}*}\mathscr{F}'$.*

Proof. Note that the canonical morphism $\mathscr{F}' \to \mathrm{R}\Psi'_* a'^{-1}\mathcal{F}'$ is an isomorphism by 3.27.(1). Thus, $\mathrm{R}f_{\mathbf{I}*}\mathscr{F}' = \mathrm{R}(\Psi\circ f_{\mathrm{v}})_* a'^{-1}\mathcal{F}'$, whose q-th cohomology is the sheaf associated to the presheaf $\widetilde{X} \mapsto H^q_{\mathrm{v}}(\widetilde{Y'}, \widetilde{a'^{-1}\mathcal{F}'}) = H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathcal{F}'})$ by 3.9, and thus (1) follows. If $Y' \to Y$ is proper, then the base change morphism $a^{-1}\mathrm{R}f_{\text{ét}*} \to \mathrm{R}f_{\mathrm{v}*}a'^{-1}$ induces an isomorphism $a^{-1}\mathcal{H} \xrightarrow{\sim} \mathrm{R}f_{\mathrm{v}*}a'^{-1}\mathcal{F}'$ by 3.10, and thus (2) follows. □

Theorem 11.13. *With the notation in* 11.3*, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. Assume that*

(i) *the morphism $X' \to X$ is proper of finite presentation, and that*
(ii) *the sheaf $\mathrm{R}^q f_{\text{ét}*}\mathbb{F}'$ is finite locally constant for each integer q and nonzero for finitely many q, and that*

(iii) *we have* $\mathrm{R}^q\psi_*\mathbb{H} = 0$ *(resp.* $\mathrm{R}^q\psi'_*\mathbb{H} = 0$*) for any finite locally constant abelian sheaf* $\mathbb{H}$ *on* $Y_{\text{ét}}$ *(resp.* $Y'_{\text{ét}}$*) and any integer* $q > 0$.

Then, the relative Faltings' comparison morphism associated to f *and* $\mathbb{F}'$ (10.7.1) *(which exists by* (iii)*) is an almost isomorphism in the derived category* $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ (5.7)*, and it induces an almost isomorphism*

$$(\psi_*\mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'^{*}\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{11.13.1}$$

of $\overline{\mathbb{Z}_p}$*-modules for each integer* q.

Proof. We follow the discussion of 11.6 and set $\mathscr{F}' = \Psi'_*a'^{-1}\mathbb{F}'$. The canonical morphism (11.6.4)

$$\mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \longrightarrow \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{11.13.2}$$

is an almost isomorphism by 11.7. It remains to show that the canonical morphism (11.6.3)

$$\mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \tag{11.13.3}$$

is also an almost isomorphism. With the notation in 11.12 by taking $\mathcal{F}' = \mathbb{F}'$, the complex $\mathcal{H}$ is a bounded complex whose cohomologies $H^q(\mathcal{H})$ are finite locally constant abelian sheaves by condition (ii). Consider the commutative diagram (11.1.1),

$$\begin{array}{ccc} (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a} & Y_{\text{ét}} \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} \\ \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} \end{array} \tag{11.13.4}$$

We set $\mathcal{L} = \mathrm{R}\psi_*\mathcal{H}$. Then, $H^q(\mathcal{L}) = \psi_*H^q(\mathcal{H})$ by Cartan-Leray spectral sequence and condition (iii). Hence, $\mathcal{L}$ is a bounded complex of abelian sheaves whose cohomologies are finite locally constant by 10.4 so that the canonical morphism

$$\mathcal{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathcal{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \tag{11.13.5}$$

is an almost isomorphism by 8.10.

On the other hand, $H^q(\mathscr{H}) = \Psi_*a^{-1}H^q(\mathcal{H})$ by Cartan-Leray spectral sequence and 3.27.(1). Thus, the base change morphism $\varepsilon^{-1}\mathrm{R}\psi_* \to \mathrm{R}\Psi_*a^{-1}$ induces an isomorphism $\varepsilon^{-1}\mathcal{L} \xrightarrow{\sim} \mathscr{H}$ by 11.2. Moreover, the canonical morphism $\mathcal{L} \to \mathrm{R}\varepsilon_*\varepsilon^{-1}\mathcal{L} = \mathrm{R}\varepsilon_*\mathscr{H} = \mathrm{R}\psi_*\mathrm{R}a_*a^{-1}\mathcal{H}$ is an isomorphism by 3.9. Thus, the canonical morphism

$$(\mathrm{R}\varepsilon_*\varepsilon^{-1}\mathcal{L}) \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathcal{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{11.13.6}$$

is an almost isomorphism by (11.13.5). In conclusion, (11.13.3) is an almost isomorphism by (11.13.6) and by the canonical isomorphisms $\varepsilon^{-1}\mathcal{L} \xrightarrow{\sim} \mathscr{H} \xrightarrow{\sim} \mathrm{R}f_{\mathbf{I}*}\mathscr{F}'$. □

Remark 11.14. We give two concrete situations where the conditions in 11.13 are satisfied:

(1) Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, $X' \to X$ a proper and finitely presented morphism of coherent $\overline{\mathbb{Z}_p}$-schemes, $Y' \to Y$ the base change of $X' \to X$ by $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Assume that $Y' \to Y$ is smooth. Then, the condition (ii) is guaranteed by [SGA 4_{III}, XVI.2.2 and XVII.5.2.8.1], and the condition (iii) is guaranteed by 10.9.

(2) Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p, $\overline{K}$ an algebraic closure of K, $X' \to X$ a proper morphism of $\mathcal{O}_K$-schemes of finite type, $Y' \to Y$ the base change of $X' \to X$ by $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\mathcal{O}_K)$. Assume that $Y' \to Y$ is smooth. Then, the condition (ii) is guaranteed by [SGA 4_{III}, XVI.2.2 and XVII.5.2.8.1], and the condition (iii) is guaranteed by 10.8.

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Tongmu He, Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

E-mail address: `hetm15@ihes.fr`