Sutherland-type Trigonometric Models, Trigonometric Invariants and Multivariate Polynomials. II. E_7 case

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Abstract

It is shown that the E_7 trigonometric Olshanetsky-Perelomov Hamiltonian, when written in terms of the Fundamental Trigonometric Invariants (FTI), is in algebraic form, i.e., has polynomial coefficients, and preserves the infinite flag of polynomial spaces with the characteristic vector $\vec{\alpha} = (1, 2, 2, 2, 3, 3, 4)$. Its flag coincides with one of the minimal characteristic vector for the E_7 rational model, which in turn coincides with the E_7 highest root.

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I. INTRODUCTION

As mentioned in [1] (thereafter addressed as I), about 30 years ago, Olshanetsky and Perelomov (for a review, see [2]) discovered a remarkable family of quantum mechanical Hamiltonians with trigonometric potentials, which are associated to the crystallographic root spaces of the classical (A_N, B_N, C_N, D_N) and exceptional $(G_2, F_4, E_{6,7,8})$ Lie algebras. The Olshanetsky and Perelomov Hamiltonians have the property of complete integrability (the number of integrals of motion in involution is equal to the dimension of the configuration space) and that of exact solvability (the spectrum can be found explicitly, in a closed analytic form that is a second-degree polynomial in the quantum numbers). The Hamiltonian associated to a Lie algebra g of rank N, with root space Δ, is

$$H_\Delta = \frac{1}{2}\sum_{k=1}^{N}\left[-\frac{\partial^2}{\partial y_k^2}\right] + \frac{\beta^2}{8}\sum_{\alpha\in R_+} g_{|\alpha|}^2 \frac{|\,\alpha|^2}{\sin^2\frac{\beta}{2}(\alpha\cdot y)} \ , \tag{1}$$

where R_+ is the set of positive roots of Δ, $\beta \in \mathbb{R}$ is a parameter introduced for convenience, $g_{|\alpha|}^2 = \mu_{|\alpha|}(\mu_{|\alpha|}-1)$ are coupling constants depending only on the root length, and $y = (y_1, y_2, \ldots, y_N)$ is the coordinate vector. The configuration space here is the Weyl alcove of the root space (see [2]). The ground state eigenfunction and its eigenvalue are

$$\Psi_0(y) \ = \ \prod_{\alpha\in R_+}\left|\sin\frac{\beta}{2}(\alpha\cdot y)\right|^{\mu_{|\alpha|}} \quad , \quad E_0 \ = \ \frac{\beta^2}{8}\rho^2 \ , \tag{2}$$

where $\rho = \sum_{\alpha\in R_+}\mu_{|\alpha|}\alpha$ is the so-called 'deformed Weyl vector' (see [2], eqs.(5.5), (6.7)). It is known that any eigenfunction Ψ has the form of (2) multiplied by a polynomial in exponential (trigonometric) coordinates, i.e. $\Psi = \Phi\Psi_0$ (see [2]). Such polynomials Φ are called (generalized) *Jack polynomials*.

Following I, we make three definitions.

Definition 1. A multivariate linear differential operator is said to be in algebraic form if its coefficients are polynomials in the independent variable(s). It is called algebraic if by an appropriate change of the independent variable(s), it can be written in an algebraic form.

Definition 2. Consider a finite-dimensional (linear) space of multivariate polynomials defined as a linear span in the following way:

$$P_{n,\{\alpha\}}^{(d)} \ = \ \langle x_1^{p_1}x_2^{p_2}\ldots x_d^{p_d}|0\le \alpha_1 p_1+\alpha_2 p_2+\ldots+\alpha_d p_d\le n\rangle \ \ ,$$

where the α's are positive integers and $n \in \mathbb{N}$. Its *characteristic vector* is the d-dimensional vector with components α_i[11]:

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots \alpha_d) \ . \tag{3}$$

For some characteristic vectors, the corresponding polynomial spaces may have a Lie-algebraic interpretation, in that they are the finite-dimensional representation spaces for some Lie algebra of differential operators.

Definition 3. Take the infinite set of spaces of multivariate polynomials $P_n \equiv P^{(d)}_{n,\{\alpha\}}$, $n \in \mathbb{N}$, defined as above, and order them by inclusion:

$$P_0 \subset P_1 \subset P_2 \subset \dots \subset P_n \subset \dots \ .$$

Such an object is called an *infinite flag (or filtration)*, and is denoted $P^{(d)}_{\{\alpha\}}$. If a linear differential operator preserves such an infinite flag, it is said to be *exactly-solvable*. It is evident that every such operator is algebraic (see [3]). If the spaces P_n can be viewed as the finite-dimensional representation spaces of some Lie algebra g, then g is called the *hidden algebra* of the exactly-solvable operator.

Any crystallographic root space Δ is characterized by its fundamental weights $w_a, a = 1, 2, \dots r$, where $r = \text{rank}(\Delta)$. One can take a fundamental weight w_a and generate its orbit Ω_a, by acting on it by all elements of the Weyl group of Δ. By averaging over this orbit, i.e. by computing

$$\tau_a(y) = \sum_{\omega \in \Omega_a} e^{i\beta(w\cdot y)} \ , \tag{4}$$

one obtains a trigonometric Weyl invariant for any specified $\beta \in \mathbb{C}$. For a given root space Δ and a fixed β, there thus exist r independent trigonometric Weyl invariants τ generated by r fundamental weights w_a. We call them *Fundamental* Trigonometric Invariants (FTI) [1].

In I it was shown that for the root spaces $A_N, BC_N, B_N, C_N, D_N, G_2, F_4$ and E_6 (i) the Jack polynomials arising from the eigenfunctions of the Hamiltonian (1), being rewritten in terms of FTI, remain polynomials in these invariants, (ii) a similarity-transformed version of (1), namely $h \propto \Psi_0^{-1}(H - E_0)\Psi_0$, acting on the space of trigonometric invariants (i.e., the space of trigonometric orbits) is an operator in algebraic form, and (iii) that h preserves an infinite flag of spaces of polynomials, with a certain characteristic vector. The goal of this paper is to show that it holds for root space E_7. Although similar results might seem to be obtainable for E_8, an analysis of this root space is absent, mainly due to great technical complications.

II. THE CASE $\Delta = E_7$

The Hamiltonian of the trigonometric E_7 model is built using the root system of the E_7 algebra (see (1)). A convenient way to represent the Hamiltonian in coordinate form is to use an 8-dimensional space with coordinates $x_1, x_2, \ldots x_8$ imposing the constraint: $x_7 = -x_8$. In terms of these coordinates,

$$H_{E_7} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{6}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right] \tag{5}$$

$$+\frac{g\beta^2}{4\sin^2\frac{\beta}{2}(x_7-x_8)} + \frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(-x_8+x_7-\sum_{j=1}^{6}(-1)^{\nu_j}x_j\right)\right]} ,$$

the second summation being one over sextuples $\{\nu_j\}$ where each $\nu_j = 0, 1$, and $\sum_{j=1}^{6}\nu_j$ is odd. Here $g = \nu(\nu-1) > -1/4$ is the coupling constant. The configuration space is the principal E_7 Weyl alcove.

In order to resolve the constraints, we introduce new variables:

$$\begin{aligned} y_i &= x_i\ , \quad i = 1\ldots 6 \\ y_7 &= x_7 - x_8\ , \qquad \text{(with the constraint } y_7 = 2x_7\text{)}, \\ Y &= \frac{1}{2}(x_7+x_8)\ , \qquad \text{(with the constraint } Y = 0\text{)}. \end{aligned} \tag{6}$$

In terms of these coordinates, the Laplacian has the representation

$$\Delta^{(8)} = \Delta_y^{(6)} + 2\frac{\partial^2}{\partial y_7^2} + \frac{1}{2}\frac{\partial^2}{\partial Y^2} , \tag{7}$$

while the potential part of (5) depends on $y_1 \ldots y_7$ only:

$$\begin{aligned} V &= \frac{g\beta^2}{4}\sum_{j<i=1}^{6}\left[\frac{1}{\sin^2\frac{\beta}{2}(y_i+y_j)^2} + \frac{1}{\sin^2\frac{\beta}{2}(y_i-y_j)}\right] + \frac{g\beta^2}{4\sin^2\frac{\beta}{2}y_7} \\ &+ \frac{g\beta^2}{4}\sum_{\nu_j,j=1}^{6}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(y_7-\sum_{j=1}^{6}(-1)^{\nu_j}y_j\right)\right]} . \end{aligned} \tag{8}$$

In this formalism, imposing the constraints requires that one should study only eigenfunctions having no dependence on Y. Hence, the Y-dependent part of the Laplacian standing in (7) can simply be dropped.

The ground state eigenfunction and its eigenvalue are

$$\Psi_0 = (\Delta_+^{(6)}\Delta_-^{(6)} \sin\frac{\beta}{2}\, y_7)^\nu \Delta_{E_7}^\nu\ , \qquad E_0 = \frac{399}{4}\beta^2\,\nu^2\ , \tag{9}$$

where

$$\Delta_\pm^{(6)} \;=\; \prod_{j<i=1}^{6} \sin\frac{\beta}{2}(y_i \pm y_j)\ , \tag{10}$$

$$\Delta_{E_7} \;=\; \prod_{\{\nu_j\}} \sin\frac{\beta}{4}(y_7 + \sum_{j=1}^{6}(-1)^{\nu_j}\, y_j)\ . \tag{11}$$

where the second product being one over sextuples $\{\nu_j\}$ where each $\nu_j = 0,1$, and $\sum_{j=1}^{6}\nu_j$ is odd. Evidently, the ground state eigenfunction (9) does not vanish in the configuration space for (5).

The main object of our study is the gauge-rotated Hamiltonian (5), with the ground state eigenfunction (9) taken as a factor, i.e.

$$h_{\mathrm{E}_7} \;=\; -\frac{2}{\beta^2}(\Psi_0)^{-1}(H_{\mathrm{E}_7} - E_0)(\Psi_0)\ , \tag{12}$$

where E_0 is given by (9).

The E_7 root space is characterized by 7 fundamental weights, which generate orbits of lengths ranging from 56 to 10080. Let us introduce an ordering of the fundamental trigonometric invariants τ_a defined by (4) following the length of the orbit, namely:

orbit variable	weight vector	orbit size	weight length (squared)
τ_1	$e_6 - e_7$	56	$\frac{3}{2}$
τ_2	$-2e_7$	126	2
τ_3	$\frac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6) - 2e_7$	576	$\frac{7}{2}$
τ_4	$e_5+e_6-2e_7$	756	4
τ_5	$-\frac{1}{2}(e_1-e_2-e_3-e_4-e_5-e_6) - 3e_7$	2016	6
τ_6	$e_4+e_5+e_6-3e_7$	4032	$\frac{15}{2}$
τ_7	$e_3+e_4+e_5+e_6-4e_7$	10080	12

It is interesting to note that, for $\beta \in \mathbb{R}$, all τ's invariants are real, $\mathrm{Im}\tau_a = 0$. After some calculations one can show that the similarity-transformed Hamiltonian (12), in terms of the FTI (τ-variables), takes on an algebraic form. This is the following:

$$h_{E_7} \;=\; \sum_{i,j=1}^{7} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{7} B_i(\tau)\frac{\partial}{\partial\tau_i}\ ,\ A_{ij} = A_{ji}\ , \tag{13}$$

where

$$\begin{aligned}
A_{11} &= 168 + 24\tau_2 + 2\tau_4 - \frac{3}{2}\tau_1^2\ ,\ A_{12} = 54\tau_1 + 7\tau_3 - \tau_1\tau_2\ ,\\
A_{13} &= 96\tau_2 + 32\tau_4 + 6\tau_5 - \frac{3}{2}\tau_1\tau_3\ ,\ A_{14} = -432\tau_1 - 77\tau_3 + 3\tau_6 - 2\tau_1\tau_4 + 20\tau_1\tau_2\ ,\\
A_{15} &= 864\tau_1 + 210\tau_3 - 10\tau_6 - 48\tau_1\tau_2 + 6\tau_2\tau_3 - 2\tau_1\tau_5\ ,\\
A_{16} &= 24192 + 5856\tau_2 + 1248\tau_4 + 146\tau_5 + 4\tau_7 - 51\tau_1\tau_3 + 16\tau_2\tau_4 - 432\tau_1^2 - \frac{5}{2}\tau_1\tau_6\ ,\\
A_{17} &= 22464\tau_1 + 4032\tau_3 - 224\tau_6 + 1440\tau_1\tau_2 + 408\tau_1\tau_4 + 80\tau_1\tau_5 + 320\tau_2\tau_3\ ,\\
A_{22} &= 504 + 96\tau_2 + 20\tau_4 + 2\tau_5 - 2\tau_2^2\ ,\ A_{23} = -576\tau_1 - 119\tau_3 + 5\tau_6 + 32\tau_1\tau_2 - 2\tau_2\tau_3\ ,\\
A_{24} &= -3024 - 600\tau_2 - 108\tau_4 - 6\tau_5 + 54\tau_1^2 + 6\tau_1\tau_3 - 2\tau_2\tau_4\ ,\\
A_{25} &= 12096 + 2976\tau_2 + 408\tau_4 - 6\tau_5 + 3\tau_7 - 432\tau_1^2 - 51\tau_1\tau_3 + 96\tau_2^2 + 16\tau_2\tau_4 - 3\tau_2\tau_5\ ,\\
A_{26} &= -3456\tau_1 - 623\tau_3 + 41\tau_6 + 80\tau_1\tau_2 - 32\tau_1\tau_4 - 5\tau_1\tau_5 + 15\tau_2\tau_3 - 3\tau_2\tau_6 + 5\tau_3\tau_4\ ,\\
A_{27} &= -217728 - 67776\tau_2 - 12096\tau_4 - 936\tau_5 - 84\tau_7 + 1296\tau_1^2 - 18\tau_1\tau_3 + 78\tau_1\tau_6 - 3456\tau_2^2\\
&\quad -1040\tau_2\tau_4 - 20\tau_2\tau_5 + 14\tau_3^2 + 4\tau_3\tau_6 - 64\tau_4^2 - 4\tau_4\tau_5 - 4\tau_2\tau_7 + 384\tau_1^2\tau_2 + 10\tau_1\tau_2\tau_3\ ,\\
A_{33} &= -192\tau_2 - 208\tau_4 - 56\tau_5 + 2\tau_7 - 144\tau_1^2 - 36\tau_1\tau_3 + 96\tau_2^2 + 16\tau_2\tau_4 - \frac{7}{2}\tau_3^2\ ,\\
A_{34} &= 5184\tau_1 + 980\tau_3 - 56\tau_6 - 176\tau_1\tau_2 + 32\tau_1\tau_4 + 5\tau_1\tau_5 - 3\tau_3\tau_4\ ,\\
A_{35} &= -13824\tau_1 - 2408\tau_3 + 152\tau_6 - 448\tau_1\tau_2 - 224\tau_1\tau_4 - 44\tau_1\tau_5 - 92\tau_2\tau_3 + 4\tau_2\tau_6 - 4\tau_3\tau_5\\
&\quad + 32\tau_1\tau_2^2\ ,\\
A_{36} &= -96768 - 19776\tau_2 - 5104\tau_4 - 824\tau_5 - 10\tau_7 + 4752\tau_1^2 + 968\tau_1\tau_3 - 44\tau_1\tau_6 + 1056\tau_2^2\\
&\quad +240\tau_2\tau_4 + 21\tau_3^2 + 32\tau_4^2 + 4\tau_4\tau_5 - \frac{9}{2}\tau_3\tau_6 - 224\tau_1^2\tau_2\ ,\\
A_{37} &= -207360\tau_1 - 35392\tau_3 + 2080\tau_6 - 15104\tau_1\tau_2 - 3232\tau_1\tau_4 - 616\tau_1\tau_5 - 12\tau_1\tau_7\\
&\quad -2720\tau_2\tau_3 + 96\tau_2\tau_6 - 88\tau_3\tau_4 - 45\tau_3\tau_5 - 6\tau_3\tau_7 + 3\tau_5\tau_6 + 864\tau_1^3 + 216\tau_1^2\tau_3 + 320\tau_1\tau_2^2\\
&\quad -96\tau_1\tau_2\tau_4 + 15\tau_1\tau_3^2\ ,
\end{aligned}$$

$$
\begin{aligned}
A_{44} &= 24192 + 4896\tau_2 + 784\tau_4 + 32\tau_5 + 4\tau_7 - 864\tau_1^2 - 122\tau_1\tau_3 + 2\tau_1\tau_6 - 8\tau_2\tau_4 - 4\tau_4^2 \\
&\quad +20\tau_1^2\tau_2 \,, \\
A_{45} &= -145152 - 36864\tau_2 - 5920\tau_4 - 320\tau_5 - 52\tau_7 + 5184\tau_1^2 + 773\tau_1\tau_3 - 5\tau_1\tau_6 - 960\tau_2^2 \\
&\quad -192\tau_2\tau_4 + 7\tau_3^2 - 4\tau_4\tau_5 - 32\tau_1^2\tau_2 + 5\tau_1\tau_2\tau_3 \,, \\
A_{46} &= 62208\tau_1 + 8288\tau_3 - 596\tau_6 + 6256\tau_1\tau_2 + 1400\tau_1\tau_4 + 131\tau_1\tau_5 + 3\tau_1\tau_7 \\
&\quad +560\tau_2\tau_3 - 32\tau_2\tau_6 + 30\tau_3\tau_4 + 5\tau_3\tau_5 - 5\tau_4\tau_6 - 432\tau_1^3 - 51\tau_1^2\tau_3 - 80\tau_1\tau_2^2 + 16\tau_1\tau_2\tau_4 \,, \\
A_{47} &= 870912 + 200448\tau_2 + 48064\tau_4 + 4256\tau_5 + 184\tau_7 + 10368\tau_1^2 + 508\tau_1\tau_3 - 484\tau_1\tau_6 \\
&\quad -5760\tau_2^2 + 128\tau_2\tau_4 - 224\tau_2\tau_5 - 40\tau_2\tau_7 - 476\tau_3^2 + 4\tau_3\tau_6 + 176\tau_4^2 + 24\tau_4\tau_5 - 6\tau_4\tau_7 \\
&\quad +6\tau_5^2 + 1856\tau_1^2\tau_2 + 384\tau_1^2\tau_4 + 78\tau_1^2\tau_5 + 624\tau_1\tau_2\tau_3 - 4\tau_1\tau_2\tau_6 + 10\tau_1\tau_3\tau_4 + 4\tau_1\tau_3\tau_5 \\
&\quad -384\tau_2^3 - 96\tau_2^2\tau_4 + 6\tau_2\tau_3^2 - 64\tau_1^2\tau_2^2 \,, \\
A_{55} &= 774144 + 206208\tau_2 + 33952\tau_4 + 2192\tau_5 + 316\tau_7 - 25056\tau_1^2 - 4304\tau_1\tau_3 + 8\tau_1\tau_6 \\
&\quad +5760\tau_2^2 + 784\tau_2\tau_4 - 152\tau_2\tau_5 + 2\tau_2\tau_7 - 126\tau_3^2 + 4\tau_3\tau_6 - 64\tau_4^2 - 24\tau_4\tau_5 - 6\tau_5^2 \\
&\quad -208\tau_1^2\tau_2 - 36\tau_1\tau_2\tau_3 + 96\tau_2^3 + 16\tau_2^2\tau_4 \,, \\
A_{56} &= -283392\tau_1 - 23576\tau_3 + 1736\tau_6 - 48640\tau_1\tau_2 - 8912\tau_1\tau_4 - 716\tau_1\tau_5 - 42\tau_1\tau_7 - 2436\tau_2\tau_3 \\
&\quad +140\tau_2\tau_6 - 72\tau_3\tau_4 + 4\tau_3\tau_5 - 6\tau_5\tau_6 + 3888\tau_1^3 + 528\tau_1^2\tau_3 - 176\tau_1\tau_2\tau_4 + 6\tau_1\tau_3^2 + 4\tau_2\tau_3\tau_4 \,, \\
A_{57} &= 387072 + 152064\tau_2 - 62848\tau_4 - 5696\tau_5 + 560\tau_7 - 65664\tau_1^2 + 6344\tau_1\tau_3 + 184\tau_1\tau_6 \\
&\quad +15360\tau_2^2 - 17472\tau_2\tau_4 - 1184\tau_2\tau_5 + 56\tau_2\tau_7 + 3087\tau_3^2 - 240\tau_3\tau_6 - 3840\tau_4^2 - 704\tau_4\tau_5 \\
&\quad -32\tau_4\tau_7 - 64\tau_5^2 - 8\tau_5\tau_7 + 5\tau_6^2 - 14144\tau_1^2\tau_2 + 192\tau_1^2\tau_4 - 312\tau_1^2\tau_5 - 2976\tau_1\tau_2\tau_3 + 104\tau_1\tau_2\tau_6 \\
&\quad +320\tau_1\tau_3\tau_4 - \tau_1\tau_3\tau_5 - 384\tau_2^3 - 576\tau_2^2\tau_4 - 9\tau_2\tau_3^2 + 3\tau_2\tau_3\tau_6 - 128\tau_2\tau_4^2 + 5\tau_3^2\tau_4 + 576\tau_1^2\tau_2^2 \,, \\
A_{66} &= -193536 - 108480\tau_2 + 9584\tau_4 - 2744\tau_5 - 166\tau_7 + 66096\tau_1^2 + 13492\tau_1\tau_3 - 376\tau_1\tau_6 \\
&\quad -13344\tau_2^2 + 4048\tau_2\tau_4 - 224\tau_2\tau_5 - 40\tau_2\tau_7 - 70\tau_3^2 + 78\tau_3\tau_6 + 1520\tau_4^2 + 224\tau_4\tau_5 \\
&\quad +2\tau_4\tau_7 + 6\tau_5^2 - \frac{15}{2}\tau_6^2 + 512\tau_1^2\tau_2 - 208\tau_1^2\tau_4 + 8\tau_1^2\tau_5 + 824\tau_1\tau_2\tau_3 - 24\tau_1\tau_2\tau_6 \\
&\quad -36\tau_1\tau_3\tau_4 + 4\tau_1\tau_3\tau_5 - 384\tau_2^3 - 96\tau_2^2\tau_4 + 6\tau_2\tau_3^2 + 16\tau_2\tau_4^2 - 64\tau_1^2\tau_2^2 \,, \\
A_{67} &= -276480\tau_1 - 166432\tau_3 + 11008\tau_6 + 71680\tau_1\tau_2 + 60704\tau_1\tau_4 + 3752\tau_1\tau_5 + 588\tau_1\tau_7 \\
&\quad -19360\tau_2\tau_3 + 992\tau_2\tau_6 + 5272\tau_3\tau_4 + 153\tau_3\tau_5 + 57\tau_3\tau_7 - 320\tau_4\tau_6 + 13\tau_5\tau_6 - 9\tau_6\tau_7 \\
&\quad +7776\tau_1^3 + 1800\tau_1^2\tau_3 - 312\tau_1^2\tau_6 - 11072\tau_1\tau_2^2 - 608\tau_1\tau_2\tau_4 - 288\tau_1\tau_2\tau_5 - 32\tau_1\tau_2\tau_7 \\
&\quad -74\tau_1\tau_3^2 - \tau_1\tau_3\tau_6 + 576\tau_1\tau_4^2 + 104\tau_1\tau_4\tau_5 + 5\tau_1\tau_5^2 - 1968\tau_2^2\tau_3 + 80\tau_2^2\tau_6 - 64\tau_2\tau_3\tau_4 \\
&\quad -20\tau_2\tau_3\tau_5 + 7\tau_3^3 + 3\tau_3\tau_4\tau_5 + 192\tau_1^3\tau_2 + 320\tau_1^2\tau_2\tau_3 - 128\tau_1\tau_2^2\tau_4 + 5\tau_1\tau_2\tau_3^2 \,,
\end{aligned}
$$

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\begin{aligned}
A_{77} \;=\; & 17031168 + 13400064\tau_2 + 1549312\tau_4 - 139264\tau_5 - 8960\tau_7 - 525312\tau_1^2 + 197824\tau_1\tau_3 \\
& +5312\tau_1\tau_6 + 2881536\tau_2^2 + 677888\tau_2\tau_4 - 16896\tau_2\tau_5 - 768\tau_2\tau_7 + 41496\tau_3^2 - 1216\tau_3\tau_6 \\
& +44416\tau_4^2 - 6464\tau_4\tau_5 - 560\tau_4\tau_7 - 1120\tau_5^2 - 128\tau_5\tau_7 - 88\tau_6^2 - 12\tau_7^2 - 270592\tau_1^2\tau_2 \\
& -11904\tau_1^2\tau_4 + 1056\tau_1^2\tau_5 + 288\tau_1^2\tau_7 - 4608\tau_1\tau_2\tau_3 - 1216\tau_1\tau_2\tau_6 + 4736\tau_1\tau_3\tau_4 \\
& +860\tau_1\tau_3\tau_5 + 36\tau_1\tau_3\tau_7 - 96\tau_1\tau_4\tau_6 + 84\tau_1\tau_5\tau_6 + 150528\tau_2^3 + 35072\tau_2^2\tau_4 - 384\tau_2^2\tau_5 \\
& +96\tau_2^2\tau_7 + 4492\tau_2\tau_3^2 - 320\tau_2\tau_3\tau_6 + 2368\tau_2\tau_4^2 - 480\tau_2\tau_4\tau_5 - 24\tau_2\tau_4\tau_7 - 24\tau_2\tau_5^2 \\
& +4\tau_2\tau_6^2 + 288\tau_3^2\tau_4 + 70\tau_3^2\tau_5 - 16\tau_3\tau_4\tau_6 + 2\tau_3\tau_5\tau_6 + 256\tau_4^3 + 32\tau_4^2\tau_5 + 4\tau_4\tau_5^2 \\
& +10368\tau_1^4 + 2592\tau_1^3\tau_3 - 23808\tau_1^2\tau_2^2 - 2112\tau_1^2\tau_2\tau_4 - 96\tau_1^2\tau_2\tau_5 + 216\tau_1^2\tau_3^2 \\
& -3680\tau_1\tau_2^2\tau_3 + 32\tau_1\tau_2^2\tau_6 + 80\tau_1\tau_2\tau_3\tau_4 - 16\tau_1\tau_2\tau_3\tau_5 + 6\tau_1\tau_3^3 + 1536\tau_2^4 + 128\tau_2^3\tau_4 \\
& -24\tau_2^2\tau_3^2 - 64\tau_2^2\tau_4^2 + 4\tau_2\tau_3^2\tau_4 + 256\tau_1^2\tau_2^3 \ ,
\end{aligned}
$$

and

$$
B_1 = -\,\frac{3}{2}(1-9\nu)\tau_1\ ,\ B_2 = 126\nu - (2-17\nu)\tau_2\ ,\ B_3 = 72\nu\tau_1 - \frac{7}{2}(1-7\nu)\tau_3\ ,
$$

$$
B_4 = 60\nu\tau_2 - 2(2-13\nu)\tau_4\ ,\ B_5 = 96\nu\tau_2 + 40\nu\tau_4 - 3(2-11\nu)\tau_5\ ,
$$

$$
B_6 = -\,1080\nu\tau_1 - 175\nu\tau_3 - \frac{15}{2}(1-5\nu)\tau_6 + 40\nu\tau_1\tau_2\ ,
$$

$$
B_7 \ = 36288\nu + 9024\nu\tau_2 + 1952\nu\tau_4 + 204\nu\tau_5 - 12(1-4\nu)\tau_7 - 648\nu\tau_1^2 - 84\nu\tau_1\tau_3 + 24\nu\tau_2\tau_4\ .
$$

It is worth mentioning that A_{ii} contains the term $-d_i^2\tau_i^2$ (see [4]) where d_i^2 is the square of the ith fundamental weight length. Also for the coefficient $B_i(\nu=0) = -d_i^2\tau_i$ [4]. The last observation leads to quite interesting consequence.

The Laplacian written in FTI variables $h_{E_7}(\nu=0)$ has infinitely many polynomial eigenfunctions φ (see a discussion below). The lowest polynomial eigenfunction $\varphi_0 = \text{const}$ with the eigenvalue $\epsilon_0 = 0$. There are seven eigenfunctions in a form of linear function in τ,

$$
\varphi_{1,2,\ldots 7} = \tau_{1,2,\ldots 7}\ ,\ \epsilon_{1,2,\ldots 7} = -d^2_{1,2,\ldots 7}\ .
$$

Thus, the variables τ's leading to the algebraic form of the Hamiltonian (5) are nothing but the seven eigenfunctions of the Laplacian next after the ground state! Their eigenvalues are square of the fundamental vector lengths.

After some analysis, one finds that the operator (13) preserves the infinite flag $P^{(7)}_{\{1,2,2,2,3,3,4\}}$. Its characteristic vector $\vec{\alpha} = (1, 2, 2, 2, 3, 3, 4)$ coincides with the minimal characteristic vector for the corresponding rational model [5]. This confirms the conjecture from I that the characteristic vector for a trigonometric model always coincides with the minimal characteristic vector for the corresponding rational model. The characteristic vector $(1, 2, 2, 2, 3, 3, 4)$ coincides with the highest root in the E_7 root system written in the basis of simple roots (Kac conjecture, see [5]). It can be checked that the flag is invariant wrt a weighted-projective transformation

$$
\begin{aligned}
&\tau_1 \to \tau_1\ , \\
&\tau_2 \to \tau_2 + a_2\tau_1^2 + b_{2,1}\tau_3 + b_{2,2}\tau_4\ , \\
&\tau_3 \to \tau_3 + a_3\tau_1^2 + b_{3,1}\tau_2 + b_{3,2}\tau_4\ , \\
&\tau_4 \to \tau_4 + a_4\tau_1^2 + b_{4,1}\tau_2 + b_{4,2}\tau_3\ , \\
&\tau_5 \to \tau_5 + a_{12}\tau_1^3 + b_{5,1}\tau_1\tau_2 + b_{5,2}\tau_1\tau_3 + b_{5,3}\tau_1\tau_4 + c_5\tau_6\ , \\
&\tau_6 \to \tau_6 + a_6\tau_1^3 + b_{6,1}\tau_1\tau_2 + b_{6,2}\tau_1\tau_3 + b_{6,3}\tau_1\tau_4 + c_6\tau_5\ , \\
&\tau_7 \to \tau_7 + a_{18}\tau_1^4 + b_{7,1}\tau_1^2\tau_2 + b_{7,2}\tau_1^2\tau_3 + b_{7,3}\tau_1^2\tau_4 + c_{7,1}\tau_1\tau_5 + c_{7,2}\tau_1\tau_6 \\
&\quad + d_{7,1}\tau_2^2 + d_{7,2}\tau_3^2 + d_{7,3}\tau_4^2 + d_{7,4}\tau_2\tau_3 + d_{7,4}\tau_2\tau_4 + d_{7,4}\tau_3\tau_4\ ,
\end{aligned}
\tag{14}
$$

where $\{a, b, c, d\}$ are arbitrary real numbers. It is a hidden invariance of the Hamiltonian (5). It is seen in a clear way in the space of orbits only.

The E_7 model depends on the parameter ν, and the nodal structure of eigenpolynomials (i.e. where they vanish) at fixed ν remains an open question.

III. CONCLUSIONS

Weyl-invariant coordinates leading to the algebraic forms of the trigonometric Olshanetsky-Perelomov Hamiltonians associated to the crystallographic root spaces A_N, BC_N, G_2, F_4, E_6 were found in I. In this paper, we have shown that the fundamental trigonometric invariants (FTI), if used as coordinates, provide a way of reducing the trigonometric Hamiltonian associated to E_7 to algebraic form. The eigenfunctions of the trigonometric Hamiltonian E_7 (i.e., the Jack polynomials) remain polynomials in the FTI. The use of FTI enabled us to find an algebraic form of the Hamiltonian associated to E_7, which did not seem feasible at all, in the past. The calculations in this paper were based on a straightforward change of variables from Cartesian coordinates to FTI. Although there are clear indications of the existence of a representation-theoretic formalism that may allow such results to be derived differently [6–9] our results were obtained very fast without any technical difficulties.

So far each of the Olshanetsky-Perelomov Hamiltonians, in algebraic form, preserves an infinite flag of polynomial spaces, with a characteristic vector $\vec{\alpha}$ that coincides with the minimal characteristic vector for the corresponding rational model (see [5]). The present study of the E_7 case confirms this correspondence. It is worth noting that the matrices A_{ij} in the algebraic-form Hamiltonians, in particular, given explicitly in Eqs. (13), with polynomial entries, correspond to flat-space metrics, in the sense that the associated Riemann tensor vanishes. The change of variables in the corresponding Laplace-Beltrami operator, from FTI to Cartesian coordinates, transforms these metrics to diagonal form. This procedure provides a set of non-trivial metrics with polynomial entries with vanishing Riemann tensor.

It should be stressed that each Hamiltonian of the form (1) is completely integrable. This implies the existence of a number of operators (the 'higher Hamiltonians') which commute with it and which are in involution forming a commutative algebra. It is evident that these commuting operators take on an algebraic form after a gauge rotation (with the corresponding ground state eigenfunction as a gauge factor), and a change of variables from Cartesian coordinates to the FTI, i.e., to the τ's. It is a consequence of the fact that all commuting operators preserve the same flag of polynomials (for a discussion, see [3]).

An analysis similar to the analysis of this paper has not yet been presented for the case of the trigonometric Olshanetsky-Perelomov Hamiltonians related to the exceptional root space E_8. We conjecture that in this case as well, the FTI taken as coordinates will yield an algebraic form for the Hamiltonian, and that the infinite flag of polynomial spaces with the same characteristic vector as the minimal [12] characteristic vector in the corresponding rational model will be preserved. In concluding, we mention that the existence of algebraic form of the E_7 trigonometric Olshanetsky-Perelomov Hamiltonian makes possible the study of their perturbations by purely algebraic means: one can develop a perturbation theory in which all corrections are found by linear-algebraic methods [10]. It also gives a hint that quasi-exactly solvable generalizations of the E_7 trigonometric Olshanetsky-Perelomov Hamiltonian may exist.

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[11] We do not think that this notation will cause a confusion with positive roots.

[12] We have doubts whether the characteristic vector found in [5] for the E_8-rational model is minimal