HOMOTOPY GRAPH-COMPLEX FOR CONFIGURATION AND KNOT SPACES

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Abstract. In the paper we prove that the primitive part of the Sinha homology spectral sequence E^2-term for the space of long knots is rationally isomorphic to the homotopy $\mathcal{E}^2$-term. We also define natural graph-complexes computing the rational homotopy of configuration and of knot spaces.

1. Introduction

In [27, 28] D. Sinha defined a cosimplicial model for the space Emb of long knots $\mathbb{R} \hookrightarrow \mathbb{R}^d$, $d \geq 4$. It was proven in [3, 21] that the associated homology Bousfield-Kan spectral sequence collapses rationally at the second term. The same result was established for the associated homotopy spectral sequence (over $\mathbb{Q}$). The proof will appear in [2]. But Emb is an H-space with a homotopy commutative product[1]. It implies in particular that the $\mathcal{E}^2$ term of the (co)homotopy spectral sequence must be rationally isomorphic to the primitive part of the E^2 (co)homology term.

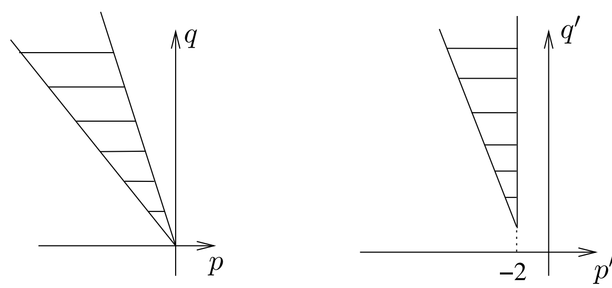

Figure 1. Homology $E^2_{p,q}$ and homotopy $\mathcal{E}^2_{p',q'}$ terms.

The (co)homology $E^2_{p,q}$-term is concentrated in the second quadrant between two lines [31]:

$$\begin{array}{lll} q = & -\frac{d-1}{2}p & \text{(lower line)} \\ q = & -(d-1)(p+1) & \text{(upper line)} \end{array}$$

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[1]It was shown recently by P. Salvatore that Emb is a double loop space [25].

The (co)homotopy $\mathcal{E}^2_{p',q'}$-term is also concentrated in the second quadrant and bounded by the lines [26]:

$$\begin{array}{rll} q' = & -(d-2)p' - d + 3 & \text{(lower line)} \\ p' = & -2 & \text{(right line)} \end{array}$$

In Part 1 of the paper we will give a simple and purely algebraic proof of this isomorphism:

$$\mathcal{E}^2_{*,*} = Prim(E^2_{*,*}).$$

In particular we will see in Section 4 how via this isomorphism the bigradings of both spectral sequences are related to each other. We will see that the lower line of the homotopy spectral sequence corresponds to the lower line of the homology one. The right line $p' = -2$ of $\mathcal{E}^2$ corresponds to the upper line $q = -(d-1)(p+1)$ of E^2. In general any vertical line $p' = -n$ corresponds to $q = -(d-1)(p+n-1)$.

This isomorphism in the case of the lower lines (on the level of the bialgebra of chord diagrams) was proved by J. Conant [9]. He gives an elegant reformulation of his result using 3-valent graphs.

Part 2, which actually gave the name to the paper, is devoted to graph-complexes. Our motivation was to produce new graph-complexes whose homology has a nice geometrical interpretation. We define a series of graph-complexes that compute the rational homotopy of configuration spaces. Building up on this series of complexes we define a bigger complex whose homology is the rational homotopy of the space of long knots. A more thorough introduction for Part 2 is Section 7.

Part 1. Isomorphism $\mathcal{E}^2 = Prim(E^2)$

2. Cosimplicial model for the space of long knots modulo immersions

The space $\overline{Emb}$ of long knots modulo immersions is the homotopy fiber of the inclusion

$$Emb \hookrightarrow Imm$$

of the space of long knots Emb in the space of long immersions Imm. By the word "long" we understand smooth map $\mathbb{R}^1 \to \mathbb{R}^d$ that coincide with a fixed linear map outside a compact subset of $\mathbb{R}^1$. (We will deliberately omit d to simplify notation, assuming that the dimension $d \geq 4$ is fixed once and forever.[2])

D. Sinha showed in [28] that $\overline{Emb}$ is homotopy equivalent to $Emb \times \Omega Imm \simeq Emb \times \Omega^2 S^{d-1}$. So, the homology and homotopy of Emb are easily related to those of $\overline{Emb}$ and the results for $\overline{Emb}$ that we obtain in the first part of the paper can be obviously reestablished for Emb.

In [27] D. Sinha defined a cosimplicial space whose homotopy totalization is $\overline{Emb}$. The n-th component C^n of the cosimplicial space is some compactification of the configuration space of points in $I \times \mathbb{R}^{d-1} = [0,1] \times \mathbb{R}^{d-1}$:

$$\left\{ (x_0, x_1, \ldots, x_{n+1}) \;\middle|\; \begin{array}{ll} x_i \in I \times \mathbb{R}^{d-1}; & x_i \neq x_j \\ x_0 = (0, \bar{0}); & x_{n+1} = (1, \bar{0}) \end{array} \right\}$$

Given $0 \leq i \leq n+1$, the coface map d^i is doubling of the i-th point x_i of the configuration in the direction $(1, \bar{0})$. The codegeneracy map s^i, for $i = 1 \ldots n$, is given by forgetting of x_i.

In [28] D. Sinha provides another "operadic" construction for the cosimplicial replacement of $\overline{Emb}$ (this cosimplicial space is homotopy equivalent to the previous one).

[2] When we consider configuration spaces we assume $d \geq 3$.

For any symmetric monoidal category $(\mathcal{C},\otimes,\mathbf{1})$ denote by $\mathcal{ASS}=\{\mathbf{1}\}_{n\geq 0}$ the associative non-Σ operad.[3]

Provided an operad $\mathcal{O}$ in $(\mathcal{C},\otimes,\mathbf{1})$ is endowed with a morphism $\mathcal{ASS}\to\mathcal{O}$, the collection $\{\mathcal{O}^n\}_{n\geq 0}=\{\mathcal{O}(n)\}_{n\geq 0}$ becomes a cosimplicial object in this category. Cofaces $d^i\colon \mathcal{O}^n\to\mathcal{O}^{n+1}$, $i=0\dots n+1$, are compositions with $m=\mathbf{1}=\mathcal{ASS}(2)$:

$$d^0(-)=m\circ_2 -;\quad d^i(-)=-\circ_i m,\ i=1\dots n;\quad d^{n+1}(-)=m\circ_1 -.$$

Codegeneracies are compositions with $e=\mathbf{1}=\mathcal{ASS}(0)$

$$s^i(-)=-\circ_i e,\ i=1\dots n.$$

Sinha applies this standard construction to an operad $\{C\langle n\rangle\}_{n\geq 0}$ homotopy equivalent to the operad of little d-cubes. Each space $C^n=C\langle n\rangle$ of this operad is the compactification in $(S^{d-1})^{\binom{n}{2}}$ of the space of reciprocal directions of n distinct points in $\mathbb{R}^d$:

$$\left\{\left(\frac{x_j-x_i}{|x_j-x_i|}\right)_{1\leq i<j\leq n}\,\middle|\,\begin{array}{c}x_i\in\mathbb{R}^d;\\ x_i\neq x_j\end{array}\right\}\subset (S^{d-1})^{\binom{n}{2}}.$$

We will assume that we work with one of these cosimplicial models. The operadic and cosimplicial structures of $C^\bullet$ induce similar structures on the (co)homology and (co)homotopy of $C^\bullet$. The cohomology simplicial algebra will be denoted by

$$A_\bullet=\{A_n\}_{n\geq 0}=\{H^*(C^n)\}_{n\geq 0}.$$

We will consider the cohomology only with rational coefficients.

The rational cohomotopy simplicial Lie coalgebra will be denoted by

$$L_\bullet=\{L_n\}_{n\geq 0}=\{Mor(\pi_*(C^n),\mathbb{Q})\}_{n\geq 0}.$$

We will also work with the rational homotopy cosimplicial Lie algebra

$$L^\bullet=\{L^n\}_{n\geq 0}=\{\pi_*(C^n)\otimes\mathbb{Q}\}_{n\geq 0}.$$

3. Explicit description of $A_n=H^*(C^n)$ and of $L^n=\pi_*(C^n)\otimes\mathbb{Q}$

The algebras A_n, $n\geq 0$, are well known [1, 7]. Being graded commutative they are generated by a_{ij}, $1\leq i\neq j\leq n$, of degree $d-1$, that satisfy the relations:

$$\begin{array}{c}\text{quadratic}\\ \text{relations}\end{array}\ \begin{cases} a_{ji}=(-1)^d a_{ij}\\ a_{ij}^2=0\\ a_{ij}a_{jk}+a_{jk}a_{ki}+a_{ki}a_{ij}=0\end{cases}\tag{3.1}$$

We assume that the component $C\langle 0\rangle$ is a point, so $A_0=\mathbb{Q}$.

Any monomial can be viewed as a directed graph on the set $\{1,2,\dots,n\}$: the directed edge (i,j) is put exactly the number of times the generator a_{ij} is represented in the monomial, see Figure 2.

It can be easily seen that a monomial in A_n is non-zero if and only if the corresponding graph is a forest. On such a graph Γ the face map $d_0(\Gamma)$ is non-zero if and only if the valence of vertex 1 is zero. In this case d_0 simply removes the vertex 1, all other vertices are shifted by 1. Face $d_i(\Gamma)$, $i=1\dots n-1$, is obtained by collapsing the segment $[i,i+1]$. If Γ contains the edge $(i,i+1)$, then $d_i(\Gamma)=0$. Face d_n acts similarly to d_0 removing the last n-th vertex.

[3] Dually we will denote by $\mathcal{COASS}=\{\mathbf{1}\}_{n\geq 0}$ the associative non-Σ cooperad in $(\mathcal{C},\otimes,\mathbf{1})$.

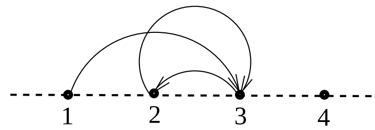


FIGURE 2. Graph corresponding to $a_{13}a_{32}a_{23} \in A_4$.

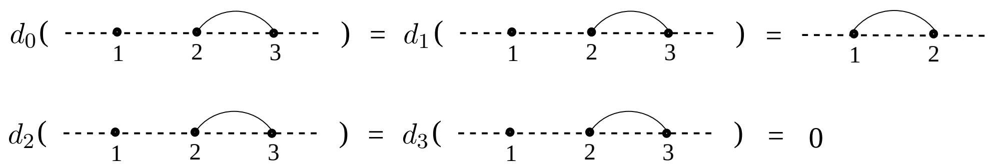


FIGURE 3

The degeneracy map s_i, for $i = 1 \dots n+1$, inserts a new vertex between $i-1$ and i.

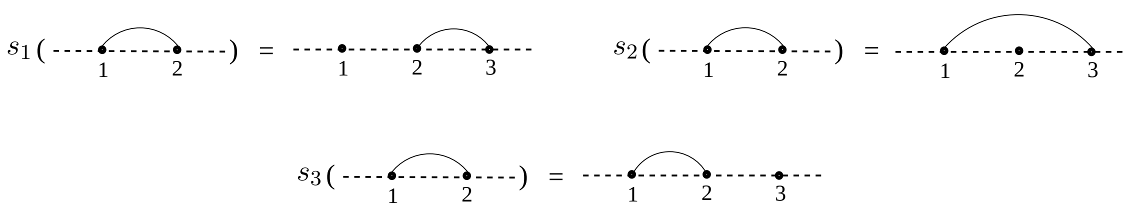


FIGURE 4

The normalized part NA_n of A_n

$$NA_n = A_n / +_{i=1}^{n} \operatorname{Im} s^i$$

is spanned by the forests with each vertex $1, \dots, n$ of positive valence.

The first term $E_1 = (\oplus_{p=0}^{\infty} E_1^{-p,*}, d)$ of the cohomology Sinha spectral sequence is the normalized complex $\operatorname{Tot} A_\bullet = (\oplus_{p=0}^{\infty} s^{-p} NA_p, d)$, where s^{-p} denotes p-fold desuspension. The differential is as usual the alternated sum of faces d_i.

The Lie algebra L^n, $n \geq 0$, is generated by α_{ij}, $1 \leq i \neq j \leq n$, of degree $d-1$ [16]. The relations are

$$\alpha_{ji} = (-1)^d \alpha_{ij}$$
$$\begin{array}{c}\text{quadratic}\\ \text{relations}\end{array} \left\{ \begin{array}{ll} [\alpha_{ij}, \alpha_{kl}] = 0, & \text{if } \#\{i,j,k,l\} = 4 \\ [\alpha_{ij}, \alpha_{jk} + \alpha_{ki}] = 0 & \end{array} \right. \tag{3.2}$$

The bracket in L^n is the Whitehead bracket which is of degree -1.

It is well known that A_n and L^n are Koszul dual [8]. This means that the $\binom{n}{2}$-dimensional space V_n of generators of A_n is dual to the space V^n of generators of L^n. And the space $R_n \subset S^2 V_n$ spanned by the quadratic relations (3.1) is orthogonal to the space $R^n \subset S^2 V^n$ of quadratic relations of L^n. Moreover A_n and L^n are Koszul which means some nice homological property of their bar-constructions. This property will be used in the proof of our main result Theorem 3.1.

The cofaces $d^k\colon L^n \to L^{n+1}$, for $k = 0 \dots n+1$; and the codegeneracies $s^k\colon L^n \to L^{n-1}$, for $k = 1 \dots n$, are defined on generators as follows:

$$d^k(\alpha_{ij}) = \begin{cases} \alpha_{ij}, & \text{if } i < j < k; \\ \alpha_{ij} + \alpha_{i,j+1}, & \text{if } i < j = k; \\ \alpha_{i,j+1}, & \text{if } i < k < j; \\ \alpha_{i,j+1} + \alpha_{i+1,j+1}, & \text{if } i = k < j; \\ \alpha_{i+1,j+1}, & \text{if } k < i < j. \end{cases} \qquad s^k(\alpha_{ij}) = \begin{cases} \alpha_{ij}, & \text{if } i < j < k; \\ 0, & \text{if } i < j = k; \\ \alpha_{i,j-1}, & \text{if } i < k < j; \\ 0, & \text{if } i = k < j; \\ \alpha_{i-1,j-1}, & \text{if } k < i < j. \end{cases}$$

In particular $d^0(\alpha_{ij}) = \alpha_{i+1,j+1}$, $d^{n+1}(\alpha_{ij}) = \alpha_{ij}$.

The normalized part

$$NL^n = \bigcap_{i=1}^{n} \ker s^i \subset L^n$$

is spanned by the monomials that use each index $i = 1 \dots n$. The space NL^n is isomorphic to a subspace of a graded free Lie algebra generated by $x_1 = \alpha_{12}$, $x_2 = \alpha_{13}$, ..., $x_{n-1} = \alpha_{1,n-1}$ spanned by the monomials using each x_i, $1 \le i \le n-1$.

The first term $\mathcal{E}^1 = (\oplus_{p=0}^{\infty} \mathcal{E}^1_{-p,*}, d)$ of the homotopy Sinha spectral sequence is the normalized complex $\operatorname{Tot} L^\bullet = (\oplus_{p=0}^{\infty} s^{-p} NL^p, d)$.

Here is our main result.

Theorem 3.1. *(i) The $\mathcal{E}^2$ term of the homotopy Sinha spectral sequence (for $\overline{Emb}$) is rationally isomorphic to the primitive part of the homology E^2 term.*

(ii) The $\mathcal{E}_2$ term of the cohomotopy Sinha spectral seqence (for $\overline{Emb}$) is rationally isomorphic to the primitive part of the cohomology E_2-term.

Since E^2-homology term is a polynomial bialgebra[4], assertions (i) and (ii) are equivalent. So, we will prove only (ii). The proof will be given in Section 6.

4. Correspondence of bigradings

In this section we describe how the homotopy spectral sequence bigradings (p', q') are related to the homology spectral sequence bigradings (p, q) via the isomorphism of Theorem 3.1.

We will give an heuristic explanation of this correspondence. But one can easily establish it by a simple analysis of the proof given in Section 6.

A monomial of degree i in NA_j is an element of $E_1^{p,q}$ with $p = -j$, $q = (d-1)i$. A monomial of degree i' in $NL^{j'}$ is an element of $\mathcal{E}^1_{p',q'}$ with $p' = -j'$, $q' = (d-1)i' - (i'-1) = (d-2)i' + 1$ (recall that the bracket in $L^\bullet$ is of degree -1).

The degree i, i' in both cases will be called *complexity*. The complexity is preserved by the differential.

Up to a shift of grading the complexes $\operatorname{Tot} A_\bullet$, $\operatorname{Tot} L^\bullet$ depend only on the parity of d. So it is natural to expect that the isomorphism of Theorem 3.1 respects this periodicity and therefore preserves the complexity. The total grading $p+q$, $p'+q'$ must be also unchanged.

[4]This is true for any field of coefficients [32, Corollary 13.4].

Let us find the bigrading $(i',j')=(i,j')$ of $\mathrm{Tot}\, L^\bullet$ that should correspond to the bigrading (i,j) of $\mathrm{Tot}\, A_\bullet$. We have

$$\begin{array}{rcl} p+q & = & (d-1)i-j \\ p'+q' & = & (d-2)i+1-j' \end{array}$$

So $p+q=p'+q'$ implies

$$j'=j-i+1. \tag{4.1}$$

For example if $j=2i$ (the case of the lower line in E_2, which corresponds to the bialgebra of chord diagrams), one has $j'=i+1$. This corresponds to the lower line in $\mathcal{E}_2$.

The case $j=i+1$ (upper line in E_2) produces $j'=2$ (right line in $\mathcal{E}_2$). This situation produces exactly the homotopy of the factor $\Omega^2 S^{d-1}$ of $\overline{Emb}$ [31].

In general for a non-trivial monomial of degree i in NA_j the number $j-i$ is the number of connected components in the corresponding forest, see Section 3. So, the correspondence 4.1 can be resumed as follows: the number of connected components of forests in the cohomological case corresponds to the number of points (of configuration spaces) minus 1 in the homotopy case.

5. Fixing notations

In this section we review some of necessary background and fix some notation.

5.1. B/B^2. By CDGA we understand the category of graded connected differential graded algebras with differential raising the degree by 1. Almost all algebras we deal with are 1-connected, *i.e.* their 1-degree component is trivial.

Consider a functor from CDGA to the category of differential graded vector spaces (complexes):

$$\begin{array}{rccc} P\colon & \mathrm{CDGA} & \longrightarrow & dg-\mathrm{Vect} \\ & B & \longmapsto & B_{>0}/(B_{>0})^2. \end{array}$$

For simplicity of notation $P(B)=B_{>0}/(B_{>0})^2$ will be denoted by B/B^2.

5.2. $\mathcal{L}(B)$. By $\mathcal{L}$ we denote the cobar construction

$$\mathcal{L}\colon \mathrm{CDGA}\to dg-\mathrm{coLie},$$

which assigns to any commutative dg-algebra B a free dg-Lie coalgebra with cobracket of degree 1:

$$\mathcal{L}(B)=\bigoplus_{n\geq 1}(\mathrm{coLie}(n)\otimes (B_{>0})^{\otimes n})_{S_n},$$

whose differential is a sum of two things — one arising from the initial differential of B, the other — from multiplication in B. A nice explicit description of this construction is given in [29]. Notice that in our construction the degree of each space $\mathrm{coLie}(n)$ is $1-n$.

One has a natural transformation

$$\begin{array}{rccc} \alpha\colon & \mathcal{L} & \longrightarrow & P \\ & \mathcal{L}(B) & \xrightarrow{\alpha_B} & B/B^2 \end{array}$$

which is a morphism of complexes sending $\bigoplus_{n\geq 2}(\mathrm{coLie}(n)\times (B_{>0})^{\otimes n})_{S_n}$ to zero and $\mathrm{coLie}(1)\otimes B_{>0}=B_{>0}$ to the quotient $B_{>0}/(B_{>0})^2$.

The following is a standard result in the rational homotopy theory [13].

Proposition 5.1. *If B is a polynomial algebra then the map $\mathcal{L}(B)\xrightarrow{\alpha} B/B^2$ is a quasi-isomorphism.*

5.3. **Totalization.** Let $V_\bullet$ be a simplicial dg-vector space. If not stated otherwise we always assume that the differential raises the degree by 1. We define $\mathrm{Tot}\, V_\bullet$ as a complex whose space is $\oplus_{n\geq 0} s^{-n} N V_n$ and the differential is the sum of inner differential of each V_n plus the alternated sum of faces. Notice that $\mathrm{Tot}\, V_\bullet$ might be negatively graded, however in all the considered cases the totalization always produces positively graded complexes.

Let $V_\bullet$ and $W_\bullet$ be two simplicial dg-spaces. Assume that $\mathrm{Tot}\, V_\bullet$ and $\mathrm{Tot}\, W_\bullet$ are left-bounded. One has the Eilenberg-MacLane quasi-isomorphism [23, § 29]:

$$\mathrm{Tot}\, V_\bullet \otimes \mathrm{Tot}\, W_\bullet \xrightarrow{EM} \mathrm{Tot}(V_\bullet \otimes W_\bullet).$$

This map permits to define a product on the totalization of any simplicial commutative dg-algebra $B_\bullet$:

$$\mathrm{Tot}\, B_\bullet \otimes \mathrm{Tot}\, B_\bullet \xrightarrow{EM} \mathrm{Tot}(B_\bullet \otimes B_\bullet) \xrightarrow{\mu_\bullet} \mathrm{Tot}\, B_\bullet.$$

The Eilenberg-MacLane map has nice properties. It is associative and S_n-equivariant. This implies that $\mathrm{Tot}\, B_\bullet$ is a commutative dg-algebra. For example, for the simplicial algebra $A_\bullet = H^*(C^\bullet)$ the product on $\mathcal{A} = \mathrm{Tot}\, A_\bullet$ is the shuffle of diagrams:

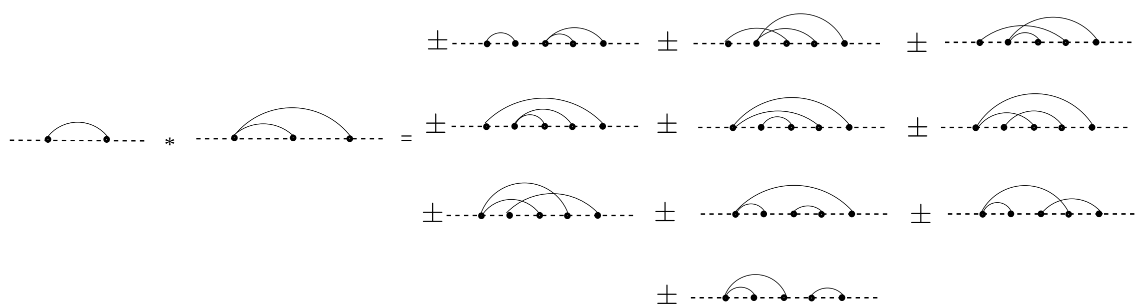


FIGURE 5. The product in $\mathcal{A}$.

Consider any polynomial functor

$$\begin{array}{cccc} \mathcal{P}\colon & dg-\mathrm{Vect} & \longrightarrow & dg-\mathrm{Vect} \\ & V & \longmapsto & \oplus_{n\geq 0}(\mathcal{P}(n)\otimes V^{\otimes n})_{S_n}. \end{array}$$

The property that Eilenberg-MacLane map is associative and S_n-equivariant permits to define a morphism:

$$\mathcal{P}(\mathrm{Tot}\, V_\bullet) \xrightarrow{EM_{\mathcal{P}}} \mathrm{Tot}\, \mathcal{P}(V_\bullet) \tag{5.1}$$

Lemma 5.2. *For a field of characteristic zero, the morphism* (5.1) *is always a quasi-isomorphism for any simplicial dg-vector space* $V_\bullet$ *(provided* $\mathrm{Tot}\, V_\bullet$ *is left-bounded).*

Proof. One has

$$(\mathrm{Tot}\, V_\bullet)^{\otimes n} \xrightarrow{EM_n} \mathrm{Tot}(V_\bullet^{\otimes n})$$

is an S_n-equivariant quasi-isomorphism. In characteristic zero it implies that

$$(\mathcal{P}(n)\otimes(\mathrm{Tot}\, V_\bullet)^{\otimes n})_{S_n} \xrightarrow{EM_{\mathcal{P}(n)}} \mathrm{Tot}((\mathcal{P}(n)\otimes V_\bullet^{\otimes n})_{S_n})$$

is also a quasi-isomorphism. $\square$

Lemma 5.3. *For any simplicial commutative dg-algebra $B_\bullet$ (provided $\mathrm{Tot}\, B_\bullet$ is positively graded) the map $EM_{\mathcal{L}}$ is a quasi-isomorphism:*

$$\mathcal{L}(\mathrm{Tot}\, B_\bullet) \xrightarrow[EM_{\mathcal{L}}]{\simeq} \mathrm{Tot}\, \mathcal{L}(B_\bullet).$$

Proof. First one has to check that $EM_{\mathcal{L}}$ is a morphism of complexes. This is so because the product part of the differential in $\mathcal{L}(\mathrm{Tot}\, B_\bullet)$ goes exactly to the product part of the differential in $\mathrm{Tot}\, \mathcal{L}(B_\bullet)$ (here one uses the fact that the product in $\mathrm{Tot}\, B_\bullet$ was defined through the Eilenberg-MacLane map). To see that $EM_{\mathcal{L}}$ is isomorphism in homology one can consider the spectral sequences for both complexes assigned to the filtration by the degree n of the polynomial functor $\mathcal{L} = \oplus_{n\geq 1}\mathcal{L}_n$. It follows from Lemma 5.2 that the induced map of spectral sequences is an isomorphism starting from the first page. □

6. Proof of Theorem 3.1

The proof is the following sequence of quasi-isomorphisms.

$$\mathcal{A}/\mathcal{A}^2 \xleftarrow[\alpha]{\simeq} \mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathrm{Tot}\, A_\bullet) \xrightarrow[EM_{\mathcal{L}}]{\simeq} \mathrm{Tot}\, \mathcal{L}(A_\bullet) \xleftarrow{\simeq} \mathrm{Tot}(L_\bullet) \tag{6.1}$$

It is well known that $\mathcal{A} = \mathrm{Tot}\, A_\bullet$ is a commutative non-cocommutative *dg*-bialgebra, moreover its homology bialgebra $H^*(\mathcal{A})$ is polynomial [30, 31]. We have by Proposition 5.1 that the first arrow α is a quasi-isomorphism. So the homology of both complexes $\mathcal{A}$, $\mathcal{L}(\mathcal{A})$ is the space of generators of $H^*(\mathcal{A})$ which is exactly the space of primitives.

The second arrow is a quasi-isomorphism by Lemma 5.3.

Let us explain the last quasi-isomorphism

$$\mathrm{Tot}(L_\bullet) \xhookrightarrow{\simeq} \mathrm{Tot}\, \mathcal{L}(A_\bullet). \tag{6.2}$$

The algebra L^n is the Lie Koszul dual of A_n [8]. One has the natural inclusion (of Lie coalgebras):

$$L_n \hookrightarrow \mathcal{L}(A_n). \tag{6.3}$$

The map (6.3) describes the so called "diagonal" homology of $\mathcal{L}(A_n)$. The property A_n is Koszul means that $\mathcal{L}(A_n)$ has only diagonal (non-trivial) homology. In other words (6.3) is a quasi-isomorphism. But (6.3) is a simplicial morphism. As a consequence (6.2) is also a quasi-isomorphism.

To see that the bigradings correspond by the way described in Section 4, one should generalize the grading *complexity* on all the intermediate complexes of the zig-zag (6.1), and to show that all the morphismes preserve it. We leave it as an exercise to the reader. □

Part 2. Graph-complexes

7. Introduction

Graph-complexes are widely used to study the homology of interesting spaces and to prove interesting theorems [10, 11, 12, 14, 15, 17, 18, 20, 24]. One series of such graph-comples (its slight modification will be denoted by $\{D_n\}_{n\geq 0}$ throughout the paper) was used by M. Kontsevich to prove the formality of the operad of little d-cubes [19]. The idea of the proof is that $\{D_n\}_{n\geq 0}$ are quasi-isomorphic to the cochains of the operad, and also one has projections inducing homology isomorphism

$$D_n \xrightarrow[\bar{I}_n]{\simeq} A_n = H^*(C\langle n\rangle). \tag{7.1}$$

A_n is considered as a commutative dg-algebra with zero differential.

A more thorough account on this result was given by I. Volic and the first author in [22].

In [5, 6] another graph-complex was defined (its slight modification will be denoted by $\mathcal{D}$ in the paper). By means of integration over configuration spaces this complex was naturally mapped to the De Rahm complex of the space *Emb* of long knots. One conjectures that this map is a quasi-isomorphism. The reason why it might be so is that $\mathcal{D}$ is quasi-isomorphic to $\mathcal{A} = \mathrm{Tot}\, A_\bullet$, see Theorem 8.6, and therefore the homology of $\mathcal{D}$ is exactly the rational homology of *Emb*.

The complexes $\{D_n\}_{n\geq 0}$ form a simplicial commutative dg-algebra. Its totalization $\mathrm{Tot}\, D_\bullet$ is exactly the complex $\mathcal{D}$.

Another motivation for us to study graph-complexes is that they generalize on higher homology of knot spaces the 3-valent diagrams calculus developed by Dr. Bar Natan [4] (in the relation with the finite type knot invariants). For example, Theorem 6 of [4], which says that the bialgebra of chord diagrams is isomorphic to the bialgebra of 3-valent diagrams, is an obvious consequence of the fact that $\mathcal{A}$ is quasi-isomorphic to $\mathcal{D}$: the lower line homology of the dual to $\mathcal{A}$ is the bialgebra of chord diagrams and the lower line homology of the dual to $\mathcal{D}$ is the bialgebra of 3-valent diagrams modulo STU, AS, and IHX relations.

In Section 9 we define a new series of graph-complexes $\{P_n\}_{n\geq 0}$ satisfying $H^*(P_n) = L_n = Hom(\pi_*(C^n), \mathbb{Q})$. We show that the totalization complex $\mathcal{P} = \mathrm{Tot}\, P_\bullet$ is quasi-isomorphic to $\mathrm{Tot}\, L_\bullet$ and therefore $H^*(\mathcal{P}) = Hom(\pi_*(\overline{Emb}), \mathbb{Q})$.

8. Cohomology graph-complex for configuration and knot spaces

Our definition of the space D_n of diagrams is very close to that of [19, 22].

A *diagram* Γ on n external and q internal vertices is any graph with n *external* vertices (lying on the line $\mathbb{R}^1$ and labeled consequently $1, 2, \ldots, n$) and q (non-labeled) internal vertices, and some number of oriented segments connecting them. Those segments that connect two external vertices are called *chords* and all others are *edges*.

The *orientation set* of a diagram is the union of the set of internal vertices (such elements are considered to be of degree $-d$) and the set of edges (such elements are of degree $d-1$). An *orientation* of a diagram is any ordering of its orientation set. The *degree* of a diagram is the total degree of the elements from the orientation set.

Definition 8.1. A diagram is called *admissible* if

(1) it does not contain an internal vertex of valence ≤ 2;
(2) it contains neither edges nor chords connecting a vertex to itself (no loops);
(3) every internal vertex is connected by a path to an external one.

Remark 8.2. The distinction between our definition and the one given in [19, 22] is that we do permit multiple edges and multiple chords. This will be important for Theorem 9.3. This difference is essential only if d is odd, for even d graphs with multiple edges/chords cancel out by the orientation relation, see below.

Definition 8.3. The space D_n is defined as the $\mathbb{Q}$-vector space spanned by the admissible diagrams Γ with n external vertices, modulo the relations

(1) if Γ_1 and Γ_2 differ only by an orientation of an edge, then

$$\Gamma_1 = (-1)^d \Gamma_2;$$

(2) if Γ_1 and Γ_2 differ only by a permutation of the orientation set, then

$$\Gamma_1 = \pm \Gamma_2$$

where the sign is the Koszul sign of permutation (taking into account the degrees of elements).

D_0 is defined to be $\mathbb{Q}$ being spanned by the empty diagram.

The differential in D_n is defined as the sum of contractions of edges.

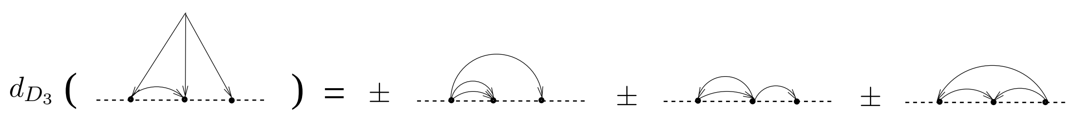


FIGURE 6. The differential in D_3.

For the signs convention, see [19, 22].

The multiplication in D_n is defined by superimposing:

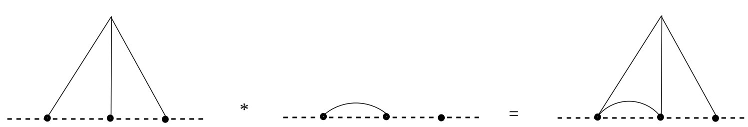


FIGURE 7. The product in D_3.

Proposition 8.4. [19, 22] *Complexes D_n, $n \geq 0$, with multiplication as above are commutative dg-algebras.*

Lemma-definition 8.5. *The morphisms*

$$\bar{I}_n \colon D_n \to A_n \tag{8.1}$$

which send any diagram having internal vertices to zero, and all others to the corresponding monomials in A_n (see Section 3), are quasi-isomorphisms of commutative dg-algebras.

Proof. Our complexes are slightly different from those used in [19, 22], but the proof is the same, see [22]. □

The complexes $\{D_n\}_{n \geq 0}$ form a cooperad in CDGA. For the definition of structure maps see [19, 22]. This cooperad is endowed with a morphism to the cooperad $\mathcal{COASS}$:

$$D_\bullet \to \mathcal{COASS}.$$

Any non-trivial diagram is sent to zero and the trivial diagram with n external vertices — to $1 \in \mathbb{Q} = \mathcal{COASS}(n)$. This endows $D_\bullet$ with a structure of a simplicial commutative dg-algebra, see Section 2. The simplicial structure of $D_\bullet$ is completely analogous to that of $A_\bullet$, see Figures 3-4. For $\Gamma \in D_n$, the face map d_0 removes the vertex 1 if it was of valence 0, otherwise $d_0(\Gamma) = 0$. Face $d_i(\Gamma)$, $i = 1 \dots n-1$, is obtained by contracting the segment $[i, i+1]$ of $\mathbb{R}^1$. And finally d_n removes the last point n (if it was of valence 0, otherwise $d_n(\Gamma) = 0$). The degeneracy s_i, $i = 1 \dots n+1$, is defined as insertion of a new external point between $i-1$ and i.

The normalized part $ND_\bullet$ of $D_\bullet$ is spanned by the diagrams whose all external vertices are of positive valence. We will define a graph-complex $\mathcal{D}$ as the totalization of $D_\bullet$.

Theorem 8.6. *The complex $\mathcal{D} = \operatorname{Tot} D_\bullet$ is quasi-isomorphic to $\mathcal{A} = \operatorname{Tot} A_\bullet$ and therefore the homology of $\mathcal{D}$ is the rational cohomology of $\overline{Emb}$:*

$$H^*(\mathcal{D}) = H^*(\overline{Emb}).$$

Proof. The map (8.1) is a quasi-isomorphism of simplicial commutative dg-algebras, which induces a quasi-isomorphism of totalizations:

$$\mathcal{D} = \operatorname{Tot}(D_\bullet) \xrightarrow{\simeq} \operatorname{Tot}(A_\bullet) = \mathcal{A}.$$

But $H^*(\mathcal{A}) = E_2^{*,*}(C^\bullet) = H^*(\overline{Emb})$. $\square$

Since d_0 and d_n act always as zero on ND_n, the differential in $\mathcal{D}$ is the sum of contractions of edges and of line segments of $\mathbb{R}^1$, see Figure 8.

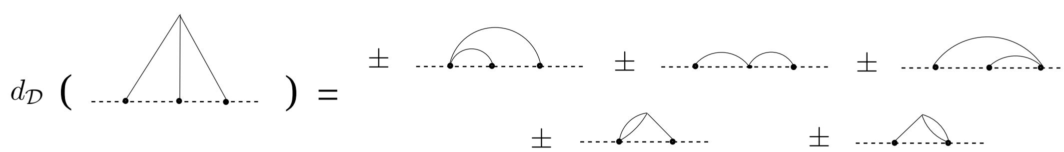


FIGURE 8. The differential in $\mathcal{D}$.

In $\mathcal{D} = \operatorname{Tot} D_\bullet$ a degree of a graph $\Gamma \in D_n$ is desuspended by n. Geometrically we add to the orientation set of Γ n elements of degree -1 that correspond to the external vertices of Γ. The product in $\mathcal{D}$, which is defined via the Eilenberg-MacLane map, acts as a shuffle of external points. For each summand the ordering of its orientation set is obtained by concatenation.

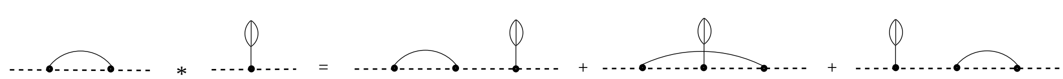


FIGURE 9. The product in $\mathcal{D}$.

The coproduct in $\mathcal{D}$ is the coconcatenation.

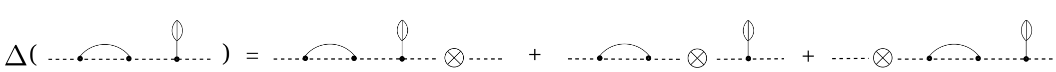


FIGURE 10. The coproduct in $\mathcal{D}$.

9. Cohomotopy graph-complex for configuration and knot spaces

A non-trivial graph $\Gamma \in D_n$ is called *non-decomposable* if it can not be represented as a product $\Gamma = \Gamma_1 \cdot \Gamma_2$ of two non-trivial graphs $\Gamma_1, \Gamma_2 \in D_n$.[5] The space spanned by non-decomposable graphs will be denoted by P_n.

Remark 9.1. In other words a graph is non-decomposable if it is connected (and non-empty) when we remove from it all the external vertices with their little neighborhoods.

Notice however that a non-decomposable graph might be disconnected: together with its main connected part it can have a number of singletons — external vertices of valence 0.

The complex P_n is a quotient-complex of D_n. It is easy to see that $P_n = D_n/D_n^2$.

Proposition 9.2. *D_n is a polynomial algebra whose space of generators is P_n.*

Proof. Obvious. It is here where it is important that we permit multiple edges/chords. □

Theorem 9.3. *The homology of P_n is the rational cohomotopy of the configuration space:*

$$H^*(P_n) = L_n = Mor(\pi_*(C^n), \mathbb{Q}).$$

Proof. We have the quasi-isomorphisms

$$P_n = D_n/D_n^2 \underset{\alpha}{\overset{\simeq}{\twoheadleftarrow}} \mathcal{L}(D_n) \overset{\simeq}{\twoheadrightarrow} \mathcal{L}(A_n) \overset{\simeq}{\hookleftarrow} L_n. \tag{9.1}$$

The first arrow α is a quasi-isomorphism by Proposition 5.1, the second one is induced by the quasi-isomorphism 8.1, the last one is due to the Koszul property. □

It follows from Remark 9.1 that $P_\bullet$ is a simplicial subspace of $D_\bullet$ — the simplicial structure maps preserve $P_\bullet$. Its noramalized part $NP_\bullet$ is spanned by the connected non-decomposable diagrams, *i.e.* by the diagrams without singletons. Now define complex $\mathcal{P}$ as the totalization of $P_\bullet$. Obviously, $\mathcal{P}$ is a quotient-complex of $\mathcal{D}$.

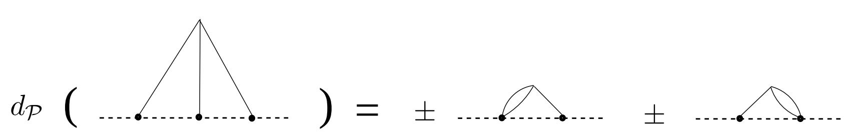


Figure 11. The differential in $\mathcal{P}$.

Theorem 9.4. *The complex $\mathcal{P} = \mathrm{Tot}\, P_\bullet$ is quasi-isomorphic to $\mathrm{Tot}\, L_\bullet$ and therefore the homology of $\mathcal{P}$ is the rational cohomotopy of $\overline{Emb}$:*

$$H^*(\mathcal{P}) = Mor(\pi_*(\overline{Emb}), \mathbb{Q}).$$

Proof. Diagram (9.1) is a sequence of quasi-isomorphisms of simplicial dg-spaces. Passing to totalization one gets the result. □

[5] We consider the inner product of D_n, see Figure 7.

Remark 9.5. $P_\bullet$ is a simplicial L_∞-coalgebra. Indeed, given B is a polynomial dg-algebra, any section $B/B^2 \hookrightarrow B$ of the projection $B \twoheadrightarrow B/B^2$ defines an L_∞-coalgebra structure on B/B^2. We have natural inclusions

$$D_n/D_n^2 = P_n \hookrightarrow D_n. \tag{9.2}$$

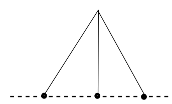
Since (9.2) is a map of simplicial vector spaces $P_\bullet \to D_\bullet$, the L_∞-coalgebra operations on P_n, $n \geq 0$, commute with the simplicial structure maps.

We finish by giving some examples of cycles in $\mathcal{P}$. Obviously the diagram

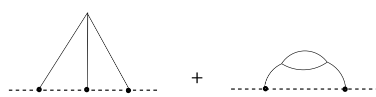
is the first non-trivial cycle (for both even and odd d). Its degree is $d-3$. It can be easily seen that the sum of diagrams (taken with appropiate signs)

is a non-trivial cycle in case when d is odd (and therefore multiple edges are possible). The degree of this cycle is $2d-5$. Recall that $\overline{Emb}$ is homotopy equivalent to $Emb \times \Omega^2 S^{d-1}$. The above cycles descibe the rational cohomotopy coming from the second factor $\Omega^2 S^{d-1}$.

The first non-trivial cohomotopy coming from the first factor Emb is of degree $2d-6$. In case of even d it is given by the diagram:



For odd d it is given by the sum:



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