

# O-MINIMAL FLOWS ON ABELIAN VARIETIES.

EMMANUEL ULLMO, ANDREI YAFAEV

ABSTRACT. Let  $A$  be an abelian variety over  $\mathbb{C}$  of dimension  $n$  and  $\pi: \mathbb{C}^n \rightarrow A$  be the complex uniformisation. Let  $X$  be an unbounded subset of  $\mathbb{C}^n$  definable in a suitable o-minimal structure. We give a description of the Zariski closure of  $\pi(X)$ .

## 1. INTRODUCTION.

Let  $A$  be a complex abelian variety of dimension  $n$ . Write  $A = \mathbb{C}^n / \Lambda$  where  $\Lambda \subset \mathbb{C}^n$  is a lattice and let  $\pi: \mathbb{C}^n \rightarrow A$  be the uniformisation map.

A subvariety  $V$  of  $A$  is called *weakly special* if  $V = P + B$  where  $P$  is a point of  $A$  and  $B$  is an abelian subvariety. The abelian Ax-Lindemann-Weierstrass theorem is the following.

**Theorem 1.1.** *Let  $Y$  be a complex algebraic subset of  $\mathbb{C}^n$ . The components of the Zariski closure of  $\pi(Y)$  are weakly special subvarieties.*

This theorem is due to Ax (see [1] and [2]) and plays an important role in the new proof by Pila and Zannier of the Manin-Mumford conjecture [7]. Note that the paper [7] provides a different proof of the abelian Ax-Lindemann-Weierstrass theorem. For a proof close in spirit to the contents of this paper, see Section 9 of [5]. In reality, in this statement,  $Y$  can be taken to be only *semialgebraic* ( $\mathbb{C}^n$  being identified with  $\mathbb{R}^{2n}$ ).

The aim of this paper is to investigate the Zariski closure of the sets  $\pi(X)$  where  $X$  is definable in an o-minimal structure which is a much wider class of objects. We refer to the book [11] for the notion of a set definable in an o-minimal structure, in particular the structures  $\mathbb{R}_{an}$  and  $\mathbb{R}_{an,exp}$ . Just recall that  $\mathbb{R}_{an}$  is the o-minimal structure generated by the restricted analytic functions and  $\mathbb{R}_{an,exp}$  is additionally generated by the graph of the real exponential. For a subset  $\Sigma$  of  $A$ , we denote by  $Zar(\Sigma)$  its Zariski closure.

To be able to prove anything, we will need to make certain additional assumptions. Firstly, the set  $X$  will be assumed to be *unbounded*. The necessity of this condition can be demonstrated by the following

example. Let  $\mathcal{F}$  be a connected bounded fundamental domain for the action of  $\Lambda$  on  $\mathbb{C}^n$ . The restriction of  $\pi$  to  $\mathcal{F}$  is definable in  $\mathbb{R}_{an}$ . Let  $V$  be any algebraic subvariety of  $A$  and let  $\tilde{V} = \pi^{-1}(V) \cap \mathcal{F}$ . Then  $\tilde{V}$  is definable in  $\mathbb{R}_{an}$  and  $Zar(\pi(\tilde{V})) = V$ .

However, when  $X$  is an unbounded real analytic manifold, we prove the following.

**Theorem 1.2.** *Let  $X$  be an unbounded real analytic manifold of  $\mathbb{C}^n = \mathbb{R}^{2n}$  definable in an o-minimal structure which is an extension of  $\mathbb{R}_{an}$ .*

*Let  $V = Zar(\pi(X))$ . For any point  $P$  of  $\pi(X)$  there is a positive dimensional abelian subvariety  $B_P$  of  $A$  such that  $P + B_P$  is contained in  $V$ .*

*In particular,  $V$  contains a Zariski dense set of weakly special subvarieties.*

To investigate general definable sets  $X$ , we will also impose some mild restrictions on the o-minimal structure. Let  $\mathcal{S}$  be an o-minimal structure over  $\mathbb{R}$ , containing  $\mathbb{R}_{an}$  and whose definable sets admit an analytic stratification (as defined in [11], Chapter 3). This condition holds for most ‘usual’ o-minimal structures, for example  $\mathbb{R}_{an}$  and  $\mathbb{R}_{an,exp}$ . We fix such a structure  $\mathcal{S}$  and in what follows and by definable, we will mean ‘definable in  $\mathcal{S}$ ’.

Next we introduce the notion of *essential Zariski closure*. Let  $X$  be an unbounded definable set as before. For  $R > 0$ , let  $B(0, R)$  be the open unit ball of centre 0 and radius  $R$ . The behavior of the set  $\pi(X \cap B(0, R))$  when  $R \rightarrow \infty$  is what we call an *o-minimal flow*. We show that for  $R$  large enough, the Zariski closure of the set  $\pi(X \setminus (X \cap B(0, R)))$  is constant. We call this Zariski closure, the *essential Zariski closure* of  $\pi(X)$  and denote it by  $Zaress(\pi(X))$ .

For an abelian subvariety  $B$  of  $A$ , write  $V_B \subset \mathbb{C}^n$  the tangent space to  $B$  at the origin and  $p_B$  the projection  $\mathbb{C}^n \rightarrow V_B$ .

We prove the following:

**Theorem 1.3.** *Let  $X$  be an unbounded definable subset of  $\mathbb{C}^n$ . Let  $V$  be  $Zaress(\pi(X))$ .*

*For each point  $P$ , in  $\pi(X)$ , there exists a positive dimensional abelian subvariety  $B_P$  of  $A$  such that  $P + B_P$  is contained in  $V$ .*

*In particular,  $V$  contains a Zariski dense set of weakly special subvarieties.*

We prove a characterisation of subvarieties of an abelian variety containing a Zariski dense set of weakly special subvarieties (see proposition 4.1) and deduce from theorem 1.3 the following.

**Theorem 1.4.** *Assume that  $X$  is a definable subset of  $\mathbb{C}^n$  such that for all abelian subvarieties  $B$  of  $A$ ,  $p_B(X)$  is unbounded. Then components of  $\text{Zar}(\pi(X))$  are weakly special.*

The strategy of the proof of the theorem 1.2 relies on the theory of o-minimality and Pila-Wilkie counting theorem. Let  $X$  be as in the statement and  $V$  be the Zariski closure of  $\pi(X)$ . Using a suitable definable set and applying Pila-Wilkie theorem, we show that there exists a positive dimensional semi-algebraic set  $W \subset \mathbb{C}^n = \mathbb{R}^{2n}$  such that  $X + W$  is contained in  $\pi^{-1}V$ . Applying the Ax-Lindemann-Weierstrass theorem, we then show that for any  $P$  of  $\pi(X)$ , there exists a weakly special subvariety  $P + B_P \subset V$ .

Finally, we would like to point out one possible application of our theorem.

Recall the following theorem of Bloch-Ochiai (see Chapter 9 of [3]) which is proved using Nevanlinna theory.

**Theorem 1.5.** *Let  $A$  be an abelian variety and  $f: \mathbb{C} \rightarrow A$  be a non-constant holomorphic map. Then the Zariski closure of  $f(\mathbb{C})$  is a translate of an abelian subvariety.*

In some cases our theorem 1.4 implies theorem 1.5.

Consider for example  $A = \mathbb{C}^n/\Lambda$  (where  $\Lambda$  is a lattice such that  $A$  is a simple abelian variety) and  $f: \mathbb{C} \rightarrow A$  given by  $f(z) = (z, \dots, z, e^z, \dots, e^z)$  with  $s$  factors of  $z$  and  $r$  times of  $e^z$  with  $r + s = n$ . Then consider the set  $X \subset \mathbb{C}^n$  given by

$$X = \{(x + iy, \dots, x + iy, e^x e^{iy}, \dots, e^x e^{iy}) : x \in \mathbb{R}, y \in [0, 2\pi]\}.$$

Clearly  $X$  is unbounded and definable in  $\mathbb{R}_{an,exp}$  and its image in  $A$  is contained in  $f(\mathbb{C})$ . By theorem 1.4, the Zariski closure of  $f(\mathbb{C})$  is  $A$  (since  $A$  is simple).

It is not however always possible to “extract” such a definable unbounded set  $X$  from  $f(\mathbb{C})$  as the example of  $(e^z, e^{iz}) \subset \mathbb{C}^2$  shows. Indeed, in this example, for any subset  $Y \subset \mathbb{C}$  such that  $f(Y)$  is definable, both the real and imaginary parts of  $z \in Y$  must be bounded.

Another (counter)-example is the following. Define the iterated exponential function  $exp_n(x)$  by  $exp_1 = exp$  and  $exp_n = exp \circ exp_{n-1}$ . By Proposition 9.10 of [4], a definable function is bounded by  $exp_n(x^m)$  for some  $n, m$ . Therefore a graph of a function which ‘grows faster’ than any  $exp_n$  will not satisfy the assumptions of our theorems.

We conclude this introduction with an open question in the spirit of [10]. It concerns the topological closure of  $\pi(X)$  rather than Zariski

closure. Recall from [10] that a *real* weakly special subvariety is defined to be a translate of a real subtorus of  $A$  (hence not necessarily algebraic).

**Conjecture 1.6.** *Let  $X$  be as before be an unbounded definable real analytic manifold. We denote by  $\overline{\pi(X)}$  the topological closure of  $\pi(X)$ .*

*There exists a real analytic submanifold  $V$  of  $A$  containing a dense subset of real weakly special subvarieties such that*

$$\overline{\pi(X)} = \pi(X) \cup V.$$

In section 4, we prove a characterisation of subvarieties of abelian varieties containing a Zariski dense subset of weakly special subvarieties, namely that such a subvariety is a union of weakly special ones. We believe this result and our argument to be of independent interest.

#### ACKNOWLEDGEMENTS.

The second author is grateful to Alex Wilkie and Gareth Jones for useful discussions at the ‘O-minimality and applications’ conference in Konstanz in July 2015. The second author gratefully acknowledges financial support of the ERC, Project 511343.

## 2. PROOF OF THEOREM 1.2.

In this section we assume that  $X$  is an unbounded real analytic submanifold of  $\mathbb{C}^n = \mathbb{R}^{2n}$  definable in some o-minimal structure which contains  $\mathbb{R}_{an}$ . Let  $V$  be the Zariski closure of  $\pi(X)$  in  $A$ .

**2.1. A definable set and point counting.** The contents of this section are essentially a reproduction of the arguments of Orr from Section 9 of [5] with slight adjustments. In this section we define a certain definable set associated with  $X$  and, using Pila-Wilkie theorem, show that this set contains a positive dimensional semi-algebraic subset.

Choose a fundamental set  $\mathcal{F}$  for the action of  $\Lambda$  on  $\mathbb{C}^n$  such that  $X \cap \mathcal{F}$  is non-empty. We choose  $\mathcal{F}$  to be an open connected subset of  $\mathbb{C}^n$  such that  $\overline{\mathcal{F}}$  is compact and  $\Lambda$ -translates of  $\overline{\mathcal{F}}$  cover  $\mathbb{C}^n$ . The set  $\mathcal{F}$  is an ‘open parallelepiped’. Since  $\mathcal{F}$  is an open subset of  $\mathbb{C}^n$ , we have that  $\dim(X \cap \mathcal{F}) = \dim(X)$ . Let  $\tilde{V}$  be  $\mathcal{F} \cap \pi^{-1}V$ . This is a definable set since the o-minimal structure contains  $\mathbb{R}_{an}$  and  $\pi$  restricted to  $\mathcal{F}$  is definable in  $\mathbb{R}_{an}$ .

Consider the definable set

$$\Sigma = \{x \in \mathbb{C}^n : \dim(X + x) \cap \tilde{V} = \dim(X)\}.$$

The argument is exactly the same as in the proof of Lemma 9.3 of [5].

We have the following lemma:

**Lemma 2.1.** *If  $\lambda \in \Lambda$  and  $X \cap (\mathcal{F} - \lambda) \neq \emptyset$ , then  $\lambda \in \Sigma$ .*

*Proof.* From  $\Lambda$ -invariance of  $\pi^{-1}V + \lambda = \pi^{-1}V$ , we see that for  $\lambda$  as in the statement (in particular for  $\lambda \in \Lambda$ ),  $X + \lambda \subset \pi^{-1}V$ .

It follows that

$$(X + \lambda) \cap \tilde{V} = (X + \lambda) \cap \mathcal{F}.$$

As  $\mathcal{F} - \lambda$  is an open subset of  $\mathbb{C}^n$ , we see that

$$\dim(X \cap (\mathcal{F} - \lambda)) = \dim(X) = \dim((X + \lambda) \cap \mathcal{F})$$

The conclusion follows.  $\square$

Fix a basis  $\lambda_1, \dots, \lambda_{2n}$  of  $\Lambda$ . Then  $\Lambda \otimes \mathbb{Q}$  is identified with  $\mathbb{Q}^{2n}$ . We define the height of an element  $\lambda = \sum a_i \lambda_i \in \Lambda$  ( $a_i \in \mathbb{Z}$ ) as

$$H(\lambda) = \max(|a_1|, \dots, |a_{2n}|).$$

This height thus coincides with the usual height on  $\mathbb{Q}^n$ .

**Proposition 2.2.** *There exists  $T_0 \geq 0$  such that for all  $T \geq T_0$ ,*

$$|\{x \in \Sigma \cap \Lambda : H(x) \leq T\}| \geq T/2.$$

*Proof.* This is essentially Lemma 9.1 of [5].

The first observation is that if  $x_1$  and  $x_2$  are two points of  $\Lambda$  such that  $X \cap (\mathcal{F} - x_1)$  and  $X \cap (\mathcal{F} - x_2)$  are both non-empty, then  $\Sigma \cap \Lambda$  contains at least one point of height  $h$  for every  $h$  between  $H(x_1)$  and  $H(x_2)$ .

Note that  $X$  is path-wise connected in the Euclidean topology. Let  $C$  be a path from a point in  $X \cap (\mathcal{F} - x_1)$  to a point in  $X \cap (\mathcal{F} - x_2)$ .

When  $C$  crosses over from  $\mathcal{F} - u_1$  to an adjacent domain  $\mathcal{F} - u_2$ , the heights of  $u_1$  and  $u_2$  change by at most one.

It follows that for any  $h$  between  $H(x_1)$  and  $H(x_2)$ , there is a  $u \in \Lambda$  of height  $\leq h$  such that  $X \cap (\mathcal{F} - u)$  is not empty. This  $u$  belongs to  $\Sigma \cap X$ .

By assumption  $X$  is unbounded. Thus as  $x$  varies in  $\Lambda$  such that  $X \cap \mathcal{F} - x$  is non-empty,  $h(x)$  goes to infinity.

It follows that there is an  $h_0$  such that for any  $h > h_0$ ,  $\Sigma \cap \Lambda$  contains at least one point of height  $h$ .

Take  $T_0 = 2h_0$ .  $\square$

We now use the following theorem of Pila and Wilkie ([6], Theorem 1.8).

For a definable subset  $\Theta \subset \mathbb{R}^n$ , we define  $\Theta^{alg}$  to be the union of all positive dimensional semi-algebraic subsets contained in  $\Theta$ . We define  $\Theta^{tr}$  to be  $\Theta \setminus \Theta^{alg}$ .

**Theorem 2.3** (Pila-Wilkie). *Let  $\Theta$  be a subset of  $\mathbb{R}^n$  definable in an  $o$ -minimal structure. Let  $\epsilon > 0$ . There exists a constant  $c = c(\Theta, \epsilon)$  such that for any  $T \geq 0$ ,*

$$|\{x \in \Theta^{tr} \cap \mathbb{Q}^n : H(x) \leq T\}| \geq cT^\epsilon.$$

From Proposition 2.2 it now follows that  $\Sigma^{alg} \cap \Lambda$  is not empty.

Let  $W$  be a connected positive dimensional semi-algebraic subset contained in  $\Sigma$ . For each  $w$  in  $W$ ,  $\dim((X + w) \cap \tilde{V}) = \dim(X)$  and hence an analytic component of  $(X + w) \cap \mathcal{F}$  is contained in  $\pi^{-1}V$ . By analytic continuation, we see that  $X + w \subset \pi^{-1}V$ . We have proved:

**Proposition 2.4.** *With the notations and assumptions of this section, there exists a positive dimensional semialgebraic subset  $W$  such that*

$$X + W \subset \pi^{-1}V.$$

**2.2. Final argument.** We use the following lemma whose proof can for example be found in [5], Lemma 8.1.

**Lemma 2.5.** *Let  $\mathcal{Z}$  be a connected complex analytic subset of  $\mathbb{C}^g$ . Let  $\mathcal{X}$  be a connected irreducible semialgebraic set contained in  $\mathcal{Z}$ . Then there is a complex algebraic variety  $\mathcal{Y}$  such that  $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$ .*

By proposition 2.4 and the above lemma, we see that for any  $x \in X$ , there exists a positive dimensional complex algebraic subset  $Y_x$  containing  $X$  and contained in  $\pi^{-1}(V)$ . By the abelian Ax-Lindemann-Weierstrass theorem 1.1, the Zariski closure of  $\pi(Y_x)$  is a union of weakly special subvarieties of  $V$ . Therefore,  $V$  contains a subvariety of the form  $P + B_P$  where  $P = \pi(x)$  and  $B_P$  is a positive dimensional abelian subvariety of  $A$ . This finishes the proof of theorem 1.2.

### 3. CELL DECOMPOSITION AND ESSENTIAL CLOSURE.

In this section we consider an unbounded definable set  $X \subset \mathbb{C}^n$ . We refer to section 8 of [4] for the definition of a real analytic cell. What is relevant to us is that a real analytic cell in  $\mathbb{R}^n$  is a definable real analytic submanifold, definable-analytically isomorphic to  $\mathbb{R}^m$  for some  $m \leq n$ . By Theorem 8.9 of [4], there is a finite number of analytic cells  $X_1, \dots, X_k$  such that  $X$  is a disjoint union of the  $X_k$ .

**Proposition 3.1.** *The essential closure  $Zaress(\pi(X))$  is the union of  $Zar(\pi(X_i))$  where  $X_i$ s are the unbounded cells.*

*Proof.* We start with a lemma.

**Lemma 3.2.** *Let  $Z$  be a real analytic manifold in  $\mathbb{C}^n$  and  $U \subset Z$  an open subset.*

Then

$$\mathrm{Zar}(\pi(U)) = \mathrm{Zar}(\pi(Z))$$

In particular, if  $Z$  is an analytic unbounded submanifold of  $\mathbb{C}^n$ , then

$$\mathrm{Zar}(\pi(Z)) = \mathrm{Zar}(\pi(Z))$$

*Proof.* One inclusion is obvious.

Write  $\mathrm{Zar}(\pi(U)) \subset \mathbb{P}^m$  for some  $m$  and let  $s \in H^0(\mathbb{P}^m, \mathcal{O}(l))$  for  $l \geq 1$  such that  $s$  is zero on  $\pi(U)$ . Then  $s \circ \pi$  is zero on  $U$  and by analytic continuation  $s \circ \pi$  is zero on  $Z$ . It follows that  $s$  is zero on  $\pi(Z)$ , hence  $\mathrm{Zar}(\pi(Z)) \subset \mathrm{Zar}(\pi(U))$ .  $\square$

Let  $X = X_1 \amalg \dots \amalg X_k$  be a cell decomposition of  $X$ . For  $R$  large enough,  $X \cap B(0, R)$  contains the union of all the bounded cells in the above decomposition.

We have

$$\mathrm{Zar}(\pi(X)) = \bigcup_{\{i: X_i \text{ unbounded}\}} \mathrm{Zar}(\pi(X_i)).$$

By Lemma 3.2, for an unbounded cell  $X_i$ ,

$$\mathrm{Zar}(\pi(X_i)) = \mathrm{Zar}(X_i).$$

The result follows.  $\square$

#### 4. CHARACTERISATION OF SUBVARIETIES CONTAINING A DENSE SET OF WEAKLY SPECIAL SUBVARIETIES.

In this section we prove a proposition which we believe to be of independent interest.

Let  $A$  be an abelian variety and  $V$  a subvariety of  $A$ . Define the stabiliser of  $V$  as

$$\mathrm{Stab}(V) = \{P \in A : P + V = V\}.$$

Recall that for an abelian subvariety  $B$  of  $A$ , there exists an abelian subvariety  $B'$  such that  $A = B + B'$  and  $B \cap B'$  is finite. We always refer to  $B$  and  $B'$  as above.

**Proposition 4.1.** *Let  $V$  be an irreducible subvariety of  $A$ .*

(1) *Assume  $\dim \mathrm{Stab}(V) > 0$ .*

*Then there exists abelian subvarieties  $B$  and  $B'$  of  $A$  such that  $A = B + B'$  and  $V = B + V'$  where  $V'$  is a subvariety of  $B'$ .*

- (2) Assume that  $\text{Stab}(V)$  is finite. Then the set of positive dimensional weakly special subvarieties contained in  $V$  is not Zariski dense.
- (3) Assume again that  $\text{Stab}(V)$  is finite. Let  $\Sigma$  be the set of all positive dimensional weakly special subvarieties contained in  $V$ . For an abelian subvariety  $B \subset A$ , denote by  $B'$  an abelian subvariety such that  $A = B + B'$ .

There exists a finite set  $B_1, \dots, B_r$  of abelian subvarieties of  $A$  and  $W_1, \dots, W_r$  of subvarieties of  $B'_i$  such that

$$\text{Zar}(\Sigma) = \bigcup_{i=1}^r B_i + W_i.$$

*Proof.* Assume  $\dim \text{Stab}(V) > 0$  and let  $B$  be the neutral component of  $\text{Stab}(V)$ .

Let  $B'$  be an abelian subvariety such that  $A = B + B'$  and let  $\psi: A \rightarrow A/B$  be the quotient. Let  $V'$  be  $\psi|_{B'}^{-1}(\psi(V))$ . Then

$$V = \{B + x : x \in V\} = \{B + x : x \in V'\} = B + V'.$$

This proves (1).

We will now prove (2). Assume that  $\text{Stab}(V)$  is finite. We start by reducing to the case where  $\text{Stab}(V) = \{0\}$ . Let  $A' = A/\text{Stab}(V)$  and let  $\phi: A \rightarrow A'$  be the quotient map and let  $V' = \phi(V)$ . Note that  $\phi^{-1}(V') = V + \text{Stab}(V) = V$ . We claim that  $\text{Stab}(V') = \{0\}$ . Let  $P \in \text{Stab}(V')$  and  $Q \in \phi^{-1}(P)$ . We have

$$\phi(Q + V) = P + V' = V'$$

It follows that  $Q + V \subset \phi^{-1}(V') = V$  and for dimension reasons  $Q + V = V$ . Hence  $Q \in \text{Stab}(V)$  and  $P = \phi(Q) = 0$ .

As the conclusion of (2) holds for  $V$  if and only if it holds for  $V'$ , we may therefore assume that  $\text{Stab}(V) = \{0\}$ .

For  $m > 1$ , consider the map

$$\phi_m: V^m \rightarrow A^{m-1}$$

defined by

$$\phi_m(x_1, \dots, x_m) = (x_1 - x_2, \dots, x_m - x_{m-1}).$$

By [12], Lemma 3.1, there exists  $m > 1$  such that the map  $\phi_m$  is a generic embedding.

Let  $P + B$  be a positive dimensional weakly special subvariety contained in  $V$ . Then  $\phi_m((P + B)^m) = B^{m-1}$ . The map  $\phi_m$  is therefore not injective on  $(P + B)^m$ . Therefore  $V$  can not contain a Zariski dense



set of positive dimensional subvarieties of the form  $P + B$ . This proves (2).

Let us now prove (3). Let  $\Sigma$  as in the statement, the set of all positive dimensional weakly special subvarieties contained in  $V$  and let  $W$  be a component of  $\text{Zar}(\Sigma)$ . Then  $W$  contains a Zariski dense set of weakly special subvarieties and by (2),  $\text{Stab}(W)$  is positive dimensional. It follows from (1) that  $W = B + W'$  where  $B$  is an abelian subvariety of  $A$  and  $W'$  a subvariety of  $B'$ . Since  $\text{Zar}(\Sigma)$  has finitely many components, the conclusion of (3) follows.  $\square$

**Remark 4.2.** *The geometric aspect of Lang's conjecture predicts that given a variety of general type  $V$ , the union subvarieties of  $W$  not of general type, is not Zariski dense. It is a known fact that a subvariety  $V$  of an abelian variety is of general type if and only if  $\text{Stab}(V)$  is finite. Therefore, our proposition 4.1 implies the geometric Lang's conjecture for subvarieties of abelian varieties.*

**Remark 4.3.** *This proposition is an abelian analogue of the result of the first author (see [9]) in the hyperbolic case which is proved by completely different methods.*

## 5. PROOF THEOREMS 1.3 AND 1.4.

In this section we deduce theorems from what precedes.

Let  $A$  and  $X$  be as in the assumptions of Theorem 1.3. Let  $V$  be a component of the essential Zariski closure of  $\pi(X)$ .

In section 3 we have seen that  $\text{Zaress}(\pi(X))$  is a finite union of Zariski closures of sets of the form  $\pi(Y)$  where  $Y$  is an unbounded definable real analytic submanifold of  $\mathbb{C}^n$ . Therefore, the conclusion of theorem 1.3 follows from theorem 1.2.

Let now  $X$  be as in 1.4. By theorem 1.3,  $V = \text{Zaress}(X)$  contains a Zariski dense set of positive dimensional weakly special subvarieties. From proposition 4.1, we deduce that  $V$  is of the form  $V = B + V'$  where  $B$  is a positive dimensional abelian subvariety of  $A$  and  $V'$  is a subvariety of  $B'$ . Reiterating the argument with  $B'$  and  $V'$ , we conclude that components of  $V$  are weakly special.

## REFERENCES

- [1] J. Ax, *On Schanuel's conjecture*, Annals of Math. **93** (1971), 1-24.
- [2] J. Ax, *Some topics in differential algebraic geometry I: Analytic subgroups of algebraic groups*, Amer. J. Math. **94** (1972), 1195-1204.
- [3] S. Kobayashi, *Hyperbolic Complex Spaces*. Vol. 318. A Series of Comprehensive studies in Mathematics, Springer 1998.

- [4] C. Miller, L. Van Den Dries, *On the real exponential field with restricted analytic functions*. Israel Journal of Math, Vol 85, 1994, 19-56.
- [5] M. Orr, *Introduction to abelian varieties and the Ax-Lindemann-Weierstrass theorem*. In “O-Minimality and Diophantine Geometry”. LMS Lecture Notes Series, 421, 2015.
- [6] J. Pila, A. Wilkie, *The rational points of a definable set*. Duke Math. Journal, 133(3), 591-616, 2006.
- [7] J. Pila, U. Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Rend. Math. Acc. Lincei **19** (2008), 149–162.
- [8] N. Ratazzi, E. Ullmo, *Galois+Equidistribution=Manin-Mumford*. Summer School Arithmetic geometry, 419-430, Clay Math. Proc., 8, Amer. Math. Soc., Providence, RI, 2009.
- [9] E. Ullmo, *Applications du théorème d’Ax-Lindemann hyperbolique*. Compositio Mathematica, Vol 150, Issue 02, 175- 190, 2014.
- [10] E. Ullmo, A. Yafaev, *Algebraic flows on abelian varieties*. Crelle’s Journal. To appear.
- [11] L. Van Den Dries, *Tame topology and o-minimal structures*. LMS Lecture Notes Series, **248**, 1998.
- [12] S-W. Zhang, *Equidistribution of small points an abelian varieties*. Annals of Maths, **147**, 159-165, 1998.

ULLMO: IHES 35 ROUTE DE CHARTRES, 91440 BURES-SUR-YVETTE, FRANCE  
*E-mail address:* ullmo@ihes.fr

YAFAEV: UCL, DEPARTMENT OF MATHEMATICS, GOWER STREET, WC1E 6BT, LONDON, UK  
*E-mail address:* yafaev@math.ucl.ac.uk