ABSOLUTE ALGEBRA
III–THE SATURATED SPECTRUM

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Abstract. We investigate the algebraic and topological preliminaries to a geometry in characteristic 1.

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1. Introduction

This work is a sequel to [2] and [3], of which, except when explicitly stated otherwise, we shall keep the definitions and notations. In particular, we shall always equip the spectrum $Spec(A)$ of a B_1–algebra with the topology defined in [3], Proposition 3.15, and the prime spectrum $Pr(A)$ of A with the topology defined in [3], Theorem 2.4.

We shall denote by $\mathcal{SP}$ the category whose objects are these B_1–spectra and whose morphisms are the continuous maps between them.

For A a B_1–algebra and S a subset of A, let $< S >$ denote the intersection of all the ideals of A containing S (there is always at least one such ideal : A itself). It is clear that $< S >$ is an ideal of A, and therefore is the smallest ideal of A containing S. As in ring theory, one may see that

$$< S >= \{\sum_{j=1}^{n} a_j s_j | n \in \mathbf{N}, (a_1, ..., a_n) \in A^n, (s_1, ..., s_n) \in S^n\}.$$

2. A new description of maximal congruences

Let A denote a B_1–algebra. For $\mathcal{P}$ a saturated prime ideal (see [3],p.1786) of A, let us define a relation $\mathcal{S}_{\mathcal{P}}$ on A by :

$$x\mathcal{S}_{\mathcal{P}}y \equiv (x \in \mathcal{P} \text{ and } y \in \mathcal{P}) \text{ or } (x \notin \mathcal{P} \text{ and } y \notin \mathcal{P}) \ .$$

Then $\mathcal{S}_{\mathcal{P}}$ is a congruence on A : if $x\mathcal{S}_{\mathcal{P}}y$ and $x^{'}\mathcal{S}_{\mathcal{P}}y'$, then one and only one of the following holds :

(i) $x \in \mathcal{P}$, $y \in \mathcal{P}$, $x^{'} \in \mathcal{P}$ and $y^{'} \in \mathcal{P}$,
(ii) $x \in \mathcal{P}$, $y \in \mathcal{P}$, $x^{'} \notin \mathcal{P}$ and $y^{'} \notin \mathcal{P}$,
(iii) $x \notin \mathcal{P}$, $y \notin \mathcal{P}$, $x^{'} \in \mathcal{P}$ and $y^{'} \in \mathcal{P}$,
(iv) $x \notin \mathcal{P}$, $y \notin \mathcal{P}$, $x^{'} \notin \mathcal{P}$ and $y^{'} \notin \mathcal{P}$.

In case (i), $x+x^{'} \in \mathcal{P}$ and $y+y^{'} \in \mathcal{P}$, whence $x+x^{'}\mathcal{S}_{\mathcal{P}}y+y^{'}$; in cases (ii) and (iv), $x+x^{'} \notin \mathcal{P}$ and $y+y^{'} \notin \mathcal{P}$ (as $\mathcal{P}$ is saturated), whence $x+x^{'}\mathcal{S}_{\mathcal{P}}y+y^{'}$. Case (iii) is symmetrical relatively to case (ii), therefore, in all cases, $x+x^{'}\mathcal{S}_{\mathcal{P}}y+y^{'}$: $\mathcal{S}_{\mathcal{P}}$ is compatible with addition.

In cases (i), (ii) and (iii), $xx^{'} \in \mathcal{P}$ and $yy^{'} \in \mathcal{P}$, whence $xx^{'}\mathcal{S}_{\mathcal{P}}yy^{'}$; in case (iv) $xx^{'} \notin \mathcal{P}$ and $yy^{'} \notin \mathcal{P}$ (as $\mathcal{P}$ is prime), whence also $xx^{'}\mathcal{S}_{\mathcal{P}}yy^{'}$: $\mathcal{S}_{\mathcal{P}}$ is compatible with multiplication, hence is a congruence on A.

As $0 \in \mathcal{P}$ and $1 \notin \mathcal{P}$, $0 \not\mathcal{S}_{\mathcal{P}}1$, therefore $\mathcal{S}_{\mathcal{P}}$ is nontrivial ; but each $x \in A$ is either in $\mathcal{P}$ (whence $x\mathcal{S}_{\mathcal{P}}0$) or not (whence $x\mathcal{S}_{\mathcal{P}}1$). It follows that

$$\frac{A}{\mathcal{S}_{\mathcal{P}}} = \{\bar{0}, \bar{1}\} \simeq B_1 \ ;$$

in particular, $\mathcal{S}_{\mathcal{P}}$ is maximal : $\mathcal{S}_{\mathcal{P}} \in MaxSpec(A)$.

Obviously, $I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}$.

Furthermore, let $(x,y) \in A^2$ be such that $x\mathcal{R}_{\mathcal{P}}y$; then there is $z \in \mathcal{P}$ such that $x+z = y+z$. If $x \in \mathcal{P}$ then $y+z = x+z \in \mathcal{P}$, whence $y \in \mathcal{P}$ (as $y+(y+z) = y+z$ and $\mathcal{P}$ is saturated) ; symmetrically, $y \in \mathcal{P}$ implies $x \in \mathcal{P}$, whence the assertions $(x \in \mathcal{P})$ and $(y \in \mathcal{P})$ are equivalent, and $x\mathcal{S}_{\mathcal{P}}y$. We have shown that

$$\mathcal{R}_{\mathcal{P}} \leq \mathcal{S}_{\mathcal{P}} \ .$$

We shall denote by α_A the mapping

$$\begin{aligned} \alpha_A \quad : \quad & Pr_s(A) \to MaxSpec(A) \\ & \mathcal{P} \mapsto \mathcal{S}_{\mathcal{P}} \ . \end{aligned}$$

Let $\mathcal{R} \in MaxSpec(A)$; then $\mathcal{R} \in Spec(A)$, whence $I(\mathcal{R})$ is prime ; by Theorem 3.8 of [3], it is saturated, *i.e.* $I(\mathcal{R}) \in Pr_s(A)$. Let us set

$$\beta_A(\mathcal{R}) := I(\mathcal{R}) \ .$$

Theorem 2.1. *The mappings*

$$\alpha_A : Pr_s(A) \mapsto MaxSpec(A)$$

and

$$\beta_A : MaxSpec(A) \mapsto Pr_s(A)$$

are bijections, inverse of one another. They are continuous for the topologies on $Pr_s(A)$ *and* $MaxSpec(A)$ *induced by the Zariski topologies on* $Pr(A)$ *and* $Spec(A)$ *defined in* [3]*(Theorem 2.4, resp. Proposition 3.15).*

Proof. Let $\mathcal{R} \in MaxSpec(A)$; then

$$\alpha_A(\beta_A(\mathcal{R})) = \alpha_A(I(\mathcal{R})) = \mathcal{S}_{I(\mathcal{R})} \ .$$

Let us assume $x\mathcal{R}y$; then, if $x \in I(\mathcal{R})$ one has $x\mathcal{R}0$, whence $y\mathcal{R}0$ and $y \in I(\mathcal{R})$; by symmetry, $y \in I(\mathcal{R})$ implies $x \in I(\mathcal{R})$, thus $(x \in I(\mathcal{R}))$ and $(y \in I(\mathcal{R}))$ are equivalent, *i.e.* $x\mathcal{S}_{I(\mathcal{R})}y$. We have proved that $\mathcal{R} \leq \mathcal{S}_{I(\mathcal{R})}$. As $\mathcal{R}$ is maximal, we have $\mathcal{R} = \mathcal{S}_{I(\mathcal{R})}$, whence

$$\alpha_A(\beta_A(\mathcal{R})) = \mathcal{S}_{I(\mathcal{R})} = \mathcal{R} \ ,$$

and

$$\alpha_A \circ \beta_A = Id_{MaxSpec(A)} \ .$$

Let now $\mathcal{P} \in Pr_s(A)$; then

$$\begin{aligned} (\beta_A \circ \alpha_A)(\mathcal{P}) &= \beta_A(\alpha_A(\mathcal{P})) \\ &= \beta_A(\mathcal{S}_\mathcal{P}) \\ &= I(\mathcal{S}_\mathcal{P}) \\ &= \mathcal{P} \ , \end{aligned}$$

whence

$$\beta_A \circ \alpha_A = Id_{Pr_s(A)} \ ,$$

and the first statement follows.

Let now F denote a closed subset of $Pr_s(A)$; then $F = G \cap Pr_s(A)$ for G a closed subset of $Pr(A)$ and $G = W(S) := \{\mathcal{P} \in Pr(A) | S \subseteq \mathcal{P}\}$ for some subset S of A. But then, for $\mathcal{R} \in MaxSpec(A)$, $\mathcal{R} \in \beta_A^{-1}(F)$ if and only if $\beta_A(\mathcal{R}) \in F$, *i.e.* $I(\mathcal{R}) \in G \cap Pr_s(A)$, that is $I(\mathcal{R}) \in G$, or $S \subseteq I(\mathcal{R})$, which means $\mathcal{R} \in V(S)$. Thus

$$\beta_A^{-1}(F) = V(S) \cap MaxSpec(A)$$

is closed in $MaxSpec(A)$. We have shown the continuity of β_A.

Let now $H \subseteq MaxSpec(A)$ be closed ; then $H = MaxSpec(A) \cap L$ for some closed subset L of $Spec(A)$and $L = V(T)$ for some subset T of A. Then a saturated prime ideal $\mathcal{P}$ of A belongs to $\alpha_A^{-1}(H)$ if and only if $\alpha_A(\mathcal{P}) \in H$, that is

$$\mathcal{S}_\mathcal{P} \in MaxSpec(A) \cap L \ ,$$

i.e.

$$\mathcal{S}_\mathcal{P} \in V(T)$$

or $T \subseteq I(\mathcal{S}_\mathcal{P})$. But $I(\mathcal{S}_\mathcal{P}) = \mathcal{P}$ whence $\mathcal{P}$ belongs to $\alpha_A^{-1}(H)$ if and only if $T \subseteq \mathcal{P}$, that is

$$\alpha_A^{-1}(H) = V(T) \cap MaxSpec(A) \ ,$$

which is closed in $MaxSpec(A)$. □

Let us consider the special case in which A is in the image of $\mathcal{F}$: $A = \mathcal{F}(M)$, for M a commutative monoid. Let P be a prime ideal of M ; as seen in [3], Theorem 4.2, $\tilde{P}$ is a saturated prime ideal in A, and one obtains in this way a bijection between $Spec_\mathcal{D}(M)$ and $Pr_s(A)$. The following is now obvious :

Theorem 2.2. *The mapping*

$$\begin{aligned}\psi_M \quad : \quad & Spec_{\mathcal{D}}(M) \to MaxSpec(\mathcal{F}(M)) \\ & P \mapsto \alpha_{\mathcal{F}(M)}(\tilde{P})\end{aligned}$$

is a bijection.

Two particular cases are of special interest :

(1) M is a group ; then $Spec_{\mathcal{D}}(M) = \{\emptyset\}$, whence $MaxSpec(\mathcal{F}(G))$ has exactly one element.

(2) $M = C_n := < x_1, ..., x_n >$ is the free monoid on n variables $x_1, ..., x_n$.
Then the elements of $Spec_{\mathcal{D}}(M)$ are the $(P_J)_{J \subseteq \{1,...,n\}}$, where

$$P_J := \bigcup_{j \in J} x_j C_n$$

(a fact that was already used in [3], Example 4.4). Then

$$\psi_M(P_J) = \alpha_{\mathcal{F}(M)}(\tilde{P_J}) = \mathcal{S}_{\tilde{P_J}}$$

whence $x\psi_M(P_J)y$ if and only if either ($x \in \tilde{P_J}$ and $y \in \tilde{P_J}$) or ($x \notin \tilde{P_J}$ and $y \notin \tilde{P_J}$). But we have seen in [3], Theorem 4.5, that $\mathcal{F}(M) = B_1[x_1, ..., x_n]$ could be identified with the set of finite formal sums of elements of M. Obviously, an element x of $\mathcal{F}(M)$ belongs to $\tilde{P_J}$ if and only if at least one of its components involves at least one factor $x_j (j \in J)$.It is now clear that, using the notation of [3], Definition 4.6 and Theorem 4.7,

$$\psi_M(P_J) = \tilde{}_J \ .$$

We hereby recover the description of $MaxSpec(B_1[x_1, ..., x_n])$ given in [1] (Theorems 4.7, 4.8 and 4.10).

The following result will be useful

Theorem 2.3. *Any proper saturated ideal of a B_1–algebra A is contained in a saturated prime ideal of A.*

Proof. Let J be a proper saturated ideal of A ; as $I(\mathcal{R}_J) = \overline{J} = J \neq A$, $\mathcal{R}_J \neq \mathcal{C}_0(A)$. By Zorn's Lemma, one has $\mathcal{R}_J \leq \mathcal{R}$ for some $\mathcal{R} \in MaxSpec(A)$. According to Theorem 1.1, $\mathcal{R} = \alpha(\mathcal{P}) = \mathcal{S}_{\mathcal{P}}$ for a saturated prime ideal $\mathcal{P}$ of A, therefore $\mathcal{R}_{\mathcal{J}} \leq \mathcal{S}_{\mathcal{P}}$ and

$$J = \overline{J} = I(\mathcal{R}_J) \subseteq I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P} \ .$$

□

3. Functorial properties of spectra

Let $\varphi : A \to C$ denote a morphism of B_1–algebras, and let $\mathcal{R} \in Spec(C)$. We define a binary relation $\tilde{\varphi}(\mathcal{R})$ on A by :

$$\forall (a, a') \in A^2 \ a\tilde{\varphi}(\mathcal{R})a' \equiv \varphi(a)\mathcal{R}\varphi(a') \ .$$

It is clear that $\tilde{\varphi}(\mathcal{R})$ is a congruence on A, and that

$$I(\tilde{\varphi}(\mathcal{R})) = \varphi^{-1}(I(\mathcal{R})) \ .$$

In particular $I(\tilde{\varphi}(\mathcal{R}))$ is a prime ideal of A, hence $\tilde{\varphi}(\mathcal{R}) \in Spec(A)$: $\tilde{\varphi}$ maps $Spec(C)$ into $Spec(A)$. Let $F := V(S)$ be a closed subset of $Spec(A)$, and let $\mathcal{R} \in Spec(C)$; then $\mathcal{R} \in \tilde{\varphi}^{-1}(F)$ if and only if $\tilde{\varphi}(\mathcal{R}) \in F$, that is $S \subseteq I(\tilde{\varphi}(\mathcal{R}))$, or $S \subseteq \varphi^{-1}(I(\mathcal{R}))$, *i.e.* $\varphi(S) \subseteq I(\mathcal{R})$, or $\mathcal{R} \in V(\varphi(S))$. Therefore $\tilde{\varphi}^{-1}(F) = V(\varphi(S))$ is closed in $Spec(C)$: $\tilde{\varphi}$ is continuous.

Furthermore, for $\varphi : A \to C$ and $\psi : C \to D$ one has

$$\widetilde{\psi \circ \varphi} = \tilde{\varphi} \circ \tilde{\psi} : Spec(D) \to Spec(A) \ .$$

It follows that the equations $\mathcal{H}(A) = Spec(A)$ and $\mathcal{H}(\varphi) = \tilde{\varphi}$ define a contravariant functor $\mathcal{H}$ from $\mathcal{Z}_a$ to $\mathcal{SP}$.

Let J denote an ideal in C, and let us assume $a\mathcal{R}_{\varphi^{-1}(J)}a'$; then there is $x \in \varphi^{-1}(J)$ with $a + x = a' + x$. Then $\varphi(x) \in J$ and

$$\begin{aligned} \varphi(a) + \varphi(x) &= \varphi(a + x) \\ &= \varphi(a' + x) \\ &= \varphi(a') + \varphi(x) \end{aligned}$$

whence $\varphi(a)\mathcal{R}_J\varphi(a')$ and $a\tilde{\varphi}(\mathcal{R}_J)a'$. We have established

Proposition 3.1. *Let A and C denote B_1–algebras, $\varphi : A \to C$ a morphism and J an ideal of C : then*

$$\mathcal{R}_{\varphi^{-1}(J)} \leq \tilde{\varphi}(\mathcal{R}_J) \ .$$

Theorem 3.2. *Let A and C denote two B_1–algebras, and $\varphi : A \to C$ a morphism. Then $\tilde{\varphi} : Spec(C) \to Spec(A)$ maps $MaxSpec(C)$ into $MaxSpec(A)$, and the diagram*

$$\begin{array}{ccc} Pr_s(C) & \stackrel{\varphi^{-1}}{\to} & Pr_s(A) \\ \downarrow \alpha_C & & \downarrow \alpha_A \\ MaxSpec(C) & \stackrel{\tilde{\varphi}}{\to} & MaxSpec(A) \end{array}$$

commutes.

Proof. Let $\mathcal{P} \in Pr_s(C)$, then, for all $(a, a') \in A^2$

$$\begin{aligned} a\tilde{\varphi}(\mathcal{S}_{\mathcal{P}})a' &\iff \varphi(a)\mathcal{S}_{\mathcal{P}}\varphi(a') \\ &\iff (\varphi(a) \in \mathcal{P} \text{ and } \varphi(a') \in \mathcal{P})\text{or}(\varphi(a) \notin \mathcal{P} \text{ and } \varphi(a') \notin \mathcal{P})) \\ &\iff (a \in \varphi^{-1}(\mathcal{P}) \text{ and } a' \in \varphi^{-1}(\mathcal{P})))\text{or}(a \notin \varphi^{-1}(\mathcal{P}) \text{ and } a' \notin \varphi^{-1}(\mathcal{P}))) \\ &\iff a\mathcal{S}_{\varphi^{-1}(\mathcal{P})}a' \ . \end{aligned}$$

Therefore

$$\begin{aligned}
(\tilde{\varphi} \circ \alpha_C)(\mathcal{P}) &= \tilde{\varphi}(\alpha_C(\mathcal{P})) \\
&= \tilde{\varphi}(\mathcal{S}_{\mathcal{P}}) \\
&= \mathcal{S}_{\varphi^{-1}(\mathcal{P})} \\
&= \alpha_A(\varphi^{-1}(\mathcal{P})) \\
&= (\alpha_A \circ \varphi^{-1})(\mathcal{P})
\end{aligned}$$

whence $\tilde{\varphi} \circ \alpha_C = \alpha_A \circ \varphi^{-1}$.

Incidentally we have proved that $\tilde{\varphi}$ maps $MaxSpec(C) = \alpha_C(Pr_s(C))$ into $\alpha_A(Pr_s(A)) = MaxSpec(A)$, *i.e.* the first assertion. $\square$

4. Nilpotent radicals and prime ideals

The usual theory generalizes without major problem to B_1–algebras.

Theorem 4.1. *In the B_1–algebra A,let us define*

$$Nil(A) := \{x \in A|(\exists n \geq 1)x^n = 0\} \ .$$

Then $Nil(A)$ is a saturated ideal of A, and one has

$$\bigcap_{\mathcal{P}\in Pr(A)} \mathcal{P} = \bigcap_{\mathcal{P}\in Pr_s(A)} \mathcal{P} = Nil(A) \ .$$

Proof. Let $M := \bigcap_{\mathcal{P}\in Pr(A)} \mathcal{P}$ and $N = \bigcap_{\mathcal{P}\in Pr_s(A)} \mathcal{P}$. If $x \in Nil(A)$ and $\mathcal{P} \in Pr(A)$, then, for some $n \geq 1$, $x^n = 0 \in \mathcal{P}$, whence (as $\mathcal{P}$ is prime) $x \in \mathcal{P}$: $Nil(A) \subseteq M$.

As $Pr_s(A) \subseteq Pr(A)$, we have $M \subseteq N$.

Let now $y \notin Nil(A)$; then

$$(\forall n \in \mathbf{N})\ y^n \neq 0 \ .$$

Define

$$\mathcal{E} := \{J \in Id_s(A)|(\forall n \geq 0)x^n \notin J\}.$$

This set is nonempty ($\{0\} \in \mathcal{E}$) and inductive for $\subseteq$, therefore, by Zorn's Lemma, there exists a maximal element $\mathcal{P}$ of $\mathcal{E}$. As $1 = x^0 \notin \mathcal{P}$, $\mathcal{P} \neq A$.

Let us assume $ab \in \mathcal{P}$, $a \notin \mathcal{P}$ and $b \notin \mathcal{P}$; then $\overline{\mathcal{P} + Aa}$ and $\overline{\mathcal{P} + Ab}$ are saturated ideals of A strictly containing $\mathcal{P}$, whence there exists two integers m and n with $x^m \in \overline{\mathcal{P} + Aa}$ and $x^n \in \overline{\mathcal{P} + Ab}$. By definition of the closure of an ideal, there are $u = p_1 + \lambda a \in \mathcal{P} + Aa$ and $v = p_2 + \mu b \in \mathcal{P} + Ab$ such that $x^m + u = u$ and $x^n + v = v$. Then

$$ub = p_1 b + \lambda(ab) \in \mathcal{P}$$

and

$$x^m b + ub = (x^m + u)b = ub \ ,$$

whence, as $\mathcal{P}$ is saturated, $x^m b \in \mathcal{P}$.

Then

$$x^m v = x^m p_2 + \mu x^m b \in \mathcal{P} \ ;$$

as

$$\begin{aligned} x^{m+n} + x^m v &= x^m(x^n + v) \\ &= x^m v \ , \end{aligned}$$

we obtain $x^{m+n} \in \mathcal{P}$, a contradiction.

Therefore $\mathcal{P}$ is prime and saturated and $x = x^1 \notin \mathcal{P}$, whence $x \notin N$. We have proved that $N \subseteq Nil(A)$, whence $M = N = Nil(A)$. □

Corollary 4.2.

$$Nil(A) = \bigcap_{\mathcal{P}\in Pr(A)} \overline{\mathcal{P}} \ .$$

Proof.

$$\begin{aligned} Nil(A) &= \bigcap_{\mathcal{P}\in Pr(A)} \mathcal{P} \text{(by Theorem 3.1)} \\ &\subseteq \bigcap_{\mathcal{P}\in Pr(A)} \overline{\mathcal{P}} \\ &\subseteq \bigcap_{\mathcal{P}\in Pr_s(A)} \overline{\mathcal{P}} \\ &= \bigcap_{\mathcal{P}\in Pr_s(A)} \mathcal{P} \\ &= Nil(A) \text{(also by Theorem 3.1).} \end{aligned}$$

□

Definition 4.3. For I an ideal of A, we define the *root* $r(I)$ of I by

$$r(I) := \{x \in A | (\exists n \geq 1) x^n \in I\}.$$

Lemma 4.4. *(i) $r(I)$ is an ideal of A.*
(ii) $\overline{r(I)} \subseteq r(\overline{I})$; in particular, if I is saturated then so is $r(I)$.
(iii) $r(\{0\}) = Nil(A)$.

Proof. (i) Obviously, $0 \in r(I)$.
If $x \in r(I)$ and $y \in r(I)$, then $x^m \in I$ for some $m \geq 1$ and $y^n \in I$ for some $n \geq 1$, whence

$$\begin{aligned} (x+y)^{m+n-1} &= \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} x^j y^{m+n-1-j} \\ (&= \sum_{j=0}^{m+n-1} x^j y^{m+n-1-j}) \\ &\in I\ , \end{aligned}$$

as $x^j \in I$ for $j \geq m$ and $y^{m+n-1-j} \in I$ for $j \leq m-1$ (as, then, $m+n-1-j \geq n$). Therefore $x+y \in r(I)$.
For $a \in A$, $(ax)^m = a^m x^m \in I$, whence $ax \in r(I)$. Therefore $r(I)$ is an ideal of A.

(ii) Let $x \in \overline{r(I)}$ then there is $u \in r(I)$ such that $x + u = u$, and there is $n \geq 1$ such that $u^n \in I$. Let us show by induction on $j \in \{0, ..., n\}$ that $u^{n-j}x^j \in \overline{I}$. This is clear for $j = 0$. Let then $j \in \{0, ..., n-1\}$, and assume that $u^{n-j}x^j \in \overline{I}$; then

$$\begin{aligned} u^{n-j-1}x^{j+1} + u^{n-j}x^j &= u^{n-j-1}x^j(x+u) \\ &= u^{n-j-1}x^j u \\ &= u^{n-j}x^j\ , \end{aligned}$$

whence $u^{n-j-1}x^{j+1} \in \overline{\overline{I}} = \overline{I}$. Thus, for $j = n$, we obtain

$$x^n = u^{n-n}x^n \in \overline{I}\ ,$$

whence $x \in r(\overline{I})$.

If I is saturated, then

$$\begin{aligned} r(I) &\subseteq \overline{r(I)} \\ &\subseteq r(\overline{I})(\text{by the above}) \\ &= r(I) \ , \end{aligned}$$

whence $r(I) = \overline{r(I)}$ is saturated.

(iii) That assertion is obvious.

□

Proposition 4.5. *For each saturated ideal I of the B_1–algebra A*

$$r(I) = \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P} \ .$$

Remark 4.6. For $I = \{0\}$, this is part of Theorem 4.1.

Proof. Let $x \in r(I)$, and let $\mathcal{P} \in Pr_s(A)$ with $I \subseteq \mathcal{P}$; then, for some $n \geq 1$ $x^n \in I$, whence $x^n \in \mathcal{P}$ and $x \in \mathcal{P}$:

$$r(I) \subseteq \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P} \ .$$

Let now $y \in A$, $y \notin r(I)$, and denote by π the canonical projection

$$\pi : A \twoheadrightarrow \tilde{A} := \frac{A}{\mathcal{R}_I} \ .$$

As I is saturated, one has

$$\forall n \geq 1 \ y^n \notin \overline{I} \ ,$$

whence

$$\forall n \geq 1 y^n \not\mathcal{R}_I 0 \ ,$$

or

$$\forall n \geq 1 \ \pi(y)^n = \pi(y^n) \neq \overline{0} \ .$$

Therefore $\pi(y) \notin Nil(\tilde{A})$, whence, according to Theorem 3.1, there exists a saturated prime ideal $\tilde{\mathcal{P}}$ of $\tilde{A}$ such that $\pi(y) \notin \tilde{\mathcal{P}}$. But then $\mathcal{P} := \pi^{-1}(\tilde{\mathcal{P}})$ is a saturated prime ideal of A containing I with $y \notin \mathcal{P}$, whence

$$y \notin \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P} \ .$$

□

5. Topology of spectra

We can now establish the basic topological properties of the spectra $Pr_s(A)$ (analogous, in our setting, to Corollary 1.1.8 and Proposition 1.1.10(ii) of [1]).

Theorem 5.1. *$Pr_s(A)$ and $MaxSpec(A)$ are T_0 and quasi–compact.*

Proof. According to Theorem 1.1, $Pr_s(A)$ and $MaxSpec(A)$ are homeomorphic, therefore it is enough to establish the result for $Pr_s(A)$.

Let $\mathcal{P}$ and $\mathcal{Q}$ denote two different points of $Pr_s(A)$; then either $\mathcal{P} \nsubseteq \mathcal{Q}$ or $\mathcal{Q} \nsubseteq \mathcal{P}$. Let us for instance assume that $\mathcal{P} \nsubseteq \mathcal{Q}$; then $\mathcal{Q} \notin W(\mathcal{P})$; set

$$O := Pr_s(A) \cap (Pr(A) \setminus W(\mathcal{P})) \ .$$

Then O is an open set in $Pr_s(A)$, $\mathcal{Q} \in O$ and, obviously, $\mathcal{P} \notin O$. Therefore $Pr_s(A)$ is T_0.

Let $(U_i)_{i\in I}$ denote an open cover of $Pr_s(A)$:

$$Pr_s(A) = \bigcup_{i\in I} U_i \ ;$$

each $Pr_s(A) \setminus U_i$ is closed, whence $Pr_s(A) \setminus U_i = Pr_s(A) \cap W(S_i)$ for some subset S_i of A. Therefore $Pr_s(A) \cap (\bigcap_{i\in I} W(S_i)) = \emptyset$, *i.e.* $Pr_s(A) \cap W(\bigcup_{i\in I} S_i) = \emptyset$. Therefore $Pr_s(A) \cap W(\overline{< \bigcup_{i\in I} S_i >}) = \emptyset$, whence, according to Theorem 2.3, $\overline{< \bigcup_{i\in I} S_i >} = A$. Let $J =< \bigcup_{i\in I} S_i >$; then $1 \in \overline{J}$, hence there is $x \in J$ such that $1 + x = x$. Furthermore, there exist $n \in \mathbf{N}$, $(i_1, ..., i_n) \in I^n$, $x_{i_k} \in S_{i_k}$ and $(a_1, ..., a_n) \in A^n$ such that $x = a_1 x_{i_1} + ... + a_n x_{i_n}$. But then

$$1 + a_1 x_{i_1} + ... + a_n x_{i_n} = a_1 x_{i_1} + ... + a_n x_{i_n}$$

whence

$$1 \in \overline{\{x_{i_1}, ..., x_{i_n}\}} \subseteq \overline{\bigcup_{j=1}^{n} S_{i_j}}$$

and

$$\overline{\bigcup_{j=1}^{n} S_{i_j}} = A \ .$$

It follows that

$$Pr_s(A) \cap W(\bigcup_{j=1}^{n} S_{i_j}) = \emptyset \ ,$$

that is

$$Pr_s(A) \cap \bigcap_{j=1}^{n} W(S_{i_j}) = \emptyset \ ,$$

or

$$Pr_s(A) = \bigcup_{j=1}^{n} U_{i_j} \ :$$

$Pr_s(A)$ is quasi–compact. □

For $f \in A$, let

$$\begin{aligned} D(f) \ &:= \ Pr_s(A) \setminus (Pr_s(A) \cap W(\{f\}) \\ &= \ \{\mathcal{P} \in Pr_s(A) | f \notin \mathcal{P}\}. \end{aligned}$$

Proposition 5.2. (1) *Each $D(f)(f \in A)$ is open and quasi–compact in $Pr_s(A)$.*
(2) *The family $(D(f))_{f\in A}$ is an open basis for $Pr_s(A)$.*

Proof. (1) The openness of $D(f)$ is obvious. Let us assume $D(f) = \bigcup_{i\in I} U_i$, where the U_i's are open sets in $D(f)$. Each U_i can be written as

$$U_i = D(f) \cap V_i \ ,$$

for V_i an open set in $Pr_s(A)$, *i.e.* $Pr_s(A) \setminus V_i = W(S_i)$ for S_i a subset of A. Then

$$D(f) \subseteq \bigcup_{i\in I} V_i = Pr_s(A) \setminus (\bigcap_{i\in I} W(S_i)) \ ,$$

whence

$$Pr_s(A) \cap W(\bigcup_{i\in I} S_i) \subseteq W(\{f\}) \ ,$$

that is, setting

$$S := \bigcup_{i\in I} S_i \ ,$$

$$f \in \bigcap_{\mathcal{P}\in W(S)\cap Pr_s(A)} \mathcal{P} = \bigcap_{\mathcal{P}\in Pr_s(A);S\subseteq\mathcal{P}} \mathcal{P} \ .$$

Therefore, by Proposition 2.4, $f \in r(\overline{<S>})$: there is $n \geq 1$ such that $f^n \in \overline{<S>}$. Thus, there is $g \in <S>$ such that $f^n + g = g$; one has $g = \sum_{j=1}^m a_j s_j$ for $a_j \in A$, $s_j \in S$; for each $j \in \{1, ..., m\}$, $s_j \in S_{i_j}$ for some $i_j \in I$. Let $S_0 = \{s_1, ..., s_m\}$; then $g \in < \bigcup_{j=1}^n S_{i_j} >$, whence $f^n \in \overline{< \bigcup_{j=1}^n S_{i_j} >}$, and reading the above argument in reverse order with S replaced by $\bigcup_{j=1}^n S_{i_j}$ yields that

$$D(f) = \bigcup_{j=1}^m U_{i_j} \ ,$$

whence the quasi–compactness of $D(f)$.

(2) Let U be an open set in $Pr_s(A)$, and $\mathcal{P} \in U$. We have $Pr_s(A) \setminus U = Pr_s(A) \cap W(S)$ for some subset S of A. As $\mathcal{P} \notin W(S)$, $S \nsubseteq \mathcal{P}$, whence there is an $s \in S$ with $s \notin \mathcal{P}$. It is now clear that $\mathcal{P} \in D(s)$ and

$$D(s) \subseteq Pr_s(A) \setminus W(S) = U \ .$$

□

6. Remarks on the one–generator case

Let us now consider the case of a nontrivial monogenic B_1–algebra containing strictly B_1, *i.e.* $A = \dfrac{B_1[x]}{\sim}$ is a quotient of the free algebra $B_1[x]$ with $x \nsim 0$, $x \nsim 1$. Denote by α the image of x in A ; then $\alpha \notin \{0, 1\}$, and α generates A as a B_1–algebra.

Let us suppose that, for some $(u, v) \in A^2$, $\alpha u = 1 + \alpha v$; then α is not nilpotent, as from $\alpha^n = 0$ would follow $0 = \alpha^n v = \alpha^{n-1}(\alpha v) = \alpha^{n-1}(1+\alpha u) = \alpha^{n-1} + \alpha^n u = \alpha^{n-1}$, whence $\alpha^{n-1} = 0$ and, by induction, $1 = \alpha^0 = 0$, a contradiction.

Therefore three cases may appear

(i) α is nilpotent.
(ii) α is not nilpotent and there does not exist $(u, v) \in A^2$ such that $\alpha u = 1+\alpha v$.
(iii) (α is not nilpotent) and there exist $(u, v) \in A^2$ such that $\alpha u = 1 + \alpha v$.

In case (i), any prime ideal of A must contain α, hence contain αA; the ideal αA is, according to the above remark, saturated, and is not contained in a strictly bigger saturated ideal other than A itself (in both cases, as any element of A not in αA is of the shape $1 + \alpha x$). Therefore $Pr_s(A) = \alpha A$, whence $Nil(A) = \alpha A$. In this case we see that

$$\frac{A}{\mathcal{R}_{Nil(A)}} \simeq B_1 \ .$$

In cases (ii) and (iii), no power of α belongs to $Nil(A)$; as $Nil(A)$ is saturated, it follows that $Nil(A) = \{0\}$. In fact, A is integral, whence $\{0\} \in Pr_s(A)$. If $\mathcal{P} \in Pr_s(A)$ and $\mathcal{P} \neq \{0\}$, then $\mathcal{P}$ contains some power of α, hence contains α, hence contains αA. As above we see that $\mathcal{P} = \alpha A$; but, in case (iii), αA is not saturated. In case (ii) it is easy to see that αA is prime and saturated. Therefore :

in case (ii), $Pr_s(A) = \{\{0\}, \alpha A\}$; $\{0\}$ is a generic point ($\overline{\{\{0\}\}} = Pr_s(A)$), and αA a "closed point "($\{\alpha A\}$ is closed) ;

in case (iii), $Pr_s(A) = \{\{0\}\}$.

One may remark that $B_1[x]$ itself falls into case (ii).

In [3], pp. 75–79, we have enumerated (up to isomorphism) monogenic B_1–algebras of cardinality ≤ 5. It is easy to see where these algebras fall in the above classification ; we keep the numbering used in [3]. Let then $3 \leq |A| \leq 5$. We have the following repartition

Case (*i*) : (6),(8),(12),(15),(18),(24)
Case (*ii*): (7),(10),(11),(16),(19),(25),(26)
Case (*iii*): (5),(9),(13),(14),(17),(20),(21),(22),(23),(27),(28)

7. Bibliography

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