

CLASS NUMBER PROBLEMS AND LANG CONJECTURES

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ABSTRACT. Given a square-free integer d we introduce an affine hypersurface whose integer points are in one-to-one correspondence with ideal classes of the quadratic number field $\mathbf{Q}(\sqrt{d})$. Using this we relate class number problems of Gauss to Lang conjectures.

1. INTRODUCTION

A binary quadratic form is a homogeneous polynomial of degree two in two variables with integer coefficients. In *Disquisitiones Arithmeticae*, [16], without the terminology and tools that are in our disposal today, Gauß defined the action of the modular group, $\mathrm{PSL}_2(\mathbf{Z})$, on binary quadratic forms and showed (among many other things) that discriminant of a binary quadratic form is invariant under this action. He then defined a binary operation on the $\mathrm{PSL}_2(\mathbf{Z})$ -classes of binary quadratic forms whose ratio of discriminants is a perfect square. Under this operation, the set of binary quadratic forms for which the ratio of the discriminant is the perfect square of a rational number forms a group.

Let K be a number field, that is a finite extension of $\mathbf{Q}$. The set of elements of K that are roots of monic polynomials in $\mathbf{Z}[x]$ is called the ring of integers of K and denoted by $\mathcal{O}_K$. $\mathcal{O}_K$ is a ring whose properties have been exploited by the works of many including Kummer, Dirichlet, Dedekind and Weber. It turns out that unlike $\mathbf{Z}$, $\mathcal{O}_K$ fails to be a unique factorization domain. It is classical to measure this failure by the ideal class group, H_K , of K , defined as the quotient of fractional ideals in K by the principal ideals. The number of elements of H_K is denoted usually by h_K and whenever is equal to 1, $\mathcal{O}_K$ is a unique factorization domain. An orientation on the set of fractional ideal, when considered as ideals generated by two elements, can be defined. This is an extension of degree at most 2 of the classical class group H_K , usually denoted by H_K^+ and called the narrow class group. Elements of H_K^+ are called narrow ideal classes, and the size of H_K^+ , denote by h_K^+ is called the narrow class number

Whenever the degree of the extension $K/\mathbf{Q}$ is 2 then K is said to be a quadratic number field. In this case, one can find a square-free integer δ so that $K \cong \mathbf{Q}(\sqrt{\delta})$. Associating each narrow ideal class a binary quadratic form and vice versa gives a one to one correspondence between narrow ideal classes in K and binary quadratic forms whose discriminant's square-free part is equal to δ .

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Using this, for negative discriminants¹ (i.e. $d < 0$) Gauß gave a list² comprising of 9 imaginary quadratic number fields with class number one which he believed to be complete. This fact was later proved by Heegner³, [21], Baker, [1], Siegel, [31] and Stark, [33]. There are only 18 imaginary quadratic number fields with class number 2 as shown by Baker, [2], and by Stark, [34, 35]. In general, given a positive integer n , determining the number of quadratic number fields having class number n is called n^{th} class number problem. As far as author is aware, class number problem for imaginary quadratic number fields is known up to $n = 100$, [41].

For positive discriminants, i.e. for real quadratic number fields, class number problems (i.e. finding the class number of $\mathbf{Q}(\sqrt{d})$) has a completely different nature, of which Gauß was aware. He conjectured the existence of infinitely many positive integers d so that the class number of $\mathbf{Q}(\sqrt{d})$ is 1. In fact, for any positive integer N , the question of whether the number of quadratic number fields with class number N is finite or not is open. Among others, the existence of $\log \varepsilon$, where ε is a fundamental unit of $\mathbf{Q}(\sqrt{d})$, as a term in the analytic class number formula is a difficulty in attacking problems concerning real quadratic number fields.

Given a complex space X the two intrinsic pseudo-metrics on X , namely Kobayashi and Carathéodory, are in a sense dual to each other⁴. These pseudo-metrics are distance decreasing under holomorphic maps and behave well under topological coverings. This property reflects the geometric nature of the two pseudo-metrics and indeed it is possible to argue that Kobayashi pseudo-metric has its roots in the uniformization theorem and Schwarz Lemma (and its generalizations), [24].

A complex space is called *Kobayashi hyperbolic* if the Kobayashi pseudo-distance is a metric. For instance, $\mathbf{C}$ is not Kobayashi hyperbolic whereas the unit disk in $\mathbf{C}$, denoted $\mathbb{D}$, is. This implies that algebraic curves of genus $g \geq 2$ are all Kobayashi hyperbolic whereas elliptic curves are not. As comes to algebraic varieties, it turns out that hyperbolicity is closely related to the degree of the algebraic variety in question. In fact, Kobayashi conjectures that if the degree of a hypersurface, X , in $\mathbf{P}^{n+1}$ is at least $2n + 2$ then X is Kobayashi hyperbolic. In [10] Demailly has claimed a proof of the Kobayashi conjecture when the hypersurface is *very general*, and in [32] Siu proved the statement for hypersurfaces of sufficiently high degree. In [26] Lang conjectures that Kobayashi hyperbolic varieties should have at most finitely many rational points. The case of algebraic curves provides a beautiful set of examples, see Table 1.

In this work, we present a completely different approach to class number problems for real quadratic number fields. The main tool, called *çark*, is an infinite bipartite ribbon graph embedded in a conformal annulus, see Figure 1 for an example. A *çark* is an infinite version of a dessin d'enfant, [20]. It has a unique cycle of finite length, called its spine. The fact that relates hyperbolicity to class number problems is that there is a one to one correspondence between the set of $\text{PSL}_2(\mathbf{Z})$ -classes of binary quadratic forms positive discriminant and *çarks*, [37]. In a work in progress, [43], it is shown that for a given square-free positive integer d indefinite binary quadratic forms whose discriminant's square-free part is equal

¹The discriminant of $\mathbf{Q}(\sqrt{d})$ is $4d$ or d depending on the class of d in $\mathbf{Z}/4\mathbf{Z}$

² $\mathbf{Q}(\sqrt{-d})$; where $d = -1, -2, -3, -7, -11, -19, -43, -67, -163$.

³Heegner's proof had some gaps, which are filled by Deuring, [12].

⁴Reader interested in intrinsic metrics on complex spaces may consult the article by Kobayashi, [22]

to δ can be realized as integral points on an affine hypersurface, $\mathcal{C}_\delta$, to which we will refer as *çark-hypersurface*. The hypersurface $\mathcal{C}_\delta$ admits an action of a certain abelian group which depends only on the integer δ . By projectivizing $\mathcal{C}_\delta$ one obtains a one to one correspondence between certain ideal classes and rational points on the projective çark-hypersurface. The degree of $\mathcal{C}_\delta$ is exactly equal to twice the dimension of its ambient space. This surface is expected to be Kobayashi hyperbolic (see Theorem 3.5), and conjecturally by Lang these surfaces have finitely many rational points.

The paper is organized as follows: In the next section we will outline basic facts and correspondences mentioned in the introduction, namely those between binary quadratic forms, narrow ideal classes and çarks. In Section 3, we provide necessary information to state Lang's conjecture relating geometry to arithmetic. More precisely, we introduce the Kobayashi pseudo-metric, investigate the hyperbolicity of algebraic curves and then review results about the hyperbolicity of hypersurfaces of high degree. In the final section we establish the one-to-one correspondence between rational points of a çark hypersurface of discriminant δ , denoted $\mathfrak{C}_\delta$ and narrow ideal classes in $\mathbf{Q}(\sqrt{\delta})$.

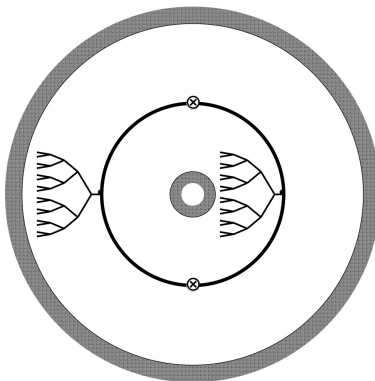


FIGURE 1. A çark

2. ÇARKS, BINARY QUADRATIC FORMS AND CLASS GROUPS

This section is devoted to introducing foundational material on çarks, quadratic number fields and binary quadratic forms. We refer to [17] and [25] for precise definitions and facts concerning finite graphs on surfaces, dessins and related issues, and to [42, §10] and [4] for statements of theorems related to quadratic number fields and binary quadratic forms.

2.1. Çarks. Let us fix the generators of the modular group, $\mathrm{PSL}_2(\mathbf{Z})$, as:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

S is of order 2 and L is of order 3 and these elements generate the modular group freely, i.e. the modular group is isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. One can define edges of *bipartite Farey tree*, denoted $\mathcal{F}$, as the set of elements of $\mathrm{PSL}_2(\mathbf{Z})$ and the set of vertices as the orbits of the subgroups $\langle S \rangle$ and $\langle L \rangle$. The incidence relation is

defined as intersection, namely there is an edge between two vertices if and only if the intersection is non-empty. As there are no relations between S and L , there are no loops. By construction $\mathcal{F}$ is a tree and bipartite. Vertices of degree 2, which are orbits of $\langle S \rangle$, are denoted by $\otimes$ and vertices of degree 3, which are orbits of $\langle L \rangle$, are denoted by $\bullet$.

For an element $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z})$ the map $M \cdot z := \frac{pz+q}{rz+s}$ defines an action of the modular group acts on the upper half plane, $\mathbb{H} := \{z \in \mathbf{C} : \mathrm{im}(z) > 0\}$. In particular, S maps each z to $-1/z$ and L maps z to $1 - 1/z$. If we denote the fixed point of S by $\otimes$ and the fixed point of L by $\bullet$ then the orbit of the geodesic, γ , in $\mathbb{H}$, connecting $\otimes$ to $\bullet$ is the topological realization of $\mathcal{F}$. Note that $\mathcal{F}$ is a ribbon graph on which the orientation around each vertex of degree 3 is induced by the ordering of $(1, L, L^2)$. The associated surface of $\mathcal{F}$ is homeomorphic to $\mathbb{H}$. For details we refer to [37].

One can apply the same construction to any subgroup, Γ , of the modular group using cosets of the subgroup in question and obtain a bipartite ribbon graph, $\Gamma \backslash \mathcal{F}$. The genus, puncture(s) and boundary component(s) of the corresponding surfaces are determined completely by the graph $\Gamma \backslash \mathcal{F}$.

If Γ is of finite index then $\Gamma \backslash \mathcal{F}$ is finite and is called a dessin d'enfant, [20]. In particular, there are no boundary components of the surface $\Gamma \backslash \mathbb{H}$ apart from the punctures. After "filling" the punctures, one obtains a compact Riemann surface or equivalently an algebraic curve, in a unique fashion. This curve admits equations whose coefficients are from a number field as a result of a well-known theorem of Belyi, [3]⁵. Conjugate subgroups give rise to isomorphic ribbon graphs. The graphs corresponding to two conjugate subgroups differ only in their base edges. However, an algebraic curve admits infinitely many dessins, a situation which depends mostly on the graph, as investigated in [18].

The absolute Galois group acts on the set of algebraic curves defined over $\overline{\mathbf{Q}}$ and thus to finite bipartite ribbon graphs. This action is known to be faithful in genus 0, [30]. It is also faithful in genus one. Indeed, if E is an elliptic curve defined over $\overline{\mathbf{Q}}$ with j -invariant $j(E)$, then $j(E)$ is also algebraic. Choose a σ in the absolute Galois group which does not fix $j(E)$. Then as the dessin corresponding to E is defined over an extension of $\mathbf{Q}(j(E))$, σ cannot fix the dessin.

If one leaves aside the action of the absolute Galois group the above construction associating a subgroup a bipartite ribbon graph can still be considered, even for subgroups of infinite index. In particular, if we consider Γ a subgroup generated by only one element, denoted $\gamma \in \mathrm{PSL}_2(\mathbf{Z})$ then the typical properties of the resulting graph is determined by the classification of elements of $\mathrm{PSL}_2(\mathbf{Z})$. Recall that elements of $\mathrm{PSL}_2(\mathbf{Z})$ are divided into three types according to the absolute value of their traces (absolute trace, in short). Namely, elements whose absolute trace is less than 2 are called *elliptic*, exactly 2 are called *parabolic* and greater than two are called *hyperbolic*. As such groups are of infinite index, the corresponding surfaces associated to these ribbon graphs has boundary. If γ is elliptic the graph $\Gamma \backslash \mathcal{F}$ is a rooted tree, see Figure 2. The Riemann surface is a disk with an orbifold point at its *center*, of stabilizer of order 2 or 3 depending on the order of γ . If γ is parabolic or hyperbolic then the graph $\Gamma \backslash \mathcal{F}$ is not a tree. It has one loop to

⁵This conclusion is rather a result of the relation between the modular group and triangle group of signature (∞, ∞, ∞) . We refer to [7] for details.

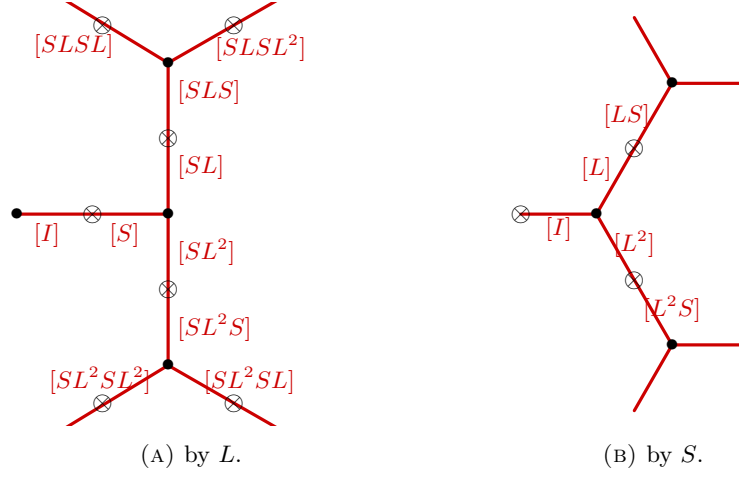


FIGURE 2. Elliptic quotients of bipartite Farey tree.

which we will refer as the *spine* of $\Gamma \backslash \mathcal{F}$. The associated Riemann surface has two boundary components in either case. If γ is parabolic, then there is one puncture and one boundary component homeomorphic to circle, S^1 . If γ is hyperbolic, then the boundary has two components each homeomorphic to S^1 . We refer to [36] for details. In the elliptic case there is a unique bipartite Farey tree attached to the root, expanding in the direction of the boundary of the disk. In parabolic and hyperbolic cases there is a finite number of bipartite Farey tree components attached to the spine from the degree 3 vertices pointing outward to the boundary components, see Figure 3a and Figure 3b.

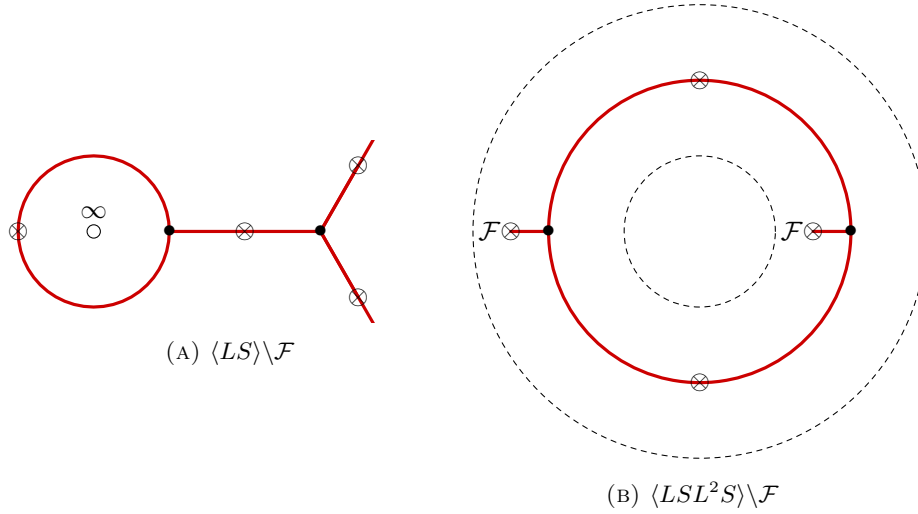


FIGURE 3. Examples of parabolic and hyperbolic subgroups.

Definition 2.1. For a subgroup, Γ , of the modular group generated by one *hyperbolic* element the ribbon graph $\Gamma \backslash \mathcal{F}$ is called a *çark*. If we mark the base edge of the çark then it is called a *base pointed çark*.

For a given çark, on traversing the spine of the çark in counterclockwise direction, one meets successive Farey branches expanding in the same (inner or outer) boundary component. Each such successive Farey branch will be referred to as a Farey bunch. We introduce the following notation: a Farey bunch of size n (i.e. a bunch containing n successive Farey branches) pointing in the inner boundary circle is denoted by a $+n$ and similarly a Farey bunch of size n pointing in the outer boundary circle is denoted by a $-n$. For instance, çark appeared in Figure 3b will be denoted by $[+1, -1]$. We will call a çark *primitive* if it does not possess any repetition when expressed in this notation. For instance, the $[-1, 2, -1, 2]$ is not a primitive çark whereas $[-1, 2]$ is.

2.2. Binary quadratic forms. An element $f(X, Y) \in \mathbf{Z}[X, Y]$ which is homogeneous of degree two will be referred to as a binary quadratic form. We also use the matrix form

$$(X \ Y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} (X \ Y)^t;$$

where a , b and c are integers and write $f = (a, b, c)$ for short. The matrix $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ is denoted by M_f . The discriminant of a binary quadratic form is defined as $\Delta(f) = b^2 - 4ac$. If $\Delta(f)$ is negative and $a > 0$ (resp. $a < 0$) then f is called positive (resp. negative) definite. Forms with positive discriminant are called indefinite. We will not consider forms with 0 discriminant as they can be written as a product of two linear forms. We will call a form *primitive* if the greatest common divisor of a , b and c is 1.

Modular group acts on the set of all binary quadratic forms as:

$$W \cdot M_f \mapsto W^t M_f W$$

The orbit under $\mathrm{PSL}_2(\mathbf{Z})$ of a binary quadratic form f is called its class and denoted by $[f]$. The discriminant of a form is invariant under this action. The stabilizer of a form under this action is called its automorphism group, which can be determined explicitly:

Theorem 2.1 ([4, Theorem 2.5.10 and Theorem 6.12.4]). *Let f be a binary quadratic form of non-zero discriminant. Then $\mathrm{Aut}(f)$ is isomorphic to*

$$\begin{aligned} &\mathbf{Z}/4\mathbf{Z} \text{ if } \Delta(f) = -4 \\ &\mathbf{Z}/6\mathbf{Z} \text{ if } \Delta(f) = -3 \\ &\mathbf{Z} \text{ if } \Delta(f) > 0 \end{aligned}$$

and trivial otherwise.

2.3. Narrow Ideal classes in quadratic number fields. Let K be a quadratic number field. Then there is an integer δ so that $K = \mathbf{Q}(\sqrt{\delta})$. The discriminant D of $K = \mathbf{Q}(\sqrt{\delta})$ is then equal to $D = 4\delta$ whenever $\delta \equiv 2, 3$ modulo 4 and $D = \delta$ whenever $\delta \equiv 1$ modulo 4. We assume without loss of generality that δ is square-free. Any element of K can be written as $\alpha = a + b\sqrt{\delta}$, with $a, b \in \mathbf{Q}$. The map sending α to $\bar{\alpha} = a - b\sqrt{\delta}$ is the unique non-trivial automorphism of K . Let $\mathcal{O}_\delta$ denote the ring of integers of K , i.e. the set of elements of K which are roots of

monic polynomials with integer coefficients. $\mathcal{O}_\delta$ is isomorphic to $\mathbf{Z} + \sqrt{\delta}\mathbf{Z}$ if d is congruent to 2 or 3 modulo 4 and is isomorphic to $\mathbf{Z} + \frac{1+\sqrt{\delta}}{2}\mathbf{Z}$ if δ is congruent to 1 modulo 4. A fractional ideal, $\mathfrak{a}$, of K is a finitely generated subgroup of K which is closed under multiplication by elements of $\mathcal{O}_\delta$, i.e. for which $\lambda \in \mathcal{O}_\delta$ and $a \in \mathfrak{a}$ implies $\lambda \cdot a \in \mathfrak{a}$. For any fractional ideal there is a non-unique natural number $n = n_{\mathfrak{a}}$ so that $n \cdot \mathfrak{a} \subset \mathcal{O}_\delta$. The norm, $N(\mathfrak{a})$ of a fractional ideal, $\mathfrak{a}$, is defined as $\frac{1}{n_{\mathfrak{a}}}N(n_{\mathfrak{a}}\mathfrak{a})$; where the norm of an ideal of $\mathcal{O}_\delta$ is defined as its index in $\mathcal{O}_\delta$. Norm of any fractional ideal is independent of the choice of $n_{\mathfrak{a}}$. Recall that ring of integers of a number field is a Dedekind domain, hence every ideal is generated by at most 2 elements. As a result, any fractional ideal is a two dimensional $\mathbf{Z}$ -module, i.e. for any fractional ideal $\mathfrak{a}$ there are elements α and β in K so that $\mathfrak{a} = \alpha\mathbf{Z} + \beta\mathbf{Z}$. The discriminant of $\mathfrak{a}$ is then defined as the square of the determinant of the matrix $\begin{pmatrix} \alpha & \beta \\ \bar{\alpha} & \bar{\beta} \end{pmatrix}$.

The product of two fractional ideals is again a fractional ideal, and in particular, the product of the ideal (1) with any other fractional ideal $\mathfrak{a}$ is equal to $\mathfrak{a}$. The inverse of a fractional ideal $\mathfrak{a}$ is given by $\frac{1}{N(\mathfrak{a})}\mathfrak{a}$. So the set of fractional ideals is an abelian group, denoted $I(\delta)$. The subset of principal ideals in $I(\delta)$ forms a subgroup of $I(\delta)$, denoted by $P(\delta)$. The quotient $H(\delta) := I(\delta)/P(\delta)$ is called the *class group* of $\mathbf{Q}(\sqrt{\delta})$ (or $\mathcal{O}_\delta$). Every element of this group is called an ideal class. The set $P^+(\delta) := \{(\xi) \in P(\delta) : N((\xi)) > 0\}$ is also subgroup of $I(\delta)$ contained in $P(\delta)$. The cardinality of the quotient $P(\delta)/P^+(\delta)$ is two whenever $\mathcal{O}_\delta$ contains a unit of norm -1 , and is one whenever $\mathcal{O}_\delta$ does not have a unit of norm -1 . In particular, if $\delta < 0$, i.e. if $\mathbf{Q}(\sqrt{\delta})$ is imaginary quadratic, then $\mathcal{O}_\delta$ cannot contain such a unit. The group $H^+(\delta) := I(\delta)/P^+(\delta)$ is called the *narrow class group* of $\mathcal{O}_\delta$. Every element in this group will be referred to as a *narrow ideal class*. The cardinality of the the class group is called *class number* and denoted by h_δ and that of the narrow class group is called the *narrow class number*, and denoted by h_δ^+ .

2.4. Çarks and Indefinite Binary Quadratic Forms. Given a base pointed çark, $\mathbb{C}$, starting from the base edge one goes in the direction of the spine, traverses once around the spine in the counterclockwise direction then comes back to the same edge. In doing so one records the following word in $\text{PSL}_2(\mathbf{Z})$: Once we arrive at the spine, on visiting a Farey bunch of size n , if the bunch expands in the direction of outer boundary then we write $(LS)^n$ and similarly on visiting a Farey bunch of size n expanding in the direction of inner boundary then we write $(L^2S)^n$ and outside the spine, on visiting a vertex of degree 3 we write LS if the turn is to left and write L^2S if the turn is to right. After completing the full turn, we obtain a word in S , L and L^2 which is an element of $\text{PSL}_2(\mathbf{Z})$, say $W = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then we define the corresponding binary quadratic form as

$$f_{\mathbb{C}}(X, Y) = \frac{r}{\delta}X^2 + \frac{s-p}{\delta}XY + \frac{-q}{\delta}Y^2;$$

where $\delta = \gcd(c, d - a, b)$. The form $(r, s - p, -q)$ will be called the corresponding binary quadratic form before normalization.

In conclusion, we have:

Theorem 2.2 ([37, Corollary 3.2]). *There is a one to one correspondence between base pointed primitive çarks and primitive indefinite binary quadratic forms.*

Since the conjugation action of $\mathrm{PSL}_2(\mathbf{Z})$ on its subgroups is realized as translation of the base edge on base pointed çarks, the element W obtained as described above is an automorphism of the corresponding indefinite binary quadratic form. Via this correspondence çarks inherit all the invariant of indefinite, in particular, we can talk about discriminant, Markov value, etc. of a çark.

Theorem 2.2 can be stated using the language of subgroups of the modular group. More precisely, there is a one-to-one correspondence between conjugacy classes of subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ generated by one hyperbolic element and çarks. Moreover, the conjugation action of $\mathrm{PSL}_2(\mathbf{Z})$ on itself amounts to translation of the base edge in the language of çarks.

Conversely, given an indefinite binary quadratic form, f , the group $\mathrm{Aut}(f)$ is a subgroup of the modular group generated by one hyperbolic element, see Theorem 2.1. The generator of this subgroup, call W , can be written as a word in S and L . Among all elements that are conjugate to W , choose word of minimal length⁶. This word is of even length and can be written as $W = (LS)^{n_1}(L^2S)^{m_1} \dots (LS)^{n_k}(L^2S)^{m_k}$; where none of n_i, m_j , $i, j \in \{1, 2, \dots, k\}$ is zero. Then the corresponding çark is the one given by $(n_1, -m_1, n_2, -m_2, \dots, n_k, -m_k)$. Not having chosen a base edge results in an action of cyclic group of order $2k$ on such tuples. We denote the corresponding class as $[n_1, -m_1, n_2, -m_2, \dots, n_k, -m_k]$. In this case, the çark is said to have *length* $2k$.

2.5. Indefinite Binary Quadratic Forms and Narrow Ideal Classes. Given a binary quadratic form $f = (a, b, c)$ of fundamental discriminant, the class containing the fractional ideal $1 \cdot \mathbf{Z} + \omega \cdot \mathbf{Z}$; where $\omega = \frac{b + \sqrt{\Delta(f)}}{2a}$ is associated to f . This map respects the action of the modular group on binary quadratic forms, i.e. equivalent forms are mapped onto equivalent narrow ideal classes.

Conversely, for a fractional ideal $\mathfrak{a}$ in a real quadratic number field, we consider the function $f_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathbf{Z}$ sending each element $\alpha \in \mathfrak{a}$ to $\frac{\alpha \bar{\alpha}}{N(\mathfrak{a})}$. If one chooses a $\mathbf{Z}$ -basis α_o, β_o for $\mathfrak{a}$ then $f_{\mathfrak{a}} = (\frac{\alpha_o \bar{\alpha}_o}{N(\mathfrak{a})}, \frac{\alpha_o \bar{\beta}_o + \bar{\alpha}_o \beta_o}{N(\mathfrak{a})}, \frac{\beta_o \bar{\beta}_o}{N(\mathfrak{a})})$ is a binary quadratic form. The discriminant of this form is equal to the discriminant of the corresponding real quadratic number field.

However, the binary quadratic form depends not only on the choice of basis, $\mathfrak{a} = (\alpha, \beta)$, but also on the ordering of the generating elements, for instance if for the basis (α, β) one gets $f = (a, b, c)$ then for the basis (β, α) one gets the form $f' = (c, b, a)$. For some forms this is indeed the case f is equivalent to f' , for example $f = (2, 8, -5)$ is equivalent to $(-5, 8, 2)$. Such forms are called ambiguous by Gauß. There are however forms, $f = (a, b, c)$ which are not equivalent under the action of $\mathrm{PSL}_2(\mathbf{Z})$ to $f' = (c, b, a)$. The equivalence of f and f' is equivalent to the existence of a unit of negative norm, i.e. a unit of norm -1 , in the group $\mathcal{O}_{\delta}^{\times}$. This is true if and only if the equation $X^2 - dY^2 = -1$ has integral solution. For instance, the form $f = (2, 8, -5)$ is of discriminant 104, i.e. it represents an element in $H(26)$. The integer $5 + \sqrt{26}$ is of norm -1 , and the pair $(5, 1)$ is a solution of the equation $X^2 - 26Y^2 = -1$. In terms of çarks this translates as the corresponding çark being symmetric with respect to its spine, i.e. using the notation introduced

⁶The length of an element is defined as the number of letters it has. An element of minimal length exists because the set of lengths is a subset of $\mathbf{N}$.

in Section 2.4 this amounts to

$$\begin{aligned} [n_1, -m_1, n_2, -m_2, \dots, n_k, -m_k] &= -[n_1, -m_1, n_2, -m_2, \dots, n_k, -m_k] \\ &:= [-n_1, m_1, -n_2, m_2, \dots, -n_k, m_k]. \end{aligned}$$

2.6. Narrow Ideal Classes and Çarks. The modular group acts on the narrow class group of any quadratic number field $K = \mathbf{Q}(\sqrt{\delta})$ of fundamental discriminant in the following manner:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot (\alpha, \beta) := (p\alpha + r\beta, q\alpha + s\beta).$$

As mentioned in the previous section, even though $(\alpha, \beta) = (\beta, \alpha)$ the corresponding binary quadratic forms may not be equivalent. To remedy this situation, one identifies the set of narrow ideal classes with the set of ordered pairs (α, β) having positive norm, i.e. those pairs (α, β) satisfying $\bar{\alpha}\beta - \alpha\bar{\beta} > 0$.

Example 2.1. For some real quadratic number fields, e.g. 2, 5, 10, 13, etc., the orbit of a fractional ideal (α, β) is equal to the orbit of (β, α) . On the other hand there are cases where the orbits of (α, β) and (β, α) are disjoint, e.g. 3, 6, 7, etc.

In fact, a characterization of this is provided by the continued fraction expansion of $\sqrt{\delta}$.

Theorem 2.3 ([29]). *The two ideal classes (α, β) and (β, α) are equivalent under the action of the modular group if and only if the length of periodic part of the continued fraction expansion of $\sqrt{\delta}$ is odd.*

Under the mentioned action, a narrow ideal class determined by $\mathfrak{a} = (\alpha, \beta)$ has stabilizer, $\text{Stab}(\mathfrak{a})$. As a result of the considerations on the automorphisms of binary quadratic forms, this group is known to be isomorphic to $\mathbf{Z}$ and has one generator, call $W_{\mathfrak{a}}$. Then writing $W_{\mathfrak{a}}$ in terms of the generators S and L of the modular group and choosing the word $W_{\mathfrak{a},o}$ in the conjugacy class of $W_{\mathfrak{a}}$ which has the shortest length and writing $W_{\mathfrak{a},o}$ in terms of the generators gives the corresponding çark.

Conversely, to a çark given as $\zeta = [n_1, -m_1, \dots, n_k, -m_k]$ we associate the element $W_{\zeta} = (LS)^{n_1}(L^2S)^{m_1} \dots (LS)^{n_k}(L^2S)^{m_k} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ in $\text{PSL}_2(\mathbf{Z})$. This element of the modular group is then mapped to the narrow ideal class, I_{ζ} , contain-

ing the two dimensional $\mathbf{Z}$ module $1 \cdot \mathbf{Z} + \omega \cdot \mathbf{Z}$ where $\omega = \frac{p - s + \sqrt{\text{Tr}(W_{\zeta}) - 4}}{2r}$. It is merely a result of the construction that the composition of two maps is identity and hence we get a one to one correspondence. So we obtain:

Theorem 2.4. *There is a one to one correspondence between narrow ideal classes of discriminant δ and primitive çarks whose discriminant's square-free part is δ ,*

3. LANG CONJECTURES

Our aim in this section is to explain some terminology around in order to be able to state Lang conjectures which are relevant to our approach. For a comprehensive treatment of the subject we invite the reader to consult [9], [40] and [8].

We first define the Kobayashi pseudo-distance and list some geometric properties. We will then mention on of the conjectures of Lang on the arithmetic properties

of Kobayashi hyperbolic algebraic varieties. We will exemplify the situation in the case of curves and then state results concerning hyperbolicity of projective hypersurfaces.

3.1. The Kobayashi Pseudo-distance. By $\text{Hol}(X, Y)$ let us denote the set of all holomorphic maps from a complex space X to a complex space Y . In particular, an element of the set $\text{Hol}(\mathbb{D}, X)$ will be referred to as a holomorphic disk in X ; where $\mathbb{D}$ denotes the unit disk in $\mathbf{C}$. A holomorphic chain in X joining $x \in X$ to $y \in X$ is a finite set of holomorphic disks, $\{f_i: \mathbb{D} \rightarrow X\}$ in X together with a finite set of distinguished points, $\{(a_i, b_i)\}$, of $\mathbb{D} \times \mathbb{D}$ satisfying

- $f_1(a_1) = x$ and $f_n(b_n) = y$
- $f_i(b_i) = f_{i+1}(a_{i+1})$ for all $i = 1, 2, \dots, n-1$

A chain will be denoted by $\alpha = ((f_1, a_1, b_1), \dots, (f_n, a_n, b_n))$, see Figure 4.

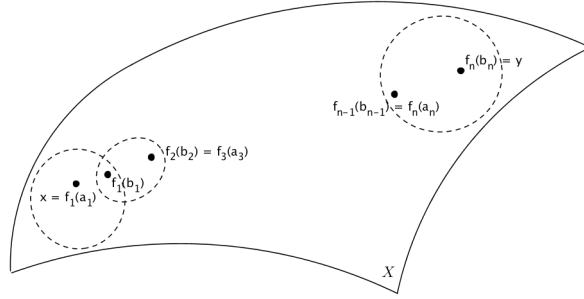


FIGURE 4. A holomorphic chain in X .

We equip $\mathbb{D}$ with the Poincaré metric, which is given in the form $\frac{4}{(1-|z|^2)^2} dz d\bar{z}$ as a metric tensor, and denote it by ρ . Then the length of a holomorphic chain α is defined as

$$\ell(\alpha) = \sum_{i=1}^n \rho(a_i, b_i).$$

Definition 3.1. Given a complex space X , the Kobayashi pseudo-distance, denoted d_X , on X between any two points $x, y \in X$ is:

$$d_X(x, y) := \inf \{\ell(\alpha)\};$$

where the infimum is taken over all holomorphic chains α joining x to y .

Any holomorphic map $\varphi: X \rightarrow Y$ between two complex spaces is distance decreasing, that is $d_X(x, y) \geq d_Y(\varphi(x), \varphi(y))$ as any holomorphic chain joining x to y in X can be used to obtain a holomorphic chain joining $\varphi(x)$ to $\varphi(y)$ in Y . In particular, d_X is invariant under the holomorphic automorphisms of X . However it is not a birational invariant, see the paragraph after Definition 3.2.

Example 3.1. Let $x, y \in \mathbf{C}$ be two distinct complex numbers. Composing a translation and a rotation, we can map x to 0 and y to $\alpha \in \mathbf{R}_{>0} \subset \mathbf{C}$. For any integer $n > 1$, the map $\varphi_n: \mathbb{D} \rightarrow \mathbf{C}$ defined as $\varphi_n(z) = \alpha n z$ gives a holomorphic disk in $\mathbf{C}$ satisfying $\varphi_n(0) = 0$ and $\varphi_n(1/n) = \alpha$. Recalling the fact that the Poincaré distance in the unit disk between origin and a point of Euclidean distance

r to the origin is given by the formula $\log(\frac{1+r}{1-r})$, we get that $d_{\mathbf{C}}(x, y) = 0$. That is, the Poincaré pseudo-distance vanishes on $\mathbf{C}$.

Example 3.2. For $x, y \in \mathbb{D}$, the unit disk in $\mathbf{C}$ since $d_{\mathbb{D}}(x, y) \leq \inf_{\alpha, |\alpha|=1} \ell(\alpha)$; where the infimum is taken over all holomorphic disks in $\mathbb{D}$ containing only one map $f: \mathbb{D} \rightarrow \mathbb{D}$. Taking f as the identity map gives us the fact that the distance between x and y is bounded above by the Poincaré distance $\rho(x, y)$. Conversely, for any α a holomorphic chain in $\mathbb{D}$,

$$\begin{aligned} \ell(\alpha) &= \sum_{i=1}^n \rho(a_i, b_i) \\ &\geq \sum_{i=1}^n \rho(f_i(a_i), f_i(b_i)) \end{aligned}$$

where the last inequality is provided by the Schwarz lemma. We conclude by triangle inequality that $\ell(\alpha) \geq \rho(x, y)$ for any x and y , and thus $d_{\mathbb{D}}(x, y) = \rho(x, y)$.

Definition 3.2. We say that a complex space is Kobayashi hyperbolic whenever its Kobayashi distance is non-degenerate.

For instance, $\mathbb{D}$ is Kobayashi hyperbolic, by Example 3.2 and therefore the Kobayashi distance restricted to $\varphi(\mathbb{D}) \subset X$ of any non-constant holomorphic map $\varphi: \mathbb{D} \rightarrow X$ is non-degenerate. On the other hand, if X is a complex space admitting a non-constant holomorphic map $\varphi: \mathbf{C} \rightarrow X$, then the Kobayashi pseudo-distance restricted to $\varphi(\mathbf{C})$ vanishes. In particular, $\mathbf{P}^1$ is not Kobayashi hyperbolic, which in turn implies that hyperbolicity is not a birational invariant.

In fact, this provides a criterion for Kobayashi hyperbolicity. Namely:

Theorem 3.1 (Brody's Theorem, [27, Theorem 2.1]). *Let X be a compact complex space. Then X is Kobayashi hyperbolic if and only if X does not admit any non-constant holomorphic map from $\mathbf{C}$ whenever X is compact.*

In addition, Kobayashi hyperbolicity behaves as expected under topological coverings, that is:

Theorem 3.2 ([22, Theorem 2.5]). *Let $\tilde{X}$ be a hyperbolic complex space and $\pi: \tilde{X} \rightarrow X$ a topological covering (universal or not) of X . X is hyperbolic if and only if $\tilde{X}$ is hyperbolic.*

Proof. Let α be a chain connecting two points x and y in X . Each map f_i can be lifted to produce a holomorphic chain in $\tilde{X}$ and hence has positive length whenever $x \neq y$. Conversely, if $\tilde{x}$ and $\tilde{y}$ are two points joined by a holomorphic chain $\tilde{\alpha}$, composing with π gives a holomorphic chain in X and hence the length is positive, unless $x = y$. $\square$

3.2. Hyperbolicity and Curvature of Differential Metrics. The answer to the question whether the Kobayashi pseudo-metric can be interpreted as a differential metric tensor turns out to be negative. Nevertheless, one can define Kobayashi pseudo-metric using infinitesimal norms. Once this is done, it is possible to define lengths of curves and passing to infimums endows the space in question with the Kobayashi pseudo-metric. In general, however, the Kobayashi pseudo-metric is smooth (may not even be continuous), yet there is a result due to Lempert, [28],

guaranteeing the smoothness of the Kobayashi metric in case when X is a strongly convex open connected subset in $\mathbf{C}^n$.

Nevertheless, this point of view provides a passage from the Kobayashi metric to smooth metrics. That is, one can relate the non-vanishing of the Kobayashi pseudo-metric on a complex space X to the existence of negatively curved metrics on X :

Theorem 3.3 ([23, Theorem 4.11]). *If X is a complex manifold admitting a metric of holomorphic sectional curvature bounded by a negative constant, then X is Kobayashi hyperbolic.*

Describing metrics of prescribed curvature on complex manifolds is equivalent to solving the complex Monge-Ampère equation on the manifold which is in general very complicated.

In [26], Lang proposes a series of conjectures relating arithmetic properties (mordellicity) of an algebraic variety to

- algebro-geometric(all subvarieties (including the variety X itself) being of general type),
- complex analytic(every map from an abelian variety or $\mathbf{C}$ or $\mathbf{P}^1$ to X being constant), and
- differential geometric(metric of negatively bounded curvature)

properties. The relation between these different categories are provided by the hyperbolicity of X . Of the many conjectures around these topics, for purposes of this work, the relevant conjecture of Lang is the following:

Conjecture 3.1 ([26]). If an algebraic variety is hyperbolic then it is mordellic⁷.

3.3. Hyperbolicity in Dimension One. Let X be a non-singular compact Riemann surface. The genus g of X determines both the universal covering space of X (an analytic property), the structure of $\text{Hol}(\mathbf{C}, X)$ (existence of non-constant maps), the sign of the curvature of the Hermitian metric that X admits (a differential geometric property) and the mordellicity of X . We summarize this in the following table:

For instance, $X = \mathbb{P}^1$ is equipped with its Fubini-Study metric, which is of constant curvature 1. Genus one surfaces are elliptic curves. The classical flat metric $ds^2 = dz \wedge d\bar{z}$ on $\mathbf{C}$ induces a flat metric on such curves whose K -rational points form a finitely generated abelian group(Mordell-Weil Theorem).

If X is of genus $g > 1$ and is defined over a number field K then by the celebrated theorem of Faltings, $X(K)$ is finite. Such curves are uniformized by the upper half plane (or the unit disk), i.e. there is a discrete subgroup, Γ , of $\text{PSL}_2(\mathbf{R})$ so that $X(\mathbf{C}) \cong \Gamma \backslash \mathbb{H}$. Poincaré metric on $\mathbb{H}$ induces an Hermitian metric of constant curvature -1 on X , which we also call the Poincaré metric on X . As a consequence of Liouville's theorem the only holomorphic maps from $\mathbf{C}$ to X are constant maps.

For an example of non-compact case, we consider $X = \mathbb{P}^1 - \{3\text{pts.}\} \cong \Gamma(2) \backslash \mathbb{H}$; where $\Gamma(2)$ is the congruence subgroup of the modular group of level 2. By Picard's theorem, we know that X does not admit any non-constant holomorphic map from

⁷Recall that an algebraic variety X is called *mordellic* if for any finitely generated field $K \subset \mathbf{C}$ the K rational points of X , denoted by $X(K)$, is not Zariski dense in $X(\mathbf{C})$

Genus(g)	Metric	Curvature	Universal cover	$X(\mathbf{Q})$	$\text{Hol}(\mathbf{C}, X)$
$g = 0$	spherical	+1	$\mathbb{P}^1$	$\emptyset$ or infinite	there are non-const. maps
$g = 1$	flat	0	$\mathbf{C}$	finitely generated abelian group	there are non-const. maps
$g \geq 2$	Poincaré	-1	$\mathbb{H}$	finite set	only const. maps

TABLE 1. Hyperbolicity in dimension 1.

C. One can also write explicit metrics of bounded negative curvature on X , for instance ([19])

$$ds^2 = \left[\frac{(1 + |z|^{1/3})^{1/2}}{|z|^{5/6}} \right] \left[\frac{(1 + |z - 1|^{1/3})^{1/2}}{|z - 1|^{5/6}} \right] dz \wedge d\bar{z}.$$

3.4. Hyperbolicity of Hypersurfaces of High Degree. Let X be a complex manifold of dimension n . The real tangent space to X at and point $x \in X$ can be defined as the set of equivalence classes of tangent vectors of differentiable curves passing through x . This can be generalized in the following sense. For $f: \mathbb{D}_r \rightarrow X$ the germ of a holomorphic mapping with $f(0) = x$; where $\mathbb{D}_r$ denotes the disk in $\mathbf{C}$ with center 0 and radius r one can write f as a power series

$$f(z) = (f_1(z), \dots, f_n(z))$$

where $f_i(z) := (z_i \circ f)(z)$ with respect to any holomorphic set of coordinates $z_1, \dots, z_n$ around x . Each $f_i(z)$ can be expanded as a power series around 0 and be written as:

$$f(z) = \left(\sum_{k=0}^{\infty} \frac{f_1^{(k)}(0)}{k!} z^k, \dots, \sum_{k=0}^{\infty} \frac{f_n^{(k)}(0)}{k!} z^k \right)$$

which will be written, by abuse of notation, as

$$(1) \quad f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \dots$$

and in particular by $f^{(k)}(z)$ we mean $(f_1^{(k)}(z), \dots, f_n^{(k)}(z))$.

Definition 3.3. Two holomorphic germs $f: \mathbb{D}_r \rightarrow X$ and $g: \mathbb{D}_r \rightarrow X$ with $f(0) = x$ and $g(0) = x$ are called k -equivalent if

$$f^{(k)}(0) = g^{(k)}(0)$$

for any $k = 1, 2, \dots, n$.

Note that this is an equivalence relation and the classes are independent of the chosen coordinate system. An equivalence class will be referred to as a k -jet and the set of k -equivalence classes of germs of holomorphic functions $f: \mathbb{D}_r \rightarrow X$ is

denoted by $J_x^k X$ and $J^k X$ is defined as the union $\cup_{x \in X} J_x^k X$. We will write $J_x^{*,k} X$ for non-constant k -jets at x . For $k = 1$ $J^1 X$ is the complex holomorphic tangent bundle and hence is a vector bundle, however for $k \geq 2$ $J^k X$ is an affine bundle over the base X with the distinguished point being the constant map. Each fiber is of dimension kn , hence the total space is of dimension $(k+1)n$.

Given a non-constant holomorphic map $f: \mathbb{D}_r \rightarrow X$, for any $t \in \mathbf{C}^*$, we define

$$\begin{aligned} (t \cdot f): \mathbb{D}_{r/|t|} &\longrightarrow X \\ z &\longmapsto tz \end{aligned}$$

For $k = 1$, the quotient of the set $J^{*,1} X$ under this action gives the classical projectivized tangent bundle. However, for $k \geq 2$, the fibers of the quotient are weighted projective spaces. Indeed, for $x \in X$, the action on the fiber of $J_x^k X$ is

$$t \cdot (f^{(1)}(z), \dots, f^{(k)}(z)) = (tf^{(1)}(z), \dots, t^k f^{(k)}(z)),$$

that is, in the notation of [13] each fiber is $\mathbf{P}(1, \dots, 1, 2, \dots, 2, \dots, k, \dots, k)$.

Fixing a weight $w \in \mathbf{Z}_{>0}$. Then a jet differential of weight w is given as a weighted polynomial with coefficients being holomorphic functions of the local coordinates $z_1, \dots, z_n$ of weight w in kn coordinates in variables dz_i^j , for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$; where the weight of dz_i^j is j . Such differentials, denoted by $\mathcal{J}_{k,w}$, form a sheaf. For instance, a 1-jet differential of weight w is a section of the w^{th} symmetric power of the sheaf of 1-forms, i.e. $\mathcal{J}_{k,w} = \text{Sym}^w \Omega_X^1$.

Given a vector sub-bundle, V of the holomorphic tangent bundle, TX , of X it is also possible to define the corresponding jet bundle. In this construction jets whose tangent vectors lie in the sub-bundle V are taken into consideration. In this case, some care must be taken in defining higher jets. We invite the reader to consult [9] for details of the construction.

Jet-differentials have proven themselves as useful tools in determining the hyperbolicity of hypersurfaces in projective space. In fact, certain sub-bundles of the jet bundle plays an important role. For instance, Demailly and ElGoul, [11], have used Semple jet bundles to prove that a very general surface in $\mathbf{P}^3$ of degree greater than or equal to 21 is Kobayashi hyperbolic. The technique of slanted vector fields and jet differentials on the universal family of hypersurfaces, due to Siu [32], is closely related to the works of Clemens, [5, 6], Ein, [14, 15], and Voisin, [38, 39], has led to the proof of the following:

Theorem 3.4. [32, Theorem 1.1] *Any generic hypersurface in $\mathbf{P}^n$ of sufficiently high degree is Kobayashi hyperbolic.*

A result in the same vein as the above is due to Demailly, [10]:

Theorem 3.5 ([10, Theorem 4.2]). *A very general hypersurface in $\mathbf{P}^{n+1}$ of degree greater than or equal to $2n+2$ is Kobayashi hyperbolic.*

Let us close this section by noting that there are a number of related research directions have been left aside. In addition to log-versions of all the discussion made in this part, for instance among many others there are results/conjectures concerning *algebraic hyperbolicity* of algebraic varieties.

4. HYPERBOLICITY OF ÇARK HYPERSURFACES

Throughout we fix $K = \mathbf{Q}(\sqrt{\delta})$ be the unique real quadratic number field; where $\delta > 0$ is a square-free integer.

4.1. Çark Hypersurfaces. Let $[n_1, -m_1]$ be a çark of length 2 (or a 2-çark for short). Then a straightforward computation tells us that the corresponding indefinite binary quadratic form before normalization is $(m_1, -n_1 m_1, n_1)$ which is of discriminant $\Delta = (n_1 n_1)(m_1 n_1 + 4)$. So any two çark of discriminant Δ must satisfy the equation $x_1 y_1 (z_1 y_1 - 4) = \Delta$. In a similar fashion, for 4-çarks the equation is $F_4(F_4 + 4)$ where

$$(2) \quad F_4 = x_1 x_2 y_1 y_2 + (x_1 + x_2)(y_1 + y_2)$$

and for 6-çarks the equation turns out to be $F_6(F_6 + 4)$ where

$$\begin{aligned} F_6 = & x_1 x_2 x_3 y_1 y_2 y_3 \\ & + x_1 x_2 y_1 y_2 + x_2 x_3 y_1 y_2 + x_1 x_2 y_1 y_3 + x_1 x_3 y_1 y_3 + x_1 x_3 y_2 y_3 + x_2 x_3 y_2 y_3 \\ & + (x_1 + x_2 + x_3)(y_1 + y_2 + y_3). \end{aligned}$$

We borrow the following result from [43]:

Theorem 4.1. *For any even positive integer $n = 2k$, there is a polynomial $F_n \in \mathbf{Z}[x_1, y_1, \dots, x_k, y_k]$ so that, any indefinite binary quadratic form $f = (a, b, c)$ whose çark is of length n is an integral point on the affine variety, denoted by $\mathcal{C}_n$ in $\mathbf{C}^{2n+1}$ (last coordinate being z), with equation $F_n(F_n + 4) = \delta z^2$; where $\Delta(f) = \delta \square$ with δ being square-free.*

The polynomial F_n is irreducible and is of degree n . For every even positive integer n , F_n enjoy other interesting properties. For instance, all the coefficients in F_n is 1. F_n does not contain any monomial of odd degree. The constant term of F_n is 0. Furthermore, for any integer $1 \leq l \leq n - 1$ the number of monomials of degree $2l$ in F_n is equal to L_l^n ; where L_l^n is the coefficient of x^{2l} in the n^{th} Lucas polynomial⁸.

On the other hand finiteness of the narrow class group implies the following:

Lemma 4.2. *There is a positive integer n (which is not unique and depends on δ) with the property that all narrow ideal classes of discriminant δ can be represented by a çark of length n .*

Example 4.1. Let $d = 7$. The narrow class number in this case is equal to 2. The corresponding ideal classes are represented by the forms $(1, 4, -3)$ and $(-1, 4, 3)$. These two forms are represented by the çarks $[-1, 1, -1, 4]$ and $[-1, 1, -4, 1]$, respectively. Since they both are represented by 4-çarks they are points on the hypersurface $F_4(F_4 + 4) = 7z^2$ with coordinates $(-1, 1, -1, 4, \pm 2)$ and $(1, -1, 1, -4, \pm 2)$, respectively. The same narrow ideal classes can be represented by the forms $(3, 10, -1)$ and $(-3, 10, 1)$, which are this time of discriminant $112 = 2^4 7$. They are represented by the çarks $[-10, 3, -2, 3]$ and $[-3, 2, -3, 10]$, respectively. This time, they correspond to $(-10, 3, -2, 3, \pm 4)$ and $(-3, 2, -3, 10, \pm 4)$, respectively.

This example shows that the correspondence stated in Theorem 4.1 is far from being one-to-one. On the other hand, if δ is not congruent to 1 modulo 4, then there are no indefinite binary quadratic forms whose discriminant is δ . Hence, there are integers $z = z_o$ for which not every ideal class is represented by a çark of discriminant δz_o^2 . But if we are given two binary quadratic forms $f = (a, b, c)$ and

⁸Recall the Lucas polynomials are defined as $L_n(x) = 2^{-n}((x - \sqrt{x^2 - 4})^n + (x + \sqrt{x^2 - 4})^n)$.

$f' = (a', b', c')$ so that $\Delta(f)/\Delta(f')$ is a perfect square of a rational number, say p/q then the forms $\tilde{f} = (qa, qb, qc)$ and $\tilde{f}' = (pa', pb', pc')$ are of the same discriminant. Since there are only finitely many narrow ideal classes, we obtain:

Lemma 4.3. *Given a square-free positive integer δ , there is a (non-unique) positive integer z_o which depends on δ so that every ideal class in of the quadratic number field $\mathbf{Q}(\sqrt{\delta})$ is represented by a binary quadratic form (or a çark) of discriminant δz_o^2 .*

Now, we fix the smallest $n = 2k$ provided by Lemma 4.2. To make the above correspondence one-to-one we first eliminate the rotational symmetry of çarks. Namely, if $\mathcal{O}_\delta$ admits a unit of negative norm, we let $G_\delta = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z}$ and if there are no units of negative norm in $\mathcal{O}_\delta$, then we set $G_\delta = \mathbf{Z}/k\mathbf{Z}$. We define the action of $G_\delta = \mathbf{Z}/k\mathbf{Z}$ on $\mathbf{C}[X, Y]$ as:

$$\begin{aligned} \mathbf{Z}/k\mathbf{Z} \times \mathbf{C}[X, Y] &\longrightarrow \mathbf{C}[X, Y] \\ p(x_1, y_1, \dots, x_k, y_k) &\mapsto p(x_k, y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}) \end{aligned}$$

in case $\mathcal{O}_\delta$ does not have any unit of negative norm; and the action of the group $G_\delta = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z}$ is defined as

$$\begin{aligned} \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z} \times \mathbf{C}[X, Y] &\longrightarrow \mathbf{C}[X, Y] \\ ((0, 1), p(x_1, y_1, \dots, x_k, y_k)) &\mapsto p(x_k, y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}) \\ ((1, 0), p(x_1, y_1, \dots, x_k, y_k)) &\mapsto p(y_1, x_1, \dots, y_k, x_k) \end{aligned}$$

Table 2 is a list of polynomials F_n ; where narrow ideal classes in $\mathbf{Q}(\sqrt{\delta})$ can be represented by çarks of length at most $n = 2k$. In the case when $h_\delta = h_\delta^+$, then there are units of norm -1 in the ring of integers, therefore $G_\delta = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z}$. In the remaining case, i.e. whenever $2h_\delta = h_\delta^+$ the group $G_\delta = \mathbf{Z}/k\mathbf{Z}$.

Observe that if an integer n satisfies the conditions of Lemma 4.3 (resp. if z_o satisfies the conditions of Lemma 4.2), then the same is true for any positive integer multiple of n (resp. z_o). Note that as n gets larger the degree of the hypersurface as well as the dimension of the ambient space gets larger.

For δ a square-free positive integer we fix an integer, z_o , provided by Lemma 4.3 and fix an n provided by Lemma 4.2. We set $C_{n, z_o} = C_n \cap H_{z_o}$; where H_{z_o} is the hyperplane in $\mathbf{C}^{n+1}$ with equation $z = z_o$. Then, the action of G_δ induces an action on C_{n, z_o} and we have:

Theorem 4.4. *Let δ be a square-free integer. There is a one-to-one correspondence between the integral points of the hypersurface $C_{n, z_o}/G_\delta$ and narrow ideal classes in the quadratic number field $\mathbf{Q}(\delta)$.*

In fact, if we let $\mathfrak{a}$ be a narrow ideal class of discriminant δ represented by the çark $[m_1, -n_1, \dots, m_k, -n_k]$ which is of discriminant δz_o^2 . Then there are two integral points on the surface C_n/G_δ representing $\mathfrak{a}$, namely: $(m_1, -n_1, \dots, m_k, -n_k, -z_o)$ and $(m_1, -n_1, \dots, m_k, -n_k, z_o)$. So C_{n, z_o} is isomorphic to $C_{n, -z_o}$ thus $\mathfrak{C}_{\delta, n, z_o}$ is isomorphic to $\mathfrak{C}_{\delta, n, -z_o}$ for any z_o .

Example 4.2. When $d = 13$, we have $h_{13} = h_{13}^+ = 1$. The unique narrow ideal class in this case can be represented by a 2-çark. So $F_{13} = x_1 y_1$. The group acting on C_{13} is $G_{13} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/1\mathbf{Z}$ interchanging y_1 and x_1 . If we introduce the variable

δ	k	F_δ	h_δ	h_δ^+	$\mathfrak{carks}$	z_o
2	2	x_1y_1	1	1	$[2, -2]$	± 2
3	2	x_1y_1	1	2	$[2, -1]$ and $[1, -2]$	± 2
7	4	$x_1x_2y_1y_2 + (x_1 + x_2)(y_1 + y_2)$	1	2	$[1, -1, 1, -4]$ and $[1, -1, 4, -1]$	± 2
13	4	$x_1x_2y_1y_2 + (x_1 + x_2)(y_1 + y_2)$	1	1	$[-3, 3]$	± 1
17	6	$x_1x_2x_3y_1y_2y_3 + x_1x_2y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 + x_1x_3y_1y_3 + x_1x_3y_2y_3 + x_2x_3y_2y_3 + (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$	1	1	$[1, -3, 1, -1, 3, -1]$	± 1
19	6	$x_1x_2x_3y_1y_2y_3 + x_1x_2y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 + x_1x_3y_1y_3 + x_1x_3y_2y_3 + x_2x_3y_2y_3 + (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$	1	2	$[2, -1, 3, -1, 2, -8]$ and $[1, -3, 1, -2, 8, -2]$	± 2

TABLE 2. Examples of F_δ and points on C_δ

$x = x_1y_1$ we obtain that an integral point on the variety defined by the zero set of the equation $x^2(x^2 + 4) = z^2 13$ gives rise to an indefinite binary quadratic form of discriminant $z^2 \cdot 13$. An immediate solution is $x = \pm 3, z = \pm 3$ $(3, 3, \pm 3) \in \mathbf{Z}^3$. Reader will immediately realize that a simple calculation modulo 13 tells us that x should be congruent to ± 3 modulo 13. Then, the problem reduces to determining integers k for which the polynomial $13l^2 + 6l + 1$ (or $13l^2 - 6l + 1$) is a perfect square. Let us be content with providing two other integral points: $(393, \pm 109)$, $(46764, \pm 129700)$.

4.2. Hyperbolicity of C_n . Theorem 4.4 motivates us to give the following:

Definition 4.1. The projectivization of the hypersurface $\mathfrak{C}_{\delta, n, z_o} = \mathfrak{C}_\delta = C_{n, z_o}/G_\delta$ in $\mathbf{P}^n$ is called a $\mathfrak{cark}$ hypersurface of discriminant δ .

In this very brief section we discuss the hyperbolicity of $\mathfrak{C}_\delta$. We note that the degree of $\mathfrak{C}_\delta$ is $2n$. Hence, in the light of Theorem 3.5 $\mathfrak{cark}$ hypersurfaces to be Kobayashi hyperbolic under the assumption that the $\mathfrak{cark}$ hypersurface is very general in the sense of Theorem 3.5. Assuming Lang conjectures on the arithmetic, for any discriminant δ , a corresponding $\mathfrak{cark}$ hypersurface $\mathfrak{C}_\delta$ is expected to have finitely many rational points, which is indeed the case according to the previous section.

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REFERENCES

- [1] A. Baker. Linear forms in the logarithmic of algebraic numbers. *Mathematika, Lond.*, 13:204–216, 1968.
- [2] A. Baker. Imaginary quadratic fields with class number 2. *Matematika, Moskva*, 16(5):3–14, 1972.

- [3] G. Belyi. On Galois extensions of a maximal cyclotomic field. *Math. USSR Izv.*, 14, No.2:247–256, 1980.
- [4] J. Buchmann and U. Vollmer. *Binary quadratic forms: An algorithmic approach*, volume 20 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2007.
- [5] H. Clemens. Curves on generic hypersurfaces. *Ann. Sci. École Norm. Sup. (4)*, 19(4):629–636, 1986.
- [6] H. Clemens. Erratum: “Curves on generic hypersurfaces” [*Ann. Sci. École Norm. Sup. (4)* **19** (1986), no. 4, 629–636; MR0875091 (88c:14037)]. *Ann. Sci. École Norm. Sup. (4)*, 20(2):281, 1987.
- [7] P. B. Cohen, C. Itzykson, and J. Wolfart. Fuchsian triangle groups and Grothendieck dessins. Variations on a theme of Belyi. *Comm. Math. Phys.*, 163(3):605–627, 1994.
- [8] I. Coskun. The arithmetic and the geometry of Kobayashi hyperbolicity. In *Snowbird lectures in algebraic geometry*, volume 388 of *Contemp. Math.*, pages 77–88. Amer. Math. Soc., Providence, RI, 2005.
- [9] J.-P. Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 285–360. Amer. Math. Soc., Providence, RI, 1997.
- [10] J.-P. Demailly. Proof of the Kobayashi conjecture on the hyperbolicity of very general hypersurfaces. This paper supersedes submission hal-01092537 / arXiv:1412.2986, Feb. 2015.
- [11] J.-P. Demailly and J. El Goul. Hyperbolicity of generic surfaces of high degree in projective 3-space. *Amer. J. Math.*, 122(3):515–546, 2000.
- [12] M. Deuring. Imaginäre quadratische Zahlkörper mit der Klassenzahl Eins. *Invent. Math.*, 5:169–179, 1968.
- [13] I. Dolgachev. Weighted projective varieties. In J. Carrell, editor, *Group Actions and Vector Fields*, volume 956 of *Lecture Notes in Mathematics*, pages 34–71. Springer Berlin Heidelberg, 1982.
- [14] L. Ein. Subvarieties of generic complete intersections. *Invent. Math.*, 94(1):163–169, 1988.
- [15] L. Ein. Subvarieties of generic complete intersections. II. *Math. Ann.*, 289(3):465–471, 1991.
- [16] C. F. Gauss. *Disquisitiones arithmeticae*. Translated into English by Arthur A. Clarke, S. J. Yale University Press, New Haven, Conn., 1966.
- [17] E. Gironde and G. González-Diez. *Introduction to compact Riemann surfaces and dessins d’enfants*, volume 79 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2012.
- [18] E. Gironde, D. Torres-Teigell, and J. Wolfart. Shimura curves with many uniform dessins. *Math. Z.*, 271(3-4):757–779, 2012.
- [19] H. Grauert and H. Reckziegel. Hermitesche Metriken und normale Familien holomorpher Abbildungen. *Math. Z.*, 89:108–125, 1965.
- [20] A. Grothendieck. Esquisse d’un programme. In *Geometric Galois Actions, 1. Around Grothendieck’s Esquisse d’un Programme*, pages 5–48. Cambridge University Press, 1997.
- [21] K. Heegner. Diophantine analysis and modular forms. (Diophantische Analysis und Modul-funktionen.). *Math. Z.*, 56:227–253, 1952.
- [22] S. Kobayashi. Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.*, 82(3):357–416, 1976.
- [23] S. Kobayashi. *Hyperbolic manifolds and holomorphic mappings*. World Scientific, 2005.
- [24] S. G. Krantz. *Complex analysis: the geometric viewpoint*, volume 23 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, second edition, 2004.
- [25] S. Lando and A. Zvonkin. *Graphs on Surfaces and Their Applications*. Encyclopaedia of Mathematical Sciences. Springer, 2004.
- [26] S. Lang. Hyperbolic and diophantine analysis. *Bull. Amer. Math. Soc.*, 14:pp. 159–205, 1986.
- [27] S. Lang. *Introduction to complex hyperbolic spaces*. Springer-Verlag, New York, 1987.
- [28] L. Lempert. La métrique de Kobayashi et la représentation des domaines sur la boule. *Bull. Soc. Math. France*, 109(4):427–474, 1981.
- [29] R. A. Mollin. A continued fraction approach to the diophantine equation $ax^2 + by^2 = 1$. *JP J. Algebra Number Theory Appl.*, pages 159–207.
- [30] L. Schneps. Dessins d’enfants on the Riemann sphere. In *The Grothendieck theory of dessins d’enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.*, pages 47–77. Cambridge Univ. Press, Cambridge, 1994.
- [31] C. L. Siegel. Zum Beweise des Starkschen Satz. *Invent. Math. (2)*, 5:180191, 1968.

- [32] Y.-T. Siu. Hyperbolicity of generic high-degree hypersurfaces in complex projective space. *Inventiones mathematicae*, pages 1–98, 2015.
- [33] H. Stark. A complete determination of the complex quadratic fields of class-number one. *Mich. Math. J.*, 14:1–27, 1967.
- [34] H. Stark. A transcendence theorem for class-number problems. *Matematika, Moskva*, 16(5):15–32, 1972.
- [35] H. Stark. A transcendence theorem for class-number problems. II. *Ann. Math. (2)*, 96:174–209, 1972.
- [36] A. M. Uludağ and A. Zeytin. Çark groupoids and Thompson’s groups. 2016. in progress.
- [37] A. M. Uludağ, A. Zeytin, and M. Durmuş. Binary quadratic forms as dessins. to appear in *J. Théor. Nombres Bordeaux*.
- [38] C. Voisin. On a conjecture of Clemens on rational curves on hypersurfaces. *J. Differential Geom.*, 44(1):200–213, 1996.
- [39] C. Voisin. A correction: “On a conjecture of Clemens on rational curves on hypersurfaces” [*J. Differential Geom.* **44** (1996), no. 1, 200–213; MR1420353 (97j:14047)]. *J. Differential Geom.*, 49(3):601–611, 1998.
- [40] C. Voisin. On some problems of Kobayashi and Lang; algebraic approaches. In *Current developments in mathematics, 2003*, pages 53–125. Int. Press, Somerville, MA, 2003.
- [41] M. Watkins. Class numbers of imaginary quadratic fields. *Math. Comput.*, 73(246):907–938, 2004.
- [42] D. B. Zagier. *Zetafunktionen und quadratische Körper. Eine Einführung in die höhere Zahlentheorie.*, 1981.
- [43] A. Zeytin. Multivariable lucas polynomials and ideal classes in quadratic number fields. 2016. in progress.

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