

A COUNTEREXAMPLE TO PREMET'S AND JOSEPH'S CONJECTURES

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INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra of rank l over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let G be the adjoint group of $\mathfrak{g}$. Given $x \in \mathfrak{g}$, we denote by $\mathfrak{g}_x$ the centraliser of x in $\mathfrak{g}$.

Conjecture 1 (Premet). *For any $x \in \mathfrak{g}$ the algebra $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ of $\mathfrak{g}_x$ -invariants is a graded polynomial algebra in l variables.*

In some particular cases the problem is simple. For example, for regular nilpotent elements the algebra $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ is known to be free. In [5], Conjecture 1 is shown to be true in types A and C . It is also verified for some nilpotent elements of orthogonal Lie algebra and for the minimal nilpotent orbits in simple Lie algebras except of type E_8 . Later, by a different method, Brown and Brundan [1] proved that Conjecture 1 holds in type A .

Suppose that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are *opposite* parabolic subalgebras of $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{p}_-$. Then the intersection $\mathfrak{q} := \mathfrak{p}_+ \cap \mathfrak{p}_-$ is called a *biparabolic* or, in other terminology, *seaweed* subalgebra. Since $\mathfrak{g}$ itself is a parabolic subalgebra, we see that parabolics are particular cases of seaweeds.

For any Lie algebra $\mathfrak{q}$ let $\mathfrak{q}' := [\mathfrak{q}, \mathfrak{q}]$ denote its derived algebra. In [3, Section 7.7], the following conjecture was made.

Conjecture 2 (Joseph). *For any seaweed subalgebra $\mathfrak{q} \subset \mathfrak{g}$ the semi-invariants $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}'}$ form a polynomial algebra.*

A formula for $\text{tr.deg } \mathcal{S}(\mathfrak{q})^{\mathfrak{q}'}$ is given in [3]. It is rather complicated and we are not going to use it in full generality. In [2] and [3], it is proved that Conjecture 2 holds for all

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parabolics and seaweeds in simple Lie algebras of types A and C . As was noticed in [5, Section 4.9], minimal nilpotent orbits provide a testing site for Joseph's conjecture as well as Premet's one. If $\mathfrak{g}$ is simple, then for each minimal nilpotent element $e \in \mathfrak{g}$ there exists a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} \cong \mathcal{S}(\mathfrak{p})^{\mathfrak{p}'}$. The detailed explanation of this construction is given below. We note only that naturally $\text{tr.deg } \mathcal{S}(\mathfrak{p})^{\mathfrak{p}'} = \text{tr.deg } \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ and $\text{tr.deg } \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = l$ by [4].

In this note, we show that Conjecture 1 does not hold for the minimal nilpotent orbit in the simple Lie algebra of type E_8 . As a consequence, a conjecture of Joseph on the semi-invariants of (bi)parabolics is not true either.

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1. THEORY

Let us say a few words about the general method of [5], which, unfortunately, does not work for the minimal nilpotent orbit in E_8 . Let $\mathfrak{g}$ be a simple Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. Suppose that $\langle e, h, f \rangle \subset \mathfrak{g}$ is an $\mathfrak{sl}_2$ -triple containing e . We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form. For each $F \in \mathcal{S}(\mathfrak{g})^G$ let eF stand for the minimal degree component of the restriction $F|_{e+\mathfrak{g}_f}$. As was shown in [5], ${}^eF \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$. A set of homogeneous generators $\{F_1, \dots, F_l\} \subset \mathcal{S}(\mathfrak{g})^G$ is said to be *good* if the eF_i 's are algebraically independent.

Given a linear function γ on $\mathfrak{g}_e$ we denote by $(\mathfrak{g}_e)_\gamma$ the stabiliser of γ in $\mathfrak{g}_e$ and set

$$(\mathfrak{g}_e^*)_{\text{sing}} := \{\gamma \in \mathfrak{g}_e^* \mid \dim(\mathfrak{g}_e)_\gamma > l\}.$$

Theorem 1. [5] *Suppose e admits a good generating system $F_1, \dots, F_l$ in $\mathcal{S}(\mathfrak{g})^G$ and assume further that $(\mathfrak{g}_e^*)_{\text{sing}}$ has codimension ≥ 2 in $\mathfrak{g}_e^*$. Then $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is a polynomial algebra in ${}^eF_1, \dots, {}^eF_l$.*

Suppose now that e is a minimal nilpotent element. Then $\dim(\mathfrak{g}_e)_\gamma = l$ for generic $\gamma \in \mathfrak{g}_e^*$, see [4]; and $(\mathfrak{g}_e^*)_{\text{sing}}$ is of codimension ≥ 2 , see [5, Section 3.10.]. If $\mathfrak{g}$ is of type E_8 , then there is no good generating system, [5, Remark 4.2.]. For that reason in Section 4.8 of [5] another approach was developed. As was proved there, Conjecture 1 holds if and only if there is a certain system of generating invariants in E_7 .

Since e is a minimal nilpotent element, the $\mathbb{Z}$ -grading defined by h is

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2),$$

with $\mathfrak{g}(2) = \mathbb{K}e$ and $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$ being a Heisenberg Lie algebra. Set $\mathfrak{l} := \mathfrak{g}(0)_e = \mathfrak{g}_e \cap \mathfrak{g}(0)$. Then $\mathfrak{g}(0) = \mathfrak{l} \oplus \mathbb{K}h$. Clearly $\mathfrak{p} := \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p}' = \mathfrak{g}(0)' \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Since $\mathfrak{p} = \mathbb{K}h \oplus \mathfrak{g}_e$ and $[h, e] = 2e$, we have

$$\mathcal{S}(\mathfrak{p})^{\mathfrak{p}'} \subset \mathcal{S}(\mathfrak{p})^e \subset \mathcal{S}(\mathfrak{g}_e).$$

If $\mathfrak{g}$ is not of type A , then $\mathfrak{l} = \mathfrak{g}(0)'$ and $\mathfrak{p}' = \mathfrak{g}_e$. Hence $\mathcal{S}(\mathfrak{p})^{\mathfrak{p}'} = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$.

Remark 1. If $\mathfrak{g}$ is of type A , then, so far, we can only say that $\mathcal{S}(\mathfrak{p})^{\mathfrak{p}'} = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e'}$. Set $\mathfrak{n} := \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Then there is an isomorphism of $\mathfrak{l}$ -modules $(\mathcal{S}(\mathfrak{g}_e)[1/e])^{\mathfrak{n}} \cong \mathcal{S}(\mathfrak{l})[e, 1/e]$, see [5, Section 4.8.] or [7, Lemma 3.]. Therefore the centre of $\mathfrak{l}$ acts on $\mathfrak{n}$ -invariants trivially and $\mathcal{S}(\mathfrak{p})^{\mathfrak{p}'} = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e'} = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$.

From now on assume that $\mathfrak{g}$ is of type E_8 . Then $\mathfrak{l}$ is of type E_7 . For generic $v \in \mathfrak{g}(1)$ the stabiliser $\mathfrak{l}_v$ is a simple Lie algebra of type E_6 . Fix such $v \in \mathfrak{g}(1)$. Let $\mathfrak{t} \subset \mathfrak{l}$ and $\hat{\mathfrak{t}} \subset \mathfrak{l}_v$ be maximal tori such that $\hat{\mathfrak{t}} \subset \mathfrak{t}$. Then there is a unique orthogonal decomposition $\mathfrak{t} = \hat{\mathfrak{t}} \oplus \mathbb{K}h_0$. Let W and W' denote the Weyl groups of $\mathfrak{l}$ and $\mathfrak{l}_v$, respectively. Each $\varphi_i \in \mathcal{S}(\mathfrak{t})^W$ can be presented uniquely as

$$\varphi_i = \sum_{j=0}^{\nu} \varphi_i^{(j)} h_0^j \quad \left(\varphi_i^{(j)} \in \mathcal{S}(\mathfrak{t})^{W'}, \quad \varphi_i^{(\nu)} \neq 0, \quad \nu = \nu(i) \right).$$

Theorem 2. [5, Theorem 4.14.] *The algebra $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is free if and only if there is a homogeneous generating system $\varphi_1, \dots, \varphi_7$ in $\mathcal{S}(\mathfrak{t})^W$ such that the elements $\varphi_1^{(\nu)} h_0^{\nu(1)}, \dots, \varphi_7^{(\nu)} h_0^{\nu(7)}$ are algebraically independent.*

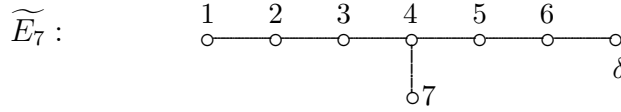
The main technical result of this paper is the following:

Proposition 1. *Suppose that $\varphi_1, \dots, \varphi_7$ is a system of homogeneous generators of $\mathcal{S}(\mathfrak{t})^W$ with $\deg \varphi_i < \deg \varphi_j$ for $i < j$. Then the elements $\varphi_1^{(\nu)} h_0^{\nu(1)}, \varphi_2^{(\nu)} h_0^{\nu(2)}, \varphi_3^{(\nu)} h_0^{\nu(3)}$ are algebraically dependent.*

Combining Theorem 2 and Proposition 1, we conclude that Conjectures 1 and 2 are false.

2. CALCULATIONS

Since it is difficult to deal with E_7 directly, we first consider a regular subalgebra $\mathfrak{sl}_8 \subset E_7$ such that $\mathfrak{t} \subset \mathfrak{sl}_8$. Let ϖ_i and ϖ'_i denote the fundamental weights of E_7 and SL_8 , respectively. We use the Vinberg–Onishchik numbering of simple roots and fundamental weights, see [6, Tables]. We may (and will) assume that the simple roots of $\mathfrak{sl}_8$ are the first six simple roots of E_7 and the lowest root δ . On the extended Dynkin diagram of E_7 , which is given below, the simple roots of $\mathfrak{sl}_8$ form the upper line. Recall that $\hat{\mathfrak{t}}$ is a maximal torus in a regular subalgebra $E_6 \subset E_7$. Without loss of generality, we may assume that $\hat{\mathfrak{t}}$ coincides with the annihilator of the weight ϖ_1 . Expressing δ as a linear combination of the simple roots one can see that $\varpi_1(\delta) = -1$. Hence the subtorus $\hat{\mathfrak{t}}$ is also the annihilator of $\varpi'_1 - \varpi'_7$.



Without loss of generality, we may assume that $\mathfrak{t}$ is the subspace of diagonal matrices of $\mathfrak{sl}_8$. The dual space $\tilde{\mathfrak{t}}^*$ is spanned by $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_8$ subject to the relation $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_8 = 0$ and the Weyl group of SL_8 permutes the ε_i 's. Since the fundamental weights ϖ'_1, ϖ'_7 can be expressed as $\varpi_1 = \varepsilon_1$ and $\varpi_7 = -\varepsilon_8$, we conclude that $\hat{\mathfrak{t}}$ is the annihilator of $\varepsilon_1 + \varepsilon_8$. Therefore $\hat{\mathfrak{t}}$ can be presented as a linear space of diagonal matrices:

$$(1) \quad \hat{\mathfrak{t}} = \{ \text{diag}(b, b_1, b_2, \dots, b_6, -b) \mid \sum_{i=1}^6 b_i = 0 \}.$$

Then

$$(2) \quad \mathbb{K}h_0 = \{ \text{diag}(a, -a/3, -a/3, -a/3, -a/3, -a/3, -a/3, a) \mid a \in \mathbb{K} \}.$$

Let us identify $\mathfrak{t}$ with $\mathfrak{t}^*$ by means of the Killing form. Then Weyl group invariants of SL_8 can be expressed in terms of variables $a, b, b_1, \dots, b_6$; the $\varphi_i^{(\nu)}$'s will be polynomials in $b, b_1, \dots, b_6$ and h_0 proportional to a . Since $\mathfrak{sl}_8$ is a maximal rank subalgebra of E_7 , one can write invariants of E_7 as polynomials in invariants of SL_8 with unknown coefficients. This will give us certain constraints on $\varphi_1^{(\nu)} h_0^{\nu(1)}$, $\varphi_2^{(\nu)} h_0^{\nu(2)}$, and $\varphi_3^{(\nu)} h_0^{\nu(3)}$.

First we concentrate on SL_8 -invariants. According to formulas (1) and (2), any diagonal matrix in $\mathfrak{sl}_8$ is of the form:

$$(a + b, -a/3 + b_1, -a/3 + b_2, -a/3 + b_3, -a/3 + b_4, -a/3 + b_5, -a/3 + b_6, a - b)$$

with $\sum_{i=1}^6 b_i = 0$. Set $\tau_k := \sum_{i=1}^6 b_i^k$, and let S_i be the trace of the $i + 1$ power of a diagonal matrix. Then

$$S_2 = 2a^2 + 2b^2 + \frac{6}{9}a^2 + \sum_{i=1}^6 \left(-\frac{2}{3}ab_i + b_i^2\right) = \frac{8}{3}a^2 + 2b^2 + \tau_2.$$

In the same way

$$\begin{aligned} S_3 &= \frac{16}{9}a^3 + 6ab^2 - a\tau_2 + \tau_3; \\ S_4 &= \left(2 + \frac{2}{27}\right)a^4 + 12a^2b^2 + \frac{2}{3}a^2\tau_2 - \frac{4}{3}a\tau_3 + 2b^4 + \tau_4; \\ S_5 &= \left(2 - \frac{2}{81}\right)a^5 + 20a^3b^2 - \frac{10}{27}a^3\tau_2 + \frac{10}{9}a^2\tau_3 + 10ab^4 - \frac{5}{3}a\tau_4 + \tau_5; \\ S_6 &= \left(2 + \frac{2}{3^5}\right)a^6 + 30a^4b^2 + \frac{5}{27}a^4\tau_2 - \frac{20}{27}a^3\tau_3 + 30a^2b^4 + \frac{5}{3}a^2\tau_4 + \dots; \\ S_8 &= \left(2 + \frac{2}{3^7}\right)a^8 + 56a^6b^2 + \frac{28}{3^6}a^6\tau_2 - \frac{56}{3^5}a^5\tau_3 + 140a^4b^4 + \frac{70}{81}a^4\tau_4 + \dots \end{aligned}$$

In principle, it is possible to calculate all of them either by hand or using computer. The expression for S_7 is of no importance for us. Also, the coefficients of smaller degrees of a play no rôle in the following calculations. Therefore they are not written down in S_6 and S_8 .

Let φ_1 , φ_2 , and φ_3 be Weyl group invariants of E_7 of degrees 2, 6, and 8, respectively. Then $\deg(\varphi_1^{(\nu)}) = 0$. Since $\mathfrak{sl}_8$ is a maximal rank subalgebra of E_7 , the E_7 -invariants are polynomials in SL_8 -invariants. Using this, we will show that $\deg(\varphi_2^{(\nu)}) \leq 2$ and $\deg(\varphi_3^{(\nu)}) \leq 4$. Since both these polynomials are invariants of the Weyl group of E_6 , they must be algebraically dependent (recall that the degrees of E_6 -invariants are 2, 5, 6, 8, 9, 12).

Lemma 1. *For the Weyl group invariant φ_2 with $\deg \varphi_2 = 6$, we have $\deg(\varphi_2^{(\nu)}) \leq 2$.*

Proof. The invariant φ_2 is a linear combination of SL_8 -invariants of degree 6. One can express this as follows:

$$\varphi_2 = x_1 S_2^3 + x_2 S_3^2 + x_3 S_2 S_4 + x_4 S_6, \quad \text{where } x_i \in \mathbb{K}.$$

Assume that $\deg(\varphi_2^{(\nu)}) > 2$. Since $\varphi_2^{(\nu)}$ is an invariant of E_6 , it cannot be of degree 3. Hence $\deg(\varphi_2^{(\nu)}) \geq 4$ and the coefficients of a^6 , a^4 , and a^3 in φ_2 are zeros. This condition gives us four linear equations on x_i .

Let us write down the polynomials in question:

$$\begin{aligned} S_2^3 &= \frac{512}{27}a^6 + \frac{64}{3}a^4(2b^2 + \tau_2) + \dots; \\ S_3^2 &= \frac{256}{81}a^6 + \frac{32}{9}a^4(6b^2 - \tau_2) + \frac{32}{9}a^3\tau_3 + \dots; \\ S_2S_4 &= \frac{448}{81}a^6 + (36 + \frac{4}{27})a^4b^2 + (2 + \frac{50}{27})a^4\tau_2 - \frac{32}{9}a^3\tau_3 + \dots; \\ S_6 &= (2 + \frac{2}{3^5})a^6 + 30a^4b^2 + \frac{5}{27}a^4\tau_2 - \frac{20}{27}a^3\tau_3 + \dots \end{aligned}$$

Again we calculate only whose coefficients, which will be used. Since b and all τ_i are algebraically independent, we indeed obtain four linear equations.

$$\begin{cases} \frac{512}{27}x_1 + \frac{256}{81}x_2 + \frac{448}{81}x_3 + (2 + \frac{2}{3^5})x_4 = 0 \\ \frac{128}{3}x_1 + \frac{64}{3}x_2 + (36 + \frac{4}{27})x_3 + 30x_4 = 0 \\ \frac{64}{3}x_1 - \frac{32}{9}x_2 + (2 + \frac{50}{27})x_3 + \frac{5}{27}x_4 = 0 \\ \frac{32}{9}x_2 - \frac{32}{9}x_3 - \frac{20}{27}x_4 = 0 \end{cases}$$

The determinant of this system is non-zero. Hence the only solution is trivial. Since $\varphi_2 \neq 0$, we have proved that $\deg(\varphi_2^{(\nu)}) \leq 2$. $\square$

Lemma 2. *For the Weyl group invariant φ_3 with $\deg \varphi_3 = 8$, we have $\deg(\varphi_3^{(\nu)}) \leq 4$.*

Proof. Argument for this invariant is essentially the same as in Lemma 1, but here calculations are more involved. Again

$$\varphi_3 = y_1S_2^4 + y_2S_2S_3^2 + y_3S_2^2S_4 + y_4S_2S_6 + y_5S_3S_5 + y_6S_4^2 + y_7S_8, \quad \text{where } y_i \in \mathbb{K}.$$

We need coefficients of this seven polynomials up to a^4 . Here they are:

$$\begin{aligned} S_2^4 &= \frac{4096}{81}a^8 + \frac{4096}{27}a^6b^2 + \frac{2048}{27}a^6\tau_2 + \frac{128}{3}a^4(4b^4 + 4b^2\tau_2 + \tau_2^2) + \dots; \\ S_2S_3^2 &= \frac{2048}{243}a^8 + \frac{5120}{81}a^6b^2 - \frac{512}{81}a^6\tau_2 + \frac{256}{27}a^5\tau_3 + \frac{416}{3}a^4b^4 - \frac{160}{9}a^4b^2\tau_2 - \frac{8}{9}a^4\tau_2^2 + \dots; \end{aligned}$$

$$\begin{aligned}
S_2^2 S_4 &= \frac{3584}{243} a^8 + \frac{8704}{81} a^6 b^2 + \frac{1280}{81} a^6 \tau_2 - \frac{256}{27} a^5 \tau_3 + \\
&\quad + \frac{4064}{3} a^4 b^4 - \frac{2144}{27} a^4 b^2 \tau_2 + \frac{152}{27} a^4 \tau_2^2 + \frac{64}{9} a^4 \tau_4 + \dots; \\
S_2 S_6 &= \frac{3904}{729} a^8 + \frac{20416}{243} a^6 b^2 + \frac{488}{243} a^6 \tau_2 - \frac{160}{81} a^5 \tau_3 + \\
&\quad + 140 a^4 b^4 + \frac{820}{27} a^4 b^2 \tau_2 + \frac{5}{27} a^4 \tau_2^2 + \frac{40}{9} a^4 \tau_4 + \dots; \\
S_3 S_5 &= \frac{2560}{729} a^8 + \frac{1280}{27} a^6 b^2 - \frac{640}{243} a^6 \tau_2 + \frac{320}{81} a^5 \tau_3 + \\
&\quad + \frac{1240}{9} a^4 b^4 - \frac{200}{9} a^4 b^2 \tau_2 + \frac{10}{27} a^4 \tau_2^2 - \frac{80}{27} a^4 \tau_4 + \dots; \\
S_4^2 &= \frac{3136}{729} a^8 + \frac{448}{9} a^6 b^2 + \frac{224}{81} a^6 \tau_2 - \frac{448}{81} a^5 \tau_3 + \frac{4112}{27} a^4 b^4 + 16 a^4 b^2 \tau_2 + \frac{4}{9} a^4 \tau_2^2 + \frac{112}{27} a^4 \tau_4 + \dots; \\
S_8 &= (2 + \frac{2}{3^7}) a^8 + 56 a^6 b^2 + \frac{28}{3^6} a^6 \tau_2 - \frac{56}{3^5} a^5 \tau_3 + 140 a^4 b^4 + \frac{70}{81} a^4 \tau_4 + \dots.
\end{aligned}$$

I calculated these expansions on the computer in “Maple”. It is quite possible to check any of the coefficients by hand, but getting them all is rather tiresome.

Assume that $\deg(\varphi_3^{(\nu)}) > 4$. Then the coefficients of a^8 , a^6 , a^5 , and a^4 in φ_3 are zeros. Therefore there are eight linear equations, corresponding to the summands

$$a^8, a^6 b^2, a^6 \tau_2, a^5 \tau_3, a^4 b^4, a^4 b^2 \tau_2, a^4 \tau_2^2, a^4 \tau_4,$$

depending on seven variables y_i . Since at least one 7×7 minor of this matrix is non-zero (it was checked on the computer), the only possible solution is zero. Thus if $\varphi_3 \neq 0$, then $\deg(\varphi_3^{(\nu)}) \leq 4$. $\square$

Proof of Proposition 1. Suppose that $\varphi_1, \dots, \varphi_7$ is a system of homogeneous generators of $\mathfrak{S}(\mathfrak{t})^W$ with $\deg \varphi_i < \deg \varphi_j$ for $i < j$. Then $\deg \varphi_1 = 2$, $\deg \varphi_2 = 6$, and $\deg \varphi_3 = 8$. Clearly $\varphi_1^{(\nu)} h_0^{\nu(1)}$ is proportional to h_0^2 . Hence the polynomials $\varphi_1^{(\nu)} h_0^{\nu(1)}$, $\varphi_2^{(\nu)} h_0^{\nu(2)}$, and $\varphi_3^{(\nu)} h_0^{\nu(3)}$ are algebraically independent if and only if $\varphi_2^{(\nu)}$ and $\varphi_3^{(\nu)}$ are. By Lemmas 1 and 2, we have $\deg \varphi_2^{(\nu)}, \deg \varphi_3^{(\nu)} \leq 4$. Recall that $\varphi_2^{(\nu)}$ and $\varphi_3^{(\nu)}$ are invariants of E_6 . Since the Weyl group of type E_6 has no basic invariants of degrees 1, 3, and 4; and only one of degree 2, these polynomials are algebraically dependent. $\square$

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