SOFTWARE MODULES AND COMPUTER-ASSISTED PROOF SCHEMES IN THE KONTSEVICH DEFORMATION QUANTIZATION

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Abstract. The Kontsevich deformation quantization combines Poisson dynamics, noncommutative geometry, number theory, and calculus of oriented graphs. To manage the algebra and differential calculus of series of weighted graphs, we present software modules: these allow generating the Kontsevich graphs, expanding the noncommutative $\star$-product by using *a priori* undetermined coefficients, and deriving linear relations between the weights of graphs. Throughout this text we illustrate the assembly of the Kontsevich $\star$-product up to order 4 in the deformation parameter $\hbar$. Already at this stage, the $\star$-product involves hundreds of graphs; expressing all their coefficients via 149 weights of basic graphs (of which 67 weights are now known exactly), we express the remaining 82 weights in terms of only 10 parameters (more specifically, in terms of only 6 parameters modulo gauge-equivalence). Finally, we outline a scheme for computer-assisted proof of the associativity, modulo $\bar{o}(\hbar^4)$, for the newly built $\star$-product expansion.

Contents

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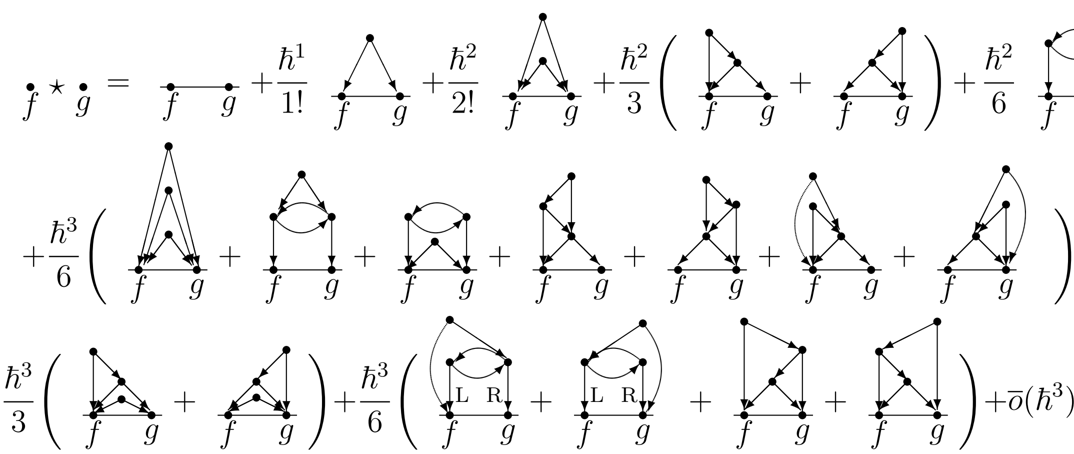
Introduction. On every finite-dimensional affine (i.e. piecewise-linear) manifold N^n, the Kontsevich star-product $\star$ is an associative but not necessarily commutative deformation of the usual product $\times$ in the algebra of functions $C^\infty(N^n)$ towards a given Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ on N^n. Specifically, whenever $\star = \times + \hbar\,\{\cdot,\cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ is an infinitesimal deformation, it can always be completed to an associative star-product $\star = \times + \hbar\,\{\cdot,\cdot\}_{\mathcal{P}} + \sum_{k\geqslant 2}\hbar^k B_k(\cdot,\cdot)$ in the space of formal power series $C^\infty(N^n)[[\hbar]]$; this was proven in [26]. An explicit calculation of the bi-linear bi-differential terms $B_k(\cdot,\cdot)$ at high orders $\hbar^k$ is a computationally hard problem. In this paper we reach the order $k = 4$ in expansion of $\star$ by using software modules for the Kontsevich graph calculus, which we presently discuss.

Convenient in practice, the idea from [26] (see also [22, 23, 25]) is to draw every derivation $\partial_i \equiv \partial/\partial x^i$ (with respect to a local coordinate x^i on a chart in the Poisson manifold N^n at hand) as decorated edge $\xrightarrow{i}$, so that large differential expressions become oriented graphs. For example, the Poisson bracket $\{f,g\}_{\mathcal{P}}(\boldsymbol{x}) = \sum_{i,j=1}^n (f)\overleftarrow{\partial_i}\big|_{\boldsymbol{x}} \cdot \mathcal{P}^{ij}(\boldsymbol{x}) \cdot \overrightarrow{\partial_j}\big|_{\boldsymbol{x}}(g)$ of two functions $f,g \in C^\infty(N^n)$ is depicted by the graph $(f) \xleftarrow{i} \mathcal{P}^{ij} \xrightarrow{j} (g)$; here $\mathcal{P}^{ij}$ is the skew-symmetric matrix of Poisson bracket coefficients and the summation over i,j running from 1 to the dimension n of N^n is implicit. In these terms, the known – from [7] – expansion of Kontsevich star-product looks as follows:[1]



(1)

By construction, every oriented edge carries its own index and every *internal* vertex (not containing the arguments f or g) is inhabited by a copy of the coefficient matrix $\mathcal{P} = (\mathcal{P}^{ij})$ of the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$. This means that expansion (1) encodes the analytic formula

$$f \star g = f \times g + \hbar \mathcal{P}^{ij}\partial_i f \partial_j g + \hbar^2\big(\tfrac{1}{2}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_k\partial_i f \partial_\ell\partial_j g + \tfrac{1}{3}\partial_\ell \mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_k\partial_i f\partial_j g$$
$$- \tfrac{1}{3}\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_i f \partial_k\partial_j g - \tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_i f\partial_k g\big) + \hbar^3\big(\tfrac{1}{6}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_n\partial_\ell\partial_j g$$

[1] The indication L and R for Left $\prec$ Right, respectively, matches the indices – which the pairs of edges carry – with the ordering of indices in the coefficients of the Poisson structure contained in the arrowtail vertex. Note that exactly *two* edges are issued from every internal vertex in every graph in formula (1); not everywhere displayed in (1), the ordering $L \prec R$ in each term is determined from same object's expansion (2).

$$
\begin{aligned}
&-\tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_k g-\tfrac{1}{6}\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad-\tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_j g-\tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_k\partial_j g\\
&\quad+\tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_j g+\tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_m\partial_k\partial_j g\\
&\quad+\tfrac{1}{3}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_\ell\partial_j g-\tfrac{1}{3}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_\ell\partial_j g\\
&\quad-\tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_i f\partial_k g+\tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_m\partial_k g\\
&\quad-\tfrac{1}{6}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g-\tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g\big)+\bar{o}(\hbar^3). \quad (2)
\end{aligned}
$$

We now see that the language of Kontsevich graphs is more intuitive and easier to percept than writing formulae. The calculation of the associator $\mathrm{Assoc}_\star(f,g,h) = (f\star g)\star h - f\star(g\star h)$ can also be done in a pictorial way (see section 2.4 on p. 18). The coefficients of graphs at $\hbar^k$ in a star-product expansion are given by the Kontsevich integrals over the configuration spaces of k distinct points in the Lobachevsky plane $\mathbb{H}$, see [26] and [9]. Although proven to exist, such weights of graphs are very hard to obtain in practice.[2] Much research has been done on deriving helpful relations between the weights in order to facilitate their calculation [11, 29, 15, 12, 2]. In Example 21 on p. 25 we explain how expansion (1) modulo $\bar{o}(\hbar^3)$ was obtained in [7]. The techniques which were then sufficient are no longer enough to build the Kontsevich $\star$-product beyond the order $\hbar^3$; clearly, extra mathematical concepts and computational tools must be developed. In this paper we present the software in which several known relations between the Kontsevich graph weights are taken into account; we express the weights of all graphs at $\hbar^4$ in terms of 10 master-parameters. (To be more precise, the ten master-parameters are reduced to just 6 by taking the quotient over certain four degrees of gauge freedom in the associative star-products.)

This paper contains three chapters. In chapter 1 we introduce the software to encode and generate the Kontsevich graphs and operate with series of such graphs. In particular, the coefficients of graphs in series can be undetermined variables. The series are then reduced modulo the skew-symmetry of graphs (under the swapping of Left $\rightleftarrows$ Right in their construction). Thirdly, a series can be evaluated at a given Poisson structure: that is, a copy of the bracket is placed at every internal vertex.

Chapter 2 is devoted to the construction of Kontsevich $\star$-product: containing a given Poisson structure in its leading deformation term, this bi-linear operation is not necessarily commutative but it is required to be associative; hence the coefficients of a power series for $\star$ must be specified. For example, at order $k=4$ of the deformation parameter $\hbar$ there are 149 parameters to be found. (The actual number of graphs at $\hbar^4$ is much greater; we here count the "basic" graphs only.) We review a number of methods to obtain the weights of Kontsevich graphs; the spectrum of techniques employed ranges from complex analysis and direct numeric integration [8] to finding linear relations between such weights by using abstract geometric reasonings. The associativity of Kontsevich $\star$-product is the main source of relations between the graph weights; at $\hbar^4$ such relations are *linear* because everything is known about the weights up to order three. We obtain these relations at order four in chapter 3 and we solve that system

[2] In fact, there are many other admissible graphs, not shown in (1), in which every internal vertex is a tail for two oriented edges, but the weights of those graphs are found to be zero.

of linear algebraic equations for 149 unknowns. The solution is expressed in terms of only 10 master-parameters, see formula (11) on pp. 34–39.[3]

The algebraic system constructed in section 3.1 was obtained by restricting the associativity for $\star$ to (a class of) specific Poisson structures. We want however to prove that for the newly found collection of graph weights, the $\star$-product is associative for *every* Poisson structure on *all* finite-dimensional affine manifolds. For that, in section 3.2 we design a computer-assisted proof scheme that is independent of the bracket (and of a manifold at hand). Specifically, in Theorem 12 on p. 30 we reveal how the associator for Kontsevich $\star$-product, taken modulo $\bar{o}(\hbar^4)$, is factorised via the Jacobiator $\mathrm{Jac}(\mathcal{P})$ or via its differential consequences that all vanish identically for Poisson structures $\mathcal{P}$ on the manifolds N^n. We discover in particular that such factorisation,

$$\mathrm{Assoc}_\star(f,g,h) = \lozenge\big(\mathcal{P},\mathrm{Jac}(\mathcal{P}),\mathrm{Jac}(\mathcal{P})\big) \mod \bar{o}(\hbar^4),$$

is quadratic and has differential order two with respect to the Jacobiator. For all Poisson brackets $\{\cdot,\cdot\}_\mathcal{P}$ on finite-dimensional affine manifolds N^n our ten-parameter expression of the $\star$-product does agree up to $\bar{o}(\hbar^4)$ with previously known results about the values of Kontsevich graph weights at some fixed values of the ten master-parameters and about the linear relations between those weights at all values of the master-parameters.[4]

The software implementation [5] consists of a `C++` library and a set of command line calls. The latter are specified in what follows; a full list of new `C++` subroutines and their call syntaxis is contained in Appendix B. Whenever a command line call refers to just one particular function in `C++`, we indicate that in the text. The current text refers to version `0.16` of the software.

[3]The values of all these ten master-parameters have recently been claimed by Panzer and Pym [27] as a result of implementation of another technique to calculate the Kontsevich weights: see Table 4 on p. 46 in Appendix A.2. In particular, the values which we conjecture in Table 3 fully agree with the exact values suggested in [27]. Based on this external input, the expansion of the Kontsevich $\star$-product becomes (13) on pp. 46–50.

[4]From Theorem 12 we also assert that the associativity of Kontsevich $\star$-product does not carry on but it can leak at orders $\hbar^{\geqslant 4}$ of the deformation parameter, should one enlarge the construction of $\star$ to an affine bundle set-up of N^n-valued fields over a given affine manifold M^m and of variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ for local functionals $F,G,H\colon C^\infty(M^m\to N^n)\to\Bbbk$, see [17, 18, 19, 20] and [21].

1. Weighted graphs

In this section we introduce the software to operate with series of graphs.

1.1. **Normal forms of graphs and their machine-readable format.** As it was explained in the introduction, we consider graphs whose vertices contain Poisson structures and whose edges represent derivatives. To be precise, the class of graphs to deal with is as follows.

Definition 1. Let us consider a class of oriented graphs on $m+n$ vertices labelled 0, …, $m+n-1$ such that the consecutively ordered vertices 0, …, $m-1$ are sinks, and each of the internal vertices m, …, $m+n-1$ is a source for two edges. For every internal vertex, the two outgoing edges are ordered using $L \prec R$: the preceding edge is labeled L (Left) and the other is R (Right). An oriented graph on m sinks and n internal vertices is a Kontsevich graph of type (m,n). We denote by $G_{m,n}$ the set of all Kontsevich graphs of type (m,n), and by $\tilde{G}_{m,n}$ the subset of $G_{m,n}$ consisting of all those graphs having neither double edges nor tadpoles.

Remark 1. The class of graphs which we consider is not the most general type considered by Kontsevich in [26]. In the construction of the Formality morphism there also appear graphs with sources for more or fewer (than two) arrows. However, in our approach to the problem at hand, which is the construction of a $\star$-product expansion that would be associative modulo $\hbar^k$ for some $k \gg 0$, we shall only meet graphs from the class of Definition 1.

Remark 2. There can be tadpoles or cycles in a graph $\Gamma \in G_{m,n}$, see Fig. 1.

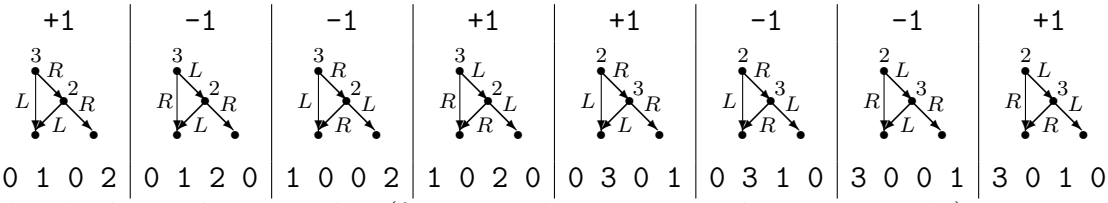
Figure 1. A tadpole and an "eye".

A Kontsevich graph $\Gamma \in G_{m,n}$ is uniquely determined by the numbers n and m together with the list of ordered pairs of targets for the internal vertices. For reasons which will become clear immediately below, we now consider a Kontsevich graph Γ together with a *sign* $s \in \{0, \pm 1\}$, denoted by concatenation of the symbols: $s\Gamma$.

Implementation 1 (encoding)**.** The format to store a signed graph $s\Gamma$ with $\Gamma \in G_{m,n}$ is the integer number $m > 0$, the integer $n \geqslant 0$, the sign s, followed by the (possibly empty, when $n = 0$) list of n ordered pairs of targets for edges issued from the internal vertices m, …, $m+n-1$, respectively. The full format is then (m, n, s; list of ordered pairs).

We recall that to every Kontsevich graph one associates a polydifferential operator by placing a copy of the Poisson bracket at each vertex. To a signed graph one associates the polydifferential operator of the graph multiplied by the sign. The skew-symmetry of the Poisson bracket implies that the same polydifferential operator may be represented by several different signed graphs, all having different encodings.

Example 1. Taken with the signs in the first row, the graphs in the second row all represent the same polydifferential operator:



In the third row the target list (for internal vertices 2 and 3, respectively) is written.

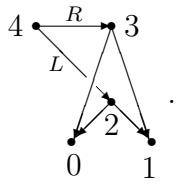
We would like to know whether two (encodings of) signed graphs specify the same topological portrait — up to a permutation of internal vertices and/or a possible swap $L \rightleftarrows R$ for some pair(s) of outgoing edges. To compare two given encodings of a signed graph, let us define its normal form. Such normal form is a way to pick the representative modulo the action of group $S_n \times (\mathbb{Z}_2)^n$ on the space $G_{m,n}$.

Definition 2 (normal form)**.** The list of targets of a graph $\Gamma \in G_{m,n}$ can be considered as a $2n$-digit integer written in base-$(n+m)$ notation. By running over the entire group $S_n \times (\mathbb{Z}_2)^n$, and by this over all the different re-labelings of Γ, we obtain many different integers written in base-$(n+m)$. The *absolute value* $|\Gamma|$ of Γ is the re-labeling of Γ such that its list of targets is *minimal* as a nonnegative base-$(n+m)$ integer. For a signed graph $s\Gamma$, the *normal form* is the signed graph $t|\Gamma|$ which represents the same polydifferential operator as $s\Gamma$. Here we let $t = 0$ if the graph is zero (see Remark 3 below).

Example 2. The minimal base-4 number in the third column of Example 1 is `0 1 0 2`. Hence the absolute value of each of the graphs in Example 1 is the first graph. The normal form of each of the signed graphs in Example 1 is the first graph taken with the appropriate sign ± 1; the encodings of the normal forms are then `2 2 ±1 0 1 0 2`.

Remark 3. The graphs $\Gamma \in G_{m,n}$ for which the associated polydifferential operator vanishes, by being equal to minus itself, are called *zero*. This property can be detected during the calculation of the normal form of a signed graph. One starts with the encoding of a signed graph. Obtain a "sorted" encoding (representing the same polydifferential operator) by sorting the outgoing edges in every pair in nondecreasing order: each swap $L \rightleftarrows R$ entails a reversion of the sign. Now run over the group S_n of permutations of the internal vertices in the graph at hand, relabeling those vertices. Should the list of targets in the sorted encoding of a relabeling be equal to the list of targets in the original sorted encoding, but the sign be opposite, then the graph is zero. We will see in Chapter 2 (specifically, in Lemma 2 on p. 14) that the weights of these graphs also vanish, this time by the anticommutativity of certain differentials under the wedge product.

Example 3. Consider the graph



.

with the encoding `2 3 1  0 1 0 1 2 3`. For the identity permutation we obtain the initial sorted encoding `2 3 1  0 1 0 1 2 3` (it was already sorted). For the permutation $2 \leftrightarrows 3$ we obtain the encoding `2 3 1  0 1 0 1 3 2`; upon sorting the pairs it becomes `2 3 -1  0 1 0 1 2 3`. The list of pairs coincides with the initial sorted encoding but the sign is opposite; hence the graph is zero.

The notion of normal form of graphs allows one to generate lists of graphs with different topological portraits (e.g., Kontsevich graph series, see section 1.2 below) by using the following algorithm. Initially, the set of generated graphs is empty. For every encoding (according to Implementation 1) in a run-through, its normal form with sign +1 or 0 is added to the list if it is not contained there (otherwise, the offered encoding is skipped).

Implementation 2. To generate all the Kontsevich graphs with `m` sinks and `n` internal vertices in $\tilde{G}_{m,n}$ (without tadpoles or double edges), the command is

```
> generate_graphs n m
```

The procedure lists all such graphs (one per line) in the standard output. The second argument `m` may be omitted: the default value is `m = 2`.
Similarly, to generate only normal forms (with sign +1 or 0), the call is

```
> generate_graphs n m --normal-forms=yes
```

The optional argument `--with-coefficients` indicates that (numbered) coefficients should be listed along with the graphs.
(Accordingly, see `KontsevichGraph::graphs` in Appendix B.)

Example 4. The Kontsevich graphs in $\tilde{G}_{m,n}$ with *one* internal vertex

```
> generate_graphs 1
2 1 1   0 1
2 1 1   1 0
```

consist of the wedge with its two different labellings. We can check that the number of Kontsevich graphs on n internal vertices and two sinks is $(n(n+1))^n$:

```
> generate_graphs 2 | wc -l
36
> generate_graphs 3 | wc -l
1728
> generate_graphs 4 | wc -l
160000
> generate_graphs 5 | wc -l
24300000
```

Here, "`| wc -l`" counts the number of lines in the output.

Let us remember that while a graph series is generated, more options can be chosen to restrict the graphs: e.g., only *prime* graphs can be taken into account, or graphs but not their mirror copies can be allowed. On the same grounds, only those graphs in which the sink(s) is or are the target(s) for at least one arrow per sink can be taken. The implementation of these conventions will be explained in the next chapter (see p. 16 below).

1.2. **Series of graphs: file format.** We now specify how formal power series expansions of graphs are implemented in software. Denote by $\hbar$ the formal parameter; in machine-readable format, a power series in $\hbar$ is a list of coefficients of $\hbar^k$, $k \geqslant 0$. The coefficients are formal sums of graphs (see `KontsevichGraphSum` in Appendix B) in which the coefficients can be of any type, e.g.,

- integer or floating point numbers (e.g., 0.333),
- rational numbers (e.g., 1/3),
- undetermined variables (resp., `OneThird`).

To be precise, the library contains the class `KontsevichGraphSeries` which depends on a template parameter `T`; it specifies the type of all the coefficients of graphs in the series. In the command line programs, the external type `GiNaC::ex`, which is the expression type of the `GiNaC` library [1], allows all of the above values (and combinations of them). Hence a series under study can contain coefficients of all types at once; the coefficient of a graph itself can be a sum of different type objects (e.g., `p16` + 0.25).

In the file format for formal power series expansions, two kinds of lines are possible: either

```
h^k:
```

or (separated by whitespace)

```
<encoding of a graph> <coefficient>
```

The precision of the formal power series expansion is indicated by the highest `k` occurring in lines of the form "`h^k:`". Hence one can control this bound by adding such a line with a high `k` at the end of the file.

Implementation 3. The substitution of undetermined coefficients by their actual values, as well as re-expression of indeterminates via other such objects, is done by using the program

```
> substitute_relations <graph-series-file> <subsitutions-file>
```

Its command line arguments are two file names: the first file contains the series and the second file consists of a list of substitutions (one per line), each substitution written in the form

```
<variable>==<what it is set equal to>
```

The command line program sends the series with all those substitutions to the standard output.

Example 5. The Kontsevich $\star$-product (see §2) is a graph series given up to the second order in the deformation parameter $\hbar$ in the file `star_product2_w.txt` which reads

```
h^0:
2 0 1              1
h^1:
2 1 1   0 1        1
h^2:
2 2 1   0 1 0 1    1/2
2 2 1   0 1 0 2    w_2_1
2 2 1   0 1 1 2    w_2_2
2 2 1   0 3 1 2    w_2_3
```

In fact, the values of the three unknowns are written in `weights2.txt`:

```
w_2_1==1/3
w_2_2==-1/3
w_2_3==-1/6
```

Whence the star-product is given modulo $\bar{o}(\hbar^2)$ as follows:

```
> substitute_relations star_product2_w.txt weights2.txt
h^0:
2 0 1              1
h^1:
2 1 1   0 1        1
h^2:
2 2 1   0 1 0 1    1/2
2 2 1   0 1 0 2    1/3
2 2 1   0 1 1 2    -1/3
2 2 1   0 3 1 2    -1/6
```

In practice one may encounter graph series containing many graphs and undetermined coefficients. To split a graph series into parts, the following command is helpful.

Implementation 4. To extract the part of a graph series proportional to a given expression, use the call

```
> extract_coefficient <graph-series-file> <expression>
```

In the standard output one obtains a modification of the original graph series: each graph coefficient `c` is now replaced by the coefficient of `<expression>` in `c`. If the coefficient of `<expression>` in `c` is identically zero, then the graph is skipped. The special value `<expression> = 1` yields the constant part of the graph series (all the undetermined variables in the input are set to zero).

Example 6. From the file in Example 5, we extract the part proportional to `w_2_1`:

```
> extract_coefficient star_product2_w.txt w_2_1
h^0:
h^1:
h^2:
2 2 1   0 1 0 2    1
```

It is just one graph.

1.3. **Reduction modulo skew-symmetry.** Let us recall that for every internal vertex in a Kontsevich graph, the pair of out-going edges is ordered by the relation Left $\prec$ Right and by a mark-up of those two edges using L and R. Starting from the vector space of formal sums of graphs with real coefficients, we pass to its quotient. Namely, we postulate that graphs which differ only by their internal vertex labeling are equal. Further, we proclaim that every reversal of the edge order in any pair entails the reversion of the graph sign. By construction, the sign is a part of the encoding for a graph, see Implementation 1 on p. 5 above; this part of a graph's description is used whenever formal sums of graphs with coefficients,

`+(sign)<coeff>` $\cdot$ Γ`raph + (sign)<coeff>` $\cdot$ Γ`raph`,

are reduced. In particular, we let

$$\texttt{+(+1)<coeff>} \cdot \Gamma \texttt{ + (-1)<coeff>} \cdot \Gamma \equiv 0$$

for a graph Γ with any coefficient `<coeff>`.

The ordering mechanism Left $\prec$ Right creates graphs that equal zero because they are equal to minus themselves (see Remark 3 and Example 3).

Remark 4. To avoid such comparison of graphs with zero all the time and so, to increase efficiency, every graph is brought to its normal form as soon as it is constructed. It is this moment when zero graphs acquire zero signs.

The algorithm to reduce a sum of graphs modulo skew-symmetry runs as follows. For the starting graph or every next graph in the list, its sign (if nonzero) is set equal to $+1$ and its coefficient is modified, if necessary, by using the rule

$$\texttt{<sign>}\cdot\texttt{<coeff> = (+1)}\cdot\texttt{<sign}\cdot\texttt{coeff>}. \tag{3}$$

Every graph with sign 0 is removed. Then the graph at hand (in its normal form, times a coefficient) is compared, disregarding signs, with all the graphs which follow in the list. A match found, its coefficient is added – using relation (3) – to the coefficient of the graph we started with; the match itself is removed. By this reduction procedure for graph sums, all vanishing graphs with zero signs are excluded from the list.

Implementation 5. To reduce a graph series expansion modulo skew-symmetry, call

```
> reduce <graph-series-file> [--print-differential-orders]
```

The resulting graph series is sent to the standard output. The optional argument `--print-differential-orders` controls whether the differential orders of the graphs (as operators acting on the sinks) are included in the output, with lines such as

```
# 2 1
```

indicating subsequent graphs have differential order $(2, 1)$. (The corresponding methods are `KontsevichGraphSeries<T>::reduce()` and `KontsevichGraphSum<T>::reduce()` in Appendix B.)

Example 7. We put the zero graph from Example 3 with the coefficient $+1$ into a file `zerograph.txt`:

```
h^3:
2 3 1  0 1 0 1 2 3    1
```

We confirm that `reduce` kills it:

```
> reduce zerograph.txt
h^3:
```

The output is an empty list of graphs.

Sometimes it is desirable to skew-symmetrize a graph series over the content of its sinks. For example, one may want to do this when dealing with first-order differential operators which represent (skew-symmetric) polyvectors (e.g., as the authors did jointly with A. Bouisaghouane in [3]).

Implementation 6. To skew-symmetrize a graph series over the content of its sinks, the command

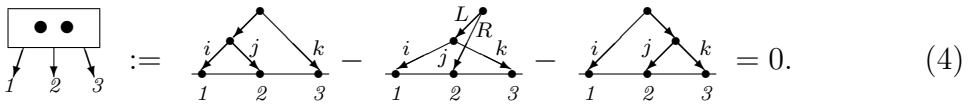
```
> skew_symmetrize <graph-series-file>
```

is available. The convention is that the sum over all permutations of the sinks is taken, with the signs of those permutations, without any pre-factor (such as $1/n!$). (Accordingly, see `KontsevichGraphSum<T>::skew_symmetrize()` in Appendix B, as well as `KontsevichGraphSeries<T>::skew_symmetrize()`, which calls the former.)

Remark 5. Sums of graphs may also be reduced modulo the (graphical) Jacobi identity and its (pictorial) differential consequences; this is the subject of section 3.2.

1.4. **Evaluate a given graph series at a given Poisson structure.** Let us recall that every Kontsevich graph contains at least one sink. Every edge (decorated with an index, say i, over which the summation runs from 1 to $n = \dim N^n$) denotes the derivation with respect to a local coordinate x^i at a given point $\boldsymbol{x}$ of the affine manifold N^n (hence the edge denotes $\partial/\partial x^i|_{\boldsymbol{x}}$). Every internal vertex (if any) encodes a copy of a given Poisson structure $\mathcal{P}$. Should the labellings of two outgoing edges be $\overset{i}{\dashrightarrow}$ and $\overset{j}{\dashrightarrow}$ so that the edge with i precedes that with j, the Poisson structure in that vertex is $\mathcal{P}^{ij}(\boldsymbol{x})$ (that is, the ordering $i \prec j$ is preserved; moreover, the reference to a point $\boldsymbol{x}$ is common to all vertices). Now, every Kontsevich graph (with a coefficient after it) represents a (poly)differential operator with respect to the content of sink(s); to build that operator, we apply the derivations (at $\boldsymbol{x} \in N^n$) to objects in the arrowhead vertices, multiply the content of all vertices at a fixed set of index values, and then sum over all the indices.

Example 8 (Jacobi identity)**.** For all Poisson structures $\mathcal{P}$ and all triples of arguments from the algebra $C^\infty(N^n)$ of functions on the Poisson manifold at hand, we have that



(4)

In formulae, by ascribing the index ℓ to the unlabeled edge, the identity reads

$$(\partial_\ell \mathcal{P}^{ij}\mathcal{P}^{\ell k} + \partial_\ell \mathcal{P}^{jk}\mathcal{P}^{\ell i} + \partial_\ell \mathcal{P}^{ki}\mathcal{P}^{\ell j})\partial_i(1)\partial_j(2)\partial_k(3) = 0.$$

Indeed, the coefficient of $\partial_i \otimes \partial_j \otimes \partial_k$ is the familiar form of the Jacobi identity.

In fact, the graph itself is the most convenient way to transcribe the formulae which one constructs from it, see [19, §2.1] for more details.[5] The computer implementation is straightforward. We acknowledge however that it is one of the most needed instruments.

[5]In the variational set-up of Poisson field models, the affine manifold N^n is realised as fibre in an affine bundle π over another affine manifold M^m equipped with a volume element. The variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ are then defined for integral functionals that take sections of such bundle π to numbers. The encoding of variational polydifferential operators by the Kontsevich graphs now reads as follows. Decorated by an index i, every edge denotes the variation with respect to the ith coordinate along the fibre. By construction, the variations act by first differentiating their argument with respect to the fibre variables (or their derivatives along the base M^m); secondly, the integrations by parts over the underlying space M^m are performed. Whenever two or more arrows arrive at a graph vertex, its content is first differentiated the corresponding number of times with respect to the jet fibre variables in $J^\infty(\pi)$ and only then it can be differentiated with respect to local coordinates on the base manifold M^m. The assumption that both the manifolds M^m and N^n be affine makes the construction coordinate-free, see [17, 21] and [18, 20].

Implementation 7. The call is

```
> poisson_evaluate <graph-series-filename> <poisson-structure>
```

and options for `<poisson-structure>` are[6]

- `2d-polar`,
- `3d-generic`,
- `3d-polynomial`,
- `4d-determinant`,
- `4d-rank2`,
- `9d-rank6`.

The output is a list of coefficients of the differential operator that the graph series represents, filtered by (a) powers of $\hbar$, (b) the differential order as an operator acting on the sinks, and (c) the actual derivatives falling on the sinks.

Example 9. Put the graph sum for the Jacobiator $\mathrm{Jac}(\mathcal{P})$ in `jacobiator.txt`:

```
3 2 1   0 1 2 3    -1
3 2 1   0 2 1 3    1
3 2 1   0 4 1 2    -1
```

We evaluate it at a Poisson structure:

```
> poisson_evaluate jacobiator.txt 2d-polar
Coordinates: r t
Poisson structure matrix:
[[0, r^(-1)]
[-r^(-1), 0]]

h^0:
# 1 1 1
# [ r ] [ r ] [ r ]
0
# [ r ] [ r ] [ t ]
0
# [ r ] [ t ] [ r ]
0
# [ r ] [ t ] [ t ]
0
# [ t ] [ r ] [ r ]
0
# [ t ] [ r ] [ t ]
0
# [ t ] [ t ] [ r ]
0
# [ t ] [ t ] [ t ]
0
```

[6]The current version of the software does not allow specification of an arbitrary Poisson structure at runtime (e.g. input as a matrix of functions); however, in the source file `util/poison_structure.hpp` the list of Poisson structures (as matrices) can be extended to one's heart's desire.

For example, the pair of lines

```
# [ r ] [ t ] [ r ]
0
```

indicates that the coefficient of $\partial_r \otimes \partial_t \otimes \partial_r$ is zero in the polydifferential operator.

Restriction of graph series to Poisson structures will be essential in section 3.1 below where systems of linear algebraic equations between the Kontsevich graph weights in $\star$ will be obtained by restricting the associativity equation $\mathrm{Assoc}_\star(f,g,h)=0$ to a given Poisson bracket.

2. The Kontsevich $\star$-product

The star-product $\star = \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ in $C^\infty(N^n)[[\hbar]]$ is an associative unital noncommutative deformation of the associative unital commutative product $\times$ in the algebra of functions $C^\infty(N^n)$ on a given affine manifold N^n of dimension $n<\infty$. The bi-linear bi-differential $\star$-product is realized as a formal power series in $\hbar$ by using the weighted Kontsevich graphs. In fact, the bi-differential operator at $\hbar^k$ is a sum of all Kontsevich graphs $\Gamma \in G_{2,k}$ without tadpoles, with k internal vertices (and two sinks) taken with some weights $w(\Gamma)$. Let us recall their original definition [26].

Definition 3. Every Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ can be embedded in the closed upper half-plane $\mathbb{H}\cup\mathbb{R}\subset\mathbb{C}$ by placing the internal vertices at pairwise distinct points in $\mathbb{H}$ and the external vertices at 0 and 1; the edges are drawn as geodesics with respect to the hyperbolic metric, i.e. as vertical lines and circular segments. The angle $\varphi(p,q)$ between two distinct points $p,q\in\mathbb{H}$ is the angle between the geodesic from p to q and the geodesic from p to ∞ (measured counterclockwise from the latter):

$$\varphi(p,q) = \mathrm{Arg}\left(\frac{q-p}{q-\bar{p}}\right),$$

and it can be extended to $\mathbb{H}\cup\mathbb{R}$ by continuity. The *weight* of a Kontsevich graph $\Gamma\in\tilde{G}_{2,k}$ is given by the integral[7]

$$w(\Gamma) = \frac{1}{(2\pi)^{2k}} \int_{C_k(\mathbb{H})} \bigwedge_{j=1}^{k} \mathrm{d}\varphi(p_j, p_{\mathrm{Left}(j)}) \wedge \mathrm{d}\varphi(p_j, p_{\mathrm{Right}(j)}), \tag{5}$$

over the *configuration space* of k points in the upper half-plane $\mathbb{H}\subset\mathbb{C}$,

$$C_k(\mathbb{H}) = \{(p_1,\dots,p_k)\in\mathbb{H}^k : p_i \text{ pairwise distinct}\};$$

the integrand is defined pointwise at $(p_1,\dots,p_k)$ by considering the embedding of Γ in $\mathbb{H}$ that sends the jth internal vertex to p_j; the numbers $\mathrm{Left}(j)$ and $\mathrm{Right}(j)$ are the left and right targets of jth vertex, respectively.

For every Poisson bi-vector $\mathcal{P}$ on N^n and an infinitesimal deformation $\times \mapsto \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ towards the respective Poisson bracket, the $\hbar$-linear star-product

$$\star = \times + \sum_{k\geqslant 1} \frac{\hbar^k}{k!} \sum_{\Gamma\in\tilde{G}_{2,k}} w(\Gamma)\, \Gamma(\mathcal{P})(\cdot,\cdot)\colon C^\infty(N^n)[[\hbar]]\times C^\infty(N^n)[[\hbar]] \to C^\infty(N^n)[[\hbar]] \tag{6}$$

[7]We omit the factor $1/k!$ that was written in [26], to make the weight multiplicative (see Lemma 5).

is associative.

Lemma 1. Permuting the internal vertex labels of a Kontsevich graph leaves the weight unchanged.

Proof. Such a permutation re-orders the factors in a wedge product of two-forms. □

Lemma 2. Swapping $L \rightleftarrows R$ at an internal vertex of a Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ implies the reversal of the sign of its weight.

Proof. Anticommutativity of wedge product of two differentials in formula (5) for $w(\Gamma)$. □

Lemma 3. The weight of a graph $\Gamma \in \tilde{G}_{2,k}$ and its reflection $\bar{\Gamma}$ are related by $w(\bar{\Gamma}) = (-)^k w(\Gamma)$.

Proof. Taking the reflection of a graph (with respect to the vertical line $\Re(z) = 1/2$) corresponds to an orientation-reversing coordinate change on each "factor" $\mathbb{H}$ in $C_n(\mathbb{H})$. □

Lemma 4 ([10])**.** For a Kontsevich graph such that at least one sink receives no edge(s), its weight is zero.[8]

Lemma 5. The map $w\colon \sqcup_k \tilde{G}_{2,k} \to \mathbb{R}$ that assigns weights to graphs is multiplicative,

$$w(\Gamma_i \bar{\times} \Gamma_j) = w(\Gamma_i) \times w(\Gamma_j), \tag{7}$$

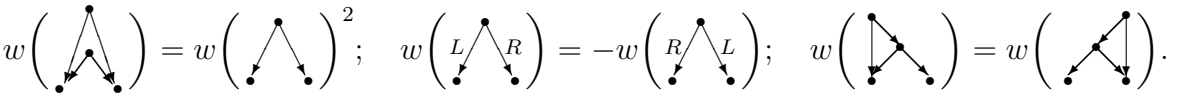
with respect to the product $\bar{\times}$ of graphs,

$$\Gamma_i \bar{\times} \Gamma_j = (\Gamma_i \sqcup \Gamma_j)\big/\{a^{\text{th}} \text{ sink in } \Gamma_i = a^{\text{th}} \text{ sink in } \Gamma_j, \quad 0 \leqslant a \leqslant 1\},$$

which identifies the respective sinks.

Proof. The integrals converge absolutely [26]; apply Fubini's theorem and linearity. □

Example 10. Some weight relations obtained from the lemmas above:



Lemma 5 motivates the following definition.

Definition 4. A Kontsevich graph $\Gamma \in \tilde{G}_{2,k}$ is called *prime* if Γ is not equal to the $\bar{\times}$ product of any Kontsevich graphs on two sinks and positive number of internal vertices in either of the co-factors. Otherwise (if such a realization is possible), the graph is called *composite*.

Using Lemma 5 and induction, we obtain that the weight of a composite graph $\Gamma = \Gamma_1 \bar{\times} \cdots \bar{\times} \Gamma_t$ is the product of the weights of its factors: $w(\Gamma) = w(\Gamma_1) \times \cdots \times w(\Gamma_t)$.

[8]The fact that the differential order of $\star$ is positive with respect to either of its arguments should be expected, in view of the required property of $\star$-product to be unital: $f \star 1 = f = 1 \star f$.

2.1. **Basic set of graphs.** We identify a set of graphs such that the weights of those graphs would suffice to determine all the other weights.

Definition 5. A *basic* set of graphs on k internal vertices is a set of pairwise distinct normal forms (the signs of which are discarded) of only those Kontsevich graphs $\Gamma \in \tilde{G}_{2,k}$ which are prime, and in which every sink receives at least one edge. By definition, the basic set contains the normal form of a graph but not its mirror reflection if it differs from the graph at hand. To decide whether a graph or its mirror-reflection $\bar{\Gamma} \neq \Gamma$ is included into a basic set, we take the graph whose absolute value is *minimal* as a base-$(k+2)$ number. Note that a basic set on $k \geqslant 3$ vertices *does* contain zero graphs.

Corollary 6. To build $\star$-product (6) up to $\bar{o}(\hbar^k)$ for some power $k \geqslant 1$, knowing the Kontsevich weights $w(\Gamma_i)$ only for a *basic* set of graphs $\Gamma_i \in \tilde{G}_{2,\ell}$ at all $\ell \leqslant k$ is enough. Indeed, the weights of all other graphs with ℓ internal vertices are calculated from Lemmas 1, 2, 3, 4, and 5.

Example 11. Consider the prime graph  and its mirror-reflection  . The encodings of their normal forms are `2 2 1 0 1 0 2` and `2 2 1 0 1 1 2` respectively. Since `0 1 0 2` < `0 1 1 2` as base-4 numbers, only the first graph is included in the basic set. The fork graph  is mirror-symmetric hence it is included anyway.

The basic set at order 3 is displayed in Figure 2.

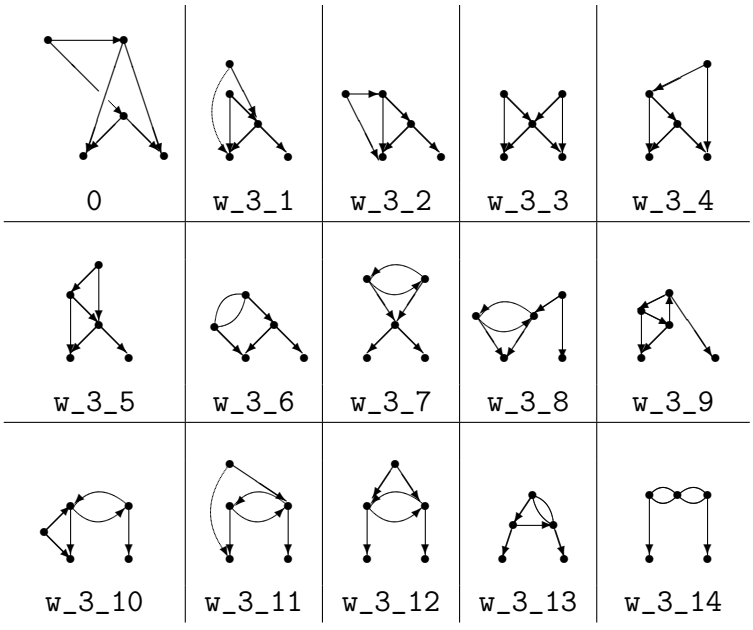


FIGURE 2. Basic set at order 3, with undetermined coefficients for nonzero graphs.

TABLE 1. How many basic graphs there are at low orders k.

Order $= k$	0	1	2	3	4	5	6
#(Basic set)	0	1	2	15	156	2307	43231
#(Nonzero in basic set)	0	1	2	14	149	2218	42050

2.2. **"All" graphs in $\star$ mod $\bar{o}(\hbar^4)$.** In Table 1 we list the number of basic graphs at every order $k \leqslant 6$ in the Kontsevich $\star$-product. The actual number of graphs with respect to which the sums in formula (6) expand is of course much greater.

Implementation 8. To obtain the list of normal forms for graphs from a basic set, the following command is available:

```
> generate_graphs k --basic=yes
```

The list of normal forms is then sent to the standard output. This command is equivalent to

```
> generate_graphs k --prime=yes --normal-forms=yes \
    --postive-differential-order=yes --modulo-mirror-images=yes
```

Starting from a basic set, the $\star$-product is built up to a certain order $k \geqslant 0$ in $\hbar$.

Implementation 9. The program

```
> star_product <basic-set-filename>
```

takes as its input a graph series with a basic set of graphs at each order; the graphs go with coefficients of any nature (i.e. number or indeterminate). The program's output is an expansion of the $\star$-product up to the order that was specified by the input. In other words, all the graphs which are produced from the ones contained in a given basic set are generated and their coefficients are (re)calculated from the ones in the input (using Lemmas 2, 3, and 5).

Example 12. To generate the star-product up to order 3 with all weights of nonzero basic graphs undetermined, one proceeds as follows:

```
$ cat > basic_graphs_undetermined3.txt
h^0:
2 0 1     1
h^1:
^D (press Ctrl+D)
$ generate_graphs 1 --basic=yes --with-coefficients=yes \
      >> basic_graphs_undetermined3.txt
$ echo 'h^2:' >> basic_graphs_undetermined3.txt
$ generate_graphs 2 --basic=yes --with-coefficients=yes \
      >> basic_graphs_undetermined3.txt
$ echo 'h^3:' >> basic_graphs_undetermined3.txt
$ generate_graphs 3 --basic=yes --with-coefficients=yes \
      >> basic_graphs_undetermined3.txt
$ star_product basic_graphs_undetermined3.txt \
      > star_product_undetermined3.txt
```

The file `star_product_undetermined3.txt` now contains the desired star-product.

2.3. **Methods to obtain the weights of basic graphs.** We deduce that to build the $\star$-product modulo $\bar{o}(\hbar^4)$ as many as 149 weights of nonzero basic graphs $\Gamma_i \in \tilde{G}_{2,4}$ at $k = 4$ must be found (or at least expressed in terms of as few master-parameters as possible). In fact, direct calculation of all of the 149 Kontsevich integrals is not needed to solve the problem in full because there exist more algebraic relations between the weights of basic graphs. In the following proposition we recall a class of such relations.[9]

Proposition 7 (cyclic weight relations [11])**.** Let Γ be a Kontsevich graph on $m = 2$ ground vertices. Let $E \subset \mathrm{Edge}(\Gamma)$ be a subset of edges in Γ such that for every $e \in E$, $\mathrm{target}(e) \neq 0$. (That is, every edge from the subset E lands on the sink 1 or an internal vertex.) For every such subset E, define the graph Γ_E as follows: let its vertices be the same as in Γ and for every edge $e \in \mathrm{Edge}(\Gamma)$, preserve it in Γ_E if $e \neq E$, but if $e \in E$ replace that edge by a new edge in Γ_E going from $\mathrm{source}(e)$ to the sink 0. By definition, the ordering $L \prec R$ of outgoing edges is inherited in Γ_E from E even if the targets of any of those edges are new. Thirdly, denote by $N_0(\Gamma_E)$ the number of edges in Γ_E such that their target is the sink 0. Then the Kontsevich weight of a graph Γ is related to the weights of all such graphs Γ_E obtained from Γ by the formula

$$w(\Gamma) = (-)^n \sum_{\substack{E \subset \mathrm{Edge}(\Gamma) \\ \forall e \in E, \mathrm{target}(e) \neq 0}} (-)^{N_0(\Gamma_E)} w(\Gamma_E). \tag{8}$$

Note that this relation is linear in the weights of all graphs.

If the graph Γ or, in practice, some of the new graphs Γ_E in (8) is composite, Lemma 5 provides a further, nonlinear reduction of $w(\Gamma)$ by using graphs with fewer internal vertices.

Example 13. Consider the graph $\Gamma_{3,8}$ in Figure 2 with weight $w(\Gamma_{3,8}) = \texttt{w\_3\_8}$. For every non-empty subset E (with $\mathrm{target}(e) \neq 0$ for every $e \in E$) the graph $(\Gamma_{3,8})_E$ is a zero-weight graph by virtue of one of the Lemmas at the beginning of this chapter. Hence the only term in the sum on the right-hand side in (8) is the weight of the graph corresponding to the empty set: $w((\Gamma_{3,8})_\varnothing) = w(\Gamma_{3,8})$. Since $n = 3$ and $N_0(\Gamma_{3,8}) = 2$ we get the cyclic relation $w(\Gamma_{3,8}) = -w(\Gamma_{3,8})$; whence $w(\Gamma_{3,8}) = 0$.

Remark 6. It is readily seen that only *prime*, that is, non-composite graphs Γ need be used to generate *all* relations (8). Indeed, every subset E of edges for a composite graph $\Gamma = \Gamma^1 \bar{\times} \Gamma^2$ splits to a disjoint union $E^1 \sqcup E^2$ of such subsets for the graphs Γ^1 and Γ^2 separately. Therefore the re-direction of edges in a composite graph would inevitably yield the composite graph $\Gamma^1_{E^1} \bar{\times} \Gamma^2_{E^2}$. Now, the multiplicativity of Kontsevich weights and the additivity of the count $N_0(\Gamma_E) = N_0(\Gamma^1_{E^1}) + N_0(\Gamma^2_{E^2})$ can be used to conclude that the relations obtained from composite graphs are redundant.

Implementation 10. The command

```
> cyclic_weight_relations <star-product-file>
```

[9] A convenient approach to calculation of Kontsevich weights (5) at order 3 by using direct integration (and for that, using methods of complex analysis such as the Cauchy residue theorem) was developed in [8], see Appendix A.1 on p. 42 below. However, we note that most successful at $k = 3$, this method is no longer effective for all graphs at $k \geqslant 4$. More progress is badly needed to allow $k \geqslant 5$.

treats the input $\star$-product as a clothesline for graphs and their weights. For each graph Γ in the $\star$-product, it outputs the relation (8) between the weights of the respective graphs in the form `LHS - RHS == 0`.

Example 14. At the order four with the $\star$-product from Table 6 in Appendix C:

```
> cyclic_weight_relations star_product4.txt
-1/1728 - 1/2*w_4_1==0
...
-1/3456-1/12*w_4_2==0
...
-1/384+1/8*w_4_6-1/4*w_4_8==0
```

Remark 7. For some (basic) graphs it happens that the weight integrand in (5), as a differential $2k$-form, vanishes identically, even if the graph is not zero due to skew-symmetry. This is the case for 21 out of 149 nonzero basic graphs at $k = 4$; see also Appendix A.1.

For calculations of particular weight integrals we refer to the literature in section 3.3.

Remark 8 (rationality). Willwacher and Felder [11] relate the questions of (ir)rationality of the Kontsevich graph weights to the questions of (ir)rationality of the value $\zeta(3)/\pi^3$ (similarly, $\zeta(5)$ and $\zeta(7)$) of the Riemann ζ-function. We note that if, by running all the algorithms from the present paper, one attains the order $\bar{o}(\hbar^7)$ in the deformation parameter (such that the associativity of the Kontsevich $\star$-product is ascertained at least mod $\bar{o}(\hbar^7)$) and if all the values $w(\Gamma_i)$ of all the graphs Γ_i involved appear to be rational numbers, then this result about the graph weight values would resolve the (ir)rationality (more precisely, algebraicity) problem for $\zeta(3)/\pi^3$.

All the above being said about methods to obtain the values $w(\Gamma)$ for Kontsevich graph weights and about the schemes to generate linear relations between these numbers, we observe that the requirement of associativity for the $\star$-product modulo $\bar{o}(\hbar^k)$, whenever that structure is completely known at all orders up to $\hbar^k$, is an ample source of relations of that kind. This will be used intensively in chapter 3 from p. 21 onwards. In particular, we mention here that the values of weights of graphs at order k may be restricted by the associativity requirement at orders $> k$, by restriction to fixed differential orders (i, j, k) (see Lemma 10 on p. 28).

2.4. **How graphs act on graphs.** Let us have a closer look at the equation of associativity for the sought-for $\star$-product:

$$\mathrm{Assoc}_\star(f, g, h) = (f \star g) \star h - f \star (g \star h) = 0.$$

We see that the graph series $f \star g$ and $g \star h$ serve as the left- and right co-multiples of h and f, respectively, in yet another copy of the star-product. To realize the associator by using the Kontsevich graphs, we now explain how graphs act on graphs (here, in every composition $\star \circ \star$ the graph series acts on a graph series by linearity).

We postulate that the action of graph series on graph series is $\Bbbk[[\hbar]]$-linear and $\Bbbk[G_{*,*}]$-linear with respect to both the graphs that act and that become the arguments.

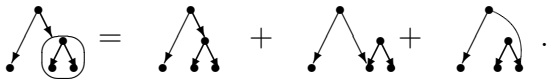
Recall that every Poisson bracket is a derivation in each of its arguments. In consequence, every derivation falling on a sink – in a graph Γ_1 that acts on a given graph Γ_2

taken as the new content of that sink – acts on the sink's content via the Leibniz rule; all the Leibniz rules for the derivations in-coming to that sink work independently from each other.

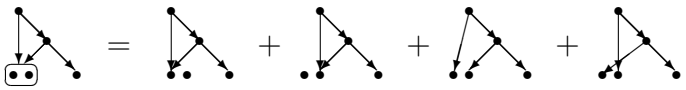
Example 15. Consider the action of a wedge graph Λ on the two-sinks graph $(\bullet\bullet) \in G_{2,0}$, taken as its second argument. We have that

The result is a sum of Kontsevich graphs of type $(3,1)$. Let us remember that the sinks are distinguished by their ordering; in particular the two Kontsevich graphs on the right-hand side are not equal.

Example 16. Now let the wedge graph act on a wedge graph (again, as the former's second argument):



Example 17. Finally, consider a graph in which two arrows fall on the first sink and let its content be $(\bullet\bullet) \in G_{2,0}$:



These three examples basically cover all the situations; we shall refer to them again, namely, from the next chapter where the restrictions by using the total differential orders are discussed.

So far, we have focused on graphs; under the action of a graph on a graph, their coefficients are multiplied. (This is why the associativity of the $\star$-product is an infinite system of quadratic equations for the coefficients of all the graphs).

Implementation 11. In the respective class, the method is

`KontsevichGraphSeries<T>::operator()`

and its argument is `std::vector` (that is, a list) of the Kontsevich graph series in $\hbar$; these are the m respective arguments for a Kontsevich graph series. The method is called for the object of the class, that is, for the graph series which is evaluated at the m specified arguments.

2.5. **Gauge transformations.** At first glance, the concept of gauge transformations for (graphs in the) $\star$-products is an extreme opposite of plugging a list of graph series as arguments of a given graph series. Namely, the idea of a gauge transformation is that a graph series (possibly of finite length) is towered over a single vertex $\bullet \in G_{1,0}$. By definition, a gauge transformation of a vertex $\bullet$ is a map of the form $\bullet \mapsto [\bullet] = \bullet + \hbar \cdot (...)$ taking $G_{1,0} \to \Bbbk[G_{1,*}][[\hbar]]$.

Example 18. The map $\bullet \mapsto \bullet + \frac{\hbar^2}{12}$  is a gauge transformation of $\bullet \in G_{1,0}$. This graph series is encoded in the following file:

```
h^0:
1 0 1     +1
h^2:
1 2 1  0 2 1 0  +1/12
```

The construction of gauge transformations is extended from $G_{1,0}$ by $\Bbbk[[\hbar]]$- and $\Bbbk[G_{*,*}]$-linearity. This effectively means that in the course of action by a gauge transformation $\mathtt{t}$ on a graph series $f \in \Bbbk[G_{*,*}][[\hbar]]$, all the arrows work over the vertices in every graph in f via the Leibniz rule (as it has been explained in the previous section). This is how one expands $[f] \star [g]$, that is, the Kontsevich $\star$-product (6) of two gauged arguments $[f]$ and $[g]$. Let us recall further that the shape $[\bullet] = \bullet + \hbar \cdot (\ldots)$, where the gauge tail of $\bullet$ is given by some graphs from $\Bbbk[G_{1,*}][[\hbar]]$, guarantees the existence of a formal left inverse $\mathtt{t}^{-1}$ to the original transformation $\mathtt{t}$, so that $(\mathtt{t}^{-1} \circ \mathtt{t})(\bullet) = \bullet$.

Lemma 8. If $\bullet \mapsto \blacksquare = \mathtt{t}(\bullet) = \bullet + \hbar\Gamma_1(\bullet) + \ldots + \hbar^\ell \Gamma_\ell(\cdot) + \bar{o}(\hbar^\ell)$ is a gauge transformation, let

$$\mathtt{t}^{-1}(\blacksquare) = \blacksquare + \hbar\gamma_1(\blacksquare) + \ldots + \hbar^\ell \gamma_\ell(\blacksquare) + \bar{o}(\hbar^\ell)$$

by setting

$$\gamma_m(\blacksquare) := -\sum_{k=0}^{m-1} \gamma_k(\Gamma_{m-k}(\blacksquare)).$$

Then $\mathtt{t}^{-1}(\mathtt{t}(\bullet)) = \bullet$, that is, the transformation $\mathtt{t}^{-1}\colon \Bbbk[G_{1,*}][[\hbar]] \to \Bbbk[G_{1,*}][[\hbar]]$ is the left inverse of $\mathtt{t}$ up to $\bar{o}(\hbar)$.

It is readily seen that the assembly of the entire $\mathtt{t}^{-1}$ can require infinitely many operations even if the direct transformation $\mathtt{t}$ took only finitely many of them, e.g., as in Example 18.

In these terms, for the Kontsevich $\star$-product (6) we obtain, by operating with gauge transformations and their formal inverses, a class of star products $\star'$ which are defined by the relation

$$\mathtt{t}(f \star' g) = \mathtt{t}(f) \star \mathtt{t}(g), \qquad f, g \in C^\infty(N^n)[[\hbar]]. \tag{9}$$

Clearly, all these gauged star-products $\star'$ remain associative (because $\star$ was) but the coefficients of graphs at an order $k \geqslant 2$ in $\hbar$ are no longer necessarily equal to the respective values in (6). The use of gauge transformations for products allows to gauge out *some* graphs, often at a certain order $\hbar^k$ in the star-product expansion.

Example 19. The graph  with a loop is gauged out from the Kontsevich $\star$-product (6) by using the gauge transformation $\mathtt{t}\colon \bullet \mapsto \bullet + \frac{\hbar^2}{12}$  , see Example 18. Note that taking the formal inverse $\mathtt{t}^{-1}$ does create loop-containing graphs at higher orders $\hbar^{\geqslant 3}$ in the gauged star-product $\star'$ which is specified by (9).

Remark 9. Not every graph taken in the Kontsevich star-product $\star$ at a particular order $\hbar^k$ can be gauged out. For example, such are the graphs $\Gamma \in \tilde{G}_{2,*}$ containing an internal vertex v with edges running from it to both the ground vertices.

Implementation 12. The command for gauge transformation is

```
> gauge <star-product-filename> <gauge-transformation-filename>
```

where

- the file `<star-product-filename>` contains a machine-format graph encoding of star-product $\star$ truncated modulo $\bar{o}(\hbar^k)$ for some $k \geqslant 0$;
- the content of `<gauge-transformation-filename>` is a gauge transformation $\mathtt{t}(\bullet)$, that is, a truncated modulo $\bar{o}(\hbar^{\ell \geqslant 0})$ series in $\hbar$ consisting of the Kontsevich graphs built over one sink vertex $\bullet$.

In the standard output one obtains the truncation, modulo $\bar{o}(\hbar^{\min(k,\ell)})$, of the graph series for the gauged star-product $\star'$ defined by $f \star' g = \mathtt{t}^{-1}(\mathtt{t}(f) \star \mathtt{t}(g))$.

(The corresponding method is `KontsevichGraphSeries<T>::gauge_transform()` in Appendix B.)

Example 20. Let the gauge transformation from Example 19 be stored in the file `gaugeloop.txt`, and recall the $\star$-product up to order two from Example 5 in the file `star_product2.txt`. The gauge transformation kills the loop graph:

```
> gauge star_product2.txt gaugeloop.txt > star_product2_gauged.txt
> reduce star_product2_gauged.txt
h^0:
2 0 1             1
h^1:
2 1 1  0 1        1
h^2:
2 2 1  0 1 0 1    1/2
2 2 1  0 1 0 2    1/3
2 2 1  0 1 1 2    -1/3
```

Indeed, we see that the line

```
2 2 1  0 3 1 2    -1/6
```

containing the loop graph has disappeared.

Let us note at once that every gauge transformation `t` given by a Kontsevich graph polynomial in $\hbar$ of degree ℓ can clearly be viewed formally as a polynomial transformation of any degree greater or equal than ℓ. This is why by using the same software we can actually obtain the gauged star-product $\star'$ modulo $\bar{o}(\hbar^4)$ starting with the Kontsevich star-product $\star$ modulo $\bar{o}(\hbar^4)$ and applying the gauge transformation of nominal degree $\ell = 2$ from Example 18. In other words, the precision in $\star'$ with respect to $\hbar$ is the same as in $\star$ even though the degree of the polynomial gauge transformation `t` is smaller. In practice, this is achieved by adding an empty list of graphs at powers $\hbar^k$ to a given gauge transformation of degree $\ell < k$.

3. Associativity of the Kontsevich $\star$-product

In the final section of this paper we explore two complementary matters. One the one hand, we analyse how the associativity postulate for the Kontsevich $\star$-product contributes to finding the values of weights $w(\Gamma)$ for graphs Γ in $\star$. On the other hand, a point is soon reached when no new information can be obtained about the values of $w(\Gamma)$: specifically, neither from the fact of associativity of the $\star$-product nor from any proven properties of the Kontsevich weights. We outline a computer-assisted

scheme of reasoning that, working uniformly over the set of all Poisson structures under study, reveals the associativity of $\star$-product on the basis of our actual knowledge about the weights $w(\Gamma)$ of graphs Γ in it.

In [7] we reported an exhaustive description of the Kontsevich $\star$-product up to $\bar{o}(\hbar^3)$. At the next expansion order $\bar{o}(\hbar^4)$ in $\star$, we now express the weights of all the $160\,000 = (5 \cdot 4)^4$ graphs $\Gamma \in \tilde{G}_{2,4}$ (of which up to $10\,000 = (5 \cdot 4/2)^4$ are different modulo signs) in terms of only 10 parameters; those ten master-parameters themselves are the (still unknown) Kontsevich weights of the four internal vertex graphs portrayed in Fig. 3. By following the second strategy we prove that for any values of those ten parameters the $\star$-product expansion modulo $\bar{o}(\hbar^4)$ is associative, also up to $\bar{o}(\hbar^4)$.

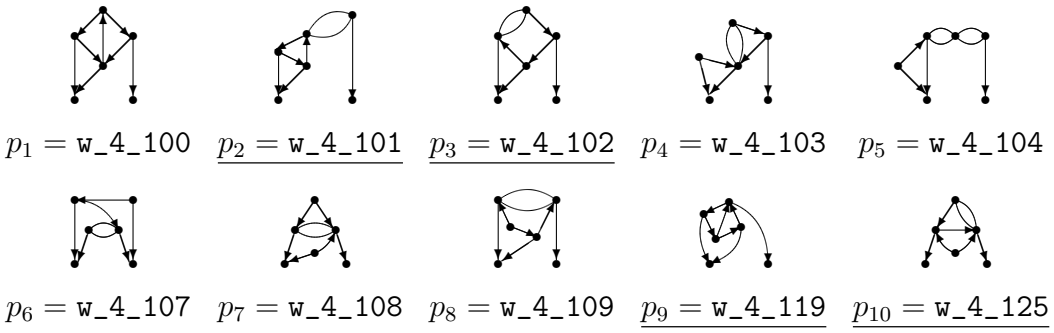


FIGURE 3. The ten graphs whose unknown weights[10] are taken as the master-parameters p_i; in fact, the four graphs whose weights are underlined can be gauged out from $\star$ so that there remain only 6 parameters that determine it modulo $\bar{o}(\hbar^4)$.

3.1. **Restriction of the $\star$-product associativity equation $\mathrm{Assoc}_\star(f,g,h) = 0$ to a Poisson structure $\mathcal{P}$.** We now view the postulate of associativity for the Kontsevich $\star$-product as an equation for coefficients in the graph expansion of $\star$. Whenever an expansion modulo $\bar{o}(\hbar^\ell)$ is known for the $\star$-product, one passes to the next order $\bar{o}(\hbar^{\ell+1})$ by taking all the graphs $\Gamma \in \tilde{G}_{2,\ell+1}$ with undetermined coefficients, and then expands (with respect to graphs) the associator $\mathrm{Assoc}_\star(f,g,h)$ up to the order $\bar{o}(\hbar^{\ell+1})$. This expansion now runs over all the graphs with at most $\ell+1$ internal vertices. It is readily seen that by construction this associativity equation $\mathrm{Assoc}_\star(f,g,h) = \bar{o}(\hbar^{\ell+1})$ is always *linear*[11] with respect to the coefficients of graphs from $\tilde{G}_{2,\ell+1}$.

Remark 10. One can still get *linear* relations between the weights $w(\Gamma)$ of graphs $\Gamma \in \tilde{G}_{2,\ell+1}$ at order $\hbar^{\ell+1}$ in $\star$ by inspecting the associativity of $\star$ at *higher* orders – ranging from $\ell+2$ till $2\ell+1$ – in $\hbar$. Indeed, a linear relation containing the unknown weights (and the already known lower-order part of $\star$ as coefficients) but not the weights of graphs with $\geqslant \ell+2$ internal vertices can appear whenever a properly chosen homogeneous component of the tri-differential operator $\mathrm{Assoc}_\star(f,g,h)$ does not contain any weights

[10] Numerical approximations of two of these weights are listed in Table 3 in Appendix A.1.

[11] Should a graph $\Gamma \in \tilde{G}_{2,\ell+1}$ be composite so that its Kontsevich weight is factorized using formula (7), the resulting nonlinearity with respect to the weights would actually involve only the graphs with at most ℓ internal vertices.

from higher orders. For instance, this is the component at homogeneity orders (i,j,k) such that prime graphs $\Gamma \in \tilde{G}_{2,\geqslant \ell+2}$ of homogeneity orders $(i+j,k)$ and $(i,j+k)$ (when viewed as bi-differential operators) do not exist or if the weights of all such graphs are known in advance.

3.1.1. Let us also note that in the graph equation $\mathrm{Assoc}_\star(f,g,h) = 0$ that holds by virtue of the Jacobi identity $\mathrm{Jac}(\mathcal{P}) = 0$, not every coefficient of every graph in the expansion should be expected to vanish. Indeed, the Jacobiator is a vanishing sum of three graphs that evaluates to zero at every Poisson structure $\mathcal{P}$ which we put into every internal vertex. This is why the restriction of associativity equation to a given Poisson structure (or to a class of Poisson structures) is a practical way to proceed in solution of the problem of finding the coefficients of graphs in $\star$. More specifically, after the restriction of associator $\mathrm{Assoc}_\star(f,g,h)$ to a structure $\mathcal{P}$ which is known to be Poisson so that all the instances and all derivatives of the Jacobiator $\mathrm{Jac}(\mathcal{P})$ are automatically trivialized, the left-hand side of the associativity equation $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}} = 0 \mod \bar{o}(\hbar^{\ell+1})$ becomes an analytic expression (*linear* with respect to the unknowns $w(\Gamma)$ for $\Gamma \in \tilde{G}_{2,\ell+1}$). At this point one can proceed in several ways.

We now outline three methods to obtain systems of linear equations upon the unknown weights $w(\Gamma)$ of basic graphs $\Gamma \in \tilde{G}_{2,\ell+1}$. Working in local coordinates, we ensure that the unknowns' coefficients in the equations which we derive are *real* numbers.[12]

The three methods which we presently describe can be compared as follows. On the one hand, as far as maximizing the rank of the algebraic system which are obtained by using the respective methods is concerned, Method 1 is the least effective whereas Method 3 is the most productive. One the other hand, Method 1 is the least computationally expensive, so it can be used effectively at the initial stage, e.g., to detect the zero values of certain graph weights: once found, such trivial values allow to decrease the number of unknowns in the further reasoning. We finally note that the linear algebraic systems which are produced by each method should be merged. Indeed, the goal is to maximize the rank and by this, reduce the number of free parameters in the solution.[13]

Method 1. Let the associator's arguments be given functions $f,g,h \in C^\infty(N^n)$. Restrict the analytic expression $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}}$ to a point $\boldsymbol{x}$ of the manifold N^n equipped with a Poisson structure $\mathcal{P}$. For every choice of $f,g,h \in C^\infty(N^n)$ and of a point $\boldsymbol{x} \in N^n$, the restriction $\mathrm{Assoc}_\star(f,g,h)\big|_{\mathcal{P}}(\boldsymbol{x}) = 0 \mod \bar{o}(\hbar^{\ell+1})$ yields *one* linear relation between the weights of graphs at order $\hbar^{\ell+1}$. Taking the restriction at several points $\boldsymbol{x}_1$, …, $\boldsymbol{x}_k \in N^n$, one obtains a system of such equations, the rank of which does not exceed

[12]From the factorization of associator for $\star$ via differential consequences of the Jacobi identity for a Poisson structure $\mathcal{P}$, which will be revealed in section 3.2 below, it will be seen in hindsight that the construction of linear relations between the graph weights is overall insensitive to a choice of local coordinates in a chart within a given Poisson manifold. Indeed, the factorization will have been achieved simultaneously for all Poisson structures on all the manifolds at once, irrespective of any local coordinates.

[13]If the rank of the resulting linear algebraic system is equal to the number of unknowns – and if all the coefficients coming from lower orders $\leqslant \ell$ within the $\star$-product expansion with respect to $\hbar$ are also rational – then all the solution components are rational numbers as well, cf. [11].

the number k of such points in N^n. Bounded by the number of unknowns $w(\Gamma)$, the rank would always stabilize as $k \to \infty$.

Examples of Poisson structures $\mathcal{P}$ – for instance, on the manifolds $\mathbb{R}^n$ – are available from [14] (here $n \geqslant 3$) and [30]; from Proposition 2.1 on p. 74 in the latter one obtains a class of Poisson (in fact, symplectic) structures with polynomial coefficients on even-dimensional affine spaces $\mathbb{R}^{2k}$. Besides, there is a regular construction (by using the R-matrix formalism, see [28, p. 287]) of Poisson brackets on the vector space of square matrices $\mathrm{Mat}(\mathbb{R}, k \times k) \cong \mathbb{R}^{k^2}$ (e.g., in this way one has a rank-six Poisson structure on $\mathbb{R}^9$).

Method 2. Now let $f, g, h \in \Bbbk[x^1, \ldots, x^n]$ be polynomials referred to local coordinates x^1, …, x^n on N^n. On that coordinate chart $U_\alpha \subset N^n$, take a Poisson structure the coefficients $\mathcal{P}^{ij}(\boldsymbol{x})$ of which would also be polynomial. In consequence, the left-hand side of the equation $\mathrm{Assoc}_\star(f, g, h)\big|_{\mathcal{P}} = 0 \mod \bar{o}(\hbar^{\ell+1})$ then becomes polynomial as well. Linear in the unknowns $w(\Gamma)$, all the coefficients of this polynomial equation vanish (independently from each other). Again, this yields a system of linear algebraic equations for the unknown weights $w(\Gamma)$ of the Kontsevich graphs $\Gamma \in \tilde{G}_{2,\ell+1}$ in the $\star$-product.

We observe that the linear equations obtained by using Method 2 better constrain the set of unknowns $w(\Gamma)$, that is, the rank of this system is typically higher than in Method 1. Intuitively, this is because the polynomials at hand are not collapsed to their values at points $\boldsymbol{x} \in N$.

Method 3. Keep the associator's arguments f, g, h unspecified and consider a class of Poisson structures $\mathcal{P}[\psi_1, \ldots, \psi_m]$ depending in a differential polynomial way on functional parameters ψ_α, that is, on arbitrary functions, whenever $\mathcal{P}$ is referred to local coordinates. (For example, let $n = 3$ and on $\mathbb{R}^3$ with Cartesian coordinates x, y, z introduce the class of Poisson brackets using the Jacobian determinants,

$$\{u, v\}_{\mathcal{P}} = p \cdot \det\bigl(\partial(q, u, v)/\partial(x, y, z)\bigr), \qquad q \in C^\infty(\mathbb{R}^3), \tag{10}$$

supposing that the density $p(x, y, z)$ is also smooth on $\mathbb{R}^3$.) Now view the associator $\mathrm{Assoc}_\star(f, g, h)\big|_{\mathcal{P}[\psi_1,\ldots,\psi_m]}$ as a polydifferential operator in the parameters f, g, h (with respect to which it is linear) and in ψ_1, …, ψ_m from $\mathcal{P}$. By splitting the associator, which is postulated to vanish modulo $\bar{o}(\hbar^{\ell+1})$, into homogeneous differential-polynomial components, we obtain a system of linear algebraic equations upon the graph weights.

It is readily seen that, whenever the parameters ψ_1, …, ψ_m are chosen to be polynomials (here let us suppose for definition that the resulting Poisson structure $\mathcal{P}(\boldsymbol{x})$ itself is polynomial), the rank of the algebraic system obtained by Method 3 can be greater than the rank of an analogous system from Method 2. This is because the analytic expression $\mathrm{Assoc}_\star(f, g, h)\big|_{\mathcal{P}[\psi_1,\ldots,\psi_m]}$ keeps track of all the parameters, whereas in Method 2 they are merged to a single polynomial.

Implementation 13. To calculate the associator $\mathrm{Assoc}_\star(f, g, h)$ for a given $\star$-product and ordered objects f, g, h, the call is

```
> star_product_associator <star-product-filename>
```

where the input file `<star-product-filename>` contains the (truncated) power series for the $\star$-product. In the standard output one obtains a (truncated at the same order in $\hbar$ as in the input) power series containing, at each power $\hbar^k$, the sums of graphs from $G_{3,k}$ with coefficients (their admissible types were introduced in §1.2 above).

The next step – namely, restriction of the associator to a given Poisson structure – can be performed by using a call `poisson_evaluate` as it has been explained in §1.4. However, the further restriction as described in the Methods has been implemented in a separate program (similar to `poisson_evaluate`) which directly outputs the desired relations, as follows.

Implementation 14. The command

```
> poisson_make_vanish <graph-series-file> <poisson-structure>
```

sends to the standard output relations such as

```
-1/24+w_3_1+4*w_3_2==0
```

between the undetermined coefficients in the input, which must hold if the input graph series is to vanish as a consequence of the Jacobi identity for the specified Poisson structure. The implementation is described in the Methods above. The choice of Poisson structure is made in the same way as in Implementation 7. If the optional extra argument `--linear-solve` is specified, the program will assume that the relations which will be obtained are linear, and attempt to solve the linear system.

Example 21. To obtain all the weights of basic graphs $\Gamma \in \tilde{G}_{2,3}$ at $\hbar^3$ in the Kontsevich star-product $\star$, it was enough to build the linear system of algebraic equations that combined (*i*) cyclic relations (8), (*ii*) the relations which Method 3 produces for generic Poisson structure (10), and (*iii*) those linear relations between the weights of $\Gamma \in \tilde{G}_{2,3}$ which – in view of Remark 10 on p. 22 – still do appear at the next power $\hbar^4$ in $\mathrm{Assoc}_\star(f,g,h) = 0$, by using the same generic Poisson structure (10). The resulting expansion of $\star$-product modulo $\bar{o}(\hbar^3)$ is shown in formula (1) on p. 2. This result is achieved by using the software as follows. Starting from the sets of basic graphs up to the order 2 (with known weights) in the file `basic_graphs_2.txt`, generate lists of basic graphs (with undetermined weights) up to the order four:

```
$ cp basic_graphs_2.txt basic_graphs_34w.txt
$ echo 'h^3:' >> basic_graphs_34w.txt
$ generate_graphs 3 --basic=yes --with-coefficients=yes \
      >> basic_graphs_34w.txt
$ echo 'h^4:' >> basic_graphs_34w.txt
$ generate_graphs 4 --basic=yes --with-coefficients=yes \
      >> basic_graphs_34w.txt
```

Build the $\star$-product expansion up to the order 4 from these basic sets:

```
$ star_product basic_graphs_34w.txt > star_product_34w.txt
```

Generate cyclic weight relations:

```
$ cyclic_weight_relations star_product_34w.txt \
      > weight_relations_34w-cyclic.txt
```

Build the associator expansion up to the order 4 from the $\star$-product expansion:

```
$ star_product_associator star_product_34w.txt > associator_34w.txt
```

Obtain relations from the requirement of associativity for the Poisson structure (10):

```
$ poisson_make_vanish associator_34w.txt 3d-generic \
      > weight_relations_34w-3d.txt
```

Merge the systems of linear relations:

```
$ cat weight_relations_34w-* > weight_relations_34w_all.txt
```

Solving the linear system in `weight_relations_34w_all.txt` yields the solution

```
w_3_1=1/24,   w_3_2=0,      w_3_3=0,  w_3_4=-1/48, w_3_5=-1/48
w_3_6=0,      w_3_7=0,      w_3_8=0,  w_3_9=0,     w_3_10=0
w_3_11=-1/48, w_3_12=-1/48, w_3_13=0, w_3_14=0.
```

Instead of evaluating the associator in full, we could also have selected (e.g. by reading the file `associator_34w.txt`, which also contains lines of the form "`# i j k`") those differential orders (i, j, k) at which only weights from order 3 appear, in view of Remark 10: such orders are $(1, 3, 2)$, $(2, 3, 1)$, $(2, 1, 3)$, $(3, 2, 1)$, $(3, 1, 2)$, $(1, 2, 3)$ and $(2, 2, 2)$.

Remark 11. A substitution of the values of certain graph weights expressed via other weights is tempting but not always effective. Namely, we do not advise repeated running of any of the three methods with such expressions taken into account in the input. Usually, the gain is disproportional to the time consumed; for instead of a coefficient to-express the program now has to handle what typically is a linear combination of several coefficients. This shows that the only types of substitutions which are effective are either setting the coefficients to fixed numeric values (e.g., to zero) or the shortest possible assignments of a weight value via a single other weight value (like $w(\Gamma_1) = -w(\Gamma_2)$ for some graphs Γ_1 and Γ_2).

3.1.2. *The $\star$-product expansion at order four.* At order four in the expansion of the Kontsevich $\star$-product with respect to $\hbar$, there are 149 basic graphs $\Gamma \in \tilde{G}_{2,4}$. The knowledge of their coefficients would completely determine the $\star$-product modulo $\bar{o}(\hbar^4)$. By using Methods 1–3 from §3.1, we found the exact values of 67 basic graphs and we expressed the remaining 82 weights in terms of the 10 master-parameters (themselves the weights of certain graphs from $\tilde{G}_{2,4}$; the other 72 weights are linear functions of these ten).

Theorem 9. *The weights of* 27 *basic Kontsevich graphs are equal to zero. Of these* 27, *the integrands of* 21 *weights are identically zero*, *and the other* 6 *weight values were found to be equal to zero. The remaining* 122 *weights of basic graphs* $\Gamma \in \tilde{G}_{2,4}$ *are arranged as follows*:

- 40 *nonzero weights are known explicitly*;
- *the values of the remaining* 82 *weights are expressed linearly in terms of the weights of those ten graphs which are shown in Fig.* 3.

• *The encoding of entire $\star$-product modulo $\bar{o}(\hbar^4)$, that is, its part up to $\bar{o}(\hbar^3)$ known from formula* (1) *plus $\hbar^4$ times the sum of all the prime and composite weighted graphs with four internal vertices, is given in Appendix* C. (*In that table the weights of composite graphs are numbers*; *for they are expressed via the known coefficients of graphs from $\tilde{G}_{2,\leqslant 3}$.*) *The weights of basic graphs at $\hbar^4$ are expressed in Table* 7 *in terms of the ten master-parameters, see p.* viii *in Appendix* C.

Moreover (as stated in Theorem 12 on p. 30 below), the associativity $\mathrm{Assoc}_\star(f,g,h) = 0 \mod \bar{o}(\hbar^4)$ is established (up to order four) for the star product $\star \mod \bar{o}(\hbar^4)$ at all values of the ten master-parameters.

Proof scheme (*for Theorem* 9). We run the software as follows. First one generates the sets of basic graphs up to order 4, with undetermined weights at order 4 (the weights at order 2 and 3 are known from e.g. Example 5 and Example 21):

```
$ cp basic_graphs_3.txt basic_graphs_4w.txt
$ echo 'h^4:' >> basic_graphs_4w.txt
$ generate_graphs 4 --basic=yes --with-coefficients=yes \
      >> basic_graphs_4w.txt
```

Build the $\star$-product expansion up to order 4:

```
$ star_product basic_graphs_4w.txt > star_product_4w.txt
```

Generate the linear cyclic weight relations at order 4:

```
$ cyclic_weight_relations star_product_4w.txt > \
      > weight_relations_4w-cyclic.txt
```

Find relations of the form `w_4_xxx==0` which hold by virtue of the weight integrand vanishing in formula (5), by using Implementation 17 in Appendix A.1, and place these relations in the file `weight_relations_4w-integrandvanishes.txt`.
Build the expansion of the associator for the $\star$-product up to the order 4:

```
$ star_product_associator star_product_4w.txt > associator_4w.txt
```

Obtain relations from the requirement of associativity for the Poisson structure (10):

```
$ poisson_make_vanish associator_4w.txt 3d-generic \
      > weight_relations_4w-3d.txt
```

Merge the systems of linear equations:

```
$ cat weight_relations_4w-* > weight_relations_4w_total.txt
```

Solve the resulting system (contained in `weight_relations_4w_total.txt`) by using any relevant software. One obtains the relations listed in Table 7 in Appendix C, e.g. in the file `weight_relations_4w_intermsof10.txt`. To express the star-product (respectively, the associator for the $\star$-product) in terms of the 10 parameters, run

```
$ substitute_relations star_product_4w.txt \
      weight_relations_4w_intermsof10.txt
```

(respectively, `associator_4w.txt`); see Implementation 3. □

Remark 12. Numerical approximations of weights are listed in Tables 2 and 3 in Appendix A.1. In particular, we have the approximate values of the master-parameters $p_4 = \mathtt{w\_4\_103} \approx -1/11520$ and $p_5 = \mathtt{w\_4\_104} \approx 1/2880$.[14]

[14]The values of ten master-parameters have been suggested by Pym and Panzer [27], see Table 4 on p. 46 in Appendix A.2 below. Their prediction completely agrees with our numeric data.

Remark 13. Out of the 149 weights of basic graphs in the Kontsevich $\star$-product, as many as 28 weights do not appear in the equation $\mathrm{Assoc}_\star(f,g,h)=0$ at $\hbar^4$. A mechanism which works towards such disappearance is that some graphs $\Gamma\in\tilde{G}_{2,4}$ which do not show up are bi-derivations with respect to the sinks. Combined at order four in the associator with only the original undeformed product $\times$, every such graph is cancelled out from $(f\star g)\star h - f\star(g\star h)$ according to the mechanism which we illustrate here:

In this way the ten master-parameters are split into the six which do show up in the associativity equation and the four weights which do not show up in $\mathrm{Assoc}_\star(f,g,h)=0$ at $\hbar^4$ but which do appear through the cyclic weight relations (see formula (8) on p. 17).

3.2. **Computer-assisted proof scheme for associativity of $\star$ for all $\{\cdot,\cdot\}_\mathcal{P}$.** In practice, the methods from §3.1 stop producing linear relations that would be new with respect to the already known constraints for the graph weights. As soon as such "saturation" is achieved, the number of master-parameters in $\star$-product expansion can in effect be minimal. That is, the $\star$-product, known so far up to a certain order $\bar{o}(\hbar^k)$, can in fact always be associative – modulo $\bar{o}(\hbar^k)$ – irrespective of a choice of the Poisson structure(s) $\mathcal{P}$.

In this section we outline a scheme of computer-assisted reasoning that allows to reveal the factorization $\mathrm{Assoc}_\star(f,g,h)=\lozenge(\mathcal{P},\mathrm{Jac}(\mathcal{P}))(f,g,h)$ of associator for $\star$ via the Jacobiator $\mathrm{Jac}(\mathcal{P})$ that vanishes by definition for every Poisson structure $\mathcal{P}$. At order $k=2$ the factorization $\lozenge(\mathrm{Jac}(\mathcal{P}))$ is readily seen; the factorizing operator $\lozenge(\mathrm{Jac}(\mathcal{P}))=\frac{2}{3}\hbar^2\,\mathrm{Jac}(\mathcal{P})+\bar{o}(\hbar^2)$ is a differential operator of order zero, acting on its argument $\mathrm{Jac}(\mathcal{P})$ by multiplication. Involving the Jacobi identity and only seven differential consequences from it at the next expansion order $k=3$, the factorization $\mathrm{Assoc}_\star(f,g,h)=\lozenge(\mathcal{P},\mathrm{Jac}(\mathcal{P}))(f,g,h)$ was established by hand in [7]. For higher orders $k\geqslant 4$ the use of software allows to extend this line of reasoning; the scheme which we now provide works uniformly at all orders $\geqslant 2$.

Let us first inspect how sums of graphs can vanish by virtue of differential consequences of the Jacobi identity $\mathrm{Jac}(\mathcal{P})=0$ for Poisson structures $\mathcal{P}$ on finite-dimensional affine real manifolds N^n.

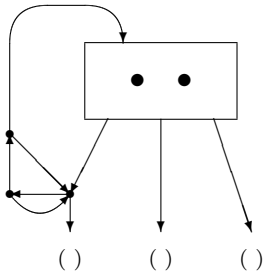
Lemma 10 ([7])**.** A tri-differential operator $C=\sum_{|I|,|J|,|K|\geqslant 0}c^{IJK}\;\partial_I\otimes\partial_J\otimes\partial_K$ with coefficients $c^{IJK}\in C^\infty(N^n)$ vanishes identically if and only if all its homogeneous components $C_{ijk}=\sum_{|I|=i,|J|=j,|K|=k}c^{IJK}\;\partial_I\otimes\partial_J\otimes\partial_K$ vanish for all differential orders (i,j,k) of the respective multi-indices (I,J,K); here $\partial_L=\partial_1^{\alpha_1}\circ\cdots\circ\partial_n^{\alpha_n}$ for a multi-index $L=(\alpha_1,\ldots,\alpha_n)$.

Lemma 10 states in practice that for every arrow falling on the Jacobiator (for which, in turn, a triple of arguments is specified), the expansion of the Leibniz rule yields four fragments which vanish separately. Namely, there is the fragment such that the derivation acts on the content $\mathcal{P}$ of the Jacobiator's two internal vertices, and there are three fragments such that the arrow falls on the first, second, or third argument of the Jacobiator. It is readily seen that the action of a derivative on an argument of the Jacobiator effectively amounts to an appropriate redefinition of its respective argument

(cf. Examples 15–17 on p. 19). Therefore, a restriction to the order $(1, 1, 1)$ is enough in the run-through over all the graphs which contain Jacobiator (4) and which stand on the three arguments f, g, h of the operator $\lozenge(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$ at hand.

Definition 6. A *Leibniz graph* is a graph whose vertices are either sinks, or the sources for two arrows, or the Jacobiator (which is a source for three arrows). There must be at least one Jacobiator vertex. The three arrows originating from a Jacobiator vertex must land on three distinct vertices (and not on the Jacobiator itself). Each edge falling on a Jacobiator works by the Leibniz rule on the two internal vertices in it.

An example of a Leibniz graph is given in Fig. 4. Every Leibniz graph can be expanded to a sum of Kontsevich graphs, by expanding both the Leibniz rule(s) and all copies of the Jacobiator. In this way (sums of) Leibniz graphs also encode (poly)differential operators $\lozenge(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$, depending on the bi-vector $\mathcal{P}$ and the tri-vector $\mathrm{Jac}(\mathcal{P})$.



- There is a cycle,
- there is a loop,
- there are no tadpoles in this graph,
- an arrow falls back on $\mathrm{Jac}(\mathcal{P})$,
- and $\mathrm{Jac}(\mathcal{P})$ does not stand on all of the three sinks.

FIGURE 4. An example of Leibniz graph.

Proposition 11. For every Poisson bi-vector $\mathcal{P}$ the value – at the Jacobiator $\mathrm{Jac}(\mathcal{P})$ – of every (poly)differential operator encoded by the Leibniz graph(s) is zero.

Hence, to show that a sum of Kontsevich graphs vanishes at every Poisson structure, it suffices to write it as a sum of Leibniz graphs.

Example 22. Consider the associator $\mathrm{Assoc}_\star(f, g, h) \mod \bar{o}(\hbar^3)$ for the $\star$-product which is fully known up to order 3. The assembly of factorizing operator $\lozenge(\mathcal{P}, \cdot)$ acting on $\mathrm{Jac}(\mathcal{P})$ is explained in [7]; linear in its argument, the operator $\lozenge$ has differential order one with respect to the Jacobiator.

Remark 14. The same technique, showing the vanishing of a sum of Kontsevich graphs by writing it as a sum of Leibniz graphs, has been used in [3].

Implementation 15 (Encoding of Leibniz graphs)**.** For a Leibniz graph with ℓ Jacobiators appearing as arguments for the Leibniz rules, and with $n - 2\ell$ internal vertices from the factorizing operator $\lozenge$, the target vertices $n-2\ell, \ldots, n-\ell-1$ are the ℓ placeholders for the Jacobiators with internal vertices $(n - 2\ell, n - 2\ell + 1), \ldots, (n - 1, n)$, respectively. (We emphasize that the (poly)differential operator $\lozenge$ can be nonlinear in $\mathrm{Jac}(\mathcal{P})$, so that $\ell \geqslant 1$.)

Example 23. The Leibniz graph from Fig. 4 (with $n = 5$ and $\ell = 1$) has the encoding

```
3 5 1   0 5 3 ⑥ 3 4 3 1 6 2
```

Here the circled vertex ⑥ is a placeholder for the Jacobiator containing the last two vertices 6 and 7; the three arguments of that Jacobiator are underlined. To expand this encoding into Kontsevich graph encodings, cyclically permute the arguments of the Jacobiator and replace ⑥ by 6 or 7 (in all possible ways). One obtains six terms:

```
3 5 1   0 5 3 6 3 4 3 1 6 2
3 5 1   0 5 3 7 3 4 3 1 6 2
3 5 1   0 5 3 6 3 4 1 2 6 3
3 5 1   0 5 3 7 3 4 1 2 6 3
3 5 1   0 5 3 6 3 4 2 3 6 1
3 5 1   0 5 3 7 3 4 2 3 6 1
```

Implementation 16. Let the input file `<graph-series-filename>` contain a graph series S with constant (e. g., rational, real or complex) coefficients; here S is supposed to vanish by virtue of the Jacobi identity and its differential consequences. Now run the command

```
> reduce_mod_jacobi <graph-series-filename>
```

The program finds a particular solution $\lozenge$ of the factorization problem

$$S(f,g,h) = \lozenge(\mathcal{P}, \mathrm{Jac}(\mathcal{P}), \ldots, \mathrm{Jac}(\mathcal{P}))(f,g,h).$$

In the standard output one obtains the list of encodings of Leibniz graphs in $\lozenge$ that specify differential consequences of the Jacobi identity; every such graph encoding is followed in the output by its sought-for nonzero coefficient.[15] Two extra options can be set equal to nonnegative integer values, by passing these two numbers as extra command-line arguments. Namely,

- the parameter `max-jacobiators` restricts the number of Jacobiators in each Leibniz graph, so that by the assignment `max-jacobiators = 1` the right-hand side $\lozenge\big(\mathcal{P}, \mathrm{Jac}(\mathcal{P})\big)$ is linear in the Jacobiator, whereas if `max-jacobiators = 2`, the right-hand side $\lozenge\big(\mathcal{P}, \mathrm{Jac}(\mathcal{P}), \mathrm{Jac}(\mathcal{P})\big)$ can be quadratic in $\mathrm{Jac}(\mathcal{P})$, and so on;
- independently, the parameter `max-jac-indegree` restricts (from above) the number of arrows falling on the Jacobiator(s) in each of the Leibniz graphs that constitute the factorizing operator $\lozenge$.

Furthermore, if `--solve` is specified as the third extra argument, the input graph series is allowed to contain undetermined coefficients; these are then added as variables to-solve-for in the linear system.

Theorem 12. *For every component $S^{(i)}$ of the associator*

$$\mathrm{Assoc}_\star(f,g,h) \mod \bar{o}(\hbar^4) =: S^{(0)} + p_1 S^{(1)} + \ldots + p_{10} S^{(10)},$$

there exists a factorizing operator $\lozenge^{(i)}$ such that

$$S^{(i)}(f,g,h) = \lozenge^{(i)}\big(\mathcal{P}, \mathrm{Jac}(\mathcal{P})\big)(f,g,h), \qquad 0 \leqslant i \leqslant 10.$$

• *At no values of the master-parameters p_i would the solution $\lozenge = \sum_i \lozenge^{(i)}$ of factorization problem be a* first*-order differential operator acting on the Jacobiator.*

[15] Sample outputs of specified type are contained in Table 9 in Appendix D.

Proof scheme. Take the associator $\mathrm{Assoc}_\star(f,g,h) \mod \bar{o}(\hbar^4)$ for the $\star$-product expansion modulo $\bar{o}(\hbar^4)$, in the file `associator4_intermsof10.txt` which was obtained in Theorem 9. The associator is linear in the ten master-parameters. Let us split it into the constant term (e.g., at the zero value of every parameter) plus the ten respective components $S^{(i)}$:

```
$ extract_coefficient associator4_intermsof10.txt 1 \
      > associator4_intermsof10_constantpart.txt
$ extract_coefficient associator4_intermsof10.txt w_4_100 \
      > associator4_intermsof10_part100.txt
$ extract_coefficient associator4_intermsof10.txt w_4_101 \
      > associator4_intermsof10_part101.txt
```

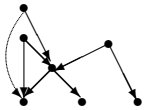
(and so on, for each parameter p_i). In fact, four of the parameters do not show up in the associator (see Remark 13): the corresponding files do not contain any graphs. Now run the command `reduce_mod_jacobi` for each input file with $S^{(i)}$, e.g., for $S^{(1)}$:

```
$ reduce_mod_jacobi associator4_intermsof10_part100.txt
```

For each $S^{(i)}$ a solution is found: the series vanishes modulo the Jacobi identity. The output for $S^{(1)}$ is written in Table 9 in Appendix D. For the second part of the theorem, we run `reduce_mod_jacobi` with the options `max-jac-indegree = 1` and `--solve`:

```
$ reduce_mod_jacobi associator4_intermsof10.txt 1 1 --solve
```

(Our setting of `max-jacobiators = 1` here makes no difference.) No solution is found. Inspecting the output, we find that the following term in the associator cannot be produced by a first-order differential consequence of the Jacobi identity:

$-\frac{2}{15}$ 

Indeed one can show this graph arises only in a differential consequence of order two. □

Corollary 13 ($\star$-product non-extendability from $\{\cdot,\cdot\}_\mathcal{P}$ to $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ at order $\hbar^4$)**.** Because there are at least two arrows falling on the object $\mathrm{Jac}(\mathcal{P})$ in $\Diamond$ at every value of the ten master-parameters p_i, the associativity can be broken at order $\hbar^4$ for extensions of the $\star$-product to infinite-dimensional set-up[5 on p. 11] of N^n-valued fields $\phi \in C^\infty(M^m \to N^n)$ over a given affine manifold M^m, of local functionals F, G, H taking such fields to numbers, and of variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ on the algebra of local functionals.

Indeed, the Jacobiator $\mathrm{Jac}(\boldsymbol{\mathcal{P}}) \cong 0$ for a variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ is a cohomologically trivial variational tri-vector on the jet space $J^\infty(M^m \to N^n)$, whence the first variation of $\mathrm{Jac}(\boldsymbol{\mathcal{P}})$ brought on it by a unique arrow would of course be vanishing identically. Nevertheless, that variational tri-vector's density is not necessarily equal to zero on $J^\infty(M^m \to N^n)$ over M^m for those variational Poisson structures whose coefficients $\mathcal{P}^{ij}$ explicitly depend on the fields ϕ or their derivatives along M^m. This is why the second and higher variations of the Jacobiator $\mathrm{Jac}(\boldsymbol{\mathcal{P}})$ would not always vanish. (Such higher-order variations of functionals are calculated by using the techniques

from [17, 21].) We know from [7] that $\mathrm{Assoc}_\star(F,G,H) \cong 0 \mod \bar{o}(\hbar^3)$, i.e. the associator is trivial up to order $\hbar^3$ for all variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ but we now see that it can contain cohomologically nontrivial terms proportional to $\hbar^4$. Consequently, it is the order four at which the associativity of $\star$-products can start to leak in the course of deformation quantization of Poisson field models.

We now claim that four master-parameters can simultaneously be gauged out of the star-product. (That is, either some of the four or all of them at once can be set equal to zero, although this may not necessarily be their true value given by formula (5).)[16]

Theorem 14. *For each $j \in \{2,3,9,10\}$ there exists a gauge transformation $\mathrm{id}+\hbar^4 p_j Z_j$ (listed in Table 10 in Appendix E) such that the master-parameter p_j is reset to zero in the deformed star-product $\star'$. This is achieved in such a way that no graph coefficients which initially did not contain the parameter to gauge out would change at all.*

- *Moreover, the gauge transformation $\mathrm{id}+\hbar^4\cdot\left(\sum_j p_j Z_j\right)$ removes at once all the four master-parameters, still preserving those coefficients of graphs in $\star$ which did not depend on any of them.*

Proof scheme. Let the $\star$-product expansion in terms of 10 parameters (obtained in Theorem 9) be contained in `star_product4_intermsof10.txt`. Construct a gauge transformation of the form $\mathrm{id}+\hbar^4 G$, where G is the sum over all possible graphs with four internal vertices over one sink which are nonzero, without double edges, without tadpoles, and with positive differential order, taken with undetermined coefficients g_i:

```
$ cat > gauge4.txt
1 0 1           1
h^4:
^D (press Ctrl+D)
$ generate_graphs 4 1 --normal-forms=yes --zero=no \
      --positive-differential-order=yes \
      --with-coefficients=yes >> gauge4.txt
$ sed -i 's/w/g/' gauge4.txt # replace coefficient prefix 'w' by 'g'
```

Obtain gauged star-product expansion $\star'$ by applying the gauge transformation to $\star$:

```
$ gauge star_product4_intermsof10.txt gauge4.txt \
      > star_product4_intermsof10_gauged.txt
```

Reduce the graph series for $\star'$ modulo skew-symmetry:

```
$ reduce star_product4_intermsof10_gauged.txt \
      > star_product4_intermsof10_gauged_reduced.txt
```

Inspect which of the 10 parameters p_j cannot be gauged out, by checking for the existence of graph coefficients containing p_j but not any g_i. For example, for $p_1 =$ `w_4_100`:

[16]Let us recall that the property of a parameter in a family of star-products to be removable by some gauge transformation is not the same as setting such parameter to zero (or any other value). Indeed, other graph coefficients, not depending on the parameter at hand, might get modified by that gauge transformation. However – and similarly to the removal of the loop graph at $\hbar^2$ in the Kontsevich $\star$-product (see Examples 19 and 20) – the trivialization of four parameters at no extra cost is the case which Theorem 14 states.

```
$ grep w_4_100 star_product4_intermsof10_gauged_reduced.txt \
      | grep -v g | wc -l
17
```

There are 17 graphs with such coefficients, so $p_1 = \texttt{w\_4\_100}$ cannot be gauged out. Following this procedure for all the 10 parameters, we find that the only candidates to be gauged out are $p_2 = \texttt{w\_4\_101}$, $p_3 = \texttt{w\_4\_102}$, $p_9 = \texttt{w\_4\_119}$, and $p_{10} = \texttt{w\_4\_125}$. Now inspect the file `star_product4_intermsof10_gauged_reduced.txt` for the lines containing these p_j and (necessarily, some) g_i. For each p_j, find a choice of g_i so that p_j is completely removed from the file. (The g_i will be of the form $g_i = \alpha_{ij} p_j$ for $\alpha_{ij} \in \mathbb{R}$.) It turns out that this is always possible. Hence this choice of g_i defines the sought-for gauge transformation $\mathrm{id} + \hbar^4 p_j Z_j$ which gauges out the parameter p_j. The gauge-transformations which kill the (four) parameters separately may be combined into the gauge-transformation $\mathrm{id} + \hbar^4(\sum_j p_j Z_j)$ that kills all (four) of them simultaneously. □

Remark 15. The master-parameters which we can gauge out are exactly the ones which do not show up in the associativity equation (see Remark 13).

Let us finally address a possible origin of so ample a freedom in the ten-parameter family of star-products (now known up to $\bar{o}(\hbar^4)$). We claim that the mechanism of vanishing via differential consequences of the Jacobi identity, which was recalled in Lemma 10 and used in Theorem 12, starts working not only for the associator built over $\star$, but it may even start working for the $\star$-product expansion itself.

Theorem 15. *The ten-parameter family of star-product expansions* $\star = ... + \hbar^4\big(\star^{(0)} + \sum_{i=1}^{10} p_i \star^{(i)}\big) + \bar{o}(\hbar^4)$ *does contain, in the ten-dimensional affine subspace parametrized by* $p_1, \ldots, p_{10}$ *in* $\Bbbk[G_{2,4}]$*, a unique one-dimensional (null or 'improper') subspace such that every point* $\alpha \cdot (\star^{(9)} - 2\star^{(6)}) = \alpha \cdot \star^{(9|6)}$ *in it admits a Leibniz graph factorization (via the Jacobiator)* $\star^{(9|6)} = \nabla(\mathcal{P}, \mathrm{Jac}(\mathcal{P})) \in \Bbbk[G_{2,4}]$*. This null space is the span of the direction* $\texttt{w\_4\_119} : \texttt{w\_4\_107} : \ldots = 1 : (-2) : 0 : ... : 0 \in \mathbb{RP}^9$*, that is, the master-parameters* p_9 *and* p_6 *occur in proportion* $1 : (-2)$ *and all the other* p_i*'s are zero.*

In effect, the respective part of the star-product always cancels out for every given Poisson structure $\mathcal{P}$. This factorization and uniqueness of the direction $\star^{(9|6)}$ is established by using the same computer-assisted scheme of reasoning which worked in the proof of Theorem 12.

3.3. **Discussion.** The values of weights for the Kontsevich graphs at orders $\hbar^3$ and $\hbar^4$ in the $\star$-product which we obtained in this paper agree with those from the literature and numerical experiment. The vanishing of three graph weights at order 3 is stated in [31]: they are `w_3_7`, `w_3_13`, `w_3_14` in Figure 2; this agrees with our calculation in Example 21. The graphs in the Bernoulli family have scaled Bernoulli numbers as weights (see [2, Corollary 6.3] or [15, Proposition 4.4.1]), e.g. $\texttt{w\_3\_2} = B_3/3! = 0$ and $\texttt{w\_4\_12} = B_4/4! = -1/720$. The weights of a family of graphs containing cycles are obtained in [2, Corollary 6.3], e.g. $\texttt{w\_3\_9} = \pm B_3/(2 \cdot 3!) = 0$ and $\texttt{w\_4\_72} = -B_4/(2 \cdot 4!) = 1/1440$. In Tables 2 and 3 in Appendix A.1 we list numerical approximations of several weights. These approximations are consistent with the exact weights (and relations) obtained in this paper; moreover, they agree with a symbolic calculation of the graph weights reported by Pym and Panzer [27] and reproduced in Table 4 on p. 46 in Appendix A.2.

Conclusion

The expansion of Kontsevich star-product modulo $\bar{o}(\hbar^4)$ is (here $f, g \in C^\infty(N^n)$)

$$
\begin{aligned}
f \star g = f \times g &+ \hbar \mathcal{P}^{ij} \partial_i f \partial_j g + \hbar^2 \big(\tfrac{1}{2} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_k \partial_i f \partial_\ell \partial_j g + \tfrac{1}{3} \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_k \partial_i f \partial_j g \\
&- \tfrac{1}{3} \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_i f \partial_k \partial_j g - \tfrac{1}{6} \partial_\ell \mathcal{P}^{ij} \partial_j \mathcal{P}^{k\ell} \partial_i f \partial_k g \big) + \hbar^3 \big(\tfrac{1}{6} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_k \partial_i f \partial_n \partial_\ell \partial_j g \\
&+ \tfrac{1}{3} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{3} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g \\
&- \tfrac{1}{6} \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_j g + \tfrac{1}{6} \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_k \partial_i f \partial_j g \\
&+ \tfrac{1}{6} \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_m \partial_k \partial_j g - \tfrac{1}{6} \partial_m \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_k \partial_i f \partial_j g \\
&- \tfrac{1}{6} \partial_m \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_k \partial_j g - \tfrac{1}{6} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_j g \\
&- \tfrac{1}{6} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_k \partial_i f \partial_m \partial_j g + \tfrac{1}{6} \partial_n \partial_\ell \mathcal{P}^{ij} \partial_j \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_m \partial_k g \\
&- \tfrac{1}{6} \partial_\ell \mathcal{P}^{ij} \partial_n \partial_j \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_m \partial_i f \partial_k g - \tfrac{1}{6} \partial_m \partial_\ell \mathcal{P}^{ij} \partial_n \partial_j \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_i f \partial_k g \big) \\
&+ \hbar^4 \big(\tfrac{1}{6} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_n \partial_\ell \partial_j g - \tfrac{1}{6} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_n \partial_\ell \partial_j g \\
&+ \tfrac{1}{6} \partial_q \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_\ell \partial_j g + \tfrac{1}{6} \partial_q \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_\ell \partial_j g \\
&- \tfrac{1}{6} \partial_p \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{6} \partial_p \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g \\
&- \tfrac{1}{6} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g - \tfrac{1}{6} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g \\
&+ \tfrac{1}{6} \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g - \tfrac{1}{6} \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_k \partial_i f \partial_m \partial_j g \\
&- \tfrac{1}{6} \mathcal{P}^{ij} \partial_p \partial_n \mathcal{P}^{k\ell} \partial_q \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_j g - \tfrac{1}{9} \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g \\
&- \tfrac{1}{9} \partial_p \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{9} \partial_p \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g \\
&+ \tfrac{1}{24} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_q \partial_n \partial_\ell \partial_j g - \tfrac{1}{12} \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_\ell \partial_j g \\
&+ \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_\ell \partial_j g - \tfrac{1}{18} \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
&+ \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_\ell \partial_j g + \tfrac{1}{18} \partial_q \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g \\
&+ \tfrac{1}{72} \partial_\ell \mathcal{P}^{ij} \partial_j \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_m \partial_i f \partial_p \partial_k g - \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_p \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_\ell \partial_j g \\
&- \tfrac{1}{18} \partial_n \mathcal{P}^{ij} \partial_p \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_m \partial_\ell \partial_j g + \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
&- \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g + \tfrac{2}{45} \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
&- \tfrac{2}{45} \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g - \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
&+ \tfrac{1}{30} \partial_q \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g - \tfrac{7}{90} \partial_p \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g \\
&+ \tfrac{7}{90} \partial_p \partial_n \partial_\ell \mathcal{P}^{ij} \partial_q \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_m \partial_k \partial_j g + \tfrac{1}{30} \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
&- \tfrac{1}{30} \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g - \tfrac{1}{45} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_p \partial_m \partial_k \partial_i f \partial_j g \\
&+ \tfrac{1}{45} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_p \partial_m \partial_k \partial_j g + \tfrac{1}{45} \partial_q \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
&- \tfrac{1}{45} \partial_n \partial_\ell \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g + \tfrac{1}{30} \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
&- \tfrac{1}{30} \partial_n \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_q \partial_\ell \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g - \tfrac{1}{90} \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_p \partial_j g \\
&+ \tfrac{1}{90} \partial_q \mathcal{P}^{ij} \mathcal{P}^{k\ell} \partial_\ell \mathcal{P}^{mn} \partial_n \mathcal{P}^{pq} \partial_k \partial_i f \partial_p \partial_m \partial_j g + \tfrac{1}{90} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g \\
&- \tfrac{1}{90} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_q \partial_n \mathcal{P}^{k\ell} \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_m \partial_k \partial_j g - \tfrac{1}{30} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g \\
&+ \tfrac{1}{30} \partial_p \partial_\ell \mathcal{P}^{ij} \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_i f \partial_m \partial_k \partial_j g - \tfrac{2}{45} \partial_\ell \mathcal{P}^{ij} \partial_p \partial_n \mathcal{P}^{k\ell} \partial_q \mathcal{P}^{mn} \mathcal{P}^{pq} \partial_m \partial_k \partial_i f \partial_j g
\end{aligned}
$$

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\begin{aligned}
&+ \tfrac{2}{45}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g - \tfrac{1}{30}\partial_q\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad + \tfrac{1}{30}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g - \tfrac{1}{15}\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad + \tfrac{1}{15}\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g - \tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad + \tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g - \tfrac{1}{45}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad + \tfrac{1}{45}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g + \tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad - \tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g + \tfrac{2}{45}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad - \tfrac{2}{45}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g + \tfrac{1}{15}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad - \tfrac{1}{15}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m\partial_j g + \tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad - \tfrac{1}{90}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g + \tfrac{1}{90}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad - \tfrac{1}{90}\partial_q\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g + \tfrac{1}{90}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad - \tfrac{1}{90}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g + \tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&\quad - \tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g - \tfrac{1}{30}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_j g\\
&\quad + \tfrac{1}{30}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k\partial_j g + \tfrac{1}{90}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&\quad + \tfrac{1}{90}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g + \tfrac{1}{90}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&\quad + \tfrac{1}{90}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g + \tfrac{1}{90}\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&\quad + \tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g - \tfrac{1}{30}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad - \tfrac{1}{30}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g - \tfrac{1}{45}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad - \tfrac{1}{45}\partial_q\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g - \tfrac{1}{90}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad - \tfrac{1}{90}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g - \tfrac{1}{60}\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad - \tfrac{1}{60}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g - \tfrac{1}{45}\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&\quad - \tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g + \tfrac{1}{30}\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad + \tfrac{1}{30}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g - \tfrac{1}{90}\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad - \tfrac{1}{45}\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g - \tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&\quad - \tfrac{1}{20}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_i f\partial_k g - \tfrac{1}{20}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&\quad - \tfrac{13}{90}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g + \tfrac{13}{90}\partial_q\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad + \tfrac{13}{90}\partial_q\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g - 16p_4\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ 16p_4\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g - 16p_5\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ 16p_5\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g - 16p_4\partial_p\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ 16p_4\partial_p\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g + 16p_4\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&- 16p_4\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g + 16p_5\partial_p\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&- 16p_5\partial_p\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g + 16p_1\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&- 16p_1\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g + 16p_2\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&- 16p_2\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + 16p_3\partial_q\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g
\end{aligned}
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\begin{aligned}
&- 16p_3\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + 16p_4\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&- 16p_4\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g + 16p_5\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&- 16p_5\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + 16p_6\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+ 16p_6\partial_q\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g + 16p_7\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&- 16p_7\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + 16p_8\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+ 16p_8\partial_n\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + 16p_9\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&- 16p_9\partial_q\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g - 32p_4\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+ 32p_4\partial_q\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + 16p_{10}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ 16p_{10}\partial_p\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g + 32p_5\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+ 32p_5\partial_q\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + (\tfrac{1}{12}+8p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (-\tfrac{1}{12}-8p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{12}-8p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (\tfrac{1}{12}+8p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{12}-8p_7)\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (\tfrac{1}{12}+8p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ (\tfrac{1}{45}+8p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ (-\tfrac{1}{60}-24p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (\tfrac{1}{60}+24p_7)\partial_p\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (\tfrac{4}{45}-16p_6)\partial_q\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+ (\tfrac{4}{45}-16p_6)\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+ (\tfrac{17}{90}+24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&+ (-\tfrac{1}{12}+16p_6+48p_5)\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&+ (\tfrac{7}{90}-16p_6-48p_5)\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+ (-\tfrac{7}{90}+16p_6+48p_5)\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+ (-\tfrac{1}{36}+16p_4-16p_5-8p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (\tfrac{1}{36}-16p_4+16p_5+8p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (-\tfrac{1}{9}+16p_6-48p_4+48p_5)\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+ (-\tfrac{1}{9}+16p_6-48p_4+48p_5)\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+ (-\tfrac{1}{36}+16p_4-16p_5-8p_7)\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ (\tfrac{1}{36}-16p_4+16p_5+8p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+ (-\tfrac{13}{180}+8p_6+24p_5-8p_1)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ (\tfrac{13}{180}-8p_6-24p_5+8p_1)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ (\tfrac{4}{45}-16p_6+48p_4-48p_5)\partial_q\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g
\end{aligned}
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\begin{aligned}
&+(\tfrac{4}{45}-16p_6+48p_4-48p_5)\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m\partial_k g\\
&\quad+(-\tfrac{1}{18}+32p_4-32p_5-16p_7)\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&\quad+(-\tfrac{1}{18}+32p_4-32p_5-16p_7)\partial_q\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&\quad+(\tfrac{17}{90}-32p_6+96p_4-96p_5)\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&\quad+(\tfrac{23}{360}+16p_4-8p_1+12p_7)\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&\quad+(-\tfrac{23}{360}-16p_4+8p_1-12p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&\quad+(\tfrac{49}{180}-16p_6-48p_5+24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&\quad+(-\tfrac{19}{180}-32p_4+16p_1-16p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&\quad+(-\tfrac{31}{180}+32p_6-96p_4+96p_5)\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&\quad+(-\tfrac{1}{90}-8p_6-24p_5+8p_1-8p_7)\partial_p\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&\quad+(\tfrac{1}{90}+8p_6+24p_5-8p_1+8p_7)\partial_p\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&\quad+(\tfrac{2}{9}-48p_8+96p_4-96p_5+48p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad+(\tfrac{2}{9}-48p_8+96p_4-96p_5+48p_7)\partial_n\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad+(\tfrac{1}{6}-32p_8+64p_4-64p_5+32p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&\quad+(-\tfrac{1}{6}+32p_8-64p_4+64p_5-32p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&\quad+(-\tfrac{1}{9}+16p_8-32p_4+32p_5-16p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g\\
&\quad+(\tfrac{1}{9}-16p_8+32p_4-32p_5+16p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&\quad+(\tfrac{1}{120}-8p_8+16p_4-24p_5+4p_7)\partial_p\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&\quad+(\tfrac{1}{120}-8p_8+16p_4-24p_5+4p_7)\partial_k\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&\quad+(\tfrac{5}{18}-48p_8+96p_4-96p_5+48p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&\quad+(\tfrac{5}{18}-48p_8+96p_4-96p_5+48p_7)\partial_q\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&\quad+(\tfrac{4}{15}-48p_8+96p_4-96p_5+48p_7)\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&\quad+(-\tfrac{4}{15}+48p_8-96p_4+96p_5-48p_7)\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&\quad+(-\tfrac{1}{18}+16p_8-32p_4+32p_5-16p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&\quad+(-\tfrac{1}{18}+16p_8-32p_4+32p_5-16p_7)\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&\quad+(-\tfrac{7}{36}+32p_8-64p_4+64p_5-32p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&\quad+(-\tfrac{7}{36}+32p_8-64p_4+64p_5-32p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&\quad+(-\tfrac{1}{12}+16p_8-32p_4+32p_5-16p_7)\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&\quad+(-\tfrac{1}{12}+16p_8-32p_4+32p_5-16p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&\quad+(-\tfrac{1}{90}-16p_8+32p_4-80p_5+16p_1)\partial_p\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&\quad+(\tfrac{1}{360}-16p_8-16p_3+32p_4-48p_5)\partial_p\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g\\
&\quad+(\tfrac{11}{120}-16p_8+32p_4-16p_5+16p_7)\partial_k\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&\quad+(\tfrac{1}{90}+8p_6+16p_4+24p_5-8p_1+8p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g
\end{aligned}
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\begin{aligned}
&+(-\tfrac{1}{90}-8p_6-16p_4-24p_5+8p_1-8p_7)\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+(-\tfrac{1}{15}+8p_8+8p_4+8p_2+16p_{10}-16p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(-\tfrac{1}{15}+8p_8+8p_4+8p_2+16p_{10}-16p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{1}{20}-8p_8+24p_4-16p_5-8p_2+8p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(-\tfrac{1}{20}+8p_8-24p_4+16p_5+8p_2-8p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(-\tfrac{1}{40}+8p_8+16p_4+8p_5+16p_{10}-12p_7)\partial_m\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(\tfrac{1}{40}-8p_8-16p_4-8p_5-16p_{10}+12p_7)\partial_p\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{11}{90}+8p_6-16p_4+40p_5-8p_1+24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+(-\tfrac{11}{90}-8p_6+16p_4-40p_5+8p_1-24p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+(\tfrac{1}{5}-32p_8-48p_5-32p_{10}+16p_1+48p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(\tfrac{1}{5}-32p_8-48p_5-32p_{10}+16p_1+48p_7)\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(-\tfrac{1}{6}+16p_8-16p_3+32p_4-16p_1-32p_7)\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+(\tfrac{1}{6}-16p_8+16p_3-32p_4+16p_1+32p_7)\partial_q\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+(\tfrac{1}{9}-16p_8+16p_4-32p_5+16p_1+16p_7)\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+(-\tfrac{1}{9}+16p_8-16p_4+32p_5-16p_1-16p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g\\
&+(-\tfrac{1}{9}+16p_8-32p_4+48p_5+16p_2-16p_7)\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+(-\tfrac{1}{9}+16p_8-32p_4+48p_5+16p_2-16p_7)\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(\tfrac{1}{9}-16p_8+32p_4-48p_5+16p_2+16p_7)\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+(-\tfrac{1}{9}+16p_8-32p_4+48p_5-16p_2-16p_7)\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g\\
&+(\tfrac{7}{90}-16p_8+40p_4-40p_5+8p_2+12p_7)\partial_k\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{7}{90}-16p_8+40p_4-40p_5+8p_2+12p_7)\partial_m\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(\tfrac{1}{180}-16p_8-16p_4-16p_5-32p_{10}+8p_7)\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{1}{180}+16p_8+16p_4+16p_5+32p_{10}-8p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(\tfrac{37}{90}-48p_8-16p_6+96p_4-96p_5+48p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(-\tfrac{37}{90}+48p_8+16p_6-96p_4+96p_5-48p_7)\partial_p\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(\tfrac{29}{360}-16p_8-16p_4-16p_{10}+8p_1+20p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(\tfrac{29}{360}-16p_8-16p_4-16p_{10}+8p_1+20p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{34}{45}-96p_8-32p_6+240p_4-288p_5+96p_7)\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(-\tfrac{34}{45}+96p_8+32p_6-240p_4+288p_5-96p_7)\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+(-\tfrac{2}{45}+8p_9+4p_6-8p_3+4p_5-4p_1-4p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(\tfrac{2}{45}-8p_9-4p_6+8p_3-4p_5+4p_1+4p_7)\partial_q\partial_m\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(\tfrac{1}{30}+8p_6+32p_4+8p_5+32p_{10}-8p_1+8p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{1}{30}-8p_6-32p_4-8p_5-32p_{10}+8p_1-8p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g
\end{aligned}
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\begin{aligned}
&+(-\tfrac{7}{90}+8p_8-16p_3+16p_4-8p_5-8p_1-16p_7)\partial_p\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(\tfrac{7}{90}-8p_8+16p_3-16p_4+8p_5+8p_1+16p_7)\partial_q\partial_k\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(-\tfrac{7}{180}+16p_8-8p_6-16p_4+8p_5+8p_1-16p_7)\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&+(\tfrac{7}{180}-16p_8+8p_6+16p_4-8p_5-8p_1+16p_7)\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(\tfrac{13}{360}-8p_8+24p_4-32p_5-8p_2+8p_1+4p_7)\partial_m\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+(-\tfrac{13}{360}+8p_8-24p_4+32p_5+8p_2-8p_1-4p_7)\partial_p\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{1}{15}-32p_8+8p_6+48p_4-72p_5+24p_1+24p_7)\partial_p\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{1}{15}+32p_8-8p_6-48p_4+72p_5-24p_1-24p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(-\tfrac{11}{180}-16p_9+16p_8-8p_6+8p_5+8p_1-16p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_n\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(-\tfrac{17}{180}+16p_8-8p_6-32p_4+40p_5-8p_1-16p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+(\tfrac{17}{180}-16p_8+8p_6+32p_4-40p_5+8p_1+16p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(\tfrac{61}{180}-48p_8-8p_6+96p_4-120p_5+24p_1+48p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(-\tfrac{61}{180}+48p_8+8p_6-96p_4+120p_5-24p_1-48p_7)\partial_q\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+(\tfrac{53}{90}-96p_8-16p_6+192p_4-240p_5+48p_1+96p_7)\partial_n\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(-\tfrac{49}{90}+48p_8+24p_6-144p_4+168p_5-24p_1-72p_7)\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(\tfrac{49}{90}-48p_8-24p_6+144p_4-168p_5+24p_1+72p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+(\tfrac{1}{90}-16p_8+8p_6-16p_3+16p_4-24p_5-8p_1+8p_7)\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{1}{90}+16p_8-8p_6+16p_3-16p_4+24p_5+8p_1-8p_7)\partial_q\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(\tfrac{3}{20}+16p_9-32p_8+8p_6+16p_4-40p_5-8p_1+32p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+(-\tfrac{3}{20}-16p_9+32p_8-8p_6-16p_4+40p_5+8p_1-32p_7)\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+(-\tfrac{7}{180}-16p_9+16p_8-8p_6+16p_4+8p_5-8p_1-16p_7)\partial_q\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&+(\tfrac{7}{180}+16p_9-16p_8+8p_6-16p_4-8p_5+8p_1+16p_7)\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+(\tfrac{7}{120}+16p_9-8p_8+8p_6+16p_4+16p_{10}-8p_1+12p_7)\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(-\tfrac{7}{120}-16p_9+8p_8-8p_6-16p_4-16p_{10}+8p_1-12p_7)\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(8p_9-8p_8+4p_6-8p_3-8p_4+4p_5-8p_2-4p_1+4p_7)\partial_p\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(-8p_9+8p_8-4p_6+8p_3+8p_4-4p_5+8p_2+4p_1-4p_7)\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_n\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+(\tfrac{23}{360}+8p_9-16p_8+4p_6-8p_3+8p_4-20p_5+8p_2-16p_{10}-4p_1+16p_7)\\
&\qquad\partial_p\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+(-\tfrac{23}{360}-8p_9+16p_8-4p_6+8p_3-8p_4+20p_5-8p_2+16p_{10}+4p_1-16p_7)\\
&\qquad\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\big)+\bar{o}(\hbar^4). \quad (11)
\end{aligned}
$$

The ten master-parameters in (11) are the still unknown weights of the prime graphs which are portrayed in Fig. 3 on p. 22. The four underlined parameters can be gauged out (*without* modifying the coefficients of any other Kontsevich graphs with four internal vertices), see Theorem 14 on p. 32. At all values of the ten master-parameters, that

is, irrespective of their true values given by formula (5), the $\star$-product is proven in Theorem 12 to be associative modulo $\bar{o}(\hbar^4)$.

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Appendix A. Approximations and conjectured values of weight integrals

The material presented here is an expanded version of section 3 of the note [8] by the authors.

A.1. **The weight integral in Cartesian coordinates.** Recall the integral formula for the weight of a graph $\Gamma \in \tilde{G}_{2,k}$ (see section 2):

$$w(\Gamma) = \frac{1}{(2\pi)^{2k}} \int_{C_k(\mathbb{H})} \bigwedge_{j=1}^{k} \mathrm{d}\varphi(p_j, p_{\mathrm{Left}(j)}) \wedge \mathrm{d}\varphi(p_j, p_{\mathrm{Right}(j)}), \tag{5}$$

such that the integral is taken over the configuration space of k points in the upper half-plane $\mathbb{H} \subset \mathbb{C}$,

$$C_k(\mathbb{H}) = \{(p_1, \ldots, p_k) \in \mathbb{H}^k : p_i \text{ pairwise distinct}\},$$

and where $\varphi\colon C_2(\mathbb{H}) \to [0, 2\pi)$ was defined by $\varphi(p, q) = \mathrm{Arg}\left(\frac{q-p}{q-\bar{p}}\right)$.

For nonzero $z = x + iy$ in $\mathbb{H}$ we have $\mathrm{Arg}(x + iy) \cong \arctan(y/x)$, where $\cong$ denotes equality of functions up to a constant. Since $\frac{d}{dt}\arctan(t) = 1/(1 + t^2)$, the weight integrand is a rational function of the Cartesian coordinates: for $p = a+ib$ and $q = x+iy$,

$$\varphi(p, q) \cong \arctan\left(\frac{2b(a - x)}{(a - x)^2 + (y + b)(y - b)}\right). \tag{12}$$

In Cartesian coordinates $(x_1, y_1, \ldots, x_k, y_k)$, the weight integrand can now be written as the Jacobian determinant of the map $\Phi_\Gamma\colon C_k(\mathbb{H}) \to [0, 2\pi)^{2k}$ defined by[17]

$$\Phi_\Gamma(p_1, \ldots, p_k) = (\varphi(p_1, p_{\mathrm{Left}(1)}), \varphi(p_1, p_{\mathrm{Right}(1)}), \ldots, \varphi(p_k, p_{\mathrm{Left}(k)}), \varphi(p_k, p_{\mathrm{Right}(k)}))$$

considered as a function of the (x_j, y_j) through $p_j = x_j + iy_j$.

Implementation 17. The command

```
> weight_integrands <graph-series-file>
```

takes as input a list of graphs $\Gamma \in \tilde{G}_{2,k}$ with (possibly undetermined) coefficients, and sends to the standard output lines of the following form:

```
(* <graph encoding>    <coefficient> *)
<weight integrand of the graph above>
```

where the weight integrands are written in Mathematica format, as `Det[...]`.

We can take integration domain to be $\mathbb{H}^k$, since for any $i \neq j$ the set $\{(p_1, \ldots, p_k) \in \mathbb{C}^k : p_i = p_j\}$ is a strict linear subspace of $\mathbb{C}^k$, which has measure zero. The weight integral is absolutely convergent [26], so by the Fubini–Tonelli theorem we may evaluate it as an iterated integral in any order. We can use the residue theorem[18] to integrate out the Cartesian coordinates corresponding to the k real parts, halving the dimension. It then remains to integrate the result (a function of the k imaginary parts) over $\mathbb{R}^k$.

[17] Called a Gauss map by M. Polyak [29].

[18] G. Dito used the residue method for one graph [10] at $k = 2$, and remarked that that it becomes unpractical for $k \geqslant 3$.

Example 24. For the wedge graph Λ we have the Cartesian coordinates $x + iy$ in the upper half-plane and the integrand (obtained using Implementation 17)

$$f(x, y) = \frac{4y}{((x-1)^2 + y^2)(x^2 + y^2)}.$$

To apply the residue theorem we interpret $f(x, y)$ as a rational function in a single *complex* variable x. Its poles are then $\pm iy$ and $1 \pm iy$, so the poles in the upper half-plane are iy and $1 + iy$ (since $y > 0$). The residues at these poles are $r_1 = 2/(i + 2y)$ and $r_2 = -2/(2y - i)$ respectively. Hence the residue theorem yields that the integral of $f(x, y)$ with respect to x over the real line is $2\pi i(r_1 + r_2) = 8\pi/(1 + 4y^2)$. When we integrate this over $y > 0$ and divide by $(2\pi)^2$ we obtain $1/2$, as desired.

This is of course a toy example. For higher k the expressions become larger, but also one has to consider more carefully which poles are in the upper half-plane. From the expression (12) for φ one can see that this issue depends on the relative position of the coordinates on the imaginary axis (y and b in that formula).

For $k = 3$ with coordinates on $\mathbb{H}^3$ given by

$$a + bi, \quad c + di, \quad e + fi,$$

let us agree to call a, c, e the real coordinates and b, d, f the imaginary coordinates. We now split the integral into a sum of integrals over $3! = 6$ regions, one for each possible ordering of the imaginary coordinates:

$$b < d < f; \quad b < f < d; \quad d < b < f; \quad d < f < b; \quad f < b < d; \quad f < d < b.$$

In each such region it is known for every (complexified) real coordinate which poles are in the upper half-plane, so we can apply the residue theorem three times. The result can be numerically integrated more effectively than the original expression, for one because we have halved the dimension of the integration domain.

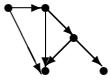
Remark 16. To integrate over the region of $\mathbb{H}^3$ defined by $b < d < f$, one can choose integration bounds as follows: $\int_0^\infty \mathrm{d}b \int_b^\infty \mathrm{d}f \int_b^f \mathrm{d}d$ (and similarly for the other permutations). For the region of $\mathbb{H}^4$ defined by $b < d < f < h$ one can choose the integration bounds $\int_0^\infty \mathrm{d}b \int_b^\infty \mathrm{d}h \int_b^h \mathrm{d}d \int_d^h \mathrm{d}f$, and so on.

Implementation 18. The strategy above is implemented by the following Mathematica code (for the order 4, but it can be adapted for others), where W is the weight integrand.

```
W = an integrand, e.g.  from list [6];

integrationvariables = {a, b, c, d, e, f, g, h};
imaginaryvariables =
 integrationvariables[[2 #1]] & /@
  Range[1, Length[integrationvariables]/2];
realvariables =
 integrationvariables[[2 #1 - 1]] & /@
  Range[1, Length[integrationvariables]/2];

basicAssumptions =
```

```
  Element[a, Reals] && Element[c, Reals] && Element[e, Reals] &&
   Element[g, Reals] && b > 0 && d > 0 && f > 0 && h > 0;

ContourIntegrate[function_, variable_, assumptions_] :=
 2*Pi*I*Total[
   Map[
    Function[{p}, (Numerator[Together[function]]/
        D[Denominator[Together[function]], variable]) /. {variable ->
        p}],
    Select[
     ReplaceList[variable,
      Assuming[assumptions,
       Flatten[FullSimplify[
         Solve[Denominator[Together[function]] == 0, variable,
          Complexes]]]],
     Function[{r}, Simplify[ComplexExpand[Im[r]] > 0, assumptions]]]]]

IteratedContourIntegrate[function_, variables_, assumptions_] :=
 Fold[ContourIntegrate[Together[#1], #2, assumptions] &, function,
  variables]

integrals = Map[
  NIntegrate[
     Simplify[
      IteratedContourIntegrate[W, realvariables,
       basicAssumptions && #1[[1]] < #1[[2]] < #1[[3]] < #1[[4]]]
     TimeConstraint -> Infinity],
     Evaluate[
      Sequence @@
       {{#1[[1]], 0, Infinity}, {#1[[3]], #1[[1]],
         Infinity}, {#1[[2]], #1[[1]], #1[[3]]}, {#1[[4]], #1[[3]],
         Infinity}}
      ],
     Method -> {GlobalAdaptive, MaxErrorIncreases -> 10^4}
     ] &, Permutations[imaginaryvariables]]

Print[integrals]
Print[Total[integrals]]
Print[Total[integrals]/N[(2 Pi)^8]]
```

Remark 17. This strategy allows effective numerical integration of all weights up to order 3. At the order 4, it works for some weights but not others: see Tables 2 and 3. The call(s) to `Map` may be replaced by `ParallelMap` to parallelize the computation.

Example 25. The second Bernoulli graph  [11] has the weight integrand

$$\frac{64bfd\left(c\left((a-c)^2+b^2\right)+d^2(c-2a)\right)\left(f^2(e-2c)+e\left((e-c)^2+d^2\right)\right)}{\left(a^2+b^2\right)\left(f^2+(e-1)^2\right)\left(f^2+e^2\right)\left(c^2+d^2\right)\left((a-c)^2+(b-d)^2\right)\left((a-c)^2+(b+d)^2\right)\left((f+d)^2+(e-c)^2\right)}$$

The residue calculation followed by the numerical integration leads to the estimate $5.71871 \times 10^{-9} - 5.92495 \times 10^{-21} i$ of the weight; this leads to the guess that it is zero and in fact it is true.

TABLE 2. Verified values

Weight	Approximation	True value
`w_4_1`	$-0.0069444401170 \pm 0.000000906189$	$-1/144 \approx -0.00694444$

TABLE 3. Conjectured values

Weight	Approximation	Conjectured true value
`w_4_103`	$-0.000086894703 \pm 0.000000681076$	$-1/11520 \approx 0.000086805$
`w_4_104`	$0.000347214860 \pm 0.000000371598$	$1/2880 \approx 0.000347222$
`w_4_112`	$-0.000347219933 \pm 0.000000042901$	$-1/2880 \approx -0.000347222$
`w_4_113`	$0.000694441623 \pm 0.000000093136$	$1/1440 \approx 0.000694444$
`w_4_133`	$0.000694443060 \pm 0.000000078774$	$1/1440 \approx 0.000694444$
`w_4_138`	$-0.001041664533 \pm 0.000000095465$	$-1/960 \approx -0.001041666$
`w_4_147`	$-0.000043376821 \pm 0.000000095465$	$-1/23040 \approx -0.000043402$
`w_4_148`	$0.000173611294 \pm 0.000000015063$	$1/5760 \approx 0.000173611$

In particular, this table lists the approximate value of the master-parameters $p_4 =$ `w_4_103` and $p_5 =$ `w_4_104`. The relation `w_4_133` $= 2 \cdot$ `w_4_104` which was found in Theorem 9 and listed in Table 7 of Appendix C is satisfied approximately. Furthermore, the relation `w_4_103` $= 2 \cdot$ `w_4_147` seems to hold approximately.

A.2. **Claimed values of the 10 master-parameters.** By using a different technique B. Pym and E. Panzer have obtained the exact values of the ten master-parameters.

Claim ([27])**.** *The values of ten master-parameters* (*which are the weights of ten graphs in Figure 3 on p. 22*) *are given in Table 4 below.*

TABLE 4. Recently suggested values of the master-parameters [27].

Master-parameter	Value
$p_1 = \texttt{w\_4\_100}$	$1/1440$
$p_2 = \texttt{w\_4\_101}$	$1/2880$
$p_3 = \texttt{w\_4\_102}$	$1/5760$
$p_4 = \texttt{w\_4\_103}$	$-1/11520$
$p_5 = \texttt{w\_4\_104}$	$1/2880$
$p_6 = \texttt{w\_4\_107}$	$13/2880$
$p_7 = \texttt{w\_4\_108}$	$-17/2880$
$p_8 = \texttt{w\_4\_109}$	$-1/1152$
$p_9 = \texttt{w\_4\_119}$	$-1/1280$
$p_{10} = \texttt{w\_4\_125}$	$-1/960$

Let it be emphasized that these ten values are conjectured via a use of software, so that more effort is needed to fully justify this result.

Remark 18. The exact values of two master-parameters `w_4_103` and `w_4_104` reproduce the values which had been conjectured in Table 3. We also note that all the weights of graphs in $\star$ mod $\bar{o}(\hbar^4)$ are rational numbers. Thirdly, the values of non-master parameters (namely, `w_4_112`, `w_4_113`, `w_4_133`, `w_4_138`, `w_4_147`, and `w_4_148`) in Table 3, whenever recalculated on the basis of conjectured values from Table 4, do all match the numerical approximations in Table 3, reproducing our conjectured rational values in its rightmost column.

In conclusion, provided that all the ten values in Table 4 are true, this is the authentic Kontsevich star-product up to $\bar{o}(\hbar^4)$:

$$
\begin{aligned}
f\star g = f\times g + \hbar\mathcal{P}^{ij}\partial_i f\partial_j g + \hbar^2\big(&-\tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_i f\partial_k g - \tfrac{1}{3}\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_i f\partial_k\partial_j g\\
&+\tfrac{1}{3}\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_k\partial_i f\partial_j g + \tfrac{1}{2}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_k\partial_i f\partial_\ell\partial_j g\big) + \hbar^3\big(-\tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_k g\\
&+\tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_m\partial_k\partial_j g - \tfrac{1}{3}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_\ell\partial_j g\\
&+\tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_j g + \tfrac{1}{3}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_\ell\partial_j g\\
&+\tfrac{1}{6}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_k\partial_i f\partial_n\partial_\ell\partial_j g - \tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_k\partial_j g\\
&+\tfrac{1}{6}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_i f\partial_m\partial_k g - \tfrac{1}{6}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_j g\\
&-\tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_m\partial_i f\partial_k g - \tfrac{1}{6}\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{6}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g - \tfrac{1}{6}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_k\partial_i f\partial_m\partial_j g\big) +
\end{aligned}
$$

$$
\begin{aligned}
&+\hbar^4\big(-\tfrac{1}{6}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_n\partial_\ell\partial_j g+\tfrac{1}{6}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_n\partial_\ell\partial_j g\\
&\quad+\tfrac{1}{24}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_q\partial_n\partial_\ell\partial_j g+\tfrac{1}{6}\partial_q\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_\ell\partial_j g\\
&\quad+\tfrac{1}{18}\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_\ell\partial_j g+\tfrac{1}{6}\partial_q\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_\ell\partial_j g\\
&\quad+\tfrac{1}{18}\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_\ell\partial_j g-\tfrac{1}{30}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k\partial_j g\\
&\quad-\tfrac{2}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k\partial_j g-\tfrac{1}{30}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k\partial_j g\\
&\quad+\tfrac{1}{45}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k\partial_j g-\tfrac{1}{12}\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_\ell\partial_j g\\
&\quad-\tfrac{1}{6}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_\ell\partial_j g-\tfrac{1}{6}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_\ell\partial_j g\\
&\quad-\tfrac{1}{9}\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_\ell\partial_j g+\tfrac{1}{30}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_j g\\
&\quad+\tfrac{2}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_j g+\tfrac{1}{30}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_j g\\
&\quad-\tfrac{1}{45}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_k\partial_i f\partial_j g-\tfrac{1}{6}\partial_p\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_\ell\partial_j g\\
&\quad+\tfrac{1}{6}\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g+\tfrac{1}{18}\partial_q\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad-\tfrac{1}{9}\partial_p\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_\ell\partial_j g-\tfrac{1}{18}\partial_n\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_\ell\partial_j g\\
&\quad+\tfrac{1}{30}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g+\tfrac{13}{90}\partial_q\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad-\tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g-\tfrac{1}{30}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad+\tfrac{1}{90}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g+\tfrac{1}{30}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad+\tfrac{1}{15}\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g+\tfrac{1}{15}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g\\
&\quad+\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m\partial_j g-\tfrac{1}{6}\partial_p\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_\ell\partial_j g\\
&\quad-\tfrac{1}{6}\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m\partial_j g-\tfrac{1}{18}\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad-\tfrac{1}{9}\partial_p\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_\ell\partial_j g-\tfrac{1}{18}\partial_n\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_\ell\partial_j g\\
&\quad-\tfrac{1}{30}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g-\tfrac{13}{90}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad+\tfrac{1}{45}\partial_q\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g+\tfrac{1}{30}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad-\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g-\tfrac{1}{30}\partial_q\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g\\
&\quad-\tfrac{1}{15}\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g-\tfrac{1}{15}\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m\partial_j g\\
&\quad-\tfrac{1}{90}\partial_q\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p\partial_j g+\tfrac{7}{90}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g\\
&\quad-\tfrac{1}{90}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g+\tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g\\
&\quad+\tfrac{2}{45}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g+\tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g\\
&\quad+\tfrac{1}{45}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g-\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g\\
&\quad-\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g-\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k\partial_j g\\
&\quad+\tfrac{1}{30}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k\partial_j g+\tfrac{1}{90}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&\quad+\tfrac{1}{90}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g+\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&\quad-\tfrac{1}{30}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g-\tfrac{1}{45}\partial_q\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{1}{60}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g - \tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&- \tfrac{1}{45}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g - \tfrac{1}{20}\partial_q\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m\partial_k g\\
&- \tfrac{7}{90}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g + \tfrac{1}{90}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&- \tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g - \tfrac{2}{45}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&- \tfrac{1}{30}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g - \tfrac{1}{45}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&+ \tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g + \tfrac{1}{90}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g\\
&+ \tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_j g - \tfrac{1}{30}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_j g\\
&+ \tfrac{1}{90}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g + \tfrac{1}{90}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g\\
&+ \tfrac{1}{90}\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_k\partial_i f\partial_p g - \tfrac{1}{30}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&- \tfrac{1}{45}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g - \tfrac{1}{60}\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&- \tfrac{1}{45}\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g - \tfrac{1}{45}\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_k\partial_i f\partial_m g\\
&- \tfrac{1}{20}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_m\partial_i f\partial_k g - \tfrac{1}{40}\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&- \tfrac{1}{72}\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g + \tfrac{1}{72}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ \tfrac{1}{360}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g - \tfrac{1}{60}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&- \tfrac{1}{60}\partial_p\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g + \tfrac{17}{720}\partial_m\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k g\\
&- \tfrac{17}{720}\partial_p\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g - \tfrac{1}{180}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ \tfrac{1}{180}\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g + \tfrac{1}{360}\partial_p\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&- \tfrac{1}{360}\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g + \tfrac{1}{160}\partial_m\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ \tfrac{1}{160}\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g - \tfrac{17}{1440}\partial_p\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ \tfrac{17}{1440}\partial_q\partial_m\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g - \tfrac{1}{360}\partial_p\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k g\\
&+ \tfrac{1}{360}\partial_q\partial_k\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g - \tfrac{13}{720}\partial_p\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&- \tfrac{13}{720}\partial_k\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g - \tfrac{1}{60}\partial_p\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m g\\
&- \tfrac{7}{720}\partial_p\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g + \tfrac{7}{720}\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_n\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g\\
&- \tfrac{1}{180}\partial_p\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g + \tfrac{1}{160}\partial_k\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m g\\
&+ \tfrac{1}{160}\partial_m\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g + \tfrac{13}{1440}\partial_m\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&- \tfrac{13}{1440}\partial_p\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g - \tfrac{1}{1440}\partial_k\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+ \tfrac{1}{1440}\partial_p\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g + \tfrac{1}{360}\partial_k\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p g\\
&+ \tfrac{1}{240}\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_n\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_m g - \tfrac{13}{360}\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&- \tfrac{1}{720}\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g + \tfrac{1}{30}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&- \tfrac{1}{720}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g - \tfrac{19}{720}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ \tfrac{1}{180}\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g + \tfrac{13}{360}\partial_m\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g
\end{aligned}
$$

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\begin{aligned}
&- \tfrac{1}{720}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g + \tfrac{1}{360}\partial_p\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&+ \tfrac{1}{720}\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g - \tfrac{1}{80}\partial_p\mathcal{P}^{ij}\partial_q\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g\\
&- \tfrac{1}{180}\partial_p\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_k\partial_j g - \tfrac{1}{36}\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ \tfrac{1}{60}\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g - \tfrac{1}{720}\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ \tfrac{1}{45}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + \tfrac{17}{720}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ \tfrac{13}{360}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + \tfrac{1}{120}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+ \tfrac{1}{240}\partial_q\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g - \tfrac{1}{80}\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&- \tfrac{1}{72}\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g - \tfrac{1}{90}\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&- \tfrac{1}{180}\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g - \tfrac{1}{360}\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ \tfrac{1}{720}\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g - \tfrac{1}{180}\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+ \tfrac{17}{180}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g - \tfrac{1}{72}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ \tfrac{7}{180}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + \tfrac{7}{180}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&- \tfrac{1}{180}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g + \tfrac{1}{90}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_m\partial_k g\\
&+ \tfrac{1}{80}\partial_q\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g + \tfrac{1}{120}\partial_q\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+ \tfrac{1}{40}\partial_q\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g - \tfrac{1}{360}\partial_q\partial_n\partial_k\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&- \tfrac{11}{720}\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + \tfrac{7}{240}\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&- \tfrac{1}{180}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_k\mathcal{P}^{mn}\partial_\ell\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + \tfrac{1}{60}\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_i f\partial_p\partial_k g\\
&+ \tfrac{1}{90}\partial_q\partial_k\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g + \tfrac{1}{60}\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_i f\partial_p\partial_m g\\
&+ \tfrac{13}{360}\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g + \tfrac{1}{720}\partial_p\partial_m\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&- \tfrac{1}{30}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g + \tfrac{1}{720}\partial_m\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ \tfrac{19}{720}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g - \tfrac{1}{180}\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&- \tfrac{13}{360}\partial_m\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g + \tfrac{1}{720}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&- \tfrac{1}{360}\partial_p\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g - \tfrac{1}{720}\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&+ \tfrac{1}{80}\partial_p\mathcal{P}^{ij}\partial_q\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g + \tfrac{1}{180}\partial_p\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_j g\\
&- \tfrac{1}{36}\partial_p\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g - \tfrac{1}{60}\partial_p\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ \tfrac{1}{720}\partial_n\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g - \tfrac{1}{45}\partial_p\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&- \tfrac{17}{720}\partial_p\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g - \tfrac{13}{360}\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&- \tfrac{1}{120}\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g - \tfrac{1}{240}\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g\\
&+ \tfrac{1}{80}\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m g + \tfrac{1}{72}\partial_q\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&+ \tfrac{1}{90}\partial_m\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g + \tfrac{1}{180}\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g\\
&+ \tfrac{1}{360}\partial_q\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g - \tfrac{1}{720}\mathcal{P}^{ij}\partial_q\partial_m\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g
\end{aligned}
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\begin{aligned}
&+\tfrac{1}{180}\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p g-\tfrac{17}{180}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&-\tfrac{1}{72}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g+\tfrac{7}{180}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&-\tfrac{7}{180}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g-\tfrac{1}{180}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+\tfrac{1}{90}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g-\tfrac{1}{80}\partial_p\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&-\tfrac{1}{120}\partial_p\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g+\tfrac{1}{40}\mathcal{P}^{ij}\partial_p\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_k g\\
&+\tfrac{1}{360}\partial_m\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g+\tfrac{11}{720}\partial_m\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&-\tfrac{7}{240}\partial_n\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g-\tfrac{1}{180}\partial_m\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&-\tfrac{1}{60}\partial_k\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g+\tfrac{1}{90}\mathcal{P}^{ij}\partial_m\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_p\partial_i f\partial_k g\\
&-\tfrac{1}{60}\partial_k\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m g-\tfrac{1}{6}\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+\tfrac{1}{72}\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g+\tfrac{17}{360}\partial_p\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&+\tfrac{1}{24}\partial_p\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g+\tfrac{1}{180}\partial_m\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_\ell\partial_j g\\
&+\tfrac{2}{45}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g-\tfrac{2}{45}\partial_p\partial_n\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{30}\partial_n\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g+\tfrac{1}{8}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{8}\partial_p\partial_n\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g+\tfrac{1}{720}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{720}\partial_p\partial_n\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g-\tfrac{11}{180}\partial_\ell\mathcal{P}^{ij}\partial_p\partial_n\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{11}{180}\partial_n\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g+\tfrac{1}{36}\partial_p\partial_\ell\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{36}\partial_q\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g+\tfrac{1}{90}\partial_p\partial_\ell\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{90}\partial_q\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g-\tfrac{1}{180}\partial_\ell\mathcal{P}^{ij}\partial_p\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{180}\partial_q\mathcal{P}^{ij}\partial_m\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g+\tfrac{1}{18}\partial_p\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{18}\partial_p\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g+\tfrac{1}{144}\partial_p\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&-\tfrac{1}{144}\partial_m\mathcal{P}^{ij}\partial_q\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g-\tfrac{1}{90}\partial_p\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_m\partial_j g\\
&+\tfrac{1}{90}\partial_m\mathcal{P}^{ij}\partial_q\partial_n\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g-\tfrac{1}{60}\partial_m\mathcal{P}^{ij}\partial_n\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g\\
&+\tfrac{1}{60}\partial_m\mathcal{P}^{ij}\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_j g-\tfrac{1}{240}\partial_q\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&-\tfrac{1}{240}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_p\partial_i f\partial_m\partial_k g-\tfrac{13}{720}\partial_n\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&-\tfrac{13}{720}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g-\tfrac{1}{90}\partial_n\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&-\tfrac{1}{90}\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g+\tfrac{1}{60}\partial_q\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+\tfrac{1}{60}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g+\tfrac{1}{30}\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+\tfrac{1}{30}\partial_\ell\mathcal{P}^{ij}\partial_q\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g-\tfrac{1}{90}\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\partial_\ell\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g\\
&+\tfrac{1}{360}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_k\partial_i f\partial_p\partial_m g+\tfrac{13}{90}\partial_q\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+\tfrac{13}{180}\partial_\ell\mathcal{P}^{ij}\partial_n\partial_j\mathcal{P}^{k\ell}\partial_q\mathcal{P}^{mn}\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g+\tfrac{13}{180}\partial_q\partial_\ell\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\\
&+\tfrac{1}{72}\partial_q\mathcal{P}^{ij}\partial_j\mathcal{P}^{k\ell}\partial_\ell\mathcal{P}^{mn}\partial_n\mathcal{P}^{pq}\partial_m\partial_i f\partial_p\partial_k g\big)+\bar{o}(\hbar^4).\quad(13)
\end{aligned}
$$

Out of 247 graphs at $\hbar^4$, as many as 138 contain two-cycles (or "eyes", as in Fig. 1 on p. 5), cf. expansion (1) up to $\bar{o}(\hbar^3)$ on p. 2.

Appendix B. C++ classes and methods

Class KontsevichGraph

Summary: a (signed) Kontsevich graph.

Data members (private):

```
size_t d_internal = 0;
size_t d_external = 0;
std::vector< std::pair<char, char> > d_targets;
int d_sign = 1;
```

Public typedefs:

```
typedef char Vertex;
typedef std::pair<Vertex, Vertex> VertexPair;
```

Constructors:

```
KontsevichGraph() = default;
KontsevichGraph(size_t internal, size_t external,
                std::vector<VertexPair> targets,
                int sign = 1, bool normalized = false);
```

Accessor methods:

```
std::vector<VertexPair> targets() const;
VertexPair targets(Vertex internal_vertex) const;
int sign() const;
int sign(int new_sign);
size_t internal() const;
size_t external() const;
```

Methods to obtain numerical information:

```
size_t vertices() const;
std::vector<Vertex> internal_vertices() const;
std::pair< size_t, std::vector<VertexPair> > abs() const;
size_t multiplicity() const;
size_t in_degree(KontsevichGraph::Vertex vertex) const;
std::vector<size_t> in_degrees() const;
std::vector<Vertex> neighbors_in(Vertex vertex) const;
KontsevichGraph mirror_image() const;
std::string as_sage_expression() const;
std::string encoding() const;
std::vector< std::tuple<KontsevichGraph, int, int> > permutations() const;
```

Methods that modify the graph:

```
void normalize();
KontsevichGraph& operator*=(const KontsevichGraph& rhs);
```

Methods that test for graph properties:

```
bool operator<(const KontsevichGraph& rhs) const;
bool is_zero() const;
```

```
bool is_prime() const;
bool positive_differential_order() const;
bool has_cycles() const;
bool has_tadpoles() const;
bool has_multiple_edges() const;
bool has_max_internal_indegree(size_t max_indegree) const;
```

Static methods:

```
static std::set<KontsevichGraph> graphs(size_t internal,
    size_t external = 2, bool modulo_signs = false,
    bool modulo_mirror_images = false,
    std::function<void(KontsevichGraph)> const& callback = nullptr,
    std::function<bool(KontsevichGraph)> const& filter = nullptr);
```

Private methods:

```
friend std::ostream& operator<<(std::ostream &os, const KontsevichGraph& g);
friend std::istream& operator>>(std::istream& is, KontsevichGraph& g);
friend bool operator==(const KontsevichGraph &lhs, const KontsevichGraph& rhs);
friend bool operator!=(const KontsevichGraph &lhs, const KontsevichGraph& rhs);
```

Functions defined outside the class:

```
KontsevichGraph operator*(KontsevichGraph lhs, const KontsevichGraph& rhs);
std::ostream& operator<<(std::ostream &os, const KontsevichGraph::Vertex v);
```

Class `KontsevichGraphSum<T>`

- Template parameter `T`: type of the coefficients (e.g. `KontsevichGraphSum<int>`).
- Publically extends: `std::vector< std::pair<T, KontsevichGraph> >`.

Summary: a sum of Kontsevich graphs, with method to reduce modulo skew-symmetry.

Data members: inherited.
Public typedefs:

```
typedef std::pair<T, KontsevichGraph> Term;
```

Constructors (inherited):

```
using std::vector< std::pair<T, KontsevichGraph> >::vector;
```

Accessor methods:

```
using std::vector< std::pair<T, KontsevichGraph> >::operator[];
KontsevichGraphSum<T> operator[](std::vector<size_t> indegrees) const;
T operator[](KontsevichGraph) const;
```

Arithmetic operators:

```
KontsevichGraphSum<T> operator()(std::vector< KontsevichGraphSum<T> >) const;
KontsevichGraphSum<T>& operator+=(const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T>& operator-=(const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T>& operator=(const KontsevichGraphSum<T>&) = default;
```

Methods:

```
std::vector< std::vector<size_t> > in_degrees(bool ascending = false) const;
KontsevichGraphSum<T> skew_symmetrization() const;
```

Methods that modify the graph sum:

```
void reduce();
```

Comparison operators:

```
bool operator==(const KontsevichGraphSum<T>& other) const;
bool operator==(int other) const;
bool operator!=(const KontsevichGraphSum<T>& other) const;
bool operator!=(int other) const;
```

Friend operators:

```
friend std::ostream& operator<< <>(std::ostream& os,
                                   const KontsevichGraphSum<T>::Term& term);
friend std::ostream& operator<< <>(std::ostream& os,
                                   const KontsevichGraphSum<T>& gs);
friend std::istream& operator>> <>(std::istream& is,
                                   KontsevichGraphSum<T>& sum);
```

Functions defined outside the class:

```
KontsevichGraphSum<T> operator+(KontsevichGraphSum<T> lhs,
                                const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T> operator-(KontsevichGraphSum<T> lhs,
                                const KontsevichGraphSum<T>& rhs);
KontsevichGraphSum<T> operator*(T lhs,
                                KontsevichGraphSum<T> rhs);
std::ostream& operator<<(std::ostream&, const std::pair<T, KontsevichGraph>&);
std::ostream& operator<<(std::ostream&, const KontsevichGraphSum<T>&);
std::istream& operator>>(std::istream&, KontsevichGraphSum<T>&);
```

Class `KontsevichGraphSeries<T>`

- Template parameter `T`: type of the coefficients (e.g. `KontsevichGraphSeries<int>`).
- Publically extends: `std::map< size_t, KontsevichGraphSum<T> >`

Summary: a formal power series expansion; sums of Kontsevich graphs as coefficients.

Data members: inherited, plus (private):

```
size_t d_precision = std::numeric_limits<std::size_t>::max();
```

Constructors (inherited):

```
using std::map< size_t, KontsevichGraphSum<T> >::map;
```

Accessor methods:

```
size_t precision() const;
size_t precision(size_t new_precision);
```

Arithmetic operators:

```
KontsevichGraphSeries<T> operator()(std::vector< KontsevichGraphSeries<T> >)
                                                                    const;
KontsevichGraphSeries<T>& operator+=(const KontsevichGraphSeries<T>& rhs);
KontsevichGraphSeries<T>& operator-=(const KontsevichGraphSeries<T>& rhs);
```

Methods:

```
KontsevichGraphSeries<T> skew_symmetrization() const;
KontsevichGraphSeries<T> inverse() const;
KontsevichGraphSeries<T> gauge_transform(const KontsevichGraphSeries<T>& gauge);
```

Comparison operators:

```
bool operator==(int other) const;
bool operator!=(int other) const;
```

Methods that modify the graph series:

```
void reduce();
```

Static methods:

```
static KontsevichGraphSeries<T> from_istream(std::istream& is,
    std::function<T(std::string)> const& parser,
    std::function<bool(KontsevichGraph, size_t)> const& filter = nullptr);
```

Friend methods:

```
friend std::ostream& operator<< <>(std::ostream& os,
                                   const KontsevichGraphSeries<T>& series);
```

Functions defined outside the class:

```
KontsevichGraphSeries<T> operator+(KontsevichGraphSeries<T> lhs,
                                   const KontsevichGraphSeries<T>& rhs);
KontsevichGraphSeries<T> operator-(KontsevichGraphSeries<T> lhs,
                                   const KontsevichGraphSeries<T>& rhs);
std::ostream& operator<<(std::ostream&, const KontsevichGraphSeries<T>&);
```

Appendix C. Encoding of the entire ⋆-product modulo $\bar{o}(\hbar^4)$

In the following two tables, containing the sets of basic graphs and the ⋆-product expansion respectively, encodings of graphs (see Implementation 1 on p. 5) are followed by their coefficients.

Table 5. Basic sets of Kontsevich graphs, up to order 4, including zero graphs.

```
h^0:
2 0 1                                 1
h^1:
2 1 1   0 1                           1/2
h^2:
2 2 1   0 1 0 2                       1/12
2 2 1   0 3 2 1                       1/24
h^3:
2 3 1   0 1 0 2 3 2                   1/48
2 3 1   0 1 0 2 0 2                   1/24
2 3 1   0 1 2 1 0 3                   1/48
2 3 1   0 3 2 1 2 1                   1/48
2 3 1   0 3 2 1 2 3                   1/48
2 3 0   0 1 0 1 2 3                   0
h^4:
2 4 1   0 1 0 1 0 2 2 3   w_4_1
2 4 1   0 1 0 1 0 2 3 4   w_4_2
2 4 0   0 1 0 1 0 5 2 3   0
2 4 1   0 1 0 1 2 3 2 3   w_4_3
2 4 1   0 1 0 1 2 3 2 4   w_4_4
2 4 1   0 1 0 1 2 5 3 4   w_4_5
2 4 1   0 1 0 2 0 2 0 2   w_4_6
2 4 1   0 1 0 2 0 2 0 3   w_4_7
2 4 1   0 1 0 2 0 2 1 2   w_4_8
2 4 1   0 1 0 2 0 2 1 3   w_4_9
2 4 1   0 1 0 2 0 2 2 3   w_4_10
2 4 0   0 1 0 2 0 2 3 4   0
2 4 1   0 1 0 2 0 3 0 3   w_4_11
2 4 1   0 1 0 2 0 3 0 4   w_4_12
2 4 1   0 1 0 2 0 3 1 2   w_4_13
2 4 1   0 1 0 2 0 3 1 3   w_4_14
2 4 1   0 1 0 2 0 3 1 4   w_4_15
2 4 1   0 1 0 2 0 3 2 3   w_4_16
2 4 1   0 1 0 2 0 3 2 4   w_4_17
2 4 1   0 1 0 2 0 3 3 4   w_4_18
2 4 1   0 1 0 2 0 5 1 2   w_4_19
2 4 1   0 1 0 2 0 5 1 3   w_4_20
2 4 1   0 1 0 2 0 5 2 3   w_4_21
2 4 1   0 1 0 2 0 5 2 4   w_4_22
2 4 1   0 1 0 2 0 5 3 4   w_4_23
2 4 1   0 1 0 2 1 2 2 3   w_4_24
2 4 1   0 1 0 2 1 2 3 4   w_4_25
2 4 1   0 1 0 2 1 3 1 3   w_4_26
2 4 1   0 1 0 2 1 3 1 4   w_4_27
2 4 1   0 1 0 2 1 3 2 3   w_4_28
2 4 1   0 1 0 2 1 3 2 4   w_4_29
2 4 1   0 1 0 2 1 3 3 4   w_4_30
2 4 1   0 1 0 2 1 5 2 3   w_4_31
2 4 1   0 1 0 2 1 5 2 4   w_4_32
2 4 1   0 1 0 2 1 5 3 4   w_4_33
2 4 1   0 1 0 2 2 3 2 3   w_4_34
2 4 1   0 1 0 2 2 3 2 4   w_4_35
2 4 1   0 1 0 2 2 3 3 4   w_4_36
2 4 1   0 1 0 2 2 5 2 4   w_4_37
2 4 1   0 1 0 2 2 5 3 4   w_4_38
2 4 1   0 1 0 2 3 5 3 4   w_4_39
2 4 1   0 1 0 4 0 3 2 3   w_4_40
2 4 1   0 1 0 4 0 5 2 3   w_4_41
2 4 1   0 1 0 4 0 5 2 4   w_4_42
2 4 1   0 1 0 4 1 3 2 3   w_4_43
2 4 1   0 1 0 4 1 5 2 3   w_4_44
2 4 1   0 1 0 4 1 5 2 4   w_4_45
2 4 1   0 1 0 4 2 3 0 4   w_4_46
2 4 1   0 1 0 4 2 3 1 4   w_4_47
2 4 1   0 1 0 4 2 3 2 3   w_4_48
2 4 1   0 1 0 4 2 3 2 4   w_4_49
2 4 1   0 1 0 4 2 3 3 4   w_4_50
2 4 1   0 1 0 4 2 5 2 3   w_4_51
2 4 1   0 1 0 4 2 5 2 4   w_4_52
2 4 1   0 1 0 4 2 5 3 4   w_4_53
2 4 1   0 1 0 4 3 5 2 3   w_4_54
2 4 1   0 1 0 4 3 5 2 4   w_4_55
2 4 1   0 1 2 4 2 3 2 3   w_4_56
2 4 0   0 1 2 4 2 3 3 4   0
2 4 1   0 1 2 4 2 5 2 3   w_4_57
2 4 1   0 1 2 4 2 5 3 4   w_4_58
2 4 0   0 1 2 4 3 5 2 4   0
2 4 1   0 1 2 4 3 5 3 4   w_4_59
2 4 1   0 3 0 2 0 2 1 2   w_4_60
2 4 1   0 3 0 2 0 2 1 3   w_4_61
2 4 1   0 3 0 2 0 2 1 4   w_4_62
2 4 1   0 3 0 2 0 5 1 2   w_4_63
2 4 1   0 3 0 2 1 2 1 2   w_4_64
2 4 1   0 3 0 2 1 2 1 3   w_4_65
2 4 1   0 3 0 2 1 2 1 4   w_4_66
2 4 1   0 3 0 2 1 2 2 3   w_4_67
2 4 1   0 3 0 2 1 2 2 4   w_4_68
2 4 1   0 3 0 2 1 2 3 4   w_4_69
2 4 0   0 3 0 2 1 5 2 3   0
2 4 1   0 3 0 2 1 5 2 4   w_4_70
2 4 1   0 3 0 4 0 2 1 2   w_4_71
2 4 1   0 3 0 4 0 5 1 2   w_4_72
2 4 1   0 3 0 4 0 5 1 3   w_4_73
2 4 1   0 3 0 4 0 5 1 4   w_4_74
2 4 1   0 3 0 4 1 2 0 3   w_4_75
2 4 1   0 3 0 4 1 2 0 4   w_4_76
2 4 1   0 3 0 4 1 2 1 2   w_4_77
2 4 1   0 3 0 4 1 2 1 3   w_4_78
2 4 1   0 3 0 4 1 2 1 4   w_4_79
2 4 1   0 3 0 4 1 2 2 3   w_4_80
2 4 1   0 3 0 4 1 2 2 4   w_4_81
2 4 1   0 3 0 4 1 2 3 4   w_4_82
2 4 1   0 3 0 4 1 3 0 3   w_4_83
2 4 1   0 3 0 4 1 3 0 4   w_4_84
2 4 1   0 3 0 4 1 3 1 2   w_4_85
2 4 1   0 3 0 4 1 3 1 3   w_4_86
2 4 1   0 3 0 4 1 3 1 4   w_4_87
2 4 1   0 3 0 4 1 3 2 3   w_4_88
2 4 1   0 3 0 4 1 3 2 4   w_4_89
2 4 1   0 3 0 4 1 3 3 4   w_4_90
2 4 1   0 3 0 4 1 5 0 4   w_4_91
2 4 1   0 3 0 4 1 5 1 2   w_4_92
2 4 1   0 3 0 4 1 5 2 3   w_4_93
2 4 1   0 3 0 4 1 5 2 4   w_4_94
2 4 1   0 3 0 4 1 5 3 4   w_4_95
2 4 1   0 3 0 4 2 3 1 2   w_4_96
2 4 1   0 3 0 4 2 3 1 3   w_4_97
2 4 1   0 3 0 4 2 3 1 4   w_4_98
2 4 1   0 3 0 4 2 5 1 2   w_4_99
2 4 1   0 3 0 4 2 5 1 3   w_4_100
2 4 1   0 3 0 4 2 5 1 4   w_4_101
2 4 1   0 3 0 4 3 5 1 2   w_4_102
2 4 1   0 3 0 4 3 5 1 3   w_4_103
2 4 1   0 3 0 4 3 5 1 4   w_4_104
2 4 1   0 3 1 2 0 3 0 3   w_4_105
2 4 1   0 3 1 2 0 3 1 2   w_4_106
2 4 1   0 3 1 2 0 3 1 4   w_4_107
2 4 1   0 3 1 2 0 3 2 3   w_4_108
2 4 1   0 3 1 2 0 3 2 4   w_4_109
2 4 1   0 3 1 2 0 3 3 4   w_4_110
2 4 1   0 3 1 2 0 5 2 3   w_4_111
2 4 1   0 3 1 2 0 5 2 4   w_4_112
2 4 1   0 3 1 2 0 5 3 4   w_4_113
2 4 1   0 3 1 2 2 3 2 3   w_4_114
2 4 1   0 3 1 2 2 3 2 4   w_4_115
2 4 1   0 3 1 2 2 5 2 4   w_4_116
2 4 1   0 3 1 2 2 5 3 4   w_4_117
2 4 1   0 3 1 4 0 5 1 2   w_4_118
2 4 1   0 3 1 4 0 5 2 3   w_4_119
2 4 1   0 3 1 4 0 5 2 4   w_4_120
2 4 1   0 3 1 4 0 5 3 4   w_4_121
2 4 1   0 3 1 4 2 3 0 3   w_4_122
2 4 1   0 3 1 4 2 3 0 4   w_4_123
2 4 1   0 3 1 4 2 3 1 4   w_4_124
2 4 1   0 3 1 4 2 3 2 3   w_4_125
2 4 1   0 3 1 4 2 3 2 4   w_4_126
2 4 1   0 3 1 4 2 3 3 4   w_4_127
2 4 0   0 3 1 4 2 5 0 3   0
2 4 1   0 3 1 4 2 5 0 4   w_4_128
2 4 1   0 3 1 4 2 5 1 4   w_4_129
2 4 1   0 3 1 4 2 5 2 3   w_4_130
2 4 1   0 3 1 4 2 5 2 4   w_4_131
2 4 1   0 3 1 4 2 5 3 4   w_4_132
2 4 1   0 3 1 4 3 5 0 4   w_4_133
2 4 1   0 3 1 4 3 5 1 4   w_4_134
2 4 1   0 3 1 4 3 5 2 3   w_4_135
2 4 1   0 3 1 4 3 5 2 4   w_4_136
2 4 1   0 3 1 4 3 5 3 4   w_4_137
2 4 0   0 3 2 4 0 3 1 3   0
2 4 1   0 3 2 4 1 3 0 3   w_4_138
2 4 1   0 3 2 4 1 3 2 3   w_4_139
2 4 1   0 3 2 4 1 3 2 4   w_4_140
2 4 1   0 3 2 4 1 5 2 3   w_4_141
2 4 1   0 3 2 4 1 5 2 4   w_4_142
2 4 1   0 3 2 4 1 5 3 4   w_4_143
2 4 1   0 3 2 4 2 3 1 3   w_4_144
2 4 1   0 3 2 4 2 3 1 4   w_4_145
2 4 1   0 3 2 4 2 5 1 3   w_4_146
2 4 1   0 3 2 4 3 5 1 3   w_4_147
2 4 1   0 3 2 4 3 5 1 4   w_4_148
2 4 1   0 3 4 5 1 5 2 3   w_4_149
```

TABLE 6. Kontsevich's star product up to order 4.

```
h^0:
2 0 1       1
h^1:
2 1 1   0 1    1
h^2:
2 2 1   0 1 0 1    1/2
2 2 1   0 1 0 2    1/3
2 2 1   0 1 1 2    -1/3
2 2 1   0 3 1 2    -1/6
h^3:
2 3 1   0 1 0 1 0 1    1/6
2 3 1   0 1 0 1 0 2    1/3
2 3 1   0 1 0 1 1 2    -1/3
2 3 1   0 1 0 4 1 3    -1/6
2 3 1   0 1 0 2 0 2    1/6
2 3 1   0 1 1 2 1 2    1/6
2 3 1   0 1 0 2 2 3    -1/6
2 3 1   0 1 1 2 2 3    -1/6
2 3 1   0 1 0 4 1 2    -1/6
2 3 1   0 1 0 2 1 3    -1/6
2 3 1   0 3 1 2 1 2    1/6
2 3 1   0 3 1 2 0 3    -1/6
2 3 1   0 3 1 2 2 3    -1/6
h^4:
2 4 1   0 1 0 1 0 1 0 1    1/24
2 4 1   0 1 0 1 0 1 0 2    1/6
2 4 1   0 1 0 1 0 1 1 2    -1/6
2 4 1   0 1 0 1 0 5 1 4    -1/12
2 4 1   0 1 0 1 0 2 0 2    1/6
2 4 1   0 1 0 1 1 2 1 2    1/6
2 4 1   0 1 0 1 0 2 2 4    -1/6
2 4 1   0 1 0 1 1 2 2 4    -1/6
2 4 1   0 1 0 1 0 5 1 2    -1/6
2 4 1   0 1 0 1 0 2 1 4    -1/6
2 4 1   0 1 0 4 1 3 1 3    1/6
2 4 1   0 1 0 4 1 3 0 4    -1/6
2 4 1   0 1 0 4 1 3 3 4    -1/6
2 4 1   0 1 0 1 0 2 0 3    1/18
2 4 1   0 1 0 1 0 2 1 3    -1/9
2 4 1   0 1 0 2 0 5 1 4    -1/18
2 4 1   0 1 0 1 1 2 1 3    1/18
2 4 1   0 1 0 4 1 3 1 2    1/18
2 4 1   0 3 1 2 0 5 1 4    1/72
2 4 1   0 1 0 1 0 2 2 3    16*w_4_1
2 4 1   0 1 0 1 1 2 2 3    16*w_4_1
2 4 1   0 1 0 1 0 2 3 4    16*w_4_2
2 4 1   0 1 0 1 1 2 3 4    16*w_4_2
2 4 1   0 1 0 1 2 3 2 3    4*w_4_3
2 4 1   0 1 0 1 2 3 2 4    16*w_4_4
2 4 1   0 1 0 1 2 5 3 4    8*w_4_5
2 4 1   0 1 0 2 0 2 0 2    8/3*w_4_6
2 4 1   0 1 1 2 1 2 1 2    -8/3*w_4_6
2 4 1   0 1 0 2 0 2 0 3    16*w_4_7
2 4 1   0 1 1 2 1 2 1 3    -16*w_4_7
2 4 1   0 1 0 2 0 2 1 2    8*w_4_8
2 4 1   0 1 0 2 1 2 1 2    -8*w_4_8
2 4 1   0 1 0 2 0 2 1 3    16*w_4_9
2 4 1   0 1 0 4 1 2 1 2    -16*w_4_9
2 4 1   0 1 0 2 0 2 2 3    16*w_4_10
2 4 1   0 1 1 2 1 2 2 3    -16*w_4_10
2 4 1   0 1 0 2 0 3 0 3    8*w_4_11
2 4 1   0 1 1 2 1 3 1 3    -8*w_4_11
2 4 1   0 1 0 2 0 3 0 4    16*w_4_12
2 4 1   0 1 1 2 1 3 1 4    -16*w_4_12
2 4 1   0 1 0 2 0 3 1 2    16*w_4_13
2 4 1   0 1 0 2 1 2 1 4    -16*w_4_13
2 4 1   0 1 0 2 0 3 1 3    16*w_4_14
2 4 1   0 1 0 4 1 2 1 4    -16*w_4_14
2 4 1   0 1 0 2 0 3 1 4    16*w_4_15
2 4 1   0 1 0 4 1 5 1 2    -16*w_4_15
2 4 1   0 1 0 2 0 3 2 3    16*w_4_16
2 4 1   0 1 1 2 1 3 2 3    -16*w_4_16
2 4 1   0 1 0 2 0 3 2 4    16*w_4_17
2 4 1   0 1 1 2 1 3 2 4    -16*w_4_17
2 4 1   0 1 0 2 0 3 3 4    16*w_4_18
2 4 1   0 1 1 2 1 3 3 4    -16*w_4_18
2 4 1   0 1 0 2 0 5 1 2    16*w_4_19
2 4 1   0 1 0 2 1 2 1 3    -16*w_4_19
2 4 1   0 1 0 2 0 5 1 3    16*w_4_20
2 4 1   0 1 0 4 1 2 1 3    -16*w_4_20
2 4 1   0 1 0 2 0 5 2 3    16*w_4_21
2 4 1   0 1 1 2 1 5 2 3    -16*w_4_21
2 4 1   0 1 0 2 0 5 2 4    16*w_4_22
2 4 1   0 1 1 2 1 5 2 4    -16*w_4_22
2 4 1   0 1 0 2 0 5 3 4    16*w_4_23
2 4 1   0 1 1 2 1 5 3 4    -16*w_4_23
2 4 1   0 1 0 2 1 2 2 3    16*w_4_24
2 4 1   0 1 0 2 1 2 2 4    -16*w_4_24
2 4 1   0 1 0 2 1 2 3 4    16*w_4_25
2 4 1   0 1 0 2 1 3 1 3    8*w_4_26
2 4 1   0 1 0 4 1 2 0 4    -8*w_4_26
2 4 1   0 1 0 2 1 3 1 4    16*w_4_27
2 4 1   0 1 0 4 0 5 1 2    -16*w_4_27
2 4 1   0 1 0 2 1 3 2 3    16*w_4_28
2 4 1   0 1 0 4 1 2 2 4    -16*w_4_28
2 4 1   0 1 0 2 1 3 2 4    16*w_4_29
2 4 1   0 1 0 4 1 2 2 3    -16*w_4_29
2 4 1   0 1 0 2 1 3 3 4    16*w_4_30
2 4 1   0 1 0 4 1 2 3 4    16*w_4_30
2 4 1   0 1 0 2 1 5 2 3    16*w_4_31
2 4 1   0 1 0 4 2 5 1 2    -16*w_4_31
2 4 1   0 1 0 2 1 5 2 4    16*w_4_32
2 4 1   0 1 0 4 2 3 1 2    -16*w_4_32
2 4 1   0 1 0 2 1 5 3 4    16*w_4_33
2 4 1   0 1 0 4 3 5 1 2    16*w_4_33
2 4 1   0 1 0 2 2 3 2 3    8*w_4_34
2 4 1   0 1 1 2 2 3 2 3    -8*w_4_34
2 4 1   0 1 0 2 2 3 2 4    16*w_4_35
2 4 1   0 1 1 2 2 3 2 4    -16*w_4_35
2 4 1   0 1 0 2 2 3 3 4    16*w_4_36
2 4 1   0 1 1 2 2 3 3 4    -16*w_4_36
2 4 1   0 1 0 2 2 5 2 4    8*w_4_37
2 4 1   0 1 1 2 2 5 2 4    -8*w_4_37
2 4 1   0 1 0 2 2 5 3 4    16*w_4_38
2 4 1   0 1 1 2 2 5 3 4    -16*w_4_38
2 4 1   0 1 0 2 3 5 3 4    8*w_4_39
2 4 1   0 1 1 2 3 5 3 4    -8*w_4_39
2 4 1   0 1 0 4 0 3 2 3    16*w_4_40
2 4 1   0 1 1 4 1 3 2 3    -16*w_4_40
2 4 1   0 1 0 4 0 5 2 3    16*w_4_41
2 4 1   0 1 1 4 1 5 2 3    -16*w_4_41
2 4 1   0 1 0 4 0 5 2 4    16*w_4_42
2 4 1   0 1 1 4 1 5 2 4    -16*w_4_42
2 4 1   0 1 0 4 1 3 2 3    16*w_4_43
2 4 1   0 1 0 4 1 3 2 4    -16*w_4_43
2 4 1   0 1 0 4 1 5 2 3    16*w_4_44
2 4 1   0 1 0 4 2 5 1 3    -16*w_4_44
2 4 1   0 1 0 4 1 5 2 4    16*w_4_45
2 4 1   0 1 0 4 2 3 1 3    -16*w_4_45
2 4 1   0 1 0 4 2 3 0 4    16*w_4_46
2 4 1   0 1 1 4 2 3 1 4    -16*w_4_46
2 4 1   0 1 0 4 2 3 1 4    16*w_4_47
2 4 1   0 1 0 4 2 5 1 4    -16*w_4_47
2 4 1   0 1 0 4 2 3 2 3    16*w_4_48
2 4 1   0 1 1 4 2 3 2 3    -16*w_4_48
2 4 1   0 1 0 4 2 3 2 4    16*w_4_49
2 4 1   0 1 1 4 2 3 2 4    -16*w_4_49
2 4 1   0 1 0 4 2 3 3 4    16*w_4_50
2 4 1   0 1 1 4 2 3 3 4    -16*w_4_50
2 4 1   0 1 0 4 2 5 2 3    16*w_4_51
2 4 1   0 1 1 4 2 5 2 3    -16*w_4_51
2 4 1   0 1 0 4 2 5 2 4    16*w_4_52
2 4 1   0 1 1 4 2 5 2 4    -16*w_4_52
2 4 1   0 1 0 4 2 5 3 4    16*w_4_53
2 4 1   0 1 1 4 2 5 3 4    -16*w_4_53
2 4 1   0 1 0 4 3 5 2 3    16*w_4_54
2 4 1   0 1 1 4 3 5 2 3    -16*w_4_54
2 4 1   0 1 0 4 3 5 2 4    16*w_4_55
2 4 1   0 1 1 4 3 5 2 4    -16*w_4_55
2 4 1   0 1 2 4 2 3 2 3    16*w_4_56
2 4 1   0 1 2 4 2 5 2 3    16/3*w_4_57
2 4 1   0 1 2 4 2 5 3 4    16*w_4_58
2 4 1   0 1 2 4 3 5 3 4    16*w_4_59
2 4 1   0 3 0 2 0 2 1 2    16*w_4_60
2 4 1   0 3 1 4 1 3 1 3    16*w_4_60
2 4 1   0 3 0 2 0 2 1 3    16*w_4_61
2 4 1   0 3 1 4 1 3 1 4    16*w_4_61
2 4 1   0 3 0 2 0 2 1 4    16*w_4_62
2 4 1   0 3 1 4 1 5 1 4    16*w_4_62
2 4 1   0 3 0 2 0 5 1 2    16*w_4_63
2 4 1   0 3 1 4 1 3 1 2    16*w_4_63
2 4 1   0 3 0 2 1 2 1 2    8*w_4_64
2 4 1   0 3 1 4 1 3 0 3    8*w_4_64
```

TABLE 6 (continued).

2	4	1	0	3	0	2	1	2	1	3	8*w_4_65
2	4	1	0	3	1	4	1	3	0	4	8*w_4_65
2	4	1	0	3	0	2	1	2	1	4	16*w_4_66
2	4	1	0	3	0	4	1	5	1	4	16*w_4_66
2	4	1	0	3	0	2	1	2	2	3	16*w_4_67
2	4	1	0	3	1	4	1	3	3	4	16*w_4_67
2	4	1	0	3	0	2	1	2	2	4	16*w_4_68
2	4	1	0	3	1	4	1	3	2	3	-16*w_4_68
2	4	1	0	3	0	2	1	2	3	4	16*w_4_69
2	4	1	0	3	1	4	1	3	2	4	-16*w_4_69
2	4	1	0	3	0	2	1	5	2	4	16*w_4_70
2	4	1	0	3	2	4	1	5	1	4	-16*w_4_70
2	4	1	0	3	0	4	0	2	1	2	16*w_4_71
2	4	1	0	3	1	4	1	5	1	3	16*w_4_71
2	4	1	0	3	0	4	0	5	1	2	16*w_4_72
2	4	1	0	3	1	4	1	5	1	2	16*w_4_72
2	4	1	0	3	0	4	0	5	1	3	16*w_4_73
2	4	1	0	3	1	4	1	2	1	3	16*w_4_73
2	4	1	0	3	0	4	0	5	1	4	16*w_4_74
2	4	1	0	3	1	2	1	3	1	4	16*w_4_74
2	4	1	0	3	0	4	1	2	0	3	16*w_4_75
2	4	1	0	3	1	4	1	2	1	4	16*w_4_75
2	4	1	0	3	0	4	1	2	0	4	16*w_4_76
2	4	1	0	3	1	4	1	2	1	2	16*w_4_76
2	4	1	0	3	0	4	1	2	1	2	16*w_4_77
2	4	1	0	3	1	4	1	2	0	3	16*w_4_77
2	4	1	0	3	0	4	1	2	1	3	16*w_4_78
2	4	1	0	3	1	4	0	5	1	3	16*w_4_78
2	4	1	0	3	0	4	1	2	1	4	16*w_4_79
2	4	1	0	3	0	4	1	5	1	3	16*w_4_79
2	4	1	0	3	0	4	1	2	2	3	16*w_4_80
2	4	1	0	3	1	4	1	2	3	4	16*w_4_80
2	4	1	0	3	0	4	1	2	2	4	16*w_4_81
2	4	1	0	3	1	4	1	2	2	3	-16*w_4_81
2	4	1	0	3	0	4	1	2	3	4	16*w_4_82
2	4	1	0	3	1	4	1	2	2	4	-16*w_4_82
2	4	1	0	3	0	4	1	3	0	3	8*w_4_83
2	4	1	0	3	1	2	1	3	1	3	8*w_4_83
2	4	1	0	3	0	4	1	3	0	4	16*w_4_84
2	4	1	0	3	1	2	1	2	1	3	16*w_4_84
2	4	1	0	3	0	4	1	3	1	2	16*w_4_85
2	4	1	0	3	1	2	0	5	1	3	16*w_4_85
2	4	1	0	3	0	4	1	3	1	3	16*w_4_86
2	4	1	0	3	1	2	0	3	1	3	16*w_4_86
2	4	1	0	3	0	4	1	3	1	4	16*w_4_87
2	4	1	0	3	0	4	1	3	2	3	16*w_4_88
2	4	1	0	3	1	2	1	3	3	4	-16*w_4_88
2	4	1	0	3	0	4	1	3	2	4	16*w_4_89
2	4	1	0	3	1	2	1	3	2	4	-16*w_4_89
2	4	1	0	3	0	4	1	3	3	4	16*w_4_90
2	4	1	0	3	1	2	1	3	2	3	-16*w_4_90
2	4	1	0	3	0	4	1	5	0	4	16*w_4_91
2	4	1	0	3	1	2	1	2	1	4	16*w_4_91
2	4	1	0	3	0	4	1	5	1	2	16*w_4_92
2	4	1	0	3	0	4	1	5	2	3	16*w_4_93
2	4	1	0	3	4	5	1	2	1	4	-16*w_4_93
2	4	1	0	3	0	4	1	5	2	4	16*w_4_94
2	4	1	0	3	2	4	1	5	1	2	-16*w_4_94
2	4	1	0	3	0	4	1	5	3	4	16*w_4_95
2	4	1	0	3	2	4	1	2	1	4	-16*w_4_95
2	4	1	0	3	0	4	2	3	1	2	16*w_4_96
2	4	1	0	3	1	4	1	5	3	4	16*w_4_96
2	4	1	0	3	0	4	2	3	1	3	16*w_4_97
2	4	1	0	3	1	4	3	5	1	3	-16*w_4_97
2	4	1	0	3	0	4	2	3	1	4	16*w_4_98
2	4	1	0	3	4	5	1	3	1	4	-16*w_4_98
2	4	1	0	3	0	4	2	5	1	2	16*w_4_99
2	4	1	0	3	1	4	1	5	2	3	-16*w_4_99
2	4	1	0	3	0	4	2	5	1	3	16*w_4_100
2	4	1	0	3	1	4	2	5	1	3	-16*w_4_100
2	4	1	0	3	0	4	2	5	1	4	16*w_4_101
2	4	1	0	3	2	4	1	5	1	3	-16*w_4_101
2	4	1	0	3	0	4	3	5	1	2	16*w_4_102
2	4	1	0	3	1	4	1	5	2	4	-16*w_4_102
2	4	1	0	3	0	4	3	5	1	3	16*w_4_103
2	4	1	0	3	1	4	2	3	1	3	-16*w_4_103
2	4	1	0	3	0	4	3	5	1	4	16*w_4_104
2	4	1	0	3	2	4	1	3	1	4	-16*w_4_104
2	4	1	0	3	1	2	0	3	0	3	8*w_4_105
2	4	1	0	3	1	2	1	2	1	2	8*w_4_105
2	4	1	0	3	1	2	0	3	1	2	16*w_4_106
2	4	1	0	3	1	2	0	3	1	4	16*w_4_107
2	4	1	0	3	1	2	0	5	1	2	16*w_4_107
2	4	1	0	3	1	2	0	3	2	3	16*w_4_108
2	4	1	0	3	1	2	1	2	2	3	-16*w_4_108
2	4	1	0	3	1	2	0	3	2	4	16*w_4_109
2	4	1	0	3	1	2	1	2	3	4	16*w_4_109
2	4	1	0	3	1	2	0	3	3	4	16*w_4_110
2	4	1	0	3	1	2	1	2	2	4	16*w_4_110
2	4	1	0	3	1	2	0	5	2	3	16*w_4_111
2	4	1	0	3	1	2	1	5	2	3	-16*w_4_111
2	4	1	0	3	1	2	0	5	2	4	16*w_4_112
2	4	1	0	3	1	2	1	5	3	4	16*w_4_112
2	4	1	0	3	1	2	0	5	3	4	16*w_4_113
2	4	1	0	3	1	2	1	5	2	4	16*w_4_113
2	4	1	0	3	1	2	2	3	2	3	8*w_4_114
2	4	1	0	3	1	2	2	3	2	4	16*w_4_115
2	4	1	0	3	1	2	2	3	3	4	-16*w_4_115
2	4	1	0	3	1	2	2	5	2	4	8*w_4_116
2	4	1	0	3	1	2	3	5	3	4	8*w_4_116
2	4	1	0	3	1	2	2	5	3	4	16*w_4_117
2	4	1	0	3	1	4	0	5	1	2	8*w_4_118
2	4	1	0	3	1	4	0	5	2	3	16*w_4_119
2	4	1	0	3	1	4	2	5	1	2	-16*w_4_119
2	4	1	0	3	1	4	0	5	2	4	16*w_4_120
2	4	1	0	3	1	4	3	5	1	2	-16*w_4_120
2	4	1	0	3	1	4	0	5	3	4	16*w_4_121
2	4	1	0	3	1	4	2	3	1	2	16*w_4_121
2	4	1	0	3	1	4	2	3	0	3	16*w_4_122
2	4	1	0	3	2	4	1	2	1	2	-16*w_4_122
2	4	1	0	3	1	4	2	3	0	4	16*w_4_123
2	4	1	0	3	2	4	1	2	1	3	-16*w_4_123
2	4	1	0	3	1	4	2	3	1	4	16*w_4_124
2	4	1	0	3	2	4	1	2	0	3	-16*w_4_124
2	4	1	0	3	1	4	2	3	2	3	16*w_4_125
2	4	1	0	3	2	4	1	2	2	4	16*w_4_125
2	4	1	0	3	1	4	2	3	2	4	16*w_4_126
2	4	1	0	3	2	4	1	2	3	4	16*w_4_126
2	4	1	0	3	1	4	2	3	3	4	16*w_4_127
2	4	1	0	3	2	4	1	2	2	3	-16*w_4_127
2	4	1	0	3	1	4	2	5	0	4	16*w_4_128
2	4	1	0	3	4	5	1	2	1	3	16*w_4_128
2	4	1	0	3	1	4	2	5	1	4	16*w_4_129
2	4	1	0	3	2	4	1	5	0	3	-16*w_4_129
2	4	1	0	3	1	4	2	5	2	3	16*w_4_130
2	4	1	0	3	4	5	1	2	2	4	-16*w_4_130
2	4	1	0	3	1	4	2	5	2	4	16*w_4_131
2	4	1	0	3	4	5	1	2	3	4	-16*w_4_131
2	4	1	0	3	1	4	2	5	3	4	16*w_4_132
2	4	1	0	3	4	5	1	2	2	3	16*w_4_132
2	4	1	0	3	1	4	3	5	0	4	16*w_4_133
2	4	1	0	3	2	4	1	3	1	2	16*w_4_133
2	4	1	0	3	1	4	3	5	1	4	16*w_4_134
2	4	1	0	3	2	4	0	3	1	2	16*w_4_134
2	4	1	0	3	1	4	3	5	2	3	16*w_4_135
2	4	1	0	3	2	4	2	5	1	2	-16*w_4_135
2	4	1	0	3	1	4	3	5	2	4	16*w_4_136
2	4	1	0	3	2	4	3	5	1	2	-16*w_4_136
2	4	1	0	3	1	4	3	5	3	4	16*w_4_137
2	4	1	0	3	2	4	2	3	1	2	16*w_4_137
2	4	1	0	3	2	4	1	3	0	3	16*w_4_138
2	4	1	0	3	2	4	1	3	1	3	-16*w_4_138
2	4	1	0	3	2	4	1	3	2	3	16*w_4_139
2	4	1	0	3	2	4	1	3	3	4	16*w_4_139
2	4	1	0	3	2	4	1	3	2	4	16*w_4_140
2	4	1	0	3	2	4	1	5	2	3	16*w_4_141
2	4	1	0	3	4	5	1	5	2	4	-16*w_4_141
2	4	1	0	3	2	4	1	5	2	4	16*w_4_142
2	4	1	0	3	2	4	1	5	3	4	16*w_4_143
2	4	1	0	3	2	4	2	5	1	4	16*w_4_143
2	4	1	0	3	2	4	2	3	1	3	16*w_4_144
2	4	1	0	3	4	5	1	3	3	4	-16*w_4_144
2	4	1	0	3	2	4	2	3	1	4	16*w_4_145
2	4	1	0	3	4	5	1	5	3	4	-16*w_4_145
2	4	1	0	3	2	4	2	5	1	3	16*w_4_146
2	4	1	0	3	4	5	1	3	2	4	-16*w_4_146
2	4	1	0	3	2	4	3	5	1	3	16*w_4_147
2	4	1	0	3	4	5	1	3	2	3	-16*w_4_147
2	4	1	0	3	2	4	3	5	1	4	16*w_4_148
2	4	1	0	3	4	5	1	5	2	3	16*w_4_149

TABLE 7. Relations between weights of $\hbar^4$-basic graphs: 149 via 10.

```
w_4_1==-1/144
w_4_2==-1/288
w_4_3==17/360 + 6*w_4_108
w_4_4==49/2880 - 3*w_4_104 - w_4_107 + (3*w_4_108)/2
w_4_5==-1/96 + 6*w_4_104 + 2*w_4_107
w_4_6==1/80
w_4_7==1/360
w_4_8==-1/240
w_4_9==-13/1440
w_4_10==-7/1440
w_4_11==1/240
w_4_12==-1/720
w_4_13==1/720
w_4_14==1/480
w_4_15==-1/1440
w_4_16==1/1440
w_4_17==-1/480
w_4_18==-1/360
w_4_19==-1/480
w_4_20==-1/240
w_4_21==-1/480
w_4_22==-1/720
w_4_23==1/1440
w_4_24==1/360
w_4_25==53/1440 + 3*w_4_100 + 12*w_4_103 - 15*w_4_104 - w_4_107 + 6*w_4_108 - 6*w_4_109
w_4_26==1/120
w_4_27==1/1440
w_4_28==-1/960 - (3*w_4_108)/2
w_4_29==-49/1440 - (3*w_4_100)/2 - 9*w_4_103 + (21*w_4_104)/2 + (3*w_4_107)/2 - (9*w_4_108)/2 + 3*w_4_109
w_4_30==1/72 + 6*w_4_103 - 6*w_4_104 + 3*w_4_108 - 3*w_4_109
w_4_31==61/2880 + (3*w_4_100)/2 + 6*w_4_103 - (15*w_4_104)/2 - w_4_107/2 + 3*w_4_108 - 3*w_4_109
w_4_32==1/1440
w_4_33==5/288 + 6*w_4_103 - 6*w_4_104 + 3*w_4_108 - 3*w_4_109
w_4_34==1/96 + w_4_108
w_4_35==-w_4_103
w_4_36==-13/2880 - w_4_100/2 + (3*w_4_104)/2 + w_4_107/2
w_4_37==0
w_4_38==1/1440 - w_4_100/2 + w_4_103 + (3*w_4_104)/2 + w_4_107/2 + w_4_108/2
w_4_39==0
w_4_40==0
w_4_41==1/1440
w_4_42==1/1440
w_4_43==37/1440 + 6*w_4_103 - 6*w_4_104 - w_4_107 + 3*w_4_108 - 3*w_4_109
w_4_44==17/360 + 15*w_4_103 - 18*w_4_104 - 2*w_4_107 + 6*w_4_108 - 6*w_4_109
w_4_45==7/1440 - 3*w_4_104 - w_4_107
w_4_46==-1/480
w_4_47==1/60 + 6*w_4_103 - 6*w_4_104 + 3*w_4_108 - 3*w_4_109
w_4_48==11/1440 - w_4_100/2 - w_4_103 + (5*w_4_104)/2 + w_4_107/2 + (3*w_4_108)/2
w_4_49==-w_4_104
w_4_50==-1/192 - w_4_108/2
w_4_51==-w_4_103
w_4_52==-1/1440 + w_4_100/2 - (3*w_4_104)/2 - w_4_107/2 - w_4_108/2
w_4_53==w_4_103
w_4_54==-1/576 + w_4_103 - w_4_104 - w_4_108/2
w_4_55==w_4_104
w_4_56==0
w_4_57==0
w_4_58==0
w_4_59==0
w_4_60==0
w_4_61==0
w_4_62==0
w_4_63==0
w_4_64==0
w_4_65==0
w_4_66==0
w_4_67==0
w_4_68==0
w_4_69==0
w_4_70==0
w_4_71==0
w_4_72==1/1440
w_4_73==1/1440
w_4_74==1/1440
w_4_75==-1/480
w_4_76==-1/720
w_4_77==1/180 + 3*w_4_103 - 3*w_4_104 - w_4_107
w_4_78==-1/144 - 3*w_4_103 + 3*w_4_104 + w_4_107
w_4_79==-1/1440
w_4_80==1/80 + w_4_100 - 3*w_4_104 + 3*w_4_108 - 2*w_4_109 - 2*w_4_125
w_4_81==1/480 - w_4_100/2 + 2*w_4_103 + w_4_104/2 + w_4_107/2 + w_4_108/2 + 2*w_4_125
```

TABLE 7 (continued).

```
w_4_82==1/2880 - w_4_103 - w_4_104 + w_4_108/2 - w_4_109 - 2*w_4_125
w_4_83==-1/480
w_4_84==-1/720
w_4_85==1/180 - w_4_107
w_4_86==1/480
w_4_87==-1/1440
w_4_88==1/96 + 4*w_4_103 - 4*w_4_104 + 2*w_4_108 - 2*w_4_109
w_4_89==1/240 + (3*w_4_100)/2 + 3*w_4_103 - (9*w_4_104)/2 + w_4_107/2 + (3*w_4_108)/2 - 2*w_4_109
w_4_90==-1/192 - w_4_108/2
w_4_91==-1/720
w_4_92==17/1440 + 6*w_4_103 - 6*w_4_104 - 2*w_4_107
w_4_93==3/320 - w_4_100/2 + w_4_103 - (5*w_4_104)/2 + w_4_107/2 + 2*w_4_108 - 2*w_4_109 + w_4_119
w_4_94==1/1440 - w_4_100/2 - w_4_102 + w_4_103 - (3*w_4_104)/2 + w_4_107/2 + w_4_108/2 - w_4_109
w_4_95==-1/576 + w_4_103 - w_4_104 - w_4_108/2
w_4_96==0
w_4_97==0
w_4_98==0
w_4_99==-7/2880 - w_4_100/2 + w_4_103 + w_4_104/2 - w_4_107/2 - w_4_108 + w_4_109 - w_4_119
w_4_105==-1/160
w_4_106==13/1440
w_4_110==-1/288 - 2*w_4_103 + 2*w_4_104 - w_4_108 + w_4_109
w_4_111==-17/2880 - w_4_100/2 - 2*w_4_103 + (5*w_4_104)/2 - w_4_107/2 - w_4_108 + w_4_109
w_4_112==-7/576 - 4*w_4_103 + 4*w_4_104 - 2*w_4_108 + 2*w_4_109
w_4_113==-1/192 - 2*w_4_103 + 2*w_4_104 - w_4_108 + w_4_109
w_4_114==1/360 + w_4_108
w_4_115==23/5760 - w_4_100/2 + w_4_103 + (3*w_4_108)/4
w_4_116==0
w_4_117==-19/2880 + w_4_100 - 2*w_4_103 - w_4_108
w_4_118==-31/1440 - 12*w_4_103 + 12*w_4_104 + 4*w_4_107
w_4_120==-1/96 - w_4_100 - w_4_102 + 2*w_4_103 - 2*w_4_108 + w_4_109
w_4_121==-1/288 + 2*w_4_103 - 2*w_4_104 - w_4_108
w_4_122==-2*w_4_103
w_4_123==-7/2880 + w_4_100/2 - w_4_103 + w_4_104/2 - w_4_107/2 - w_4_108 + w_4_109
w_4_124==1/144 + w_4_100 + w_4_103 - 2*w_4_104 + w_4_108 - w_4_109
w_4_126==29/5760 + w_4_100/2 - w_4_103 + (5*w_4_108)/4 - w_4_109 - w_4_125
w_4_127==-1/640 + w_4_103 + w_4_104/2 - (3*w_4_108)/4 + w_4_109/2 + w_4_125
w_4_128==-1/144 + w_4_101 - 2*w_4_103 + 3*w_4_104 - w_4_108 + w_4_109
w_4_129==1/144 + w_4_101 + 2*w_4_103 - 3*w_4_104 + w_4_108 - w_4_109
w_4_130==7/1920 - w_4_100/2 + w_4_103 + w_4_107/2 + (3*w_4_108)/4 - w_4_109/2 + w_4_119 + w_4_125
w_4_131==23/5760 - w_4_100/4 + w_4_101/2 - w_4_102/2 + w_4_103/2 - (5*w_4_104)/4 + w_4_107/4 + w_4_108 - w_4_109 +
         w_4_119/2 - w_4_125
w_4_132==-1/240 + w_4_101/2 + w_4_103/2 - w_4_108 + w_4_109/2 + w_4_125
w_4_133==2*w_4_104
w_4_134==0
w_4_135==-1/360 - w_4_100/4 - w_4_102/2 + w_4_104/4 + w_4_107/4 - w_4_108/4 + w_4_119/2
w_4_136==-7/1440 - w_4_100/2 - w_4_102 + w_4_103 - w_4_104/2 - w_4_108 + w_4_109/2
w_4_137==0
w_4_138==-1/144 - 2*w_4_103 + 2*w_4_104 - w_4_108 + w_4_109
w_4_139==1/1920 + w_4_103 - (3*w_4_104)/2 + w_4_108/4 - w_4_109/2
w_4_140==-1/1440 + w_4_100 + 2*w_4_103 - 5*w_4_104 - w_4_109
w_4_141==-w_4_100/4 - w_4_101/2 - w_4_102/2 - w_4_103/2 + w_4_104/4 + w_4_107/4 + w_4_108/4 - w_4_109/2 + w_4_119/2
w_4_142==1/5760 - w_4_102 + 2*w_4_103 - 3*w_4_104 - w_4_109
w_4_143==7/1440 + w_4_101/2 + (5*w_4_103)/2 - (5*w_4_104)/2 + (3*w_4_108)/4 - w_4_109
w_4_144==0
w_4_145==0
w_4_146==13/5760 + w_4_100/2 - w_4_101/2 + (3*w_4_103)/2 - 2*w_4_104 + w_4_108/4 - w_4_109/2
w_4_147==1/320 - w_4_101/2 + (3*w_4_103)/2 - w_4_104 + w_4_108/2 - w_4_109/2
w_4_148==11/1920 + 2*w_4_103 - w_4_104 + w_4_108 - w_4_109
w_4_149==-11/2880 + w_4_100/2 + w_4_104/2 - w_4_107/2 - w_4_108 + w_4_109 - w_4_119
```

Appendix D. Encoding of the associator of the $\star$-product modulo $\bar{o}(\hbar^4)$

Encodings of graphs (see Implementation 1 on p. 5) are followed by their coefficients, in the following table containing the expansion of the associator $(f\star g)\star h - f\star(g\star h)$.

Table 8. The associator of $\star$ up to order 4 in terms of 149 parameters.

```
h^0:
h^1:
h^2:
# 1 1 1
3 2 1   0 1 2 3    -2/3
3 2 1   0 2 1 3    2/3
3 2 1   0 4 1 2    -2/3
h^3:
# 1 2 2
3 3 1   0 1 1 2 2 3    -2/3
3 3 1   0 2 1 2 1 3    2/3
3 3 1   0 4 1 2 1 2    -2/3
# 2 1 2
3 3 1   0 1 0 2 2 3    -2/3
3 3 1   0 2 0 2 1 3    2/3
3 3 1   0 2 0 5 1 2    -2/3
# 2 2 1
3 3 1   0 1 0 1 2 3    -2/3
3 3 1   0 1 0 2 1 4    2/3
3 3 1   0 1 0 5 1 2    -2/3
# 1 2 1
3 3 1   0 1 1 3 2 4    1/3
3 3 1   0 1 1 5 2 3    -1/3
3 3 1   0 4 1 5 1 2    -1/3
3 3 1   0 4 1 2 1 3    1/3
# 1 1 1
3 3 1   0 4 1 3 2 4    1/6
3 3 1   0 1 2 3 3 4    -1/6
3 3 1   0 1 2 5 3 4    -1/6
3 3 1   0 4 1 2 3 4    -1/6
3 3 1   0 4 2 3 1 4    -1/6
3 3 1   0 4 1 5 2 4    1/6
3 3 1   0 4 3 5 1 2    -1/6
3 3 1   0 4 2 3 1 3    -1/6
# 1 1 2
3 3 1   0 1 2 3 2 3    1/3
3 3 1   0 1 2 3 2 4    1/3
3 3 1   0 2 1 5 2 3    -1/6
3 3 1   0 4 2 5 1 2    1/6
3 3 1   0 2 1 3 2 4    -1/6
3 3 1   0 4 1 2 2 3    1/6
3 3 1   0 4 1 2 2 4    1/3
3 3 1   0 2 1 2 3 4    -1/3
3 3 1   0 2 1 3 2 3    -1/3
# 2 1 1
3 3 1   0 1 0 3 2 3    -1/3
3 3 1   0 1 0 3 2 4    -1/6
3 3 1   0 1 0 2 3 4    1/3
3 3 1   0 1 0 5 2 3    -1/6
3 3 1   0 2 0 3 1 3    1/3
3 3 1   0 4 1 2 0 4    -1/3
3 3 1   0 4 0 5 1 2    -1/3
3 3 1   0 2 0 5 1 3    1/6
3 3 1   0 2 0 3 1 4    1/6
h^4:
# 3 3 1
3 4 1   0 1 0 1 0 1 2 3    -1/3
3 4 1   0 1 0 1 0 2 1 5    1/3
3 4 1   0 1 0 1 0 6 1 2    -1/3
# 3 2 2
3 4 1   0 1 0 1 0 2 2 3    -2/3
3 4 1   0 1 0 2 0 2 1 4    2/3
3 4 1   0 1 0 2 0 6 1 2    -2/3
# 3 1 3
3 4 1   0 1 0 2 0 2 2 3    -1/3
3 4 1   0 2 0 2 0 2 1 3    1/3
3 4 1   0 2 0 2 0 6 1 2    -1/3
# 2 3 2
3 4 1   0 1 0 1 1 2 2 3    -2/3
3 4 1   0 1 0 2 1 2 1 4    2/3
3 4 1   0 1 0 5 1 2 1 2    -2/3
# 1 3 3
3 4 1   0 1 1 2 1 2 2 3    -1/3
3 4 1   0 2 1 2 1 2 1 3    1/3
3 4 1   0 4 1 2 1 2 1 2    -1/3
# 2 2 3
3 4 1   0 1 0 2 1 2 2 3    -2/3
3 4 1   0 2 0 2 1 2 1 3    2/3
3 4 1   0 2 0 5 1 2 1 2    -2/3
# 1 3 2
3 4 1   0 1 1 2 1 3 2 5    1/3
3 4 1   0 1 1 2 1 4 2 3    -2/9
3 4 1   0 1 1 2 1 6 2 3    -1/3
3 4 1   0 2 1 2 1 3 1 4    2/9
3 4 1   0 4 1 2 1 2 1 5    -2/9
3 4 1   0 4 1 5 1 2 1 2    -1/3
3 4 1   0 4 1 2 1 2 1 3    1/3
# 2 3 1
3 4 1   0 1 0 1 1 3 2 4    2/9
3 4 1   0 1 0 1 1 3 2 5    1/3
3 4 1   0 1 0 1 1 6 2 3    -1/3
3 4 1   0 1 0 2 1 3 1 4    -2/9
3 4 1   0 1 0 5 1 2 1 3    2/9
3 4 1   0 1 0 5 1 6 1 2    -1/3
3 4 1   0 1 0 5 1 2 1 4    1/3
# 2 1 3
3 4 1   0 1 0 2 2 3 2 3    1/3
3 4 1   0 1 0 2 2 3 2 4    2/9
3 4 1   0 1 0 2 2 3 2 5    1/3
3 4 1   0 2 0 2 1 6 2 3    -1/6
3 4 1   0 2 0 5 2 6 1 2    1/6
3 4 1   0 2 0 2 1 3 2 5    -1/6
3 4 1   0 2 0 5 1 2 2 4    1/6
3 4 1   0 2 0 2 1 3 2 4    -2/9
3 4 1   0 2 0 5 1 2 2 3    2/9
3 4 1   0 2 0 5 1 2 2 5    1/3
3 4 1   0 2 0 2 1 2 3 5    -1/3
3 4 1   0 2 0 2 1 3 2 3    -1/3
# 3 2 1
3 4 1   0 1 0 1 0 3 2 3    -1/3
3 4 1   0 1 0 1 0 3 2 4    -2/9
3 4 1   0 1 0 1 0 3 2 5    -1/6
3 4 1   0 1 0 1 0 2 3 5    1/3
3 4 1   0 1 0 1 0 6 2 3    -1/6
3 4 1   0 1 0 2 0 3 1 4    2/9
3 4 1   0 1 0 3 0 6 1 2    -2/9
3 4 1   0 1 0 2 0 4 1 4    1/3
3 4 1   0 1 0 5 1 2 0 5    -1/3
3 4 1   0 1 0 5 0 6 1 2    -1/3
3 4 1   0 1 0 2 0 6 1 4    1/6
3 4 1   0 1 0 2 0 4 1 5    1/6
# 3 1 2
3 4 1   0 1 0 2 0 3 2 3    -1/3
3 4 1   0 1 0 2 0 3 2 5    -1/6
3 4 1   0 1 0 2 0 2 3 4    1/3
3 4 1   0 1 0 2 0 4 2 3    -2/9
3 4 1   0 1 0 2 0 6 2 3    -1/6
3 4 1   0 2 0 2 0 3 1 3    1/3
3 4 1   0 2 0 5 1 2 0 5    -1/3
3 4 1   0 2 0 2 0 3 1 4    2/9
3 4 1   0 2 0 3 0 6 1 2    -2/9
3 4 1   0 2 0 5 0 6 1 2    -1/3
3 4 1   0 2 0 2 0 6 1 3    1/6
3 4 1   0 2 0 2 0 3 1 5    1/6
# 1 2 3
3 4 1   0 1 1 2 2 3 2 3    1/3
3 4 1   0 1 1 2 2 3 2 4    2/9
3 4 1   0 1 1 2 2 3 2 5    1/3
3 4 1   0 2 1 2 1 6 2 3    -1/6
3 4 1   0 4 2 5 1 2 1 2    1/6
3 4 1   0 2 1 2 1 3 2 5    -1/6
3 4 1   0 4 1 2 1 2 2 3    1/6
3 4 1   0 2 1 2 1 3 2 4    -2/9
3 4 1   0 4 1 2 1 2 2 5    2/9
3 4 1   0 4 1 2 1 2 2 4    1/3
3 4 1   0 2 1 2 1 2 3 4    -1/3
3 4 1   0 2 1 2 1 3 2 3    -1/3
# 2 2 2
3 4 1   0 1 0 1 2 3 2 3    1/3
3 4 1   0 1 0 1 2 3 2 4    4/9
3 4 1   0 1 0 3 1 2 2 3    -1/3
3 4 1   0 1 0 3 1 2 2 4    -1/6
3 4 1   0 1 0 2 1 3 2 5    1/3
3 4 1   0 1 0 1 2 3 2 5    1/3
3 4 1   0 1 0 2 1 2 3 4    1/3
3 4 1   0 1 0 2 1 4 2 3    -4/9
3 4 1   0 1 0 2 1 6 2 3    -1/3
3 4 1   0 1 0 5 2 3 1 2    -1/6
3 4 1   0 1 0 2 1 6 2 4    -1/6
3 4 1   0 1 0 5 2 6 1 2    1/6
3 4 1   0 1 0 2 1 4 2 5    -1/6
3 4 1   0 1 0 5 1 2 2 4    1/6
3 4 1   0 2 0 3 1 2 1 3    1/3
3 4 1   0 4 1 2 0 4 1 2    -1/3
3 4 1   0 2 0 5 1 2 1 3    4/9
3 4 1   0 4 1 2 0 6 1 2    -4/9
3 4 1   0 2 0 5 1 6 1 2    -1/3
3 4 1   0 1 0 5 1 2 2 5    1/3
3 4 1   0 4 0 5 1 2 1 2    -1/3
3 4 1   0 1 0 2 1 2 4 5    -1/3
3 4 1   0 2 0 5 1 2 1 4    1/3
3 4 1   0 2 0 5 1 3 1 2    1/6
3 4 1   0 2 0 3 1 2 1 4    1/6
3 4 1   0 1 0 2 1 4 2 4    -1/3
# 1 3 1
3 4 1   0 1 1 3 1 3 2 3    -1/6+8*w_4_6
3 4 1   0 1 1 3 1 3 2 4    -1/3+16*w_4_7
3 4 1   0 1 1 2 1 3 3 4    2/9
3 4 1   0 1 1 2 1 3 4 5    1/9
3 4 1   0 1 1 3 1 6 2 3    1/9+16*w_4_7
3 4 1   0 1 1 3 1 6 2 4    1/9+16*w_4_12
3 4 1   0 1 1 2 1 4 3 4    2/9
3 4 1   0 1 1 5 2 3 1 5    -1/6+8*w_4_11
3 4 1   0 2 1 3 1 3 1 3    16/3*w_4_6
3 4 1   0 4 1 2 1 4 1 4    -1/6+8*w_4_6
3 4 1   0 2 1 3 1 3 1 4    32*w_4_7
3 4 1   0 4 1 2 1 3 1 4    1/9+16*w_4_7
3 4 1   0 4 1 2 1 4 1 5    16*w_4_7
3 4 1   0 4 1 5 1 2 1 5    -1/3+16*w_4_7
3 4 1   0 2 1 3 1 4 1 4    16*w_4_11
3 4 1   0 4 1 2 1 3 1 3    -1/6+8*w_4_11
3 4 1   0 4 1 5 1 2 1 4    16*w_4_11
3 4 1   0 2 1 3 1 4 1 5    32*w_4_12
3 4 1   0 4 1 2 1 3 1 5    16*w_4_12
3 4 1   0 4 1 5 1 2 1 3    1/9+16*w_4_12
3 4 1   0 4 1 5 1 6 1 2    16*w_4_12
3 4 1   0 1 1 2 1 4 3 5    -1/9
3 4 1   0 1 1 3 1 4 2 3    16*w_4_7
3 4 1   0 1 1 3 1 4 2 4    16*w_4_11
3 4 1   0 1 1 3 1 4 2 5    16*w_4_12
3 4 1   0 1 1 5 1 6 2 3    16*w_4_12
# 3 1 1
3 4 1   0 1 0 3 0 3 2 3    -1/6-8*w_4_8
3 4 1   0 1 0 3 0 3 2 4    -1/3-16*w_4_9
3 4 1   0 1 0 2 0 3 3 4    1/9-16*w_4_1
3 4 1   0 1 0 2 0 3 4 5    -1/9-16*w_4_2
3 4 1   0 1 0 3 0 6 2 3    -1/9-16*w_4_19
3 4 1   0 1 0 3 0 6 2 4    -1/9-16*w_4_20
3 4 1   0 1 0 2 0 4 3 4    1/3+16*w_4_1
3 4 1   0 1 0 5 2 3 0 5    -1/6+8*w_4_26
3 4 1   0 2 0 3 0 3 1 3    -8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 0 4 0 4    -1/6+8/3*w_4_6
3 4 1   0 2 0 3 0 3 1 4    -16*w_4_9+16*w_4_7
3 4 1   0 2 0 3 0 4 1 3    -16*w_4_13+16*w_4_7
3 4 1   0 2 0 3 0 6 1 3    -16*w_4_19+16*w_4_7
3 4 1   0 4 0 5 1 2 0 5    -1/3+16*w_4_7
3 4 1   0 2 0 3 0 4 1 4    16*w_4_11-16*w_4_14
3 4 1   0 2 0 5 1 3 0 5    8*w_4_11+8*w_4_26
3 4 1   0 4 0 5 1 2 0 4    -1/6+8*w_4_11
```

TABLE 8 (part 2).

```
3 4 1   0 2 0 3 0 4 1 5    -16*w_4_15+16*w_4_12
3 4 1   0 2 0 3 0 6 1 4    16*w_4_12-16*w_4_20
3 4 1   0 2 0 5 0 6 1 3    16*w_4_12+16*w_4_27
3 4 1   0 4 0 5 0 6 1 2    16*w_4_12
3 4 1   0 1 0 2 0 4 3 5    -16*w_4_2
3 4 1   0 1 0 3 0 4 2 3    -16*w_4_13
3 4 1   0 1 0 3 0 4 2 4    -16*w_4_14
3 4 1   0 1 0 3 0 4 2 5    -16*w_4_15
3 4 1   0 1 0 5 0 6 2 3    16*w_4_27
# 1 1 3
3 4 1   0 1 2 3 2 3 2 3    -1/6+8/3*w_4_6
3 4 1   0 1 2 3 2 3 2 4    -1/3+16*w_4_7
3 4 1   0 1 2 3 2 4 2 4    -1/6+8*w_4_11
3 4 1   0 2 1 2 2 3 3 4    1/3+16*w_4_1
3 4 1   0 2 1 2 2 4 3 4    1/9-16*w_4_1
3 4 1   0 2 1 2 2 3 4 5    16*w_4_2
3 4 1   0 2 1 2 2 4 3 5    1/9+16*w_4_2
3 4 1   0 2 1 3 2 3 2 3    -8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 2 4 2 4    -1/6-8*w_4_8
3 4 1   0 2 1 5 2 3 2 3    -16*w_4_9+16*w_4_7
3 4 1   0 4 2 5 1 2 2 5    -1/3-16*w_4_9
3 4 1   0 2 1 3 2 3 2 5    -16*w_4_13+16*w_4_7
3 4 1   0 4 1 2 2 4 2 5    -16*w_4_13
3 4 1   0 2 1 5 2 3 2 5    16*w_4_11-16*w_4_14
3 4 1   0 4 2 5 1 2 2 4    -16*w_4_14
3 4 1   0 2 1 5 2 6 2 3    -16*w_4_15+16*w_4_12
3 4 1   0 4 2 5 2 6 1 2    -16*w_4_15
3 4 1   0 2 1 3 2 3 2 4    -16*w_4_19+16*w_4_7
3 4 1   0 4 1 2 2 3 2 4    -1/9-16*w_4_19
3 4 1   0 2 1 5 2 3 2 4    16*w_4_12-16*w_4_20
3 4 1   0 4 2 5 1 2 2 3    -1/9-16*w_4_20
3 4 1   0 2 1 3 2 4 2 4    8*w_4_11+8*w_4_26
3 4 1   0 4 1 2 2 3 2 3    -1/6+8*w_4_26
3 4 1   0 2 1 3 2 4 2 5    16*w_4_12+16*w_4_27
3 4 1   0 4 1 2 2 3 2 5    16*w_4_27
3 4 1   0 1 2 3 2 4 2 5    16*w_4_12
# 2 2 1
3 4 1   0 1 0 5 1 4 2 3    1/9
3 4 1   0 1 0 5 1 4 2 5    1/6
3 4 1   0 1 0 5 1 3 2 3    1/6+16*w_4_9
3 4 1   0 1 0 5 1 3 2 4    1/6+16*w_4_20
3 4 1   0 1 0 5 1 3 2 5    1/6+16*w_4_14
3 4 1   0 1 0 3 1 4 2 3    1/6+16*w_4_19
3 4 1   0 1 0 3 1 4 2 4    1/6-16*w_4_26
3 4 1   0 1 0 3 1 4 2 5    1/6-16*w_4_27
3 4 1   0 1 0 1 2 3 3 5    -1/6
3 4 1   0 1 0 1 2 3 4 5    -1/3-16*w_4_2
3 4 1   0 1 0 3 1 2 3 5    1/9
3 4 1   0 1 0 3 1 2 4 5    -1/18
3 4 1   0 1 0 3 1 6 2 3    -1/9+16*w_4_13
3 4 1   0 1 0 3 1 6 2 4    -1/9-16*w_4_27
3 4 1   0 1 0 2 1 3 3 4    -1/9-16*w_4_1
3 4 1   0 1 0 2 1 3 4 5    1/9-16*w_4_2
3 4 1   0 1 0 5 2 3 1 3    1/9+16*w_4_9
3 4 1   0 1 0 5 2 6 1 3    1/9+16*w_4_15
3 4 1   0 2 0 5 1 4 1 3    -1/9
3 4 1   0 4 1 2 0 6 1 5    1/9
3 4 1   0 1 0 1 2 6 3 5    -1/6
3 4 1   0 1 0 2 1 4 3 4    1/3+16*w_4_1
3 4 1   0 1 0 5 2 3 1 5    -1/3+16*w_4_14
3 4 1   0 1 0 5 1 2 4 5    -1/6
3 4 1   0 1 0 5 2 4 1 5    -1/6
3 4 1   0 2 0 3 1 3 1 3    8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 0 4 1 4    -1/6+8*w_4_6
3 4 1   0 2 0 3 1 3 1 4    16*w_4_19+16*w_4_7
3 4 1   0 2 0 3 1 3 1 5    16*w_4_13+16*w_4_7
3 4 1   0 2 0 5 1 3 1 3    16*w_4_9+16*w_4_7
3 4 1   0 4 1 2 0 4 1 3    1/3+16*w_4_7
3 4 1   0 4 0 5 1 2 1 5    -1/9+16*w_4_7
3 4 1   0 4 1 2 0 6 1 4    -1/3+16*w_4_7
3 4 1   0 2 0 3 1 4 1 4    8*w_4_11-8*w_4_26
3 4 1   0 2 0 5 1 3 1 5    16*w_4_11+16*w_4_14
3 4 1   0 4 0 5 1 2 1 4    16*w_4_11
3 4 1   0 4 1 5 1 2 0 4    -1/6+8*w_4_11
3 4 1   0 2 0 3 1 4 1 5    16*w_4_12-16*w_4_27
3 4 1   0 2 0 5 1 3 1 4    16*w_4_12+16*w_4_20
3 4 1   0 2 0 5 1 6 1 3    16*w_4_15+16*w_4_12
3 4 1   0 4 0 5 1 2 1 3    1/6+16*w_4_12
3 4 1   0 4 1 2 0 6 1 3    1/6+16*w_4_12
3 4 1   0 4 0 5 1 6 1 2    -1/9+16*w_4_12
3 4 1   0 1 0 5 1 6 2 5    1/6
3 4 1   0 1 0 5 4 6 1 2    -1/6
3 4 1   0 1 0 5 3 6 1 2    -1/6
3 4 1   0 1 0 5 2 4 1 4    -1/6
3 4 1   0 1 0 1 2 3 3 4    -16*w_4_1
3 4 1   0 1 0 2 1 4 3 5    -16*w_4_2
3 4 1   0 1 0 3 1 3 2 3    16*w_4_8
3 4 1   0 1 0 3 1 3 2 5    16*w_4_13
3 4 1   0 1 0 5 1 6 2 3    16*w_4_15
3 4 1   0 1 0 3 1 3 2 4    16*w_4_19
3 4 1   0 1 0 5 2 3 1 4    16*w_4_20
# 1 2 2
3 4 1   0 1 1 3 2 3 2 3    -1/6+8*w_4_6
3 4 1   0 1 1 3 2 3 2 4    -1/3+16*w_4_7
3 4 1   0 1 1 3 2 4 2 4    -1/6+8*w_4_11
3 4 1   0 4 1 3 1 2 2 4    1/6
3 4 1   0 1 1 3 2 3 2 5    -1/9+16*w_4_7
3 4 1   0 1 1 3 2 4 2 5    -1/9+16*w_4_12
3 4 1   0 1 1 2 2 3 3 5    -1/6
3 4 1   0 1 1 5 2 3 2 3    1/3+16*w_4_7
3 4 1   0 1 1 2 2 6 3 5    -1/6
3 4 1   0 1 1 5 2 4 2 3    1/9
3 4 1   0 1 1 2 2 4 3 5    -1/18
3 4 1   0 1 1 5 2 6 2 3    1/6+16*w_4_12
3 4 1   0 1 1 2 2 6 3 4    -1/6
3 4 1   0 1 1 5 2 3 2 4    1/6+16*w_4_12
3 4 1   0 4 1 2 1 2 3 4    -1/6
3 4 1   0 4 2 3 1 2 1 4    -1/6
3 4 1   0 2 1 3 1 6 2 5    -1/9
3 4 1   0 4 1 2 1 6 2 5    1/9
3 4 1   0 2 1 2 1 3 3 4    1/3+16*w_4_1
3 4 1   0 2 1 2 1 4 3 4    -1/9-16*w_4_1
3 4 1   0 4 1 2 1 2 4 5    16*w_4_1
3 4 1   0 2 1 2 1 3 4 5    16*w_4_2
3 4 1   0 2 1 2 1 4 3 5    -1/9+16*w_4_2
3 4 1   0 4 1 2 1 2 3 5    -1/3-16*w_4_2
3 4 1   0 2 1 3 1 3 2 3    8*w_4_8+8*w_4_6
3 4 1   0 4 1 2 1 4 2 4    16*w_4_8
3 4 1   0 2 1 3 1 3 2 4    16*w_4_9+16*w_4_7
3 4 1   0 4 1 2 1 4 2 3    1/9+16*w_4_9
3 4 1   0 4 1 2 1 4 2 5    1/6+16*w_4_9
3 4 1   0 2 1 3 1 4 2 3    16*w_4_13+16*w_4_7
3 4 1   0 4 1 2 1 3 2 4    -1/9+16*w_4_13
3 4 1   0 4 1 5 1 2 2 5    16*w_4_13
3 4 1   0 2 1 3 1 4 2 4    16*w_4_11+16*w_4_14
3 4 1   0 4 1 2 1 3 2 3    -1/3+16*w_4_14
3 4 1   0 4 1 5 1 2 2 4    1/6+16*w_4_14
3 4 1   0 2 1 3 1 4 2 5    16*w_4_15+16*w_4_12
3 4 1   0 4 1 2 1 3 2 5    16*w_4_15
3 4 1   0 4 1 5 1 2 2 3    1/9+16*w_4_15
3 4 1   0 2 1 3 1 6 2 3    16*w_4_19+16*w_4_7
3 4 1   0 4 1 2 1 6 2 4    1/6+16*w_4_19
3 4 1   0 4 2 5 1 2 1 5    16*w_4_19
3 4 1   0 2 1 3 1 6 2 4    16*w_4_12+16*w_4_20
3 4 1   0 4 1 2 1 6 2 3    16*w_4_20
3 4 1   0 4 2 5 1 6 1 2    1/6+16*w_4_20
3 4 1   0 2 1 5 2 3 1 5    8*w_4_11-8*w_4_26
3 4 1   0 4 2 5 1 2 1 4    1/6-16*w_4_26
3 4 1   0 2 1 5 1 6 2 3    16*w_4_12-16*w_4_27
3 4 1   0 4 1 5 2 6 1 2    1/6-16*w_4_27
3 4 1   0 4 2 5 1 2 1 3    -1/9-16*w_4_27
3 4 1   0 4 1 5 2 4 1 2    1/6
3 4 1   0 1 1 2 2 4 3 4    1/9
3 4 1   0 4 3 5 1 2 1 2    -1/6
3 4 1   0 4 2 3 1 2 1 3    -1/6
3 4 1   0 1 1 5 2 3 2 5    16*w_4_11
# 2 1 2
3 4 1   0 1 0 3 2 3 2 3    1/6+8*w_4_8
3 4 1   0 1 0 3 2 3 2 4    1/3+16*w_4_19
3 4 1   0 1 0 3 2 4 2 4    1/6-8*w_4_26
3 4 1   0 2 0 5 1 4 2 5    1/6
3 4 1   0 1 0 3 2 3 2 5    1/9+16*w_4_13
3 4 1   0 1 0 3 2 4 2 5    1/9-16*w_4_27
3 4 1   0 1 0 2 2 3 3 4    -1/3-16*w_4_1
3 4 1   0 1 0 2 2 3 3 5    -1/6
3 4 1   0 1 0 5 2 3 2 3    1/3+16*w_4_9
3 4 1   0 1 0 2 2 6 3 5    -1/6
3 4 1   0 1 0 5 2 4 2 3    1/9
3 4 1   0 1 0 2 2 4 3 5    -1/6-16*w_4_2
3 4 1   0 1 0 5 2 6 2 3    1/6+16*w_4_15
3 4 1   0 1 0 2 2 6 3 4    -1/6
3 4 1   0 1 0 5 2 3 2 4    1/6+16*w_4_20
3 4 1   0 2 0 5 1 2 4 5    -1/6
3 4 1   0 2 0 5 2 4 1 5    -1/6
3 4 1   0 2 0 5 2 4 1 3    -1/9
3 4 1   0 4 1 2 0 6 2 5    1/9
3 4 1   0 2 0 3 1 2 3 5    16*w_4_1
3 4 1   0 2 0 5 1 2 3 5    -1/3-16*w_4_1
```

TABLE 8 (part 3).

```
3 4 1   0 2 0 3 1 2 4 5    -1/6-16*w_4_2
3 4 1   0 2 0 5 1 2 3 4    16*w_4_2
3 4 1   0 2 0 3 1 3 2 3    32*w_4_8
3 4 1   0 4 1 2 0 4 2 4    1/6+8*w_4_8
3 4 1   0 2 0 3 1 3 2 4    16*w_4_9+16*w_4_19
3 4 1   0 2 0 3 1 3 2 5    16*w_4_9+16*w_4_13
3 4 1   0 4 1 2 0 4 2 3    1/3+16*w_4_9
3 4 1   0 2 0 3 1 4 2 3    16*w_4_19+16*w_4_13
3 4 1   0 2 0 5 1 3 2 3    16*w_4_9+16*w_4_13
3 4 1   0 4 0 5 1 2 2 5    1/9+16*w_4_13
3 4 1   0 2 0 3 1 4 2 4    -16*w_4_26+16*w_4_14
3 4 1   0 2 0 5 1 3 2 5    32*w_4_14
3 4 1   0 4 0 5 1 2 2 4    16*w_4_14
3 4 1   0 2 0 3 1 4 2 5    16*w_4_15-16*w_4_27
3 4 1   0 2 0 5 1 3 2 4    16*w_4_15+16*w_4_20
3 4 1   0 4 0 5 1 2 2 3    1/6+16*w_4_15
3 4 1   0 2 0 3 1 6 2 3    16*w_4_19+16*w_4_13
3 4 1   0 2 0 5 2 3 1 3    16*w_4_9+16*w_4_19
3 4 1   0 4 1 2 0 6 2 4    1/3+16*w_4_19
3 4 1   0 2 0 3 1 6 2 4    16*w_4_20-16*w_4_27
3 4 1   0 2 0 5 2 6 1 3    16*w_4_15+16*w_4_20
3 4 1   0 4 1 2 0 6 2 3    1/6+16*w_4_20
3 4 1   0 2 0 5 2 3 1 5    -16*w_4_26+16*w_4_14
3 4 1   0 4 2 5 1 2 0 4    1/6-8*w_4_26
3 4 1   0 2 0 5 1 6 2 3    16*w_4_15-16*w_4_27
3 4 1   0 2 0 5 2 3 1 4    16*w_4_20-16*w_4_27
3 4 1   0 4 0 5 2 6 1 2    1/9-16*w_4_27
3 4 1   0 2 0 5 1 6 2 5    1/6
3 4 1   0 2 0 5 4 6 1 2    -1/6
3 4 1   0 2 0 5 3 6 1 2    -1/6
3 4 1   0 2 0 5 2 4 1 4    -1/6
3 4 1   0 1 0 2 2 4 3 4    16*w_4_1
3 4 1   0 1 0 2 2 3 4 5    -16*w_4_2
3 4 1   0 1 0 5 2 3 2 5    16*w_4_14
# 1 2 1
3 4 1   0 1 1 3 2 3 3 4    1/6+16*w_4_10
3 4 1   0 1 1 3 2 4 3 4    1/6+16*w_4_16
3 4 1   0 1 1 3 2 6 3 4    1/6+16*w_4_21
3 4 1   0 4 1 3 1 3 2 3    -1/6-16*w_4_105+16*w_4_60
3 4 1   0 4 1 3 1 3 2 4    -1/6+16*w_4_61-16*w_4_84
3 4 1   0 4 1 3 1 3 2 5    -1/6-16*w_4_91+16*w_4_62
3 4 1   0 1 1 3 2 3 3 5    1/9+16*w_4_10
3 4 1   0 1 1 3 2 3 4 5    1/9
3 4 1   0 1 1 3 2 4 3 5    1/9+16*w_4_17
3 4 1   0 1 1 3 2 4 4 5    1/9+16*w_4_18
3 4 1   0 4 1 3 1 2 3 5    1/9-16*w_4_40
3 4 1   0 4 1 3 1 2 4 5    -1/18-16*w_4_40
3 4 1   0 4 1 3 1 6 2 3    1/18+16*w_4_63-16*w_4_91
3 4 1   0 4 1 3 1 6 2 4    1/18+16*w_4_63-16*w_4_74
3 4 1   0 1 1 3 2 6 3 5    1/18+16*w_4_22
3 4 1   0 1 1 3 2 6 4 5    1/18+16*w_4_23
3 4 1   0 1 1 5 2 3 3 5    -1/3+16*w_4_16
3 4 1   0 1 1 2 3 4 4 5    -1/6
3 4 1   0 1 1 5 2 3 4 5    -1/6-16*w_4_18
3 4 1   0 1 1 5 2 4 3 5    -1/9+16*w_4_40
3 4 1   0 1 1 5 2 6 3 5    -1/6+16*w_4_42
3 4 1   0 2 1 3 1 3 3 4    32*w_4_10
3 4 1   0 4 1 2 1 4 3 4    -1/9-16*w_4_10
3 4 1   0 4 1 2 1 4 4 5    1/6+16*w_4_10
3 4 1   0 2 1 3 1 4 3 4    32*w_4_16
3 4 1   0 4 1 2 1 3 3 4    1/3-16*w_4_16
3 4 1   0 4 1 5 1 2 4 5    -1/6-16*w_4_16
3 4 1   0 2 1 3 1 4 3 5    32*w_4_17
3 4 1   0 4 1 2 1 3 4 5    16*w_4_17
3 4 1   0 4 1 5 1 2 3 5    -1/9-16*w_4_17
3 4 1   0 2 1 3 1 4 4 5    32*w_4_18
3 4 1   0 4 1 2 1 3 3 5    1/6+16*w_4_18
3 4 1   0 4 1 5 1 2 3 4    -1/9-16*w_4_18
3 4 1   0 2 1 3 1 6 3 4    32*w_4_21
3 4 1   0 4 1 2 1 6 3 4    -16*w_4_21
3 4 1   0 4 5 6 1 2 1 5    1/6+16*w_4_21
3 4 1   0 2 1 3 1 6 3 5    32*w_4_22
3 4 1   0 4 1 2 1 6 4 5    16*w_4_22
3 4 1   0 4 3 5 1 2 1 5    -1/18-16*w_4_22
3 4 1   0 2 1 3 1 6 4 5    32*w_4_23
3 4 1   0 4 1 2 1 6 3 5    16*w_4_23
3 4 1   0 4 3 5 1 6 1 2    -1/18-16*w_4_23
3 4 1   0 2 1 5 1 4 3 4    32*w_4_40
3 4 1   0 2 1 5 1 6 3 4    32*w_4_41
3 4 1   0 4 1 5 3 6 1 2    -16*w_4_41
3 4 1   0 4 5 6 1 2 1 3    16*w_4_41
3 4 1   0 2 1 5 1 6 3 5    32*w_4_42
3 4 1   0 4 1 5 4 6 1 2    -16*w_4_42
3 4 1   0 4 3 5 1 2 1 3    1/6-16*w_4_42
3 4 1   0 2 1 5 3 4 1 5    32*w_4_46
3 4 1   0 4 3 5 1 2 1 4    -16*w_4_46
3 4 1   0 4 5 6 1 2 1 4    16*w_4_46
3 4 1   0 4 1 3 1 4 2 4    16*w_4_60-16*w_4_83
3 4 1   0 4 1 3 1 4 2 3    16*w_4_61-16*w_4_84
3 4 1   0 4 1 3 1 4 2 5    -16*w_4_74+16*w_4_62
3 4 1   0 4 1 5 1 4 2 3    -16*w_4_63+16*w_4_62
3 4 1   0 4 2 5 1 6 1 5    16*w_4_63-16*w_4_62
3 4 1   0 4 1 5 1 3 2 3    -16*w_4_76+16*w_4_71
3 4 1   0 4 1 5 1 3 2 5    16*w_4_71-16*w_4_75
3 4 1   0 4 1 5 1 3 2 4    -16*w_4_73+16*w_4_71
3 4 1   0 4 1 5 1 6 2 4    16*w_4_73-16*w_4_71
3 4 1   0 4 1 5 2 3 1 3    -16*w_4_76+16*w_4_73
3 4 1   0 4 2 5 1 3 1 5    16*w_4_73-16*w_4_75
3 4 1   0 4 1 5 1 6 2 5    16*w_4_74-16*w_4_62
3 4 1   0 4 1 5 2 4 1 3    -1/18-16*w_4_63+16*w_4_74
3 4 1   0 4 2 3 1 3 1 5    16*w_4_74-16*w_4_91
3 4 1   0 4 1 5 2 3 1 4    -16*w_4_73+16*w_4_75
3 4 1   0 4 2 5 1 3 1 3    -16*w_4_76+16*w_4_75
3 4 1   0 4 1 5 2 6 1 4    -16*w_4_71+16*w_4_75
3 4 1   0 4 1 5 2 3 1 5    16*w_4_76-16*w_4_75
3 4 1   0 4 2 5 1 3 1 4    16*w_4_76-16*w_4_73
3 4 1   0 4 2 5 1 6 1 4    16*w_4_76-16*w_4_71
3 4 1   0 4 1 5 2 4 1 4    -16*w_4_60+16*w_4_83
3 4 1   0 4 2 3 1 3 1 3    -8*w_4_105+8*w_4_83
3 4 1   0 4 1 5 2 4 1 5    1/6-16*w_4_61+16*w_4_84
3 4 1   0 4 2 5 1 4 1 5    -16*w_4_61+16*w_4_84
3 4 1   0 4 1 5 2 6 1 5    1/6+16*w_4_91-16*w_4_62
3 4 1   0 4 2 5 1 4 1 3    -1/18-16*w_4_63+16*w_4_91
3 4 1   0 4 2 3 1 4 1 5    -16*w_4_74+16*w_4_91
3 4 1   0 4 2 3 1 4 1 4    8*w_4_105-8*w_4_83
3 4 1   0 4 2 5 1 4 1 4    1/6+16*w_4_105-16*w_4_60
3 4 1   0 4 1 2 1 4 3 5    -1/9
3 4 1   0 1 1 5 2 4 3 4    1/18+16*w_4_40
3 4 1   0 1 1 2 3 4 3 5    -1/6
3 4 1   0 1 1 5 2 3 3 4    16*w_4_17
3 4 1   0 1 1 5 3 6 2 3    16*w_4_21
3 4 1   0 1 1 5 3 4 2 3    16*w_4_22
3 4 1   0 1 1 5 4 6 2 3    -16*w_4_23
3 4 1   0 1 1 5 2 6 3 4    16*w_4_41
3 4 1   0 1 1 5 3 6 2 4    16*w_4_41
3 4 1   0 1 1 5 3 4 2 4    16*w_4_42
3 4 1   0 1 1 5 3 4 2 5    16*w_4_46
3 4 1   0 1 1 5 3 6 2 5    16*w_4_46
# 2 1 1
3 4 1   0 1 0 3 2 3 3 4    1/6-16*w_4_24
3 4 1   0 1 0 3 2 4 3 4    1/6-16*w_4_28
3 4 1   0 1 0 3 2 6 3 4    1/6-16*w_4_31
3 4 1   0 4 1 3 0 4 2 3    1/6-16*w_4_106+16*w_4_61
3 4 1   0 4 1 3 0 4 2 4    1/6+16*w_4_60-16*w_4_86
3 4 1   0 4 1 3 0 4 2 5    1/6-16*w_4_107+16*w_4_62
3 4 1   0 1 0 3 2 3 3 5    -1/9+16*w_4_24
3 4 1   0 1 0 3 2 3 4 5    -1/9-16*w_4_25
3 4 1   0 1 0 3 2 4 3 5    -1/9-16*w_4_29
3 4 1   0 1 0 3 2 4 4 5    -1/9-16*w_4_30
3 4 1   0 2 0 5 1 4 3 4    1/18+16*w_4_40-16*w_4_43
3 4 1   0 2 0 5 1 4 3 5    1/18+16*w_4_40+16*w_4_43
3 4 1   0 4 1 3 0 6 2 3    1/18+16*w_4_63-16*w_4_107
3 4 1   0 4 1 3 0 6 2 4    1/18+16*w_4_63-16*w_4_85
3 4 1   0 1 0 3 2 6 3 5    -1/18-16*w_4_32
3 4 1   0 1 0 3 2 6 4 5    -1/18-16*w_4_33
3 4 1   0 1 0 2 3 4 3 4    1/6-8*w_4_3
3 4 1   0 1 0 5 2 3 3 5    -1/3+16*w_4_28
3 4 1   0 1 0 2 3 4 4 5    -1/6+16*w_4_4
3 4 1   0 1 0 5 2 3 4 5    -1/6-16*w_4_30
3 4 1   0 1 0 5 2 4 3 5    -1/6+16*w_4_43
3 4 1   0 1 0 5 2 6 3 5    -1/6-16*w_4_45
3 4 1   0 2 0 3 1 3 3 4    -16*w_4_24+16*w_4_10
3 4 1   0 2 0 3 1 3 3 5    16*w_4_24+16*w_4_10
3 4 1   0 4 1 2 0 4 3 4    -1/3-16*w_4_10
3 4 1   0 2 0 3 1 4 3 4    -16*w_4_28+16*w_4_16
3 4 1   0 2 0 5 1 3 3 5    16*w_4_28+16*w_4_16
3 4 1   0 4 0 5 1 2 4 5    -16*w_4_16
3 4 1   0 2 0 3 1 4 3 5    -16*w_4_29+16*w_4_17
3 4 1   0 2 0 5 1 3 3 4    16*w_4_29+16*w_4_17
3 4 1   0 4 0 5 1 2 3 5    -1/6-16*w_4_17
3 4 1   0 2 0 3 1 4 4 5    16*w_4_18-16*w_4_30
3 4 1   0 2 0 5 1 3 4 5    -16*w_4_18-16*w_4_30
3 4 1   0 4 0 5 1 2 3 4    -1/6-16*w_4_18
3 4 1   0 2 0 3 1 6 3 4    16*w_4_21-16*w_4_31
3 4 1   0 2 0 5 3 6 1 3    16*w_4_21+16*w_4_31
3 4 1   0 4 1 2 0 6 3 4    -1/6-16*w_4_21
3 4 1   0 2 0 3 1 6 3 5    16*w_4_22-16*w_4_32
3 4 1   0 2 0 5 3 4 1 3    16*w_4_22+16*w_4_32
3 4 1   0 4 1 2 0 6 4 5    1/6+16*w_4_22
```

TABLE 8 (part 4).

```
3 4 1   0 2 0 3 1 6 4 5    16*w_4_23-16*w_4_33
3 4 1   0 2 0 5 4 6 1 3    -16*w_4_23-16*w_4_33
3 4 1   0 4 1 2 0 6 3 5    16*w_4_23
3 4 1   0 4 0 3 1 2 3 5    -16*w_4_40
3 4 1   0 2 0 5 1 6 3 4    -16*w_4_44+16*w_4_41
3 4 1   0 2 0 5 3 6 1 4    16*w_4_44+16*w_4_41
3 4 1   0 4 0 5 3 6 1 2    -16*w_4_41
3 4 1   0 2 0 5 1 6 3 5    -16*w_4_45+16*w_4_42
3 4 1   0 2 0 5 3 4 1 4    16*w_4_45+16*w_4_42
3 4 1   0 4 0 5 4 6 1 2    -16*w_4_42
3 4 1   0 2 0 5 3 4 1 5    -16*w_4_47+16*w_4_46
3 4 1   0 2 0 5 3 6 1 5    16*w_4_47+16*w_4_46
3 4 1   0 4 3 5 1 2 0 4    -1/6-16*w_4_46
3 4 1   0 4 0 3 1 3 2 3    16*w_4_60-16*w_4_64
3 4 1   0 4 0 5 1 4 2 4    16*w_4_60-16*w_4_86
3 4 1   0 4 0 3 1 3 2 4    16*w_4_61-16*w_4_65
3 4 1   0 4 0 5 1 4 2 5    16*w_4_61-16*w_4_87
3 4 1   0 4 0 3 1 3 2 5    -16*w_4_66+16*w_4_62
3 4 1   0 4 0 5 1 4 2 3    16*w_4_62-16*w_4_85
3 4 1   0 4 0 3 1 6 2 3    -16*w_4_66+16*w_4_63
3 4 1   0 4 0 5 1 3 2 3    -16*w_4_77+16*w_4_71
3 4 1   0 4 0 5 1 3 2 4    16*w_4_71-16*w_4_78
3 4 1   0 4 0 5 1 3 2 5    -16*w_4_79+16*w_4_71
3 4 1   0 4 0 5 1 6 2 3    -16*w_4_92+16*w_4_72
3 4 1   0 4 1 5 0 6 2 3    -16*w_4_118+16*w_4_72
3 4 1   0 4 0 5 2 6 1 3    -16*w_4_92+16*w_4_72
3 4 1   0 4 0 5 1 6 2 4    -16*w_4_79+16*w_4_73
3 4 1   0 4 1 5 0 6 2 4    16*w_4_73-16*w_4_78
3 4 1   0 4 0 5 2 3 1 3    16*w_4_73-16*w_4_77
3 4 1   0 4 0 5 1 6 2 5    1/18-16*w_4_66+16*w_4_74
3 4 1   0 4 1 5 0 6 2 5    16*w_4_74-16*w_4_107
3 4 1   0 4 0 5 2 4 1 3    16*w_4_74-16*w_4_85
3 4 1   0 4 0 5 2 3 1 4    -16*w_4_78+16*w_4_75
3 4 1   0 4 1 5 2 3 0 4    -16*w_4_77+16*w_4_75
3 4 1   0 4 0 5 2 6 1 4    -16*w_4_79+16*w_4_75
3 4 1   0 4 0 5 2 3 1 5    -16*w_4_79+16*w_4_76
3 4 1   0 4 1 5 2 3 0 5    16*w_4_76-16*w_4_78
3 4 1   0 4 2 5 1 3 0 4    16*w_4_76-16*w_4_77
3 4 1   0 4 0 5 2 4 1 4    -16*w_4_86+16*w_4_83
3 4 1   0 4 1 5 2 4 0 4    1/12+8*w_4_83-8*w_4_64
3 4 1   0 4 0 5 2 4 1 5    -16*w_4_87+16*w_4_84
3 4 1   0 4 1 5 2 4 0 5    1/6+16*w_4_84-16*w_4_65
3 4 1   0 4 2 3 0 4 1 3    -16*w_4_106+16*w_4_84
3 4 1   0 4 0 5 2 6 1 5    1/18-16*w_4_66+16*w_4_91
3 4 1   0 4 1 5 2 6 0 5    16*w_4_91-16*w_4_85
3 4 1   0 4 2 3 0 4 1 5    -16*w_4_107+16*w_4_91
3 4 1   0 4 2 3 0 4 1 4    16*w_4_105-16*w_4_86
3 4 1   0 4 2 5 1 4 0 4    1/12+8*w_4_105-8*w_4_64
3 4 1   0 1 0 2 3 4 3 5    -16*w_4_4
3 4 1   0 1 0 2 3 6 4 5    -16*w_4_5
3 4 1   0 2 0 3 1 3 4 5    -16*w_4_25
3 4 1   0 1 0 5 2 3 3 4    16*w_4_29
3 4 1   0 1 0 5 3 6 2 3    16*w_4_31
3 4 1   0 1 0 5 3 4 2 3    16*w_4_32
3 4 1   0 1 0 5 4 6 2 3    -16*w_4_33
3 4 1   0 1 0 5 2 4 3 4    -16*w_4_43
3 4 1   0 1 0 5 2 6 3 4    -16*w_4_44
3 4 1   0 1 0 5 3 6 2 4    16*w_4_44
3 4 1   0 1 0 5 3 4 2 4    16*w_4_45
3 4 1   0 1 0 5 3 4 2 5    -16*w_4_47
3 4 1   0 1 0 5 3 6 2 5    16*w_4_47
# 1 1 2
3 4 1   0 4 1 3 2 3 2 3    -1/12-8*w_4_105+8*w_4_64
3 4 1   0 4 1 3 2 3 2 4    -1/6-16*w_4_84+16*w_4_65
3 4 1   0 4 1 3 2 4 2 4    -1/12-8*w_4_83+8*w_4_64
3 4 1   0 4 1 3 2 3 2 5    -1/18+16*w_4_66-16*w_4_91
3 4 1   0 4 1 3 2 4 2 5    -1/18+16*w_4_66-16*w_4_74
3 4 1   0 1 2 3 2 3 3 4    1/3+16*w_4_10
3 4 1   0 1 2 3 2 6 3 5    1/6+16*w_4_22
3 4 1   0 1 2 3 2 4 4 5    1/6+16*w_4_18
3 4 1   0 1 2 3 2 4 3 5    1/6+16*w_4_17
3 4 1   0 1 2 3 2 6 3 4    1/6+16*w_4_21
3 4 1   0 1 2 5 3 4 2 5    1/6+16*w_4_46
3 4 1   0 2 1 2 3 4 3 4    -1/6+8*w_4_3
3 4 1   0 2 1 2 3 4 3 5    -1/6+16*w_4_4
3 4 1   0 2 1 2 3 4 4 5    -16*w_4_4
3 4 1   0 2 1 2 3 6 4 5    16*w_4_5
3 4 1   0 2 1 3 2 3 3 4    16*w_4_24+16*w_4_10
3 4 1   0 4 1 2 2 4 3 4    1/9-16*w_4_24
3 4 1   0 2 1 3 2 3 3 5    -16*w_4_24+16*w_4_10
3 4 1   0 4 1 2 2 4 4 5    1/6-16*w_4_24
3 4 1   0 2 1 3 2 3 4 5    16*w_4_25
3 4 1   0 4 1 2 2 4 3 5    1/9+16*w_4_25
3 4 1   0 2 1 3 2 4 3 4    16*w_4_28+16*w_4_16
3 4 1   0 4 1 2 2 3 3 4    1/3-16*w_4_28
3 4 1   0 2 1 5 2 3 3 5    -16*w_4_28+16*w_4_16
3 4 1   0 4 2 5 1 2 4 5    -1/6+16*w_4_28
3 4 1   0 2 1 3 2 4 3 5    16*w_4_29+16*w_4_17
3 4 1   0 4 1 2 2 3 4 5    16*w_4_29
3 4 1   0 2 1 5 2 3 3 4    -16*w_4_29+16*w_4_17
3 4 1   0 4 2 5 1 2 3 5    1/9+16*w_4_29
3 4 1   0 2 1 3 2 4 4 5    16*w_4_18+16*w_4_30
3 4 1   0 4 1 2 2 3 3 5    1/6+16*w_4_30
3 4 1   0 2 1 5 2 3 4 5    -16*w_4_18+16*w_4_30
3 4 1   0 4 2 5 1 2 3 4    1/9+16*w_4_30
3 4 1   0 2 1 3 2 6 3 4    16*w_4_21+16*w_4_31
3 4 1   0 4 1 2 2 6 3 4    -16*w_4_31
3 4 1   0 2 1 5 3 6 2 3    16*w_4_21-16*w_4_31
3 4 1   0 4 5 6 1 2 2 5    1/6-16*w_4_31
3 4 1   0 2 1 3 2 6 3 5    16*w_4_22+16*w_4_32
3 4 1   0 4 1 2 2 6 4 5    16*w_4_32
3 4 1   0 2 1 5 3 4 2 3    16*w_4_22-16*w_4_32
3 4 1   0 4 3 5 1 2 2 5    1/18+16*w_4_32
3 4 1   0 2 1 3 2 6 4 5    16*w_4_23+16*w_4_33
3 4 1   0 4 1 2 2 6 3 5    16*w_4_33
3 4 1   0 2 1 5 4 6 2 3    -16*w_4_23+16*w_4_33
3 4 1   0 4 3 5 2 6 1 2    1/18+16*w_4_33
3 4 1   0 2 1 5 2 4 3 4    1/18+16*w_4_40+16*w_4_43
3 4 1   0 4 2 3 1 2 3 5    1/6-16*w_4_43
3 4 1   0 2 1 5 2 4 3 5    1/18+16*w_4_40-16*w_4_43
3 4 1   0 4 2 3 1 2 4 5    16*w_4_43
3 4 1   0 2 1 5 2 6 3 4    16*w_4_44+16*w_4_41
3 4 1   0 4 2 5 3 6 1 2    -16*w_4_44
3 4 1   0 2 1 5 3 6 2 4    -16*w_4_44+16*w_4_41
3 4 1   0 4 5 6 1 2 2 3    -16*w_4_44
3 4 1   0 2 1 5 2 6 3 5    16*w_4_45+16*w_4_42
3 4 1   0 4 2 5 4 6 1 2    -16*w_4_45
3 4 1   0 2 1 5 3 4 2 4    -16*w_4_45+16*w_4_42
3 4 1   0 4 3 5 1 2 2 3    1/6+16*w_4_45
3 4 1   0 2 1 5 3 4 2 5    16*w_4_47+16*w_4_46
3 4 1   0 4 3 5 1 2 2 4    -16*w_4_47
3 4 1   0 2 1 5 3 6 2 5    -16*w_4_47+16*w_4_46
3 4 1   0 4 5 6 1 2 2 4    -16*w_4_47
3 4 1   0 4 2 5 2 4 1 4    -16*w_4_60+16*w_4_64
3 4 1   0 4 2 5 2 4 1 5    -16*w_4_61+16*w_4_65
3 4 1   0 4 1 5 2 6 2 5    16*w_4_66-16*w_4_62
3 4 1   0 4 2 5 2 4 1 3    16*w_4_66-16*w_4_63
3 4 1   0 4 1 5 2 3 2 3    -16*w_4_76+16*w_4_77
3 4 1   0 4 2 5 1 3 2 5    16*w_4_77-16*w_4_75
3 4 1   0 4 2 5 2 3 1 4    -16*w_4_73+16*w_4_77
3 4 1   0 4 2 5 2 6 1 4    16*w_4_77-16*w_4_71
3 4 1   0 4 1 5 2 3 2 4    -16*w_4_73+16*w_4_78
3 4 1   0 4 2 5 1 3 2 3    -16*w_4_76+16*w_4_78
3 4 1   0 4 2 5 1 6 2 4    -16*w_4_71+16*w_4_78
3 4 1   0 4 2 5 2 3 1 5    16*w_4_78-16*w_4_75
3 4 1   0 4 1 5 2 3 2 5    16*w_4_79-16*w_4_75
3 4 1   0 4 2 5 1 3 2 4    16*w_4_79-16*w_4_73
3 4 1   0 4 1 5 2 6 2 4    16*w_4_79-16*w_4_71
3 4 1   0 4 2 5 2 3 1 3    16*w_4_79-16*w_4_76
3 4 1   0 4 1 5 2 4 2 3    -1/18-16*w_4_63+16*w_4_85
3 4 1   0 4 2 3 1 3 2 5    -16*w_4_91+16*w_4_85
3 4 1   0 4 2 3 1 6 2 4    -16*w_4_74+16*w_4_85
3 4 1   0 4 2 5 2 6 1 5    -16*w_4_62+16*w_4_85
3 4 1   0 4 1 5 2 4 2 4    -1/6-16*w_4_60+16*w_4_86
3 4 1   0 4 2 3 1 3 2 3    -16*w_4_105+16*w_4_86
3 4 1   0 4 2 3 1 4 2 4    16*w_4_86-16*w_4_83
3 4 1   0 4 2 5 1 4 2 4    -16*w_4_60+16*w_4_86
3 4 1   0 4 1 5 2 4 2 5    -16*w_4_61+16*w_4_87
3 4 1   0 4 2 3 1 3 2 4    16*w_4_87-16*w_4_84
3 4 1   0 4 1 5 2 6 2 3    16*w_4_92-16*w_4_72
3 4 1   0 4 2 5 2 6 1 3    16*w_4_92-16*w_4_72
3 4 1   0 4 2 3 1 4 2 3    16*w_4_106-16*w_4_84
3 4 1   0 4 2 5 1 4 2 5    -1/6+16*w_4_106-16*w_4_61
3 4 1   0 4 2 3 1 4 2 5    -16*w_4_74+16*w_4_107
3 4 1   0 4 2 5 1 4 2 3    -1/18-16*w_4_63+16*w_4_107
3 4 1   0 4 2 3 1 6 2 3    16*w_4_107-16*w_4_91
3 4 1   0 4 2 5 1 6 2 5    -1/6+16*w_4_107-16*w_4_62
3 4 1   0 4 2 5 1 6 2 3    16*w_4_118-16*w_4_72
3 4 1   0 1 2 3 2 4 3 4    16*w_4_16
3 4 1   0 1 2 3 2 6 4 5    16*w_4_23
3 4 1   0 1 2 5 2 4 3 4    16*w_4_40
3 4 1   0 1 2 5 2 6 3 4    16*w_4_41
3 4 1   0 1 2 5 2 6 3 5    16*w_4_42
# 1 1 1
3 4 1   0 4 1 3 2 3 3 4    1/6+16*w_4_108+16*w_4_67
3 4 1   0 4 1 3 2 4 3 4    1/6-16*w_4_67+16*w_4_90
3 4 1   0 4 1 3 2 6 3 4    1/6+16*w_4_111
3 4 1   0 4 1 3 2 3 3 5    1/18-16*w_4_110+16*w_4_68
3 4 1   0 4 1 3 2 3 4 5    1/18-16*w_4_109+16*w_4_69
3 4 1   0 4 1 3 2 4 3 5    1/18+16*w_4_89+16*w_4_69
3 4 1   0 4 1 3 2 4 4 5    1/18+16*w_4_68+16*w_4_88
```

TABLE 8 (part 5).

3	4	1	0	4	1	3	2	6	3	5	1/36+16*w_4_70-16*w_4_113
3	4	1	0	4	1	3	2	6	4	5	1/36-16*w_4_112+16*w_4_70
3	4	1	0	1	2	3	3	4	3	4	-1/6+8*w_4_34
3	4	1	0	1	2	3	3	4	4	5	1/6+16*w_4_36
3	4	1	0	1	2	5	3	4	3	4	-1/6+16*w_4_48
3	4	1	0	1	2	5	3	4	4	5	1/6+16*w_4_50
3	4	1	0	2	1	3	3	4	3	4	16*w_4_34
3	4	1	0	4	1	2	3	4	3	4	-1/6+8*w_4_34
3	4	1	0	2	1	3	3	4	3	5	32*w_4_35
3	4	1	0	4	1	2	3	4	4	5	-16*w_4_35
3	4	1	0	2	1	3	3	4	4	5	32*w_4_36
3	4	1	0	4	1	2	3	4	3	5	-1/6-16*w_4_36
3	4	1	0	2	1	3	3	6	3	5	16*w_4_37
3	4	1	0	4	1	2	4	6	4	5	8*w_4_37
3	4	1	0	2	1	3	3	6	4	5	32*w_4_38
3	4	1	0	4	1	2	3	6	4	5	16*w_4_38
3	4	1	0	2	1	3	4	6	4	5	16*w_4_39
3	4	1	0	4	1	2	3	6	3	5	8*w_4_39
3	4	1	0	2	1	5	3	4	3	4	32*w_4_48
3	4	1	0	4	3	5	1	2	3	5	-1/6+16*w_4_48
3	4	1	0	2	1	5	3	4	3	5	32*w_4_49
3	4	1	0	4	3	5	1	2	4	5	16*w_4_49
3	4	1	0	2	1	5	3	4	4	5	32*w_4_50
3	4	1	0	4	3	5	1	2	3	4	-1/6-16*w_4_50
3	4	1	0	2	1	5	3	6	3	4	32*w_4_51
3	4	1	0	4	5	6	1	2	3	5	-16*w_4_51
3	4	1	0	2	1	5	3	6	3	5	32*w_4_52
3	4	1	0	4	5	6	1	2	4	5	-16*w_4_52
3	4	1	0	2	1	5	3	6	4	5	32*w_4_53
3	4	1	0	4	5	6	1	2	3	4	16*w_4_53
3	4	1	0	2	1	5	4	6	3	4	32*w_4_54
3	4	1	0	4	3	5	3	6	1	2	-16*w_4_54
3	4	1	0	2	1	5	4	6	3	5	32*w_4_55
3	4	1	0	4	3	5	4	6	1	2	-16*w_4_55
3	4	1	0	4	1	5	2	3	3	4	16*w_4_80+16*w_4_81
3	4	1	0	4	2	5	1	3	3	5	-16*w_4_80+16*w_4_82
3	4	1	0	4	1	5	2	3	3	5	16*w_4_81+16*w_4_82
3	4	1	0	4	2	5	1	3	4	5	-16*w_4_80-16*w_4_81
3	4	1	0	4	1	5	2	3	4	5	-16*w_4_80+16*w_4_82
3	4	1	0	4	2	5	1	3	3	4	16*w_4_81+16*w_4_82
3	4	1	0	4	1	5	2	4	3	4	1/18+16*w_4_68+16*w_4_88
3	4	1	0	4	2	3	1	3	3	5	-16*w_4_110-16*w_4_88
3	4	1	0	4	1	5	2	4	3	5	1/18+16*w_4_89+16*w_4_69
3	4	1	0	4	2	3	1	3	4	5	-16*w_4_89-16*w_4_109
3	4	1	0	4	1	5	2	4	4	5	1/6-16*w_4_67+16*w_4_90
3	4	1	0	4	2	3	1	3	3	4	16*w_4_108+16*w_4_90
3	4	1	0	4	1	5	2	6	3	4	16*w_4_99+16*w_4_93
3	4	1	0	4	2	5	3	6	1	3	-16*w_4_93+16*w_4_119
3	4	1	0	4	1	5	2	6	3	5	16*w_4_102+16*w_4_94
3	4	1	0	4	2	5	4	6	1	3	-16*w_4_94+16*w_4_120
3	4	1	0	4	1	5	2	6	4	5	16*w_4_95-16*w_4_96
3	4	1	0	4	2	5	3	4	1	3	16*w_4_95-16*w_4_121
3	4	1	0	4	1	5	3	4	2	3	-16*w_4_121+16*w_4_96
3	4	1	0	4	3	5	1	3	2	5	16*w_4_95-16*w_4_96
3	4	1	0	4	1	5	3	4	2	4	16*w_4_103+16*w_4_97
3	4	1	0	4	3	5	1	3	2	3	16*w_4_122-16*w_4_97
3	4	1	0	4	1	5	3	4	2	5	16*w_4_98-16*w_4_124
3	4	1	0	4	3	5	1	3	2	4	16*w_4_123-16*w_4_98
3	4	1	0	4	1	5	3	6	2	3	16*w_4_99+16*w_4_119
3	4	1	0	4	5	6	1	3	2	5	16*w_4_99+16*w_4_93
3	4	1	0	4	1	5	3	6	2	4	32*w_4_100
3	4	1	0	4	5	6	1	3	2	3	16*w_4_100
3	4	1	0	4	1	5	3	6	2	5	-16*w_4_129+16*w_4_101
3	4	1	0	4	5	6	1	3	2	4	-16*w_4_128+16*w_4_101
3	4	1	0	4	1	5	4	6	2	3	16*w_4_102+16*w_4_120
3	4	1	0	4	3	5	2	6	1	3	16*w_4_102+16*w_4_94
3	4	1	0	4	1	5	4	6	2	4	16*w_4_103+16*w_4_97
3	4	1	0	4	3	5	2	3	1	3	16*w_4_122+16*w_4_103
3	4	1	0	4	1	5	4	6	2	5	-16*w_4_134+16*w_4_104
3	4	1	0	4	3	5	2	4	1	3	-16*w_4_133+16*w_4_104
3	4	1	0	4	2	3	1	4	3	4	16*w_4_108+16*w_4_90
3	4	1	0	4	2	5	1	4	4	5	-1/6-16*w_4_108-16*w_4_67
3	4	1	0	4	2	3	1	4	3	5	16*w_4_89+16*w_4_109
3	4	1	0	4	2	5	1	4	3	5	1/18-16*w_4_109+16*w_4_69
3	4	1	0	4	2	3	1	4	4	5	16*w_4_110+16*w_4_88
3	4	1	0	4	2	5	1	4	3	4	1/18-16*w_4_110+16*w_4_68
3	4	1	0	4	2	3	1	6	3	4	32*w_4_111
3	4	1	0	4	5	6	1	6	2	5	1/6+16*w_4_111
3	4	1	0	4	2	3	1	6	3	5	16*w_4_112-16*w_4_113
3	4	1	0	4	3	5	1	6	2	5	1/36-16*w_4_112+16*w_4_70
3	4	1	0	4	2	3	1	6	4	5	-16*w_4_112+16*w_4_113
3	4	1	0	4	3	5	2	6	1	5	1/36+16*w_4_70-16*w_4_113
3	4	1	0	4	2	5	1	6	3	4	16*w_4_99+16*w_4_119
3	4	1	0	4	5	6	1	6	2	3	-16*w_4_93+16*w_4_119
3	4	1	0	4	2	5	1	6	3	5	16*w_4_102+16*w_4_120
3	4	1	0	4	3	5	1	6	2	3	16*w_4_94-16*w_4_120
3	4	1	0	4	2	5	1	6	4	5	16*w_4_121-16*w_4_96
3	4	1	0	4	3	5	2	3	1	5	16*w_4_95-16*w_4_121
3	4	1	0	4	2	5	3	4	1	4	16*w_4_122+16*w_4_103
3	4	1	0	4	2	5	4	6	1	4	-16*w_4_122+16*w_4_97
3	4	1	0	4	2	5	3	4	1	5	16*w_4_123-16*w_4_124
3	4	1	0	4	5	6	1	6	2	4	16*w_4_123-16*w_4_98
3	4	1	0	4	3	5	2	3	1	4	16*w_4_123-16*w_4_124
3	4	1	0	4	5	6	1	4	2	5	16*w_4_98-16*w_4_124
3	4	1	0	4	2	5	3	6	1	5	16*w_4_128-16*w_4_129
3	4	1	0	4	3	5	1	6	2	4	-16*w_4_128+16*w_4_101
3	4	1	0	4	3	5	2	6	1	4	-16*w_4_129+16*w_4_101
3	4	1	0	4	5	6	1	4	2	3	16*w_4_128-16*w_4_129
3	4	1	0	4	2	5	4	6	1	5	-16*w_4_134+16*w_4_133
3	4	1	0	4	3	5	2	4	1	5	-16*w_4_133+16*w_4_104
3	4	1	0	4	3	5	1	4	2	3	16*w_4_134-16*w_4_133
3	4	1	0	4	3	5	1	4	2	5	-16*w_4_134+16*w_4_104
3	4	1	0	4	3	5	2	4	1	4	32*w_4_138
3	4	1	0	4	5	6	1	4	2	4	16*w_4_138
3	4	1	0	1	2	3	3	4	3	5	16*w_4_35
3	4	1	0	1	2	3	3	6	3	5	8*w_4_37
3	4	1	0	1	2	3	3	6	4	5	16*w_4_38
3	4	1	0	1	2	3	4	6	4	5	8*w_4_39
3	4	1	0	1	2	5	3	4	3	5	16*w_4_49
3	4	1	0	1	2	5	3	6	3	4	16*w_4_51
3	4	1	0	1	2	5	3	6	3	5	16*w_4_52
3	4	1	0	1	2	5	3	6	4	5	16*w_4_53
3	4	1	0	1	2	5	4	6	3	4	16*w_4_54
3	4	1	0	1	2	5	4	6	3	5	16*w_4_55
3	4	1	0	4	2	5	3	6	1	4	16*w_4_100
3	4	1	0	4	3	5	1	4	2	4	16*w_4_138

The table below contains the output of the command

```
$ reduce_mod_jacobi associator4_intermsof10_part100.txt
```

as described in Implementation 16.

TABLE 9. Sample output of `reduce_mod_jacobi`.

```
3 4 1   0 1 0 3 2 6 3 4    -24+c_1_1221_211
3 4 1   0 4 1 3 2 6 3 4    -8+c_1_1240_111
3 4 1   0 1 0 3 2 3 4 5    -48+c_1_1221_211
                              -c_1_513_211
3 4 1   0 1 0 3 2 4 3 5    24-c_1_1221_211
3 4 1   0 4 1 3 2 4 3 5    24-c_1_1240_111
                              -c_1_540_111
3 4 1   0 1 2 3 3 4 4 5    -8-c_1_1228_111
3 4 1   0 1 2 5 3 4 3 4    -8-c_1_1228_111
3 4 1   0 2 0 3 1 4 3 5    24-c_1_1005_211
3 4 1   0 2 0 5 1 3 3 4    -24-c_1_516_211
3 4 1   0 2 0 3 1 6 3 4    -24+c_1_1005_211
3 4 1   0 2 0 5 3 6 1 3    24+c_1_516_211
3 4 1   0 2 1 3 2 3 4 5    48+c_1_1230_112
                              -c_1_1008_112
3 4 1   0 4 1 2 2 4 3 5    48-c_1_525_112
                              -c_1_1239_112
3 4 1   0 2 1 3 2 4 3 5    -24-c_1_1230_112
3 4 1   0 4 1 2 2 3 4 5    -24+c_1_1239_112
3 4 1   0 2 1 5 2 3 3 4    24-c_1_1008_112
3 4 1   0 4 2 5 1 2 3 5    -24+c_1_525_112
3 4 1   0 2 1 3 2 6 3 4    24+c_1_1230_112
3 4 1   0 4 1 2 2 6 3 4    -24+c_1_1239_112
3 4 1   0 2 1 5 3 6 2 3    -24+c_1_1008_112
3 4 1   0 4 5 6 1 2 2 5    -24+c_1_525_112
3 4 1   0 2 1 3 3 4 4 5    -16-c_1_1012_111
3 4 1   0 4 1 2 3 4 3 5    8+c_1_529_111
3 4 1   0 2 1 3 3 6 4 5    -16-c_1_1012_111
3 4 1   0 4 1 2 3 6 4 5    -8-c_1_529_111
3 4 1   0 2 1 5 3 4 3 4    -16-c_1_1012_111
3 4 1   0 4 3 5 1 2 3 5    -8-c_1_529_111
3 4 1   0 2 1 5 3 6 3 5    16+c_1_1012_111
3 4 1   0 4 5 6 1 2 4 5    -8-c_1_529_111
3 4 1   0 4 1 5 2 3 3 4    8-c_1_1023_111
3 4 1   0 4 2 5 1 3 3 5    -16+c_1_540_111
3 4 1   0 4 1 5 2 3 3 5    -8-c_1_538_111
3 4 1   0 4 2 5 1 3 4 5    -8+c_1_1021_111
3 4 1   0 4 1 5 2 3 4 5    -16+c_1_1242_111
3 4 1   0 4 2 5 1 3 3 4    -8-c_1_1245_111
3 4 1   0 4 1 5 2 4 3 5    24-c_1_536_111
                              -c_1_1242_111
3 4 1   0 4 2 3 1 3 4 5    -24-c_1_1245_111
                              +c_1_1019_111
3 4 1   0 4 1 5 2 6 3 4    -16+c_1_1242_111
3 4 1   0 4 2 5 3 6 1 3    8+c_1_1245_111
3 4 1   0 4 1 5 2 6 3 5    -8-c_1_538_111
3 4 1   0 4 2 5 4 6 1 3    -8+c_1_1021_111
3 4 1   0 4 1 5 3 4 2 5    -16+c_1_1242_111
3 4 1   0 4 3 5 1 3 2 4    8+c_1_1245_111
3 4 1   0 4 1 5 3 6 2 3    -8+c_1_1023_111
3 4 1   0 4 5 6 1 3 2 5    -16+c_1_540_111
3 4 1   0 4 1 5 3 6 2 4    32-c_1_1242_111
                              -c_1_540_111
3 4 1   0 4 5 6 1 3 2 3    16-c_1_1023_111
                              +c_1_1245_111
3 4 1   0 4 1 5 4 6 2 3    -16+c_1_1242_111
3 4 1   0 4 3 5 2 6 1 3    -8-c_1_1245_111
3 4 1   0 4 2 3 1 4 3 5    24+c_1_538_111
                              -c_1_1019_111
3 4 1   0 4 2 3 1 6 3 4    -16+c_1_1019_111
3 4 1   0 4 5 6 1 6 2 5    -8+c_1_536_111
3 4 1   0 4 2 5 1 6 3 4    -8+c_1_1021_111
3 4 1   0 4 5 6 1 6 2 3    8+c_1_538_111
3 4 1   0 4 2 5 1 6 3 5    -16+c_1_540_111
3 4 1   0 4 3 5 1 6 2 3    8-c_1_1023_111
3 4 1   0 4 2 5 3 4 1 5    -8+c_1_1021_111
3 4 1   0 4 5 6 1 6 2 4    8+c_1_538_111
3 4 1   0 4 3 5 2 3 1 4    -8+c_1_1023_111
3 4 1   0 4 5 6 1 4 2 5    -16+c_1_540_111
3 4 1   0 2 0 3 1 3 4 5    -48+c_1_1005_211-c_1_516_211
3 4 1   0 1 0 5 2 3 3 4    -24-c_1_513_211
3 4 1   0 1 0 5 3 6 2 3    24+c_1_513_211
3 4 1   0 1 2 3 3 6 4 5    -8-c_1_1228_111
3 4 1   0 1 2 5 3 6 3 5    8+c_1_1228_111
3 4 1   0 4 2 5 3 6 1 4    16+c_1_538_111-c_1_1021_111
3 4 1   0 4 1 3 2 3 4 5    -c_1_1023_111+c_1_1240_111
3 4 1   0 4 2 5 1 4 3 5    c_1_536_111-c_1_1021_111

3 4 1   0 1 2 3 0 3 5 4    c_1_513_211==-24
3 4 1   0 2 1 3 0 3 5 4    c_1_516_211==-24
3 4 1   1 2 2 3 0 3 5 4    c_1_525_112==24
3 4 1   1 2 3 5 0 3 5 4    c_1_529_111==-8
3 4 1   1 4 2 3 0 3 5 4    c_1_536_111==8
3 4 1   1 4 2 5 0 3 5 4    c_1_538_111==-8
3 4 1   1 5 2 3 0 3 5 4    c_1_540_111==16
3 4 1   0 2 0 3 1 3 5 4    c_1_1005_211==24
3 4 1   0 2 2 3 1 3 5 4    c_1_1008_112==24
3 4 1   0 2 3 5 1 3 5 4    c_1_1012_111==-16
3 4 1   0 4 2 3 1 3 5 4    c_1_1019_111==16
3 4 1   0 4 2 5 1 3 5 4    c_1_1021_111==8
3 4 1   0 5 2 3 1 3 5 4    c_1_1023_111==8
3 4 1   0 1 0 3 2 3 5 4    c_1_1221_211==24
3 4 1   0 1 3 5 2 3 5 4    c_1_1228_111==-8
3 4 1   0 2 1 3 2 3 5 4    c_1_1230_112==-24
3 4 1   0 4 1 2 2 3 5 4    c_1_1239_112==24
3 4 1   0 4 1 3 2 3 5 4    c_1_1240_111==8
3 4 1   0 4 1 5 2 3 5 4    c_1_1242_111==16
3 4 1   0 5 1 3 2 3 5 4    c_1_1245_111==-8
```

The first part of the output lists the graph series $S^{(1)} - \Diamond$, reduced modulo skew-symmetry, wherein the coefficients of $\Diamond$ are still undetermined. The second part of the ouput (after the blank line) specifies the coefficients such that $S^{(1)} = \Diamond$. Every coefficient in the second part is preceded by the encoding of the graph that specifies a differential consequence of the Jacobi identity. Such a differential consequence expands into a sum of graphs that can be read in the first part of the output.

Appendix E. Gauge transformation that removes 4 master-parameters out of 10

Encodings of graphs (see Implementation 1 on p. 5) built over one sink vertex are followed by their coefficients, in the following table containing the gauge transformation which was claimed to exist in Theorem 14.

Table 10. Gauge transformation that removes 4 master-parameters out of 10.

```
h^0:
1 0 1                       1
h^4:
1 4 1   0 2 0 3 1 4 0 3     16*w_4_101
1 4 1   0 2 0 3 1 4 1 3     8*w_4_101
1 4 1   0 2 0 3 1 4 2 3     8*w_4_101
1 4 1   0 2 1 3 0 4 1 2     -8*w_4_101
1 4 1   0 2 1 3 0 4 2 3     8*w_4_101
1 4 1   0 2 1 3 1 4 0 2     -8*w_4_101
1 4 1   0 2 1 3 2 4 0 2     -8*w_4_101
1 4 1   0 2 0 3 0 4 1 3     -16*w_4_102
1 4 1   0 2 0 3 1 4 1 3     -8*w_4_102
1 4 1   0 2 0 3 2 4 1 2     -8*w_4_102
1 4 1   0 2 0 3 2 4 1 3     -16*w_4_102
1 4 1   0 2 1 3 0 4 1 2     -8*w_4_102
1 4 1   0 2 1 3 0 4 1 3     -8*w_4_102
1 4 1   0 2 0 3 0 4 1 2     16*w_4_119
1 4 1   0 2 0 3 1 4 1 2     16*w_4_119
1 4 1   0 2 0 3 1 4 1 3     8*w_4_119
1 4 1   0 2 0 3 2 4 1 2     8*w_4_119
1 4 1   0 2 1 3 0 4 1 2     8*w_4_119
1 4 1   0 2 3 4 0 4 1 2     -8*w_4_119
1 4 1   0 2 0 3 0 1 1 2     -32*w_4_125
1 4 1   0 2 0 3 1 2 1 2     16*w_4_125
1 4 1   0 2 0 3 1 2 1 3     -16*w_4_125
1 4 1   0 2 0 3 1 2 2 3     16*w_4_125
1 4 1   0 2 0 3 1 4 1 2     16*w_4_125
1 4 1   0 2 0 3 1 4 1 3     -16*w_4_125
1 4 1   0 2 0 3 1 4 2 3     16*w_4_125
```