DERIVED EQUIVALENCES FOR CLUSTER-TILTED ALGEBRAS OF DYNKIN TYPE D

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Abstract. We provide a far reaching derived equivalence classification of cluster-tilted algebras of Dynkin type D. We introduce another notion of equivalence called *good mutation* equivalence which is slightly stronger than derived equivalence but is algorithmically more tractable, and give a complete classification together with normal forms. We also suggest normal forms for the derived equivalence classes, but some subtle questions in the derived equivalence classification remain open.

Contents

Key words and phrases. Cartan matrix, Cartan determinant, cluster tilted algebra, cluster tilting object, derived category, derived equivalence, Dynkin diagram, finite representation type, quiver mutation, tilting complex.

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Introduction

Cluster categories have been introduced in [9] (see also [14] for Dynkin type A) as a representation-theoretic approach to Fomin and Zelevinsky's cluster algebras without coefficients having skew-symmetric exchange matrices (so that matrix mutation becomes the combinatorial recipe of mutation of quivers). This highly successful approach allows to use deep algebraic and representation-theoretic methods in the context of cluster algebras. A crucial role is played by the so-called cluster tilting objects in the cluster category which model the clusters in the cluster algebra. The endomorphism algebras of these cluster tilting objects are called cluster-tilted algebras.

Cluster-tilted algebras are particularly well-understood if the quiver underlying the cluster algebra, and hence the cluster category, is of Dynkin type. Cluster-tilted algebras of Dynkin type can be described as quivers with relations where the possible quivers are precisely the quivers in the mutation class of the Dynkin quiver, and the relations are uniquely determined by the quiver in an explicit way [11]. By a result of Fomin and Zelevinsky [18], the mutation class of a Dynkin quiver is finite. Moreover, the quivers in the mutation classes of Dynkin quivers are explicitly known; for type A_n they can be found in [13], for type D_n in [31] and for type $E_{6,7,8}$ they can be enumerated using a computer, for example by the Java applet [22].

However, despite knowing the cluster-tilted algebras of Dynkin type as quivers with relations, many structural properties are not understood yet. One important structural aspect is to understand the derived module categories of the cluster-tilted algebras. In particular, one would want to know when two cluster-tilted algebras have equivalent derived categories. A derived equivalence classification has been achieved so far for cluster-tilted algebras of Dynkin type A_n by Buan and Vatne [13], and for Dynkin type $E_{6,7,8}$ by the authors [5]. Moreover, a complete derived equivalence classification has also been given by the first author for cluster-tilted algebras of extended Dynkin type $\tilde{A}_n$ [4].

In the present paper we are going to address this problem for cluster-tilted algebras of Dynkin type D_n. We shall obtain a far reaching derived equivalence classification, see Theorem 2.3. This classification is complete up to D_{14}, but it will turn out to be surprisingly subtle to distinguish certain of the cluster-tilted algebras up to derived equivalence.

There are two natural approaches to address derived equivalence classification problems of a given collection of algebras arising from some combinatorial data. The bottom-to-top approach is to systematically construct, based on the combinatorial data, derived equivalences between pairs of these algebras and then to arrange these algebras into groups where any two algebras are related by a sequence of such derived equivalences. The top-to-bottom approach is to divide the algebras into equivalence classes according to some invariants of derived equivalence, so that algebras belonging to different classes are not derived equivalent. To obtain a complete derived equivalence classification one has to combine these approaches and hope that the two resulting partitions of the entire collection of algebras coincide.

Since any two quivers in a mutation class are connected by a sequence of mutations, it is natural to ask when mutation of quivers leads to derived equivalence of their corresponding cluster-tilted algebras. The third author [25] has presented a procedure to determine when two cluster-tilted algebras whose quivers are related by a mutation are also related by Brenner-Butler (co-)tilting, which is a particular kind of derived equivalence. We call such quiver mutations *good mutations*. Obviously, the cluster-tilted algebras of quivers connected by a sequence of good mutations (i.e. *good mutation equivalent*) are derived equivalent. The explicit knowledge of the relations for cluster-tilted algebras of Dynkin type implies that good mutation equivalence is decidable for these algebras. In particular, we can achieve a complete classification of the cluster-tilted algebras of Dynkin type D_n, see Theorem 2.32.

Whereas for cluster-tilted algebras of Dynkin types A_n and $E_{6,7,8}$ the notions of good mutation equivalence and derived equivalence coincide (see Theorem 2.2 below for type A, and [5, Theorem 1.1] for type E) allowing for a complete derived equivalence classification, this is not the case for Dynkin type D which underlines and explains why the situation in type D is much more complicated.

One aspect of this complication is the fact that cluster-tilted algebras of Dynkin type D might be derived equivalent without being connected by a sequence of good mutations. This occurs already for types D_6 and D_8, see Examples 2.14 and 2.15 below. Although we have been able to find further systematic derived equivalences, one cannot be sure that these are all. Another aspect demonstrating the latter point is that when trying to apply the top-to-bottom approach by using the equivalence class of the integral bilinear form defined by the Cartan matrix as derived invariant to distinguish the different derived equivalence classes, one encounters arbitrarily large sets of cluster-tilted algebras with the same derived invariant but for which we cannot determine their derived equivalence, see Section 2.5.

The paper is organized as follows. In Section 1 we collect some preliminaries about invariants of derived equivalences, mutations of algebras and fundamental properties of cluster-tilted algebras, particularly of Dynkin types A and D. These are needed for the statements of our main results, which are given in Section 2. In particular, Theorem 2.3 gives the far reaching derived equivalence classification of cluster-tilted algebras of type D, and Theorem 2.32 the complete classification up to good mutation equivalence. Examples and some open questions are also given in that section. In Section 3 we determine all the good mutations for cluster-tilted algebras of Dynkin types A and D, whereas in Section 4 we present further derived equivalences between cluster-tilted algebras of type D which are not given by good mutations. Building on these results we provide in Section 5, which is purely combinatorial, standard forms for derived equivalence as well as ones for good mutation equivalence of cluster-tilted algebras of type D, thus proving Theorem 2.3 and Theorem 2.32. We also describe an explicit algorithm which decides on good mutation equivalence. Finally, the appendix contains the proof of the formulae for the determinants of the Cartan matrices of cluster-tilted algebras of type D, as given in Theorem 2.5. This invariant is used in the paper to distinguish some cluster-tilted algebras up to derived equivalence.

1. Preliminaries

1.1. **Derived equivalences and tilting complexes.** Throughout this paper let K be an algebraically closed field. All algebras are assumed to be finite-dimensional K-algebras.

For a K-algebra A, we denote the bounded derived category of right A-modules by $\mathcal{D}^b(A)$. Two algebras A and B are called *derived equivalent* if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

A famous theorem of Rickard [29] characterizes derived equivalence in terms of the so-called tilting complexes, which we now recall. Denote by $\operatorname{per} A$ the full triangulated subcategory of $\mathcal{D}^b(A)$ consisting of the *perfect* complexes of A-modules, that is, complexes (quasi-isomorphic) to bounded complexes of finitely generated projective A-modules.

Definition 1.1. A *tilting complex* T over A is a complex $T \in \operatorname{per} A$ with the following two properties:

(i) It is *exceptional*, i.e. $\operatorname{Hom}_{\mathcal{D}^b(A)}(T, T[i]) = 0$ for all $i \neq 0$, where $[1]$ denotes the shift functor in $\mathcal{D}^b(A)$;

(ii) It is a *compact generator*, that is, the minimal triangulated subcategory of $\operatorname{per} A$ containing T and closed under taking direct summands, equals $\operatorname{per} A$.

Theorem 1.2 (Rickard [29])**.** *Two algebras A and B are derived equivalent if and only if there exists a tilting complex T over A such that* $\operatorname{End}_{\mathcal{D}^b(A)}(T) \simeq B$.

Although Rickard's theorem gives us a criterion for derived equivalence, it does not give a decision process nor a constructive method to produce tilting complexes. Thus, given two algebras A and B in concrete form, it is sometimes still unknown whether they are derived equivalent or not, as we do not know how to construct a suitable tilting complex or to prove the non-existence of such, see Sections 2.4 and 2.5 for some concrete examples.

1.2. **Invariants of derived equivalence.** Let $P_1, \dots, P_n$ be a complete collection of pairwise non-isomorphic indecomposable projective A-modules (finite-dimensional over K). The *Cartan matrix* of A is then the $n \times n$ matrix C_A defined by $(C_A)_{ij} = \dim_K \operatorname{Hom}(P_j, P_i)$. An important invariant of derived equivalence is given by the following well known proposition. For a proof see the *proof* of Proposition 1.5 in [6], and also [5, Prop. 2.6].

Proposition 1.3. *Let A and B be two finite-dimensional, derived equivalent algebras. Then the matrices C_A and C_B represent equivalent bilinear forms over $\mathbb{Z}$, that is, there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PC_AP^T = C_B$, where n denotes the number of indecomposable projective modules of A and B (up to isomorphism).*

In general, to decide whether two integral bilinear forms are equivalent is a very subtle arithmetical problem. Therefore, it is useful to introduce somewhat weaker invariants that are computationally easier to handle. In order to do this, assume further that C_A is invertible over $\mathbb{Q}$. In this case one can consider the rational matrix $S_A = C_A C_A^{-T}$ (here C_A^{-T} denotes the inverse of the transpose of C_A), known in the theory of non-symmetric bilinear forms as the *asymmetry* of C_A.

Proposition 1.4. *Let A and B be two finite-dimensional, derived equivalent algebras with invertible (over $\mathbb{Q}$) Cartan matrices. Then we have the following assertions, each implied by the preceding one:*

(a) *There exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PC_AP^T = C_B$.*

(b) *There exists* $P \in \mathrm{GL}_n(\mathbb{Z})$ *such that* $PS_AP^{-1} = S_B$.
(c) *There exists* $P \in \mathrm{GL}_n(\mathbb{Q})$ *such that* $PS_AP^{-1} = S_B$.
(d) *The matrices* S_A *and* S_B *have the same characteristic polynomial.*

For proofs and discussion, see for example [24, Section 3.3]. Since the determinant of an integral bilinear form is also invariant under equivalence, we obtain the following discrete invariant of derived equivalence.

Definition 1.5. For an algebra A with invertible Cartan matrix C_A over $\mathbb{Q}$, we define its *associated polynomial* as $(\det C_A) \cdot \chi_{S_A}(x)$, where $\chi_{S_A}(x)$ is the characteristic polynomial of the asymmetry matrix $S_A = C_A C_A^{-T}$.

Remark 1.6. The matrix S_A (or better, minus its transpose $-C_A^{-1}C_A^T$) is related to the *Coxeter transformation* which has been widely studied in the case when A has finite global dimension (so that C_A is invertible over $\mathbb{Z}$), see [28]. It is the K-theoretic shadow of the Serre functor and the related Auslander-Reiten translation in the derived category. The characteristic polynomial is then known as the *Coxeter polynomial* of the algebra.

Remark 1.7. In general, S_A might have non-integral entries. However, when the algebra A is *Gorenstein*, the matrix S_A is integral, which is an incarnation of the fact that the injective modules have finite projective resolutions. By a result of Keller and Reiten [23], this is the case for cluster-tilted algebras.

1.3. **Mutations of algebras.** We recall the notion of mutations of algebras from [25]. These are local operations on an algebra A producing new algebras derived equivalent to A.

Let $A = KQ/I$ be an algebra given as a quiver with relations. For any vertex i of Q, there is a trivial path e_i of length 0; the corresponding indecomposable projective $P_i = e_iA$ is spanned by the images of the paths starting at i. Thus an arrow $i \xrightarrow{\alpha} j$ gives rise to a map $P_j \to P_i$ given by left multiplication with α.

Let k be a vertex of Q without loops. Consider the following two complexes of projective A-modules

$$T_k^-(A) = \big(P_k \xrightarrow{f} \bigoplus_{j \to k} P_j\big) \oplus \big(\bigoplus_{i \neq k} P_i\big), \qquad\qquad T_k^+(A) = \big(\bigoplus_{k \to j} P_j \xrightarrow{g} P_k\big) \oplus \big(\bigoplus_{i \neq k} P_i\big)$$

where the map f is induced by all the maps $P_k \to P_j$ corresponding to the arrows $j \to k$ ending at k, the map g is induced by the maps $P_j \to P_k$ corresponding to the arrows $k \to j$ starting at k, the term P_k lies in degree -1 in $T_k^-(A)$ and in degree 1 in $T_k^+(A)$, and all other terms are in degree 0.

Definition 1.8. Let A be an algebra given as a quiver with relations and k a vertex without loops.

(a) We say that the negative mutation of A at k is *defined* if $T_k^-(A)$ is a tilting complex over A. In this case, we call $\mu_k^-(A) = \mathrm{End}_{\mathcal{D}^b(A)} T_k^-(A)$ the *negative mutation* of A at the vertex k.
(b) We say that the positive mutation of A at k is *defined* if $T_k^+(A)$ is a tilting complex over A. In this case, we call $\mu_k^+(A) = \mathrm{End}_{\mathcal{D}^b(A)} T_k^+(A)$ the *positive mutation* of A at the vertex k.

Remark 1.9. By Rickard's Theorem 1.2, the negative and the positive mutations of an algebra A at a vertex, when defined, are always derived equivalent to A.

There is a combinatorial criterion to determine whether a mutation at a vertex is defined, see [25, Prop. 2.3]. Since the algebras we will be dealing with in this paper are schurian, we state here the criterion only for this case, as it takes a particularly simple form. Recall that an algebra is *schurian* if the entries of its Cartan matrix are only 0 or 1.

Proposition 1.10. *Let A be a schurian algebra.*

(a) *The negative mutation $\mu_k^-(A)$ is defined if and only if for any non-zero path $k \rightsquigarrow i$ starting at k and ending at some vertex i, there exists an arrow $j \to k$ such that the composition $j \to k \rightsquigarrow i$ is non-zero.*
(b) *The positive mutation $\mu_k^+(A)$ is defined if and only if for any non-zero path $i \rightsquigarrow k$ starting at some vertex i and ending at k, there exists an arrow $k \to j$ such that the composition $i \rightsquigarrow k \to j$ is non-zero.*

Remark 1.11. It follows from [25, Remark 2.10], that when A is schurian, the negative mutation of A at k is defined if and only if one can associate with k the corresponding Brenner-Butler tilting module. Moreover, in this case, $T_k^-(A)$ is isomorphic in $\mathcal{D}^b(A)$ to that Brenner-Butler tilting module.

1.4. **Cluster-tilted algebras.** In this section we assume that all quivers are without loops and 2-cycles. Given such a quiver Q and a vertex k, we denote by $\mu_k(Q)$ the Fomin-Zelevinsky quiver mutation [17] of Q at k. Two quivers are called *mutation equivalent* if one can be reached from the other by a finite sequence of quiver mutations. The *mutation class* of a quiver Q is the set of all quivers which are mutation equivalent to Q.

For a quiver Q' without oriented cycles, the corresponding cluster category $\mathcal{C}_{Q'}$ was introduced in [9]. A *cluster-tilted algebra* of *type* Q' is an endomorphism algebra of a cluster-tilting object in $\mathcal{C}_{Q'}$, see [10]. It is known by [10] that for any quiver Q mutation equivalent to Q', there is a cluster-tilted algebra whose quiver is Q. Moreover, by [8], it is unique up to isomorphism. Hence, there is a bijection between the quivers in the mutation class of an acyclic quiver Q' and the isomorphism classes of cluster-tilted algebras of type Q'. This justifies the following notation.

Notation 1.12. Throughout the paper, for a quiver Q which is mutation equivalent to an acyclic quiver, we denote by Λ_Q the corresponding cluster-tilted algebra and by C_Q its Cartan matrix C_{Λ_Q}.

When Q' is a Dynkin quiver of types A, D or E, the corresponding cluster-tilted algebras are said to be of Dynkin type. These algebras have been investigated in [11], where it is shown that they are schurian and moreover they can be defined by using only zero and commutativity relations that can be extracted from their quivers in an algorithmic way.

1.5. **Good quiver mutations.** For cluster-tilted algebras of Dynkin type, the statement of Theorem 5.3 in [25], linking more generally mutation of cluster-tilting objects in 2-Calabi-Yau categories with mutations of their endomorphism algebras, takes the following form.

Proposition 1.13. *Let Q be mutation equivalent to a Dynkin quiver and let k be a vertex of Q.*

(a) $\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q)$ *if and only if the two algebra mutations* $\mu_k^-(\Lambda_Q)$ *and* $\mu_k^+(\Lambda_{\mu_k(Q)})$ *are defined.*

(b) $A_{\mu_k(Q)} \simeq \mu_k^+(\Lambda_Q)$ *if and only if the two algebra mutations* $\mu_k^+(\Lambda_Q)$ *and* $\mu_k^-(\Lambda_{\mu_k(Q)})$ *are defined.*

This motivates the following definition.

Definition 1.14. When one of the conditions in the proposition holds, we say that the quiver mutation of Q at k, is *good*, since it implies the derived equivalence of the corresponding cluster-tilted algebras Λ_Q and $\Lambda_{\mu_k(Q)}$. When none of the conditions in the proposition hold, we say that the quiver mutation is *bad*.

Remark 1.15. In view of Propositions 1.10 and 1.13, there is an algorithm which decides, given a quiver which is mutation equivalent to a Dynkin quiver, whether a mutation at a vertex is good or not.

Whereas in Dynkin types A and E, the quivers of any two derived equivalent cluster-tilted algebras are connected by a sequence of good mutations [5], this is no longer the case in type D. Therefore, we need also to consider mutations of algebras going beyond the family of cluster-tilted algebras (which is obviously not closed under derived equivalence).

Definition 1.16. Let Q and Q' be quivers with vertices k and k' such that $\mu_k(Q) = \mu_{k'}(Q')$. We call the sequence of the two mutations from Q to Q' (first at k and then at k') a *good double mutation* if both algebra mutations $\mu_k^-(\Lambda_Q)$ and $\mu_{k'}^+(\Lambda_{Q'})$ are defined and moreover, they are isomorphic to each other.

By definition, for quivers Q and Q' related by a good double mutation, the cluster-tilted algebras Λ_Q and $\Lambda_{Q'}$ are derived equivalent. Note, however, that we do not require the intermediate algebra $\mu_k^-(\Lambda_Q) \simeq \mu_{k'}^+(\Lambda_{Q'})$ to be a cluster-tilted algebra.

1.6. **Cluster-tilted algebras of Dynkin types A and D.** In this section we recall the explicit description of cluster-tilted algebras of Dynkin types A and D, which are our main objects of study, as quivers with relations.

Recall that a the quiver A_n is the following directed graph on $n \geq 1$ vertices

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n \ .$$

The quivers which are mutation equivalent to A_n have been explicitly determined in [13]. They can be characterized as follows.

Definition 1.17. The *neighborhood* of a vertex x in a quiver Q is the full subquiver of Q on the subset of vertices consisting of x and the vertices which are targets of arrows starting at x or sources of arrows ending at x.

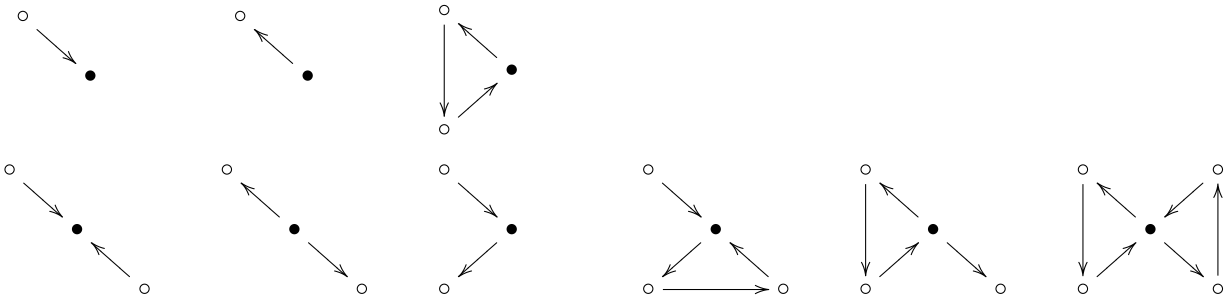


FIGURE 1. The 9 possible neighborhoods of a vertex $\bullet$ in a quiver which is mutation equivalent to A_n, $n \geq 2$. The three at the top row are the possible neighborhoods of a *root* in a rooted quiver of type A.

Proposition 1.18. *Let $n \geq 2$. A quiver is mutation equivalent to A_n if and only if it has n vertices, the neighborhood of each vertex is one of the nine depicted in Figure 1, and there are no cycles in its underlying graph apart from those induced by oriented cycles contained in neighborhoods of vertices.*

Definition 1.19. Let Q be a quiver mutation equivalent to A_n. A *triangle* is an oriented 3-cycle in Q, and a *line* is an arrow in Q which is not part of a triangle. We denote by $s(Q)$ and $t(Q)$ the number of lines and triangles in Q, respectively.

Remark 1.20. We have $n = 1 + s(Q) + 2t(Q)$.

Remark 1.21. Given a quiver Q mutation equivalent to A_n, the *relations* defining the corresponding cluster-tilted algebra Λ_Q (which has Q as its quiver) are obtained as follows [11, 14, 15]; any triangle

$$\begin{tikzcd} & \bullet \arrow[dr, "\beta"] & \\ \bullet \arrow[ur, "\alpha"] & & \bullet \arrow[ll, "\gamma"] \end{tikzcd}$$

in Q gives rise to three zero relations $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, and there are no other relations.

Recall that the quiver D_n is the following quiver

$$\begin{tikzcd} \bullet_1 \arrow[dr] & & & \\ & \bullet_3 \arrow[r] & \cdots \arrow[r] & \bullet_n \\ \bullet_2 \arrow[ur] & & & \end{tikzcd}$$

on $n \geq 4$ vertices. We now recall the description by Vatne [31] of the quivers which are mutation equivalent to D_n, and the relations defining the corresponding cluster-tilted algebras following [11]. It would be most convenient to use the language of gluing of rooted quivers.

Definition 1.22. A *rooted quiver of type A* is a pair (Q, v) where Q is a quiver which is mutation equivalent to A_n for some $n \geq 1$, and v is a vertex of Q (the *root*) whose neighborhood is one of the three appearing in the first row of Figure 1 if $n \geq 2$.

By abuse of notation, we shall sometimes refer to such a rooted quiver (Q, v) just by Q.

Definition 1.23. Let Q_0 be a quiver, called a *skeleton*, and let $c_1, c_2, \dots, c_k$ be $k \geq 0$ distinct vertices of Q_0. The *gluing* of k rooted quivers of type A, say $(Q_1, v_1), (Q_2, v_2), \dots, (Q_k, v_k)$, to Q_0 at the vertices $c_1, \dots, c_k$ is defined as the quiver obtained from the disjoint union $Q_0 \sqcup Q_1 \sqcup \dots \sqcup Q_k$ by identifying each vertex c_i with the corresponding root v_i, for $1 \leq i \leq k$.

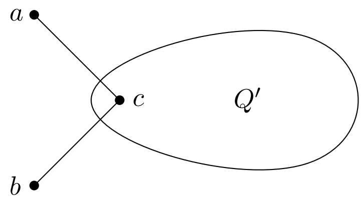
Remark 1.24. Given relations (i.e. linear combinations of parallel paths) on the skeleton Q_0, they induce relations on the gluing, namely by taking the union of all the relations on $Q_0, Q_1, \dots, Q_k$, where the relations on the rooted quivers of type A are those stated in Remark 1.21.

A cluster-tilted algebra of Dynkin type D belongs to one of the following four families, which are called *types* and are defined as gluing of rooted quivers of type A to certain skeleta. Note that in view of Remark 1.24, it is enough to specify the relations on the skeleton. For each type, we define *parameters* which will be useful in the sequel when referring to the cluster-tilted algebras of that type.

Type I. The gluing of a rooted quiver Q' of type A at the vertex c of one of the three skeleta

$$
\begin{tikzcd}[column sep=small] \bullet_a \arrow[dr] & \\ & \bullet_c \arrow[dl] \\ \bullet_b & \end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small] \bullet_a \arrow[dr] & \\ & \bullet_c \\ \bullet_b \arrow[ur] & \end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small] \bullet_a & \\ & \bullet_c \arrow[ul] \arrow[dl] \\ \bullet_b & \end{tikzcd}
$$

as in the following picture:

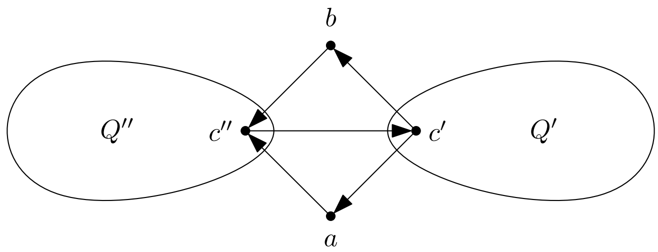


The *parameters* are $\big(s(Q'), t(Q')\big)$.

Type II. The gluing of two rooted quivers Q' and Q'' of type A at the vertices c' and c'', respectively, of the following skeleton

$$
\begin{tikzcd} & \bullet_b \arrow[dl, "\beta"'] & \\ \bullet_{c''} \arrow[rr, "\varepsilon"] & & \bullet_{c'} \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & \bullet_a \arrow[ul, "\delta"] & \end{tikzcd}
$$

with the commutativity relation $\alpha\beta - \gamma\delta$ and the zero relations $\varepsilon\alpha$, $\varepsilon\gamma$, $\beta\varepsilon$, $\delta\varepsilon$ as in the following picture:

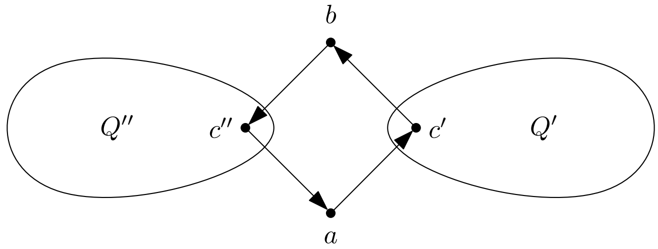


The *parameters* are $\big(s(Q'), t(Q'), s(Q''), t(Q'')\big)$.

Type III. The gluing of two rooted quivers Q' and Q'' of type A at the vertices c' and c'', respectively, of the following skeleton

$$
\begin{tikzcd} & \bullet_b \arrow[dl, "\beta"'] & \\ \bullet_{c''} \arrow[dr, "\gamma"'] & & \bullet_{c'} \arrow[ul, "\alpha"'] \\ & \bullet_a \arrow[ur, "\delta"'] & \end{tikzcd}
$$

with the four zero relations $\alpha\beta\gamma$, $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, as in the following picture:



As in type II, the *parameters* are $\big(s(Q'), t(Q'), s(Q''), t(Q'')\big)$.

Type IV. The gluing of $r \geq 0$ rooted quivers $Q^{(1)}, \dots, Q^{(r)}$ of type A at the vertices $c_1, \dots, c_r$ of a skeleton $Q(m, \{i_1, \dots, i_r\})$ defined below, see Figure 2.

Definition 1.25. Given integers $m \geq 3$, $r \geq 0$ and an increasing sequence $1 \leq i_1 < i_2 < \cdots < i_r \leq m$, we define the following quiver $Q(m, \{i_1, \dots, i_r\})$ with relations.

(a) $Q(m, \{i_1, \dots, i_r\})$ has $m + r$ vertices, labeled $1, 2, \dots, m$ together with $c_1, c_2, \dots, c_r$, and its arrows are

$$\big\{i \to (i+1)\big\}_{1 \leq i \leq m} \cup \big\{c_j \to i_j, (i_j + 1) \to c_j\big\}_{1 \leq j \leq r},$$

where $i + 1$ is considered modulo m, i.e. 1, if $i = m$.

The full subquiver on the vertices $1, 2, \dots, m$ is thus an oriented cycle of length m, called the *central cycle*, and for every $1 \leq j \leq r$, the full subquiver on the vertices $i_j, i_j + 1, c_j$ is an oriented 3-cycle, called a *spike*.

(b) The *relations* on $Q(m, \{i_1, \dots, i_r\})$ are as follows:
- The paths $i_j, i_j + 1, c_j$ and $c_j, i_j, i_j + 1$ are zero for all $1 \leq j \leq r$;
- For any $1 \leq j \leq r$, the path $i_j + 1, c_j, i_j$ equals the path $i_j + 1, \dots, 1, \dots, i_j$ of length $m - 1$ along the central cycle;
- For any $i \notin \{i_1, \dots, i_r\}$, the path $i + 1, \dots, 1, \dots, i$ of length $m - 1$ along the central cycle is zero.

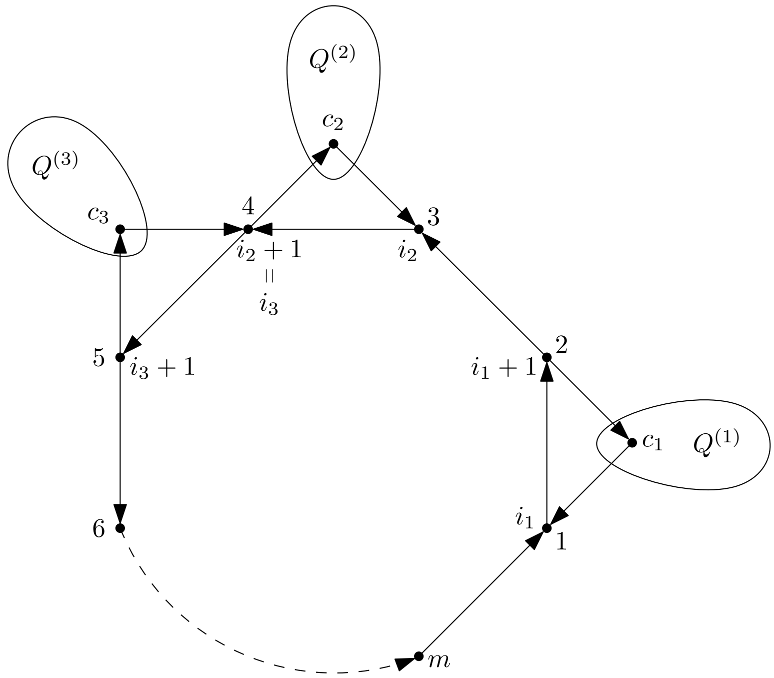


FIGURE 2. A quiver of a cluster-tilted algebra of type IV.

The *parameters* are encoded as follows. If $r = 0$, that is, there are no spikes hence no attached rooted quivers of type A, the quiver is just an oriented cycle, thus parameterized by its length $m \geq 3$. In all other cases, due to rotational symmetry, we define the *distances* $d_1, d_2, \dots, d_r$ by

$$d_1 = i_2 - i_1, d_2 = i_3 - i_2, \dots, d_r = i_1 + m - i_r$$

so that $m = d_1 + d_2 + \cdots + d_r$, and encode the cluster-tilted algebra by the sequence of triples

$$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big) \tag{1.1}$$

where $s_j = s(Q^{(j)})$, $t_j = t(Q^{(j)})$ are the numbers of lines and triangles of the rooted quiver $Q^{(j)}$ of type A glued at the vertex c_j of the j-th spike.

Remark 1.26. Note that the cluster-tilted algebras in type III can be viewed as a degenerate version of type IV, namely corresponding to the skeleton $Q(2, \{1, 1\})$ with central cycle of length 2 (hence it is "invisible") with all spikes present. It turns out that this point of view is consistent with the constructions of good mutations and double mutations as well as with the determinant computations presented later in this paper. However, for simplicity, the proofs that we give for type III will not rely on this observation.

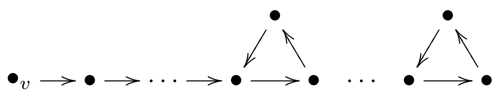
2. Main results

In this section we describe the main results of the paper.

2.1. **Derived equivalences.** We start by providing standard forms for derived equivalence. Since rooted quivers of Dynkin type A are important building blocks of the quivers of cluster-tilted algebras of type D, we recall the results on derived equivalence classification of cluster-tilted algebras of type A, originally due to Buan and Vatne [13].

Definition 2.1. Let Q be a quiver of a cluster-tilted algebra of type A. The *standard form* of Q is the following quiver consisting of $s(Q)$ lines and $t(Q)$ triangles arranged as follows:

(2.1)



The *standard form* of a rooted quiver (Q, v) of type A is a rooted quiver of type A as in (2.1) consisting of $s(Q)$ lines and $t(Q)$ triangles with the vertex v as the root.

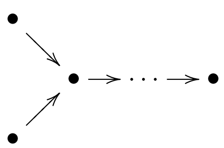
The name "standard form" is justified by the next theorem which follows from the results of [13], see also Section 3.

Theorem 2.2. *Let Q be a quiver of a cluster-tilted algebra of Dynkin type A. Then Q can be transformed via a sequence of good mutations to its standard form. Moreover, two standard forms are derived equivalent if and only if they coincide.*

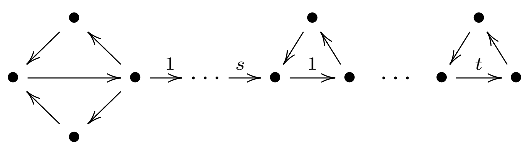
In Dynkin type D, we suggest the following standard forms.

Theorem 2.3. *A cluster-tilted algebra of type D with n vertices is derived equivalent to one of the cluster-tilted algebras with the following quivers, which we call "standard forms" for derived equivalence:*

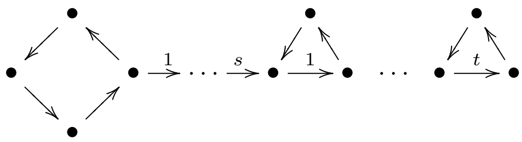
(a) *D_n (i.e. type I with a linearly oriented A_{n-2} quiver attached);*



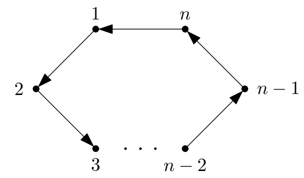
(b) *Type II as in the following figure, where $s, t \geq 0$ and $s + 2t = n - 4$;*



(c) *Type III as in the following figure, where* $s, t \geq 0$ *and* $s + 2t = n - 4$*;*



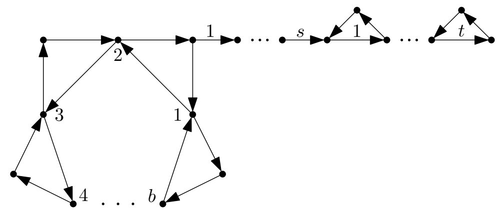
(d_1) *(only when* n *is odd) Type IV with a central cycle of length* n *without spikes, as in the following picture:*



(d_2) *Type IV with parameter sequence*

$$\big((1, s, t), (1, 0, 0), \dots, (1, 0, 0)\big)$$

of length $b \geq 3$*, with* $s, t \geq 0$ *such that* $n = 2b + s + 2t$*, and the attached rooted quiver of type A is in standard form;*



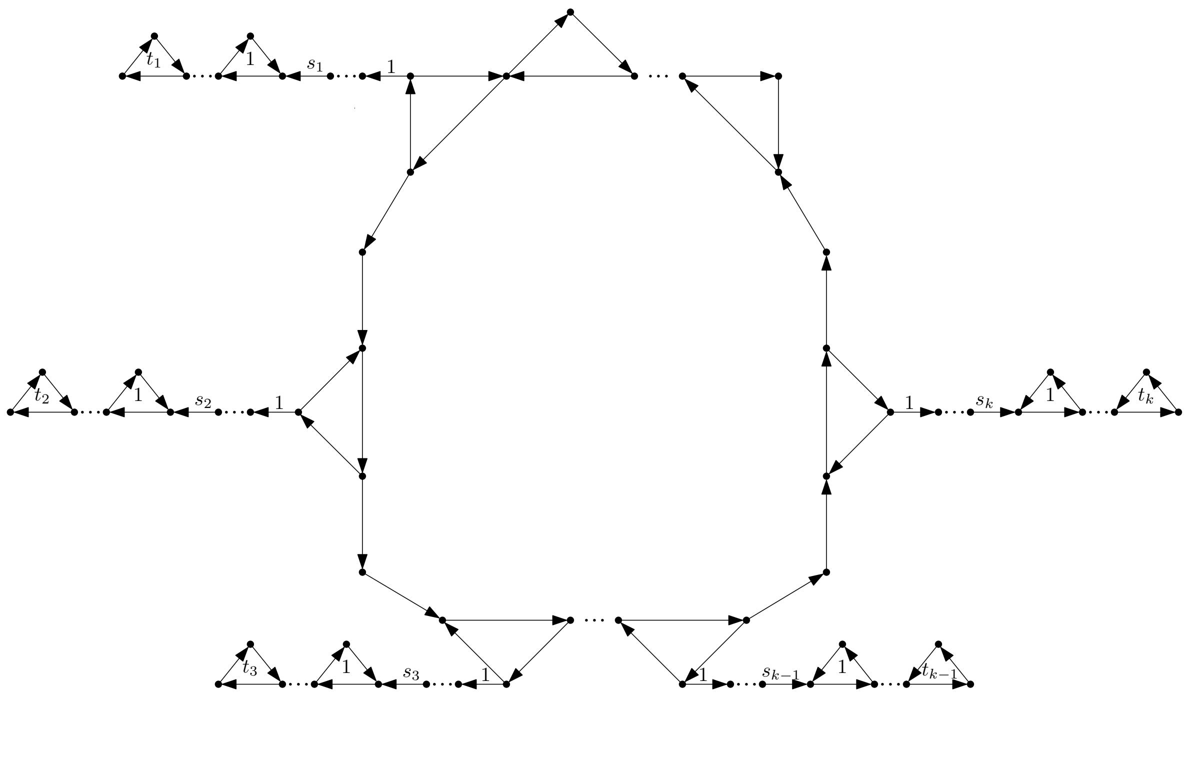
(d_3) *Type IV with parameter sequence*

$$\big((1, 0, 0), (1, 0, 0), \dots, (1, 0, 0), (3, s_1, t_1), (3, s_2, t_2), \dots, (3, s_k, t_k)\big)$$

for some $k > 0$*, where the number of triples* $(1, 0, 0)$ *is* $b \geq 0$*, the non-negative integers* $s_1, t_1, \dots, s_k, t_k$ *are considered up to rotation of the sequence*

$$\big((s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)\big),$$

$n = 4k + 2b + s_1 + 2t_1 + \dots + s_k + 2t_k$ *and the attached rooted quivers of type A are in standard form.*



Moreover, any two distinct standard forms which are not of the class (d_3) *are not derived equivalent.*

Remark 2.4. There is no known example of two distinct standard forms which are derived equivalent.

2.2. **Numerical invariants.** The main tool for distinguishing the various standard forms appearing in Theorem 2.3 is the computation of their numerical invariants of derived equivalence described in Section 1.2. We start by giving the formulae for the determinants of the Cartan matrices of all cluster-tilted algebras of type D.

Theorem 2.5. *Let Q be a quiver which is mutation equivalent to D_n for $n \geq 4$. Using the notation from Section 1.6 we have the following formulae for the determinants of the Cartan matrices.*

(I) *If Q is of type I, then* $\det C_Q = 2^{t(Q')} = \det C_{Q'}$.
(II) *If Q is of type II, then* $\det C_Q = 2 \cdot 2^{t(Q')+t(Q'')} = 2 \cdot \det C_{Q'} \cdot \det C_{Q''}$.
(III) *If Q is of type III, then* $\det C_Q = 3 \cdot 2^{t(Q')+t(Q'')} = 3 \cdot \det C_{Q'} \cdot \det C_{Q''}$.
(IV) *For a quiver Q of type IV with central cycle of length $m \geq 3$, let $Q^{(1)}, \ldots, Q^{(r)}$ be the rooted quivers of type A glued to the spikes and let $c(Q)$ be the number of vertices on the central cycle which are part of two (consecutive) spikes, i.e. $c(Q) = |\{1 \leq j \leq r \,:\, d_j = 1\}|$, cf.* (1.1). *Then*
$$\det C_Q = (m + c(Q) - 1) \cdot \prod_{j=1}^{r} 2^{t(Q^{(j)})} = (m + c(Q) - 1) \cdot \prod_{j=1}^{r} \det C_{Q^{(j)}}.$$

We immediately obtain the following.

Corollary 2.6.

(a) *A cluster-tilted algebra in type II is not derived equivalent to any cluster-tilted algebra in type III.*
(b) *A cluster-tilted algebra in type II is not derived equivalent to any cluster-tilted algebra in type IV whose Cartan determinant is not a power of* 2.

Note that the determinant alone is not enough to distinguish types II and IV, the smallest example occurs already in type D_5.

Example 2.7. The Cartan matrices of the cluster-tilted algebra in type II with parameters $(1, 0, 0, 0)$ and the one in type IV with parameters $\big((3, 1, 0)\big)$ whose quivers are given by

have both determinant 2, but the characteristic polynomials of their asymmetries differ, namely $x^5 - x^3 + x^2 - 1$ and $x^5 - 2x^3 + 2x^2 - 1$, respectively.

Since the determinants of the Cartan matrices of all cluster-tilted algebras of type D do not vanish, one can consider their asymmetry matrices and the corresponding characteristic polynomials. We list below the characteristic polynomials of the asymmetry matrices for cluster-tilted algebras of subtypes I, II and III of Dynkin type D and for certain cases in type IV. Combining this with Theorem 2.5, we get the corresponding associated polynomials. Using these it is possible to distinguish several further standard forms of Theorem 2.3 up to derived equivalence. However, in the present paper we are not embarking on this aspect, and therefore only list these polynomials for the sake of completeness. For the proofs we refer to the forthcoming thesis of the first author [3] as well as to the note [26] containing a general method for computing the associated polynomial of an algebra obtained by gluing rooted quivers of type A to a given quiver with relations.

Notation 2.8. For a quiver Q mutation equivalent to a Dynkin quiver, we denote by $\chi_Q(x)$ the characteristic polynomial of the asymmetry matrix of the Cartan matrix C_Q of the cluster-tilted algebra corresponding to Q.

Remark 2.9. Let Q be the quiver of a cluster-tilted algebra of type A. Then
$$\chi_Q(x) = (x+1)^{t-1}\Big(x^{s+t+2} + (-1)^{s+1}\Big)$$
where $s = s(Q)$ and $t = t(Q)$.

Remark 2.10. Consider a cluster-tilted algebra in type D with quiver Q of type I, II, III or IV and parameters as defined in Section 1.6.

(I) If Q is of type I, then

$$\chi_Q(x) = (x+1)^t(x-1)\Big(x^{s+t+2} + (-1)^s\Big)$$

where $s = s(Q')$ and $t = t(Q')$.

(II/III) If Q is of type II or type III, then

$$\chi_Q(x) = (x+1)^{t+1}(x-1)\Big(x^{s+t+2} + (-1)^{s+1}\Big)$$

where $s = s(Q') + s(Q'')$ and $t = t(Q') + t(Q'')$.

(IV) If Q is of type IV, then we have the following.

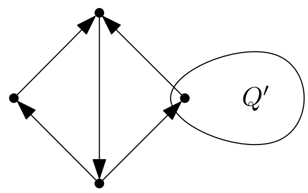
(a) If Q is an oriented cycle of length n without spikes then

$$\chi_Q(x) = \begin{cases} x^n - 1 & \text{if } n \text{ is odd,} \\ \left(x^{\frac{n}{2}} - 1\right)^2 & \text{if } n \text{ is even.} \end{cases}$$

(b) If Q has parameters $\big((1,s,t),(1,0,0),\dots,(1,0,0)\big)$ with $b \geq 3$ spikes, then

$$\chi_Q(x) = (x+1)^t(x^b-1)\Big(x^{s+t+b} + (-1)^{s+1}\Big).$$

(c) If Q has parameters $\big((3,s,t)\big)$ as in the picture below



then

$$\chi_Q(x) = (x+1)^{t-1}(x-1)\Big(x^{s+t+4} + 2\cdot x^{s+t+3} + (-1)^{s+1}\cdot 2x + (-1)^{s+1}\Big).$$

(d) If Q has parameters $\big((3,s_1,t_1),(3,s_2,t_2)\big)$ then

$$\begin{aligned}\chi_Q(x) = (x+1)^{t_1+t_2-2}\cdot(x-1)\cdot\Big(& x^{s_1+t_1+s_2+t_2}\cdot(x^9+3x^8+4x^7+4x^6) \\ &+ \big((-1)^{s_1}x^{s_2+t_2} + (-1)^{s_2}x^{s_1+t_1}\big)(x^5-x^4) \\ &+ (-1)^{s_1+s_2+1}(1+3x+4x^2+4x^3)\Big).\end{aligned}$$

2.3. **Complete derived equivalence classification up to D_{14}.** Fixing the number of vertices n, it is possible to enumerate over all the standard forms of derived equivalence with n vertices as given in Theorem 2.3, and compute the Cartan matrices of the corresponding cluster-tilted algebras and their associated polynomials. As long as the resulting polynomials (or any other derived invariants) do not coincide for two distinct standard forms, the derived equivalence classification is complete since then we know that any cluster-tilted algebra in type D is derived equivalent to one of the standard forms, and moreover any two distinct such forms are not derived equivalent.

By carrying out this procedure on a computer using the MAGMA system [7], we have been able to obtain a complete derived equivalence classification of the cluster-tilted algebras of type D_n for $n \leq 14$. Table 1 lists, for $4 \leq n \leq 14$, the number of such algebras (using the formula given in [12]) and the number of their derived equivalence classes.

As a consequence of our methods, we deduce the following characterization of derived equivalence for cluster-tilted algebras of type D_n with $n \leq 14$.

Proposition 2.11. *Let Λ and Λ' be two cluster-tilted algebras of type D_n with $n \leq 14$. Then the following conditions are equivalent:*

(a) *Λ and Λ' have the same associated polynomials;*
(b) *The Cartan matrices of Λ and Λ' represent equivalent bilinear forms over $\mathbb{Z}$;*
(c) *Λ and Λ' are derived equivalent;*

n	Algebras	Classes
4	6	3
5	26	5
6	80	9
7	246	10
8	810	17
9	2704	18
10	9252	29
11	32066	31
12	112720	49
13	400024	53
14	1432860	81

TABLE 1. The numbers of cluster-tilted algebras of type D_n and their derived equivalence classes, $n \leq 14$.

(d) *Either Λ and Λ' are both self-injective, or none of them is self-injective and they are connected by a sequence of good mutations or good double mutations.*

Remark 2.12. The implications (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) always hold.

Remark 2.13. It follows from their derived equivalence classification (see [13] for type A and [5] for type E) that a statement analogous to Corollary 2.11 is true also for cluster-tilted algebras of Dynkin types A and E, replacing the condition (d) by:

(d') Λ *and* Λ' *are connected by a sequence of good mutations.*

However, for Dynkin type D the following two examples in types D_6 and D_8 demonstrate that one must replace condition (d') by the weaker one (d). Thus, in some sense the derived equivalence classification in type D_8 is more complicated than that in type E_8.

Example 2.14. For any $b \geq 3$, the cluster-tilted algebra in type IV with a central cycle of length $2b$ without any spike is derived equivalent to that in type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$ of length b (see Lemma 4.5). There is no sequence of good mutations connecting these two self-injective [30] algebras. Indeed, none of the algebra mutations at any of the vertices is defined. The smallest such pair occurs in type D_6 and the corresponding quivers are shown below.

Example 2.15. The two cluster-tilted algebras of type D_8 with quivers

of type III are derived equivalent but cannot be connected by a sequence of good mutations. Indeed, the mutations at all the vertices of the right quiver are bad (see cases II.1, III.3 in Tables 5 and 6 in Section 3).

2.4. **Opposite algebras.** The smallest example of two distinct standard forms (as in Theorem 2.3) for which it is unknown whether they are derived equivalent or not arises for $n = 15$ vertices. It is related to the question of derived equivalence of an algebra and its opposite which we briefly discuss below.

It follows from the description of the quivers and relations given in Section 1.6 that if Λ is a cluster-tilted algebra of Dynkin type D, then so is its opposite algebra Λ^{op}. Moreover, a careful analysis of the classes given in Theorem 2.3 shows that any cluster-tilted algebra with standard form in the classes (a),(b),(c),(d_1) or (d_2) is derived equivalent to its opposite, and the opposite of a cluster-tilted algebra with standard from in class (d_3) and parameter sequence

$$\big((1,0,0),(1,0,0),\dots,(1,0,0),(3,s_1,t_1),(3,s_2,t_2),\dots,(3,s_k,t_k)\big)$$

is derived equivalent to a standard form in the same class, but with parameter sequence

$$\big((1,0,0),(1,0,0),\dots,(1,0,0),(3,s_k,t_k),\dots,(3,s_2,t_2),(3,s_1,t_1)\big)$$

which may not be equivalent to the original one when $k \geq 3$. The smallest such pairs of rotation-inequivalent standard forms occur when $k = 3$, and the number of vertices is then 15.

The equivalence class of the integral bilinear form defined by the Cartan matrix can be a very tricky derived invariant when it comes to assessing the derived equivalence of an algebra Λ and its opposite Λ^{op}. Indeed, the Cartan matrix of Λ^{op} is the transpose of that of Λ. Since the bilinear forms defined by any square matrix (over any field) and its transpose are always equivalent over that field [16, 19, 32], it follows that the bilinear forms defined by the integral matrices C_Λ and $C_{\Lambda^{op}}$ are equivalent over $\mathbb{Q}$ as well as over all prime fields $\mathbb{F}_p$. Hence determining their equivalence over $\mathbb{Z}$ becomes a delicate arithmetical question. Moreover, it follows that the asymmetry matrices (when defined) are similar over any field, and hence the associated polynomials corresponding to Λ and Λ^{op} always coincide.

To illustrate these difficulties, we present here the two smallest examples occurring in type D_{15}. In one example, the equivalence of the bilinear forms is known, whereas in the other it is unknown. In both cases, we are not able to tell whether the algebra is derived equivalent to its opposite.

Example 2.16. The Cartan matrices of the opposite cluster-tilted algebras of type D_{15} with standard forms

$$\big((3,1,0),(3,0,1),(3,0,0)\big), \qquad\qquad \big((3,0,0),(3,0,1),(3,1,0)\big)$$

define equivalent bilinear forms over the integers. This has been verified by computer search (using MAGMA [7]). One can even choose the matrix inducing equivalence to have all its entries in $\{0, \pm 1, \pm 2\}$.

Example 2.17. For the opposite cluster-tilted algebras of type D_{15} with standard forms

$$\big((3,1,0),(3,2,0),(3,0,0)\big), \qquad\qquad \big((3,0,0),(3,2,0),(3,1,0)\big)$$

it is unknown whether their Cartan matrices define equivalent bilinear forms over $\mathbb{Z}$.

The above discussion motivates the following question.

Question 2.18. Let Q be an acyclic quiver such that its path algebra KQ is derived equivalent to its opposite, and let Λ be a cluster-tilted algebra of type Q. Is it true that Λ is derived equivalent to its opposite Λ^{op}?

Remark 2.19. The answer to the above question is affirmative for cluster-tilted algebras of Dynkin types A, E and affine type $\widetilde{A}$, as well as for cluster-tilted algebras of type D with at most 14 simples. This follows from the corresponding derived equivalence classifications.

Remark 2.20. If the answer to the question is positive for cluster-tilted algebras of Dynkin type D, then in class (d_3) of Theorem 2.3, one has to consider the non-negative integers $s_1, t_1, \dots, s_k, t_k$ in the k-term sequence

$$\big((s_1,t_1),(s_2,t_2),\dots,(s_k,t_k)\big)$$

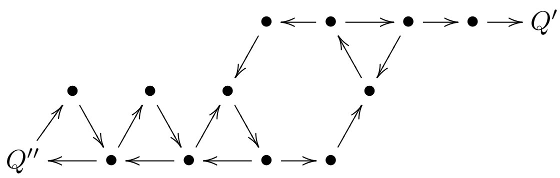
up to rotation as well as order reversal.

2.5. **Other open questions for D_n, $n \geq 15$.** An immediate consequence of part (IV)(d) of Remark 2.10 is the following systematic construction of standard forms with the same associated polynomial.

Remark 2.21. Let $s_1, t_1, s_2, t_2 \geq 0$. Then the cluster-tilted algebras with standard forms

$$\big((3,2+s_1,t_1),(3,s_2,2+t_2)\big) \qquad\qquad \big((3,2+s_2,t_2),(3,s_1,2+t_1)\big) \tag{2.2}$$

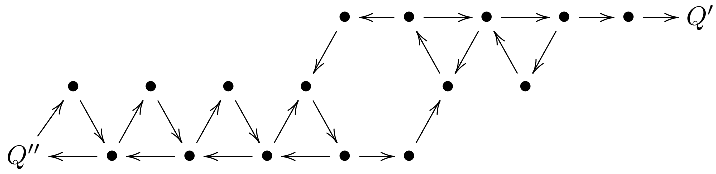
have the same associated polynomial. In other words, for a cluster-tilted algebra with quiver



where Q' and Q'' are rooted quivers of type A, exchanging the rooted quivers Q' and Q'' does not change the associated polynomial.

The following proposition, which is a specific case of a statement in the note [27], shows that moreover, in most cases, the Cartan matrices of the corresponding standard forms define equivalent bilinear forms (over $\mathbb{Z}$).

Proposition 2.22. *Let $s_1, s_2 \geq 0$ and $t_1, t_2 > 0$. Then the bilinear forms defined by the Cartan matrices of the cluster-tilted algebras with standard forms as in* (2.2) *are equivalent over $\mathbb{Z}$. In other words, for a cluster-tilted algebra with quiver*



where Q' and Q'' are rooted quivers of type A, exchanging the rooted quivers Q' and Q'' does not change the equivalence class of the bilinear form defined by the Cartan matrix.

Question 2.23. Are any two cluster-tilted algebras as in Proposition 2.22 derived equivalent?

Let $m \geq 1$. By considering the $m+1$ standard forms with parameters

$$\big((3, 2i, 2m+1-2i), (3, 2m+1-2i, 1+2i)\big) \qquad\qquad 0 \leq i \leq m$$

and invoking Proposition 2.22, we obtain the following.

Corollary 2.24. *Let $m \geq 1$. Then one can find $m+1$ distinct standard forms of cluster-tilted algebras of type D_{6m+13} whose Cartan matrices define equivalent bilinear forms.*

Remark 2.25. The smallest case occurs when $m = 1$, giving a pair of cluster-tilted algebras of type D_{19} whose Cartan matrices define equivalent bilinear forms but their derived equivalence is unknown. They correspond to the choice of $Q' = A_1$ and $Q'' = A_2$ in Proposition 2.22.

In some of the remaining cases in Remark 2.21, despite the collision of the associated polynomials, it is still possible to distinguish the standard forms by using the bilinear forms of their Cartan matrices, whereas in other cases this remains unsettled. We illustrate this by two examples.

Example 2.26. The two standard forms $\big((3,2,0),(3,1,2)\big)$ and $\big((3,3,0),(3,0,2)\big)$ in type D_{15} corresponding to the choice of $Q' = A_1$ and $Q'' = A_2$ in Remark 2.21 have the same associated polynomial, namely

$$20 \cdot \big(x^{15} + 2x^{14} + x^{13} - 4x^{11} + x^9 - 3x^8 + 3x^7 - x^6 + 4x^4 - x^2 - 2x - 1\big)$$

but their asymmetry matrices are not similar over the finite field $\mathbb{F}_3$ (and hence not over $\mathbb{Z}$). Thus the bilinear forms defined by their Cartan matrices are not equivalent so the two algebras are not derived equivalent.

This is the smallest non-trivial example of Remark 2.21, and it also shows that the implication (a) $\Rightarrow$ (b) in Proposition 2.11 does not hold in general for cluster-tilted algebras of type D.

Example 2.27. The two standard forms $\big((3,2,0),(3,3,2)\big)$ and $\big((3,5,0),(3,0,2)\big)$ in type D_{17} corresponding to the choice of $Q' = A_1$ and $Q'' = A_4$ in Remark 2.21 have the same associated polynomials and the asymmetries are similar over $\mathbb{Q}$ as well as over all finite fields $\mathbb{F}_p$, but it is unknown whether the bilinear forms are equivalent over $\mathbb{Z}$ or not.

Table 2 lists for $15 \leq n \leq 20$ the number of cluster-tilted algebras of type D_n (according to the formula in [12]) together with upper and lower bounds on the number of their derived equivalence classes. There are two upper bounds which are obtained by counting the number of standard forms given in Theorem 2.3 with ("max$^{\text{op}}$") or without ("max") the assumption of affirmative answer to Question 2.18 concerning the derived equivalence of opposite cluster-tilted algebras. The lower bound ("min") is obtained by considering the number of standard forms with distinct numerical invariants of derived equivalence. It follows that in types D_{15}, D_{16} and D_{18} there are no unsettled cases apart from those arising as opposites of algebras.

n	Algebras	Classes min	$\max^{\mathrm{op}}$	max
15	5170604	91	91	93
16	18784170	136	136	139
17	68635478	156	157	167
18	252088496	231	231	248
19	930138522	273	275	312
20	3446167860	395	401	461

TABLE 2. The number of cluster-tilted algebras of type D_n together with lower and upper bounds on the number of their derived equivalence classes, $15 \leq n \leq 20$.

2.6. **Good mutation equivalence classification.** Whereas the classification according to derived equivalence becomes subtle when the number of vertices grows, leaving some questions still undecided, this does not happen for the stronger, but algorithmically more tractable relation of good mutation equivalence.

Definition 2.28. Two cluster-tilted algebras (of Dynkin type) with quivers Q' and Q'' are called *good mutation equivalent* if one can move from Q' to Q'' by performing a sequence of good mutations. In other words, there exists a sequence of vertices $k_1, k_2, \dots, k_m$ such that if we set $Q_0 = Q'$ and $Q_j = \mu_{k_j}(Q_{j-1})$ for $1 \leq j \leq m$, and denote by Λ_j the cluster-tilted algebra with quiver Q_j, then $Q'' = Q_r$ and for any $1 \leq j \leq m$ we have $\Lambda_j = \mu^-_{k_j}(\Lambda_{j-1})$ or $\Lambda_j = \mu^+_{k_j}(\Lambda_{j-1})$.

Remark 2.29. Any two cluster-tilted algebras which are good mutation equivalent are also derived equivalent. The converse is true in Dynkin types A and E but not in type D, see Section 2.3 above.

Let Q be mutation equivalent to a Dynkin quiver. When assessing whether its quiver mutation at a vertex k is good or not, one needs to consider which of the algebra mutations at k of the two cluster-tilted algebras Λ_Q and $\Lambda_{\mu_k(Q)}$ corresponding to Q and its mutation $\mu_k(Q)$ are defined.

A-priori, there may be 16 possibilities as there are four algebra mutations to consider (negative and positive for Λ_Q and $\Lambda_{\mu_k(Q)}$) and each of them can be either defined or not. However, our following result shows that the question which of the algebra mutations of Λ_Q at k is defined and the analogous question for $\Lambda_{\mu_k(Q)}$ are not independent of each other, and the number of possibilities that can actually occur is only 5.

Proposition 2.30. *Let Q be mutation equivalent to a Dynkin quiver and let k be a vertex of Q. Consider the algebra mutations $\mu^-_k(\Lambda_Q)$ and $\mu^+_k(\Lambda_Q)$ of the corresponding cluster-tilted algebra Λ_Q.*

(a) *If none of these mutations is defined, then both algebra mutations $\mu^-_k(\Lambda_{\mu_k(Q)})$ and $\mu^+_k(\Lambda_{\mu_k(Q)})$ are defined. Obviously, the quiver mutation at k is bad.*
(b) *If $\mu^-_k(\Lambda_Q)$ is defined but $\mu^+_k(\Lambda_Q)$ is not, then $\mu^+_k(\Lambda_{\mu_k(Q)})$ is defined and $\mu^-_k(\Lambda_{\mu_k(Q)})$ is not, hence the quiver mutation at k is good.*
(c) *If $\mu^+_k(\Lambda_Q)$ is defined but $\mu^-_k(\Lambda_Q)$ is not, then $\mu^-_k(\Lambda_{\mu_k(Q)})$ is defined and $\mu^+_k(\Lambda_{\mu_k(Q)})$ is not, hence the quiver mutation at k is good.*
(d) *If both algebra mutations of Λ_Q at k are defined, then either both mutations of $\Lambda_{\mu_k(Q)}$ at k are defined or none of them is defined. Accordingly, the quiver mutation at k may be good or bad.*

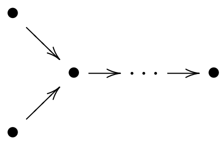
We now present our results concerning good mutations in type D.

Theorem 2.31. *There is an algorithm which decides for two quivers which are mutation equivalent to D_n given in parametric notation (i.e. specified by their type I,II,III,IV and the parameters), whether the corresponding cluster-tilted algebras are good mutation equivalent or not. The running time of this algorithm is at most quadratic in the number of parameters.*

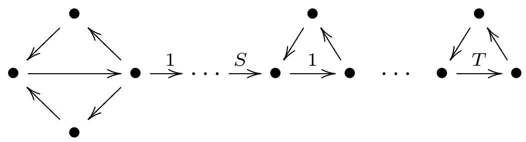
We provide also a list of "canonical forms" for good mutation equivalence.

Theorem 2.32. *A cluster-tilted algebra of type D with n vertices is good mutation equivalent to one (and only one!) of the cluster-tilted algebras with the following quivers:*

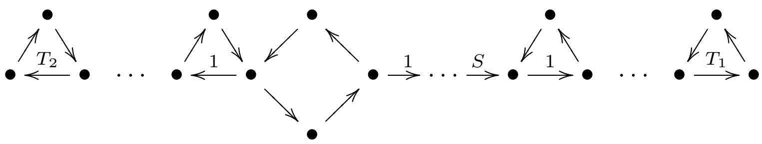
(a) D_n *(i.e. type I with a linearly oriented A_{n-2} quiver attached);*



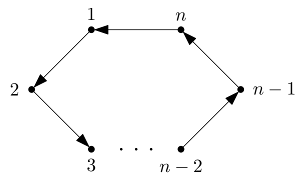
(b) *Type II as in the following figure, where $S, T \geq 0$ and $S + 2T = n - 4$;*



(c) *Type III as in the following figure, with $S \geq 0$, the non-negative integers T_1, T_2 are considered up to rotation of the sequence (T_1, T_2), and $S + 2(T_1 + T_2) = n - 4$;*



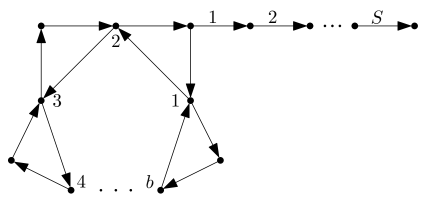
(d_1) *Type IV with a central cycle of length n without any spikes;*



($\mathrm{d}_{2,1}$) *Type IV with parameter sequence*

$$\big((1, S, 0), (1, 0, 0), \dots, (1, 0, 0)\big)$$

of length $b \geq 3$, with $S \geq 0$ such that $n = 2b + S$ and the attached rooted quiver of type A is linearly oriented A_{S+1};



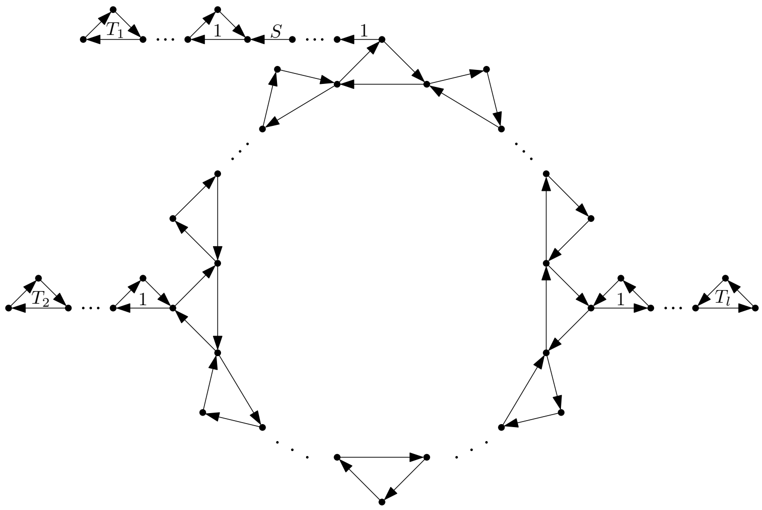
($\mathrm{d}_{2,2}$) *Type IV with parameter sequence*

$$\big((1, S, T_1), (1, 0, 0), \dots, (1, 0, 0), (1, 0, T_2), (1, 0, 0), \dots, (1, 0, 0), \dots, (1, 0, T_l), (1, 0, 0), \dots, (1, 0, 0)\big)$$

which is a concatenation of $l \geq 1$ blocks of positive lengths $b_1, b_2, \dots, b_l$ whose sum is not smaller than 3, with $S \geq 0$ and $T_1, \dots, T_l > 0$ considered up to rotation of the sequence

$$\big((b_1, T_1), (b_2, T_2), \dots, (b_l, T_l)\big),$$

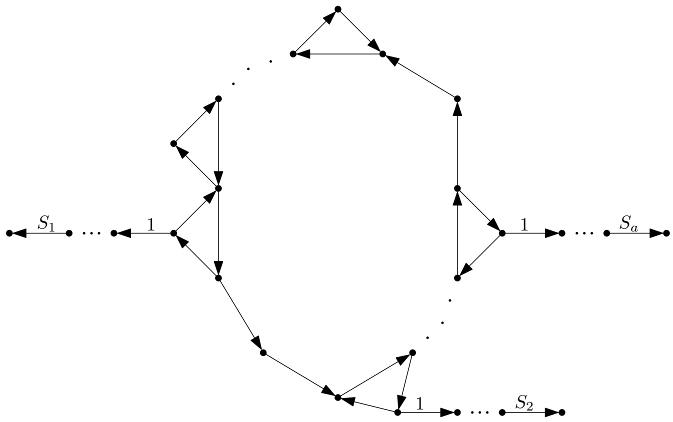
$n = 2(b_1 + \dots + b_l + T_1 + \dots + T_l) + S$ and the attached quivers of type A are in standard form;



(d$_{3,1}$) *Type IV with parameter sequence*

$$\big((1,0,0),\dots,(1,0,0),(3,S_1,0),\dots,(3,S_a,0)\big)$$

for some $a>0$, where the number of the triples $(1,0,0)$ is $b\geq 0$, the sequence of non-negative integers $(S_1,\dots,S_a)$ is considered up to a cyclic permutation, $n=4a+2b+S_1+\dots+S_a$ and the attached rooted quivers of type A are in standard form (i.e. linearly oriented $A_{S_1+1},\dots,A_{S_a+1}$);



(d$_{3,2}$) *Type IV with parameter sequence which is a concatenation of $l\geq 1$ sequences of the form*

$$\gamma_j=\begin{cases}\big((1,0,T_j),(1,0,0),\dots,(1,0,0),(3,S_{j,1},0),(3,S_{j,2},0),\dots,(3,S_{j,a_j})\big) & \text{if } b_j>0,\\ \big((3,S_{j,1},T_j),(3,S_{j,2},0),\dots,(3,S_{j,a_j},0) & \text{otherwise,}\end{cases}$$

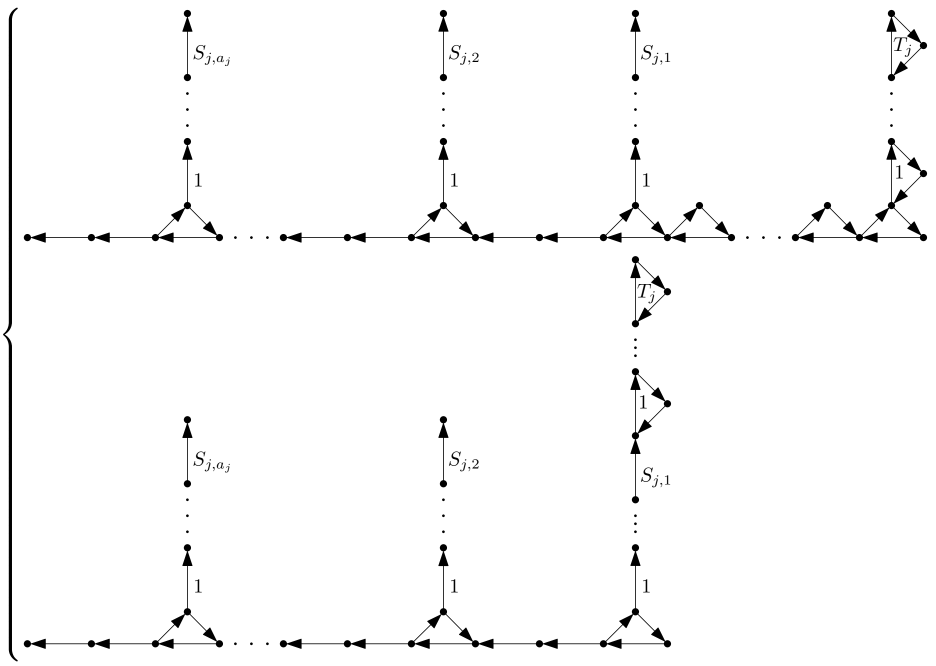
where each sequence γ_j for $1\leq j\leq l$ is defined by non-negative integers a_j, b_j not both zero, a sequence of a_j non-negative integers $S_{j,1},\dots,S_{j,a_j}$ and a positive integer T_j, and not all the a_j are zero. All these numbers are considered up to rotation of the l-term sequence

$$\Big(\big(b_1,(S_{1,1},\dots,S_{1,a_1}),T_1\big),\big(b_2,(S_{2,1},\dots,S_{2,a_2}),T_2\big),\dots,\big(b_l,(S_{l,1},\dots,S_{l,a_l}),T_l\big)\Big),$$

they satisfy $n=\sum_{j=1}^{l}(4a_j+2b_j+S_{j,1}+\dots+S_{j,a_j}+2T_j)$, and the attached rooted quivers of type A are in standard form.

That is, the quiver is a concatenation of $l \geq 1$ quivers γ_j of the form

$\gamma_j =$



if $b_j > 0$,

if $b_j = 0$,

where the last vertex of γ_l is glued to the first vertex of γ_1.

3. Good mutation equivalences

In this section we determine all the good mutations for cluster-tilted algebras of Dynkin types A and D.

3.1. **Rooted quivers of type** A. For a rooted quiver (Q, v) of type A, we call a mutation at a vertex other than the root v a mutation *outside the root*.

Proposition 3.1. *Any two rooted quivers of type A with the same numbers of lines and triangles can be connected by a sequence of good mutations outside the root.*

Remark 3.2. It is enough to show that a rooted quiver of type A can be transformed to its standard form via good mutations outside the root.

We begin by characterizing the good mutations in Dynkin type A.

Lemma 3.3. *Let Q be a quiver mutation equivalent to A_n. Then a mutation of Q is good if and only if it does not change the number of triangles.*

Proof. Each row of Table 3 displays a pair of neighborhoods of a vertex $\bullet$ in such a quiver related by a mutation (at $\bullet$). Using the description of the relations of the corresponding cluster-tilted algebras as in Remark 1.21, we can use Proposition 1.10 and easily determine, for each entry in the table, which of the negative $\mu_\bullet^-$ or the positive $\mu_\bullet^+$ mutations is defined. Then Proposition 1.13 tells us if the quiver mutation is good or not.

By examining the entries in the table, we see that the only bad mutation occurs in row $2b$, where a triangle is created (or destroyed). □

Proof of Proposition 3.1. In view of Remark 3.2, we give an algorithm for the mutation to the standard form above (similar to the procedures in [4] and [13]): Let Q be a rooted quiver of type A which has at least one triangle (otherwise we get the desired orientation of a standard form by sink/source mutations as in 1 and 2a in Table 3). For any triangle C in Q denote by v_C the unique vertex of the triangle having minimal distance to the root c. Choose a triangle C_1 in Q such that to the vertices of the triangle $\neq v_1 := v_{C_1}$ only linear parts are attached; denote them by p_1 and p_2, respectively.

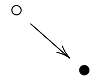
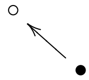
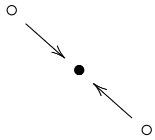
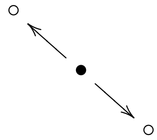
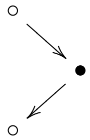
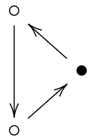
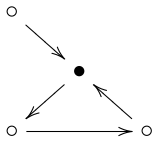
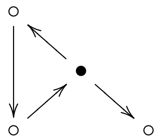
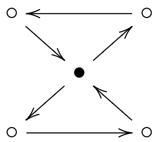
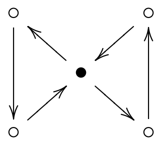
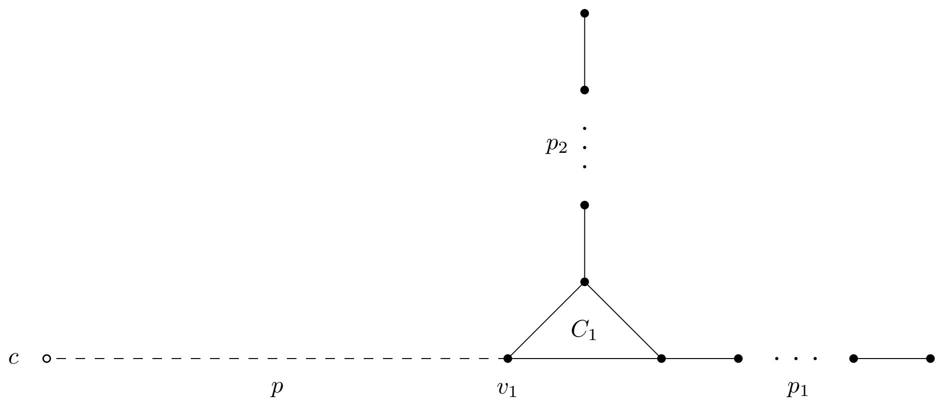
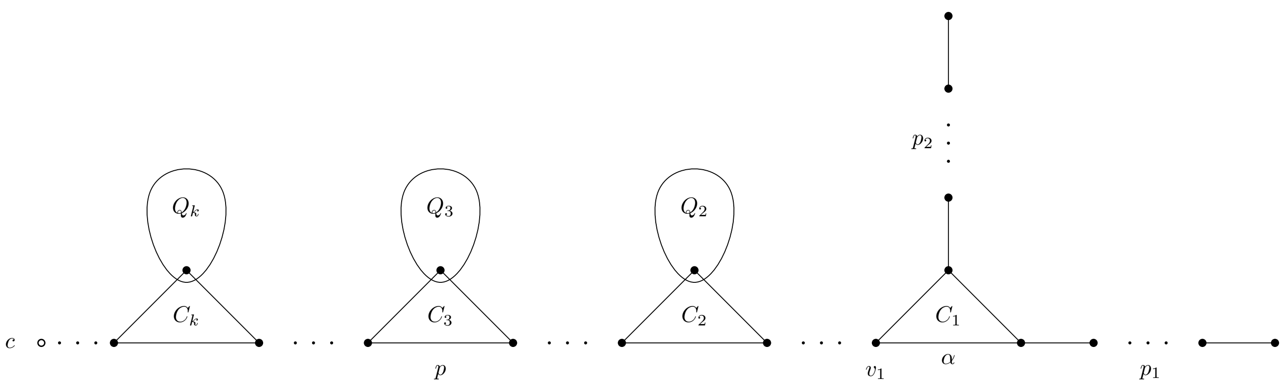
1		$\mu_\bullet^-$		$\mu_\bullet^+$	good
$2a$		$\mu_\bullet^-$		$\mu_\bullet^+$	good
$2b$		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
3		$\mu_\bullet^-$		$\mu_\bullet^+$	good
4		$\mu_\bullet^-, \mu_\bullet^+$		$\mu_\bullet^-, \mu_\bullet^+$	good

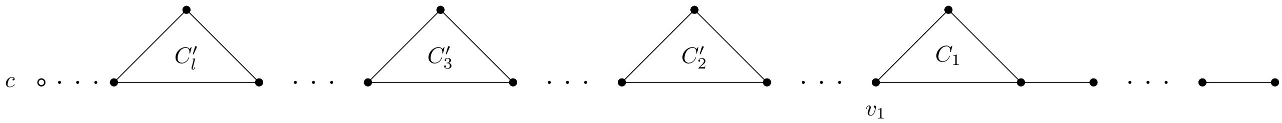
TABLE 3. The neighborhoods in Dynkin type A and their mutations.



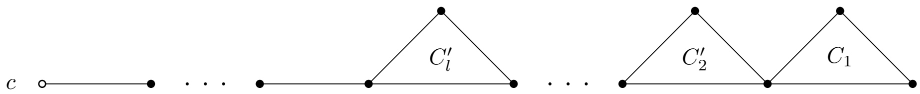
Denote by $C_2, \ldots, C_k$ the (possibly) other triangles along the path p from v_1 to c.



Now we move all subquivers p_2, $Q_2, \dots, Q_k$ onto the path $p_1 \alpha p$. For this we use the same mutations as in the steps 1 and 2 in in [4, Lemma 3.10.]. Note that this can be done with the good mutations presented in Table 3. Thus, we get a new complete set of triangles $\{C_1, C'_2, \dots, C'_l\}$ on the path from v_1 to c:



We move all the triangles along the path to the right side. For this we use the same mutations as in step 4 in [4, Lemma 3.10.]. We then obtain a quiver of the form



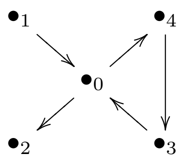
□

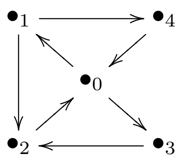
3.2. **Good mutations in types I and II.** The good mutations involving quivers in types I and II are given in Tables 4 and 5 below. In each row of these tables, we list:

(a) The quiver, where Q, Q', Q'' and Q''' are rooted quivers of type A;
(b) Which of the algebra mutations (negative $\mu^-_\bullet$, or positive $\mu^+_\bullet$) at the distinguished vertex $\bullet$ are defined;
(c) The (Fomin-Zelevinsky) mutation of the quiver at the vertex $\bullet$; and for the corresponding cluster-tilted algebra:
(d) Which of the algebra mutations at the vertex $\bullet$ are defined.
(e) Based on these, we determine whether the mutation is good or not, see Proposition 1.13.

To check whether a mutation is defined or not, we use the criterion of Proposition 1.10. Observe that since the gluing process introduces no new relations, it is enough to assume that each rooted quiver of type A consists of just a single vertex. The finite list of quivers we obtain can thus be examined by using a computer. We illustrate the details of these checks on a few examples. Since there is at most one arrow between any two vertices, we indicate a path by the sequence of vertices it traverses.

Example 3.4. Consider the case I.4c in Table 4. We look at the two cluster-tilted algebras Λ and Λ' with the following quivers



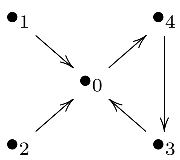


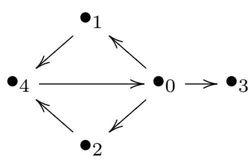
and examine their mutations at the vertex 0.

Since the arrow $1 \to 0$ does not appear in any relation of Λ, its composition with any non-zero path starting at 0 is non-zero. Thus the negative mutation $\mu_0^-(\Lambda)$ is defined. Similarly, since $0 \to 2$ does not appear in any relation of Λ, its composition with any non-zero path that ends at 0 is non-zero, and the positive mutation $\mu_0^+(\Lambda)$ is also defined.

Consider now Λ'. The path $0, 1, 2$ is non-zero, as it equals $0, 3, 2$, but both compositions $2, 0, 1, 2$ and $4, 0, 1, 2$ vanish because of the zero relations $2, 0, 1$ and $4, 0, 1$, hence $\mu_0^-(\Lambda')$ is not defined. Similarly, the path $1, 2, 0$ is non-zero, as it equals $1, 4, 0$, but both compositions $1, 2, 0, 1$ and $1, 2, 0, 3$ vanish because of the zero relations $2, 0, 1$ and $2, 0, 3$, showing that $\mu_0^+(\Lambda')$ is not defined.

Example 3.5. Consider the case I.5a in Table 4. We look at the two cluster-tilted algebras Λ and Λ' with the following quivers





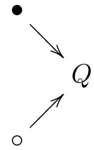
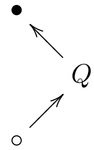
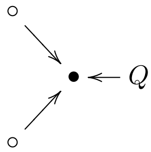
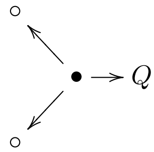
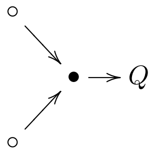
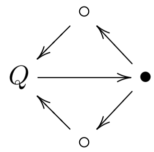
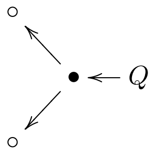
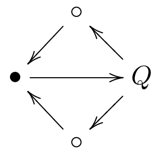
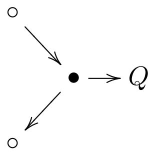
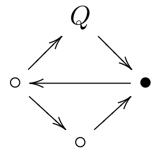
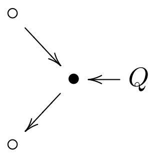
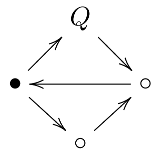
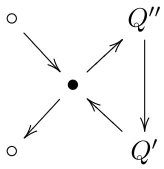
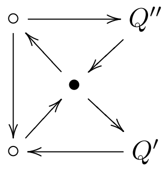
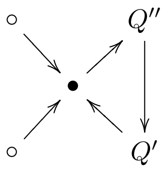
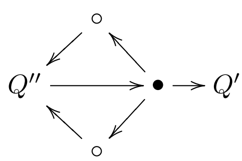
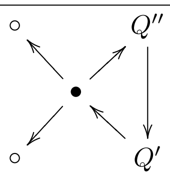
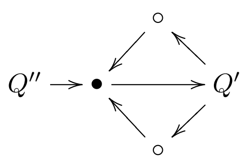
I.1		$\mu_\bullet^+$		$\mu_\bullet^-$	good
I.2		$\mu_\bullet^-$		$\mu_\bullet^+$	good
I.3a		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.3b		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.4a		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.4b		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.4c		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.5a		$\mu_\bullet^-$		$\mu_\bullet^+$	good
I.5b		$\mu_\bullet^+$		$\mu_\bullet^-$	good

TABLE 4. Mutations involving type I quivers.

II.1	$\begin{tikzcd} & \bullet \arrow[dl] & \\ Q'' \arrow[rr] & & Q' \arrow[ul] \arrow[dl] \\ & \circ \arrow[ul] & \end{tikzcd}$	$\mu_\bullet^-, \mu_\bullet^+$	$\begin{tikzcd} & \bullet \arrow[dr] & \\ Q'' \arrow[ur] & & Q' \arrow[dl] \\ & \circ \arrow[ul] & \end{tikzcd}$	none	bad
II.2	$\begin{tikzcd} & \circ \arrow[dl] & & \\ Q'' \arrow[rr] & & \bullet \arrow[ul] \arrow[dl] & Q' \arrow[l] \\ & \circ \arrow[ul] & & \end{tikzcd}$	$\mu_\bullet^-$	$\begin{tikzcd} & & \circ \arrow[dl] & \\ Q'' & \bullet \arrow[l] \arrow[rr] & & Q' \arrow[ul] \arrow[dl] \\ & & \circ \arrow[ul] & \end{tikzcd}$	$\mu_\bullet^+$	good
II.3	$\begin{tikzcd} & \circ \arrow[dl] & & Q' \arrow[dl] \\ Q'' \arrow[rr] & & \bullet \arrow[ul] \arrow[dl] \arrow[dr] & \\ & \circ \arrow[ul] & & Q''' \arrow[uu] \end{tikzcd}$	$\mu_\bullet^-, \mu_\bullet^+$	$\begin{tikzcd} Q''' \arrow[dr] & & \circ \arrow[dl] & \\ & \bullet \arrow[dl] \arrow[rr] & & Q' \arrow[ul] \arrow[dl] \\ Q'' \arrow[uu] & & \circ \arrow[ul] & \end{tikzcd}$	$\mu_\bullet^-, \mu_\bullet^+$	good

TABLE 5. Mutations involving type II quivers.

and examine their mutations at the vertex 0.

As in the previous example, since the arrow $1 \to 0$ (or $2 \to 0$) does not appear in any relation of Λ, the negative mutation $\mu_0^-(\Lambda)$ is defined. But $\mu_0^+(\Lambda)$ is not defined since the composition of the arrow $3 \to 0$ with $0 \to 4$ vanishes. Similarly for Λ', the positive mutation $\mu_0^+(\Lambda')$ is defined since the arrow $0 \to 3$ does not appear in any relation, and $\mu_0^-(\Lambda')$ is not defined since the composition of the arrow $4 \to 0$ with the arrow $0 \to 1$ vanishes.

Example 3.6. Consider the case II.3 in Table 5. We will show that if Λ is one of the cluster-tilted algebras with the quivers given below

$$\begin{tikzcd} & \bullet_1 \arrow[dl] & & \bullet_3 \arrow[dl, "\alpha"'] \\ \bullet_4 \arrow[rr, "\alpha'"] & & \bullet_0 \arrow[ul] \arrow[dl, "\beta'"] \arrow[dr, "\beta"] & \\ & \bullet_2 \arrow[ul] & & \bullet_5 \arrow[uu] \end{tikzcd} \qquad \begin{tikzcd} \bullet_5 \arrow[dr, "\alpha"] & & \bullet_1 \arrow[dl, "\alpha'"'] & \\ & \bullet_0 \arrow[dl, "\beta"] \arrow[rr] & & \bullet_3 \arrow[ul] \arrow[dl] \\ \bullet_4 \arrow[uu] & & \bullet_2 \arrow[ul, "\beta'"] & \end{tikzcd}$$

then both algebra mutations $\mu_0^-(\Lambda)$ and $\mu_0^+(\Lambda)$ are defined.

Indeed, let $p = \gamma_1\gamma_2 \dots \gamma_r$ be a non-zero path starting at 0 written as a sequence of arrows. If $\gamma_1 \neq \beta$, then the composition $\alpha \cdot p$ is not zero, whereas otherwise the composition $\alpha' \cdot p$ is not zero, hence $\mu_0^-(\Lambda)$ is defined. Similarly, if $p = \gamma_1 \dots \gamma_r$ is a non-zero path ending at 0, then composition $p \cdot \beta$ is not zero if $\gamma_r \neq \alpha$, and otherwise $p \cdot \beta'$ is not zero, hence $\mu_0^+(\Lambda)$ is defined as well.

3.3. **Good mutations in types III and IV.** These are given in Tables 6 and 7. Table 6 is computed in a similar way to Tables 4 and 5. In Table 7, the dotted lines indicate the central cycle, and the two vertices at the sides may be identified (for the right quivers in IV.2a and IV.2b, these identifications lead to the left quivers of III.1 and III.2). The proof that all the mutations listed in that table are good relies on the lemmas below.

Mutations at vertices on the central cycle are discussed in Lemmas 3.7, 3.10 and 3.13, whereas mutations at the spikes are discussed in Lemmas 3.8 and 3.11. The moves IV.1a and IV.1b in Table 7 follow from Corollary 3.9. The moves IV.2a and IV.2b follow from Corollary 3.12. Lemma 3.13 implies that there are no additional good mutations involving type IV quivers.

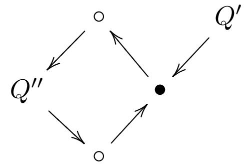
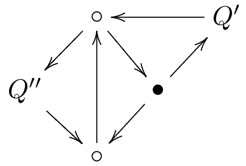
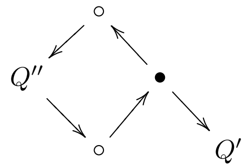
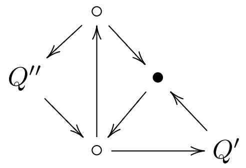
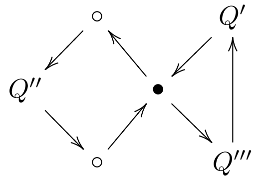
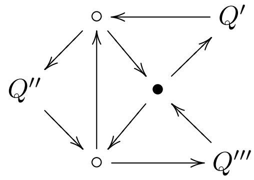
III.1		$\mu_\bullet^-$		$\mu_\bullet^+$	good
III.2		$\mu_\bullet^+$		$\mu_\bullet^-$	good
III.3		$\mu_\bullet^-, \mu_\bullet^+$		none	bad

TABLE 6. Mutations involving type III quivers.

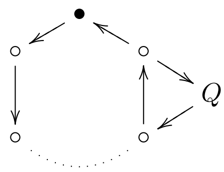
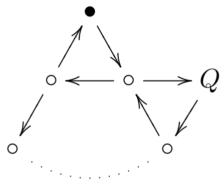
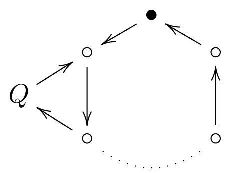
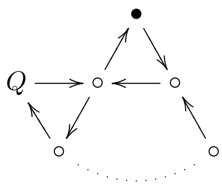
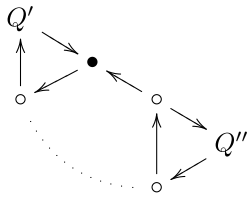
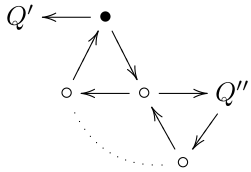
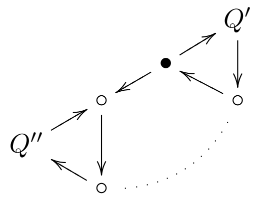
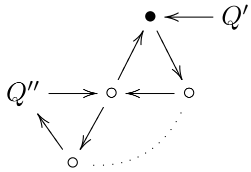
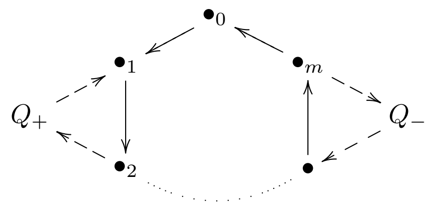
IV.1a		$\mu_\bullet^-$		$\mu_\bullet^+$
IV.1b		$\mu_\bullet^+$		$\mu_\bullet^-$
IV.2a		$\mu_\bullet^-$		$\mu_\bullet^+$
IV.2b		$\mu_\bullet^+$		$\mu_\bullet^-$

TABLE 7. Good mutations involving type IV quivers.

Lemma 3.7. *Let $m \geq 2$ and consider a cluster-tilted algebra Λ of type IV with the quiver*

(3.1)



having a central cycle $0, 1, \dots, m$ and optional spikes Q_- and Q_+ (which coincide when $m = 2$). Then:

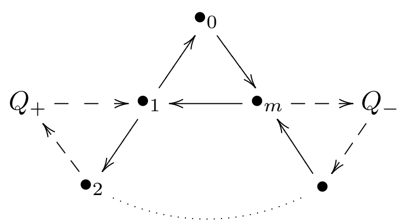
(a) *$\mu_0^-(\Lambda)$ is defined if and only if the spike Q_- is present.*
(b) *$\mu_0^+(\Lambda)$ is defined if and only if the spike Q_+ is present.*

Proof. We prove only the first assertion, as the proof of the second is similar.

We use the criterion of Proposition 1.10. The negative mutation $\mu_0^-(\Lambda)$ is defined if and only if the composition of the arrow $m \to 0$ with any non-zero path starting at 0 is not zero. This holds for all such paths of length smaller than $m-1$, so we only need to consider the path $0, 1, \dots, m-1$. Now, the composition $m, 0, 1, \dots, m-1$ vanishes if Q_- is not present, and otherwise equals the (non-zero) path $m, v_-, m-1$ where v_- denotes the root of Q_-. □

Lemma 3.8. *Let $m \geq 3$ and consider a cluster-tilted algebra Λ of type IV with the quiver*

(3.2)



having a central cycle $1, 2, \dots, m$ and optional spikes Q_- and Q_+. Then:

(a) *$\mu_0^-(\Lambda)$ is defined if and only if the spike Q_- is not present.*
(b) *$\mu_0^+(\Lambda)$ is defined if and only if the spike Q_+ is not present.*

Proof. We prove only the first assertion, as the proof of the second is similar.

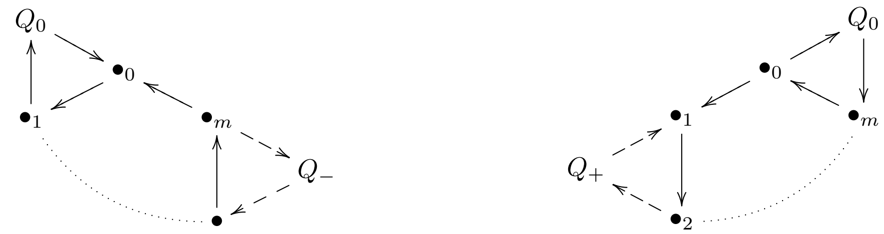
We use the criterion of Proposition 1.10. The negative mutation $\mu_0^-(\Lambda)$ is defined if and only if the composition of the arrow $1 \to 0$ with any non-zero path starting at 0 is not zero. For the path $0, m$, the composition $1, 0, m$ equals the path $1, 2, \dots, m$ and hence it is non-zero. This shows that $\mu_0^-(\Lambda)$ is defined when Q_- is not present. When Q_- is present, the path $0, m, v_-$ to the root v_- of Q_- is non-zero, but the composition $1, 0, m, v_-$ equals the path $1, 2, \dots, m, v_-$ which is zero since the path $m-1, m, v_-$ vanishes □

Corollary 3.9. *Let Λ be a cluster-tilted algebra corresponding to a quiver as in (3.1) and let Λ' be the one corresponding to its mutation at 0, as in (3.2). The following table lists which of the algebra mutations at 0 are defined for Λ and Λ' depending on whether the optional spikes Q_- or Q_+ are present ("yes") or not ("no").*

Q_-	Q_+	Λ	Λ'	
yes	*yes*	μ^-, μ^+	*none*	*bad*
yes	*no*	μ^-	μ^+	*good*
no	*yes*	μ^+	μ^-	*good*
no	*no*	*none*	μ^-, μ^+	*bad*

Lemma 3.10. *Let $m \geq 2$ and consider cluster-tilted algebras Λ_- and Λ_+ of type IV with the following quivers*

(3.3)



having a central cycle $0, 1, \dots, m$ and optional spikes Q_- or Q_+. Then:

(a) $\mu_0^+(\Lambda_-)$ *is never defined;*
(b) $\mu_0^-(\Lambda_-)$ *is defined if and only if the spike Q_- is present;*
(c) $\mu_0^-(\Lambda_+)$ *is never defined;*
(d) $\mu_0^+(\Lambda_+)$ *is defined if and only if the spike Q_+ is present.*

Proof. We prove only the first two assertions, the proof of the others is similar.

(a) Let v_0 denote the root of Q_0. Then the path $v_0, 0$ is non-zero whereas $v_0, 0, 1$ is zero.
(b) Since the path $v_0, 0, 1$ is zero, the composition of the arrow $v_0 \to 0$ with any non-trivial path starting at 0 is zero. Therefore the negative mutation at 0 is defined if and only if the composition of the arrow $m \to 0$ with any non-zero path starting at 0 is not zero, and the proof goes in the same manner as in Lemma 3.7.

□

Lemma 3.11. *Let $m \geq 3$ and consider cluster-tilted algebras Λ_- and Λ_+ of type IV with the following quivers*

(3.4)

having a central cycle $1, \dots, m$ and optional spikes Q_- or Q_+. Then:

(a) $\mu_0^+(\Lambda_-)$ *is always defined;*
(b) $\mu_0^-(\Lambda_-)$ *is defined if and only if the spike Q_- is not present;*
(c) $\mu_0^-(\Lambda_+)$ *is always defined;*
(d) $\mu_0^+(\Lambda_+)$ *is defined if and only if the spike Q_+ is not present.*

Proof. We prove only the first two assertions, the proof of the others is similar.

(a) Let v_0 denote the root of Q_0. Then the composition of any non-zero path ending at 0 with the arrow $0 \to v_0$ is not zero.
(b) Since the composition of the arrow $1 \to 0$ with any non-zero path whose first arrow is $0 \to v_0$ is not zero, we only need to consider paths whose first arrow $0 \to m$. The proof is then the same as in Lemma 3.8.

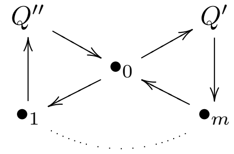
□

Corollary 3.12. *Let Λ_- and Λ_+ be cluster-tilted algebras corresponding to quivers as in* (3.3) *and let Λ'_- and Λ'_+ be the ones corresponding to their mutations at* 0*, as in* (3.4)*. The following tables list which of the algebra mutations at* 0 *are defined for Λ_-, Λ'_-, Λ_+ and Λ'_+ depending on whether the optional spikes Q_- or Q_+ are present ("yes") or not ("no").*

Q_-	Λ_-	Λ'_-	
yes	μ^-	μ^+	*good*
no	*none*	μ^-, μ^+	*bad*

Q_+	Λ_+	Λ'_+	
yes	μ^+	μ^-	*good*
no	*none*	μ^-, μ^+	*bad*

Lemma 3.13. *Let $m \geq 2$ and consider a cluster-tilted algebra Λ of type IV with the following quiver*



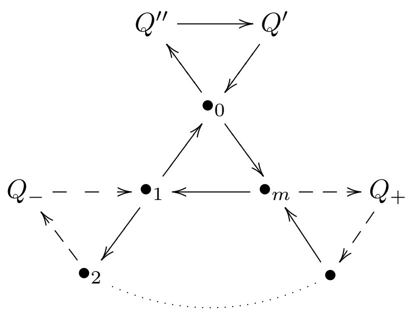
having a central cycle $0, 1, \dots, m$. Then the algebra mutations $\mu_0^-(\Lambda)$ and $\mu_0^+(\Lambda)$ are never defined.

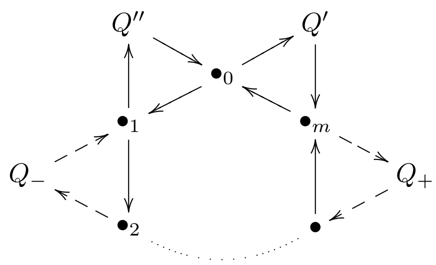
Proof. Denote by v', v'' the roots of Q' and Q'', respectively, and consider the path $0, 1, \dots, m$. It is non-zero, since it equals the path $0, v', m$. However, its composition with the arrow $v'' \to 0$ is zero since the path $v'', 0, 1$ vanishes, and its composition with the arrow $m \to 0$ is zero as well, since it equals $m, 0, v', m$ and the path $m, 0, v'$ vanishes. By Proposition 1.10, the mutation $\mu_0^-(\Lambda)$ is not defined. The proof for $\mu_0^+(\Lambda)$ is similar. □

4. Further derived equivalences in types III and IV

4.1. **Good double mutations in types III and IV.** The good double mutations we consider in this section consist of two algebra mutations. The first takes a cluster-tilted algebra Λ to a derived equivalent algebra which is not cluster-tilted, whereas the second takes that algebra to another cluster-tilted algebra Λ', thus obtaining a derived equivalence of Λ and Λ'. As already demonstrated in Example 2.15, these derived equivalences cannot in general be obtained by performing sequences consisting of only good mutations.

Lemma 4.1. *Let $m \geq 3$ and consider a cluster-tilted algebra $\Lambda = \Lambda_{\widetilde{Q}}$ of type IV with the quiver $\widetilde{Q}$ as in the left picture*





having a central cycle $1, \dots, m$ and optional spikes Q_- and Q_+. Let $\mu_0(\widetilde{Q})$ denote the mutation of $\widetilde{Q}$ at the vertex 0, as in the right picture. Then:

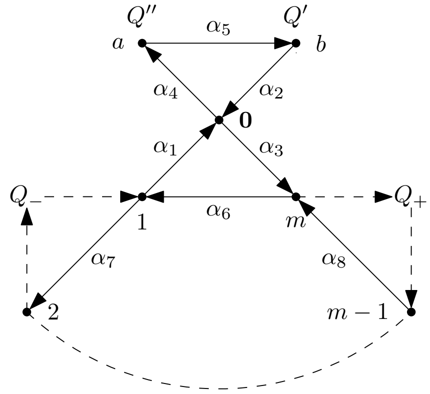
(a) *$\mu_0^-(\Lambda)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $\Lambda_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path p given by*

$$p = \begin{cases} 1, 2, \dots, m, 0 & \textit{if the spike } Q_- \textit{ is present,} \\ 2, \dots, m, 0 & \textit{otherwise.} \end{cases}$$

(b) *$\mu_0^+(\Lambda)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $\Lambda_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path p given by*

$$p = \begin{cases} 0, 1, \dots, m & \textit{if the spike } Q_+ \textit{ is present,} \\ 0, 1, \dots, m-1 & \textit{otherwise.} \end{cases}$$

Proof. We prove only the first assertion and leave the second to the reader. Let $\Lambda = \Lambda_{\widetilde{Q}}$ be the cluster-tilted algebra corresponding to the quiver $\widetilde{Q}$ depicted as



It is easily seen using Proposition 1.10 that the negative mutation $\mu_0^-(\Lambda)$ is defined. In order to describe it explicitly, we recall that $\mu_0^-(\Lambda) = \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T_0^-(\Lambda))$, where

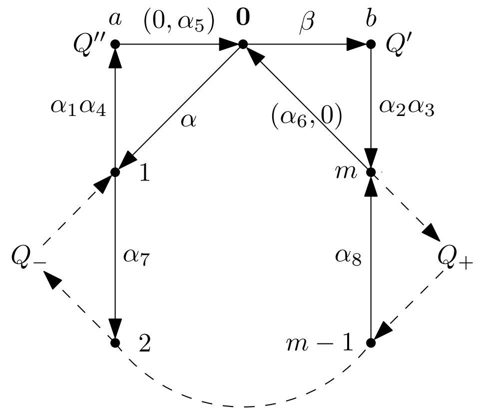
$$T_0^-(\Lambda) = \big(P_0 \xrightarrow{(\alpha_1,\alpha_2)} (P_1 \oplus P_b)\big) \oplus \big(\bigoplus_{i>0} P_i\big) = L_0 \oplus \big(\bigoplus_{i>0} P_i\big).$$

Using an alternating sum formula of Happel [20] we can compute the Cartan matrix of $\mu_0^-(\Lambda)$ to be

$$C_{\mu_0^-(\Lambda)} = \begin{array}{c|ccccc:ccc}
 & 0 & 1 & m & a & b & 2 & (m-1) & \cdots \\ \hline
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \cdots \\
1 & \boxed{0} & 1 & 1 & 1 & 0 & 1 & 1 & \cdots \\
m & 1 & 1 & 1 & 0 & 0 & 1 & ? & \cdots \\
a & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
b & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ \hdashline
2 & 1/\boxed{0} & 1/0 & 1 & 0 & 0 & 1 & 1 & \cdots \\
(m-1) & 1 & 1 & 1 & 0 & 0 & ? & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

where $1/0$ means 1 if Q_- is present and 0 if Q_- is not present.

Now to each arrow of the following quiver we define a homomorphism of complexes between the summands of $T_0^-(\Lambda)$.

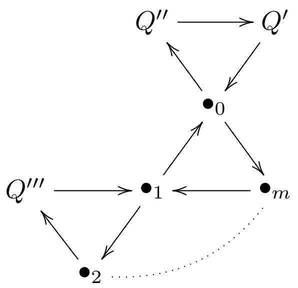


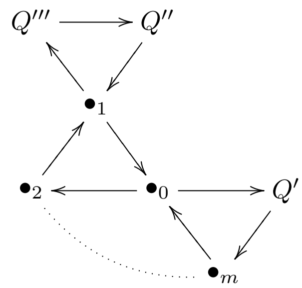
First we have the embeddings $\alpha := (\mathrm{id}, 0) : P_1 \to L_0$ and $\beta := (0, \mathrm{id}) : P_b \to L_0$ (in degree zero). Moreover, we have the homomorphisms $\alpha_1\alpha_4 : P_a \to P_1$, $\alpha_2\alpha_3 : P_m \to P_b$, $(\alpha_6, 0) : L_0 \to P_m$ and $(0, \alpha_5) : L_0 \to P_a$. All the other homomorphisms are as before.

Now we have to show that these homomorphisms satisfy the defining relations of the algebra $\Lambda_{\mu_0(\tilde{Q})}/I(p)$, up to homotopy, where $I(p)$ is the ideal generated by the path p stated in the lemma. Clearly, the concatenation of $(0, \alpha_5)$ and α and the concatenation of $(\alpha_6, 0)$ and β are zero-relations. The concatenation of $\alpha_2\alpha_3$ and $(\alpha_6, 0)$ is zero as before. It is easy to see that the two paths from vertex 0 to vertex m are the same since $(0, \alpha_2\alpha_3)$ is homotopic to $(\alpha_7 \ldots \alpha_8, 0)$ (and $\alpha_7 \ldots \alpha_8 = \alpha_1\alpha_3$ in Λ). The path from vertex 0 to vertex a is zero since $(\alpha_1\alpha_4, 0)$

is homotopic to zero. There is no non-zero path from vertex 1 to vertex 0 since $(0, \alpha_1\alpha_4\alpha_5) = 0 = (\alpha_7 \dots \alpha_8\alpha_6, 0)$. This corresponds to the path p in the case if Q_- is present and is marked in the Cartan matrix by a box. If Q_- is not present then the path from vertex 2 to vertex 0 is already zero since $(\dots \alpha_8\alpha_6, 0) = 0$ which is also marked in the Cartan matrix above. Thus, $\mu_0^-(\Lambda)$ is isomorphic to the quotient of the cluster-tilted algebra $\Lambda_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path p. $\square$

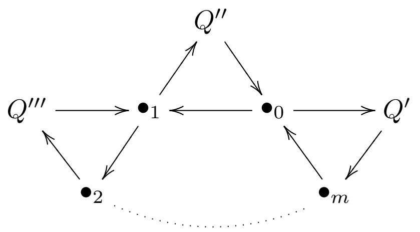
Corollary 4.2. *The two cluster-tilted algebras with quivers*





(where Q', Q'' and Q''' are rooted quivers of type A) are related by a good double mutation (at the vertex 0 and then at 1).

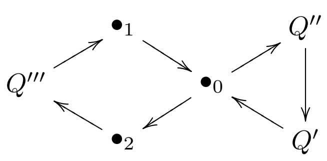
Proof. Denoting the left algebra by Λ_L and the right one by Λ_R, we see that $\mu_0^-(\Lambda_L) \simeq \mu_1^+(\Lambda_R)$, as by Lemma 4.1 these algebra mutations are isomorphic to quotient of the cluster-tilted algebra of the quiver

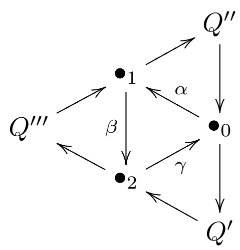


by the ideal generated by the path $1, 2, \dots, m, 0$. $\square$

There is an analogous version of Lemma 4.1 for cluster-tilted algebras in type III, corresponding to the case where $m = 2$, and the spikes Q_- and Q_+ coincide (and are present). That is, there is a central cycle of length $m = 2$ (hence it is "invisible") with all spikes present.

Lemma 4.3. *Consider the cluster-tilted algebra $\Lambda_{\widetilde{Q}}$ of type III whose quiver $\widetilde{Q}$ is shown in the left picture, where Q', Q'' and Q''' are rooted quivers of type A.*





Let $\mu_0(\widetilde{Q})$ denote the mutation of $\widetilde{Q}$ at the vertex 0, as in the right picture. Then:

(a) *$\mu_0^-(\Lambda)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $\Lambda_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path $\beta\gamma$.*
(b) *$\mu_0^+(\Lambda)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $\Lambda_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path $\alpha\beta$.*

Corollary 4.4. *The cluster-tilted algebras of type III with quivers*

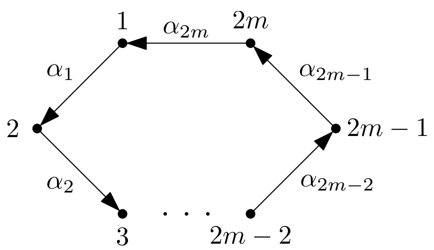
(where Q', Q'' and Q''' are rooted quivers of type A) are related by a good double mutation (at 0 *and then at* 1*).*

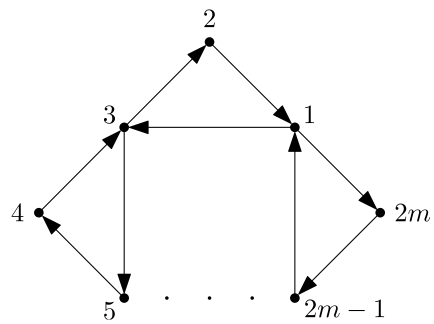
4.2. **Self-injective cluster-tilted algebras.** The self-injective cluster-tilted algebras have been determined by Ringel in [30]. They are all of Dynkin type D_n, $n \geq 3$. Fixing the number n of vertices, there are one or two such algebras according to whether n is odd or even. Namely, there is the algebra corresponding to the cycle of length n without spikes, and when $n = 2m$ is even, there is also the one of type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$ of length m.

The following lemma shows that these two algebras are in fact derived equivalent. Note that this could also be deduced from the derived equivalence classification of self-injective algebras of finite representation type [2].

Lemma 4.5. *Let $m \geq 3$. Then the cluster-tilted algebra of type IV with a central cycle of length $2m$ without any spike is derived equivalent to that in type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$ of length m.*

Proof. Let Λ be the cluster-tilted algebra corresponding to a cycle of length $2m$ as in the left picture.





We leave it to the reader to verify that the following complex of projective Λ-modules

$$T = \Big(\bigoplus_{i=1}^{m}(P_{2i} \xrightarrow{\alpha_{2i-1}} P_{2i-1})\Big) \oplus \big(\bigoplus_{i=1}^{m} P_{2i-1}\big)$$

(where the terms P_{2i-1} are always at degree 0) is a tilting complex whose endomorphism algebra $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is isomorphic to the cluster-tilted algebra whose quiver is given in the right picture. $\square$

5. Algorithms and standard forms

In this section we provide standard forms for derived equivalence (Theorem 2.3) as well as ones for good mutation equivalence (Theorem 2.32) of cluster-tilted algebras of type D. We also describe an explicit algorithm which decides on good mutation equivalence (Theorem 2.31).

5.1. **Good mutations and double mutations in parametric form.** We start by describing all the good mutations determined in Section 3 using the parametric notation of Section 1.6 which would be useful in the sequel. Note that by Proposition 3.1 two quivers with the same type and parameters are indeed equivalent by good mutations so this notation makes sense.

Each row of Table 8 describes a good mutation between two quivers of cluster-tilted algebras of type D given in parametric form. The numbers $s', s'', s''', t', t'', t'''$ are arbitrary non-negative and correspond to the parameters of the rooted quivers Q', Q'', Q''' of type A appearing in the corresponding pictures (referenced by the column "Move").

By looking at the first four rows of the table we immediately draw the following conclusions.

Lemma 5.1. *Consider quivers of type I or II.*

(a) *The subset consisting of the quivers of type I or II is closed under good mutations.*

Move	Type	Parameters	Type	Parameters	Remarks
I.5a	I	$(s'+s'', t'+t''+1)$	II	$(s'+1, t', s'', t'')$	
I.5b	I	$(s'+s'', t'+t''+1)$	II	$(s', t', s''+1, t'')$	
II.2	II	$(s'+1, t', s'', t'')$	II	$(s', t', s''+1, t'')$	
II.3	II	$(s'+s''', t'+t'''+1, s'', t'')$	II	$(s', t', s''+s''', t'+t'''+1)$	
III.1	III	$(s'+1, t', s'', t'')$	IV	$\big((2, s', t'), (1, s'', t'')\big)$	
III.2	III	$(s'+1, t', s'', t'')$	IV	$\big((1, s', t'), (2, s'', t'')\big)$	
IV.1a	IV	$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((1, s_1, t_1), (d_1-2, 0, 0), (d_2, s_2, t_2), \dots\big)$	$d_1 \geq 4$
IV.1b	IV	$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((d_1-2, s_1, t_1), (1, 0, 0), (d_2, s_2, t_2), \dots\big)$	$d_1 \geq 4$
IV.2a	IV	$\big((2, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((1, s_1, t_1), (d_2, s_2+1, t_2), \dots\big)$	
IV.2b	IV	$\big((2, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((1, s_1+1, t_1), (d_2, s_2, t_2), \dots\big)$	

TABLE 8. All good mutations in parametric form.

(b) *The subset consisting of all the orientations of a D_n diagram is closed under good mutations*
(c) *A quiver in type I with parameters $(s, t+1)$ for some $s, t \geq 0$ is equivalent by good mutations to one in type II with parameters $(s+1, t, 0, 0)$.*
(d) *A quiver in type II with parameters (s', t', s'', t'') is equivalent by good mutations to one in type II with parameters $(s'+s'', t'+t'', 0, 0)$.*
(e) *Two quivers in type II with parameters $(s_1, t_1, 0, 0)$ and $(s_2, t_2, 0, 0)$ are equivalent by good mutations if and only if $s_1 = s_2$ and $t_1 = t_2$.*

Lemma 5.2. *Consider quivers of type III.*
(a) *A quiver of type III with parameters $(s'+1, t', s'', t'')$ is good mutation equivalent to one of type III with parameters $(s', t', s''+1, t'')$.*
(b) *A quiver of type III with parameters (s', t', s'', t'') is good mutation equivalent to one of type III with parameters $(s'+s'', t', 0, t'')$.*

Proof. (a) We have

$$\begin{aligned} \mathrm{III}(s'+1, t', s'', t'') &\xrightarrow{\mathrm{III.1}} \mathrm{IV}\big((2, s', t'), (1, s'', t'')\big) \simeq \mathrm{IV}\big((1, s'', t''), (2, s', t')\big) \xrightarrow{\mathrm{III.2}} \mathrm{III}(s''+1, t'', s', t') \\ &\simeq \mathrm{III}(s', t', s''+1, t'') \end{aligned}$$

where the isomorphisms follow from rotational symmetries.
(b) Follows from the first part.

□

Remark 5.3. The previous lemma shows that in type III, it is possible to move linear parts in the rooted quivers of type A from side to side by using good mutations. It is not possible, however, to move triangles by good mutations (see III.3 in Table 6 and Example 2.15).

For the next two lemmas we need the following terminology on spikes in quivers of type IV. Spikes are *consecutive* if the distance (d_i) between them is 1. A spike is *free* if the attached rooted quiver of type A consists of just a single vertex.

Lemma 5.4. *Consider quivers of type IV. A free spike at the end of a group of at least two consecutive spikes can be moved by good mutations to the next group of consecutive spikes. In other words, the two quivers with parameters $\big((1, s, t), (d, 0, 0), \dots\big)$ and $\big((d, s, t), (1, 0, 0), \dots\big)$ are connected by good mutations.*

Proof. We assume $d \geq 2$, otherwise there is nothing to prove. Then

$$\big((1, s, t), (d, 0, 0), \dots\big) \xrightarrow{\mathrm{IV.1a}} \big((d+2, s, t), \dots\big) \xrightarrow{\mathrm{IV.1b}} \big((d, s, t), (1, 0, 0), \dots\big).$$

□

Lemma 5.5. *Lines in a rooted quiver of type A attached to a spike in a group of consecutive spikes in a quiver of type IV can be moved by good mutations to a rooted quiver attached to any spike in that group.*

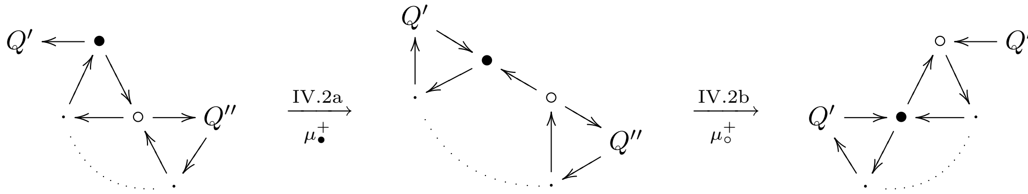
Type	Parameters	Type	Parameters
III	$(s'+s'', t'+t''+1, s''', t''')$	III	$(s', t', s''+s''', t''+t'''+1)$
IV	$\big((1, s'+s'', t'+t''+1), (d_2, s''', t'''), \dots\big)$	IV	$\big((1, s', t'), (d_2, s''+s''', t''+t'''+1), \dots\big)$

TABLE 9. Good double mutations in parametric form.

Proof. It suffices to show that the two quivers with parameters $\big((1, s_1, t_1), (d_2, s_2+1, t_2), \dots\big)$ and $\big((1, s_1+1, t_1), (d_2, s_2, t_2), \dots\big)$ are good mutation equivalent. Indeed,

$$\big((1, s_1, t_1), (d_2, s_2+1, t_2), \dots\big) \xrightarrow{\text{IV.2a}} \big((2, s_1, t_1), (d_2, s_2, t_2), \dots\big) \xrightarrow{\text{IV.2b}} \big((1, s_1+1, t_1), (d_2, s_2, t_2), \dots\big).$$

In pictures,



□

We now turn to good double mutations as determined in Section 4. They are presented in parametric form in Table 9, based on Corollaries 4.2 and 4.4. Using them, we can achieve the following further transformations of quivers in types III and IV described in the next lemmas.

Lemma 5.6. *Consider quivers of type III.*

(a) *A quiver of type III with parameters $(s', t'+1, s'', t'')$ is equivalent by good double mutations to one of type III with parameters $(s', t', s'', t''+1)$.*
(b) *A quiver of type III with parameters (s', t', s'', t'') can be transformed using good mutations and good double mutations to one of type III with parameters $(s'+s'', t'+t'', 0, 0)$.*

Proof. (a) Follows from the first row in Table 9.
(b) Follows from the first part together with Lemma 5.2.

□

Lemma 5.7. *Triangles in a rooted quiver of type A attached to a spike in a group of consecutive spikes in a quiver of type IV can be moved by good double mutations to a rooted quiver attached to any spike in that group.*

Proof. It suffices to show that the two quivers with parameters $\big((1, s_1, t_1), (d_2, s_2, t_2+1), \dots\big)$ and $\big((1, s_1, t_1+1), (d_2, s_2, t_2), \dots\big)$ are equivalent by good double mutation. Indeed, setting $(s'', t'') = (0, 0)$ in the second row of Table 9 shows this equivalence. □

Remark 5.8. A careful look at Tables 8 and 9 shows that one can regard type III quivers with parameters (s', t', s'', t'') as "formal" type IV quivers with parameters $\big((1, s', t'), (1, s'', t'')\big)$. Indeed, the good mutation moves III.1 and III.2 in Table 8 then become just specific cases of moves IV.2b and IV.2a, respectively, and the first row in Table 9 becomes a specific instance of the second.

5.2. **Standard forms for derived equivalence.** In this section we prove Theorem 2.3. Namely, given a quiver Q of a cluster-tilted algebra of Dynkin type D, we show how to find a quiver in one of the standard forms of Theorem 2.3 whose cluster-tilted algebra is derived equivalent to that of Q.

Indeed, if Q is of type I or II, then by Lemma 5.1 we can transform it by good mutations to a quiver in the classes (a) or (b) in Theorem 2.3, thus proving the derived equivalence for this case. Similarly, if Q is of type III, then by Lemma 5.2 and Lemma 5.6 the corresponding cluster-tilted algebra is derived equivalent to a one in class (c) of the theorem.

Let Q be a quiver of type IV. If it is a cycle without any spikes, we distinguish two cases. If the number of vertices is even, then by Lemma 4.5 the corresponding cluster-tilted algebra is derived equivalent to another one in type IV with spikes. Otherwise, it gives rise to the standard form (d_1).

If Q is of type IV with some spikes, let $\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big)$ be its parameters. By iteratively applying the good mutations IV.1a (or IV.1b) we can repeatedly shorten by 2 all the distances $d_i \geq 4$ until they become 2 or 3. By applying the good mutations IV.2a (or IV.2b) we can shorten further any distance 2 to a distance of 1. Thus we get a parameter sequence where all distances d_i are either 1 or 3.

If all the distances are 1, we are in class (d$_2$) when they sum up to at least 3 or in class (c) otherwise. The latter case can be dealt with Lemma 5.2 and Lemma 5.6 yielding the required standard form, whereas for the former we note that by Lemma 5.5 and Lemma 5.7 we can successively move all the lines and triangles of the attached rooted quivers of type A and concentrate them on a single spike, yielding the standard form of (d$_2$).

Otherwise, when there is at least one distance of 3, we observe the following. Consider a group of consecutive spikes. By Lemma 5.5 and Lemma 5.7, inside such a group one can always concentrate the attached type A quivers at one of the spikes in the group, thus creating a free spike at the end of the group. By Lemma 5.4 this free spike can then be moved to the beginning of the next group. In this way, one can move all spikes of the group except one to the next group, thus creating a single spike with some rooted quiver of type A attached.

Continuing in this way, we can eventually merge all groups of at least two consecutive spikes into one large group, with all the other spikes being single spikes. In other words, the sequence of distances will look like $(1, 1, \dots, 1, 3, 3, \dots, 3)$. At this large group of consecutive spikes, one can concentrate the rooted quivers of type A at the last spike, yielding exactly the standard form occurring in (d$_3$) of Theorem 2.3.

To complete the proof of Theorem 2.3, we show how to distinguish among standard forms which are not of the class (d$_3$). First, observe that when the number n of vertices is odd, the form in the class (d$_1$) corresponds to the unique self-injective cluster-tilted algebra with n vertices [30], hence it is distinguished by the fact that being self-injective is invariant under derived equivalence [1].

The standard forms in all other classes (except (d$_3$)) can be distinguished by the determinants of their Cartan matrices. Indeed, these are given in the list below

1	2^{t+1}	$3 \cdot 2^t$	$(2b-1) \cdot 2^t$
(a)	(b)	(c)	(d_2)

from which it is clear how to distinguish among the standard forms.

5.3. **Algorithm for good mutation equivalence.** In this section we prove Theorem 2.31 and Theorem 2.32. We first observe that by Lemma 5.1 the set of quivers of types I or II is closed under good mutations, and moreover that lemma completely characterizes good mutation equivalence among these quivers in terms of their parameters, leading to the classes (a) and (b) in Theorem 2.32. We observe also that the cyclic quiver of type IV without any spikes does not admit any good mutations, thus it falls into a separate equivalence class (d$_1$) in that theorem. Therefore we are left to deal only with quivers of types III and IV (with spikes). Before describing the algorithm, we introduce a few notations.

Notation 5.9. Given $r \geq 1$ and a non-empty subset $\phi \neq \mathcal{I} \subseteq \{1, 2, \dots, r\}$, we define the following two partitions of the set $\{1, 2, \dots, r\}$. Write the elements of $\mathcal{I}$ in an increasing order $1 \leq i_1 < i_2 < \dots < i_l \leq r$ for $l = |\mathcal{I}|$, and define the intervals

$$\begin{aligned}
i_1^+ &= \{i_1, i_1+1, \dots, i_2 - 1\} & \qquad i_1^- &= \{i_l + 1, \dots, r, 1, \dots, i_1\} \\
i_2^+ &= \{i_2, i_2+1, \dots, i_3 - 1\} & \qquad i_2^- &= \{i_1 + 1, \dots, i_2 - 1, i_2\} \\
&\cdots & &\cdots \\
i_l^+ &= \{i_l, \dots, r, 1, \dots, i_1 - 1\} & \qquad i_l^- &= \{i_{l-1} + 1, \dots, i_l - 1, i_l\}
\end{aligned}$$

We call the partition $i_1^+ \cup i_2^+ \cup \dots \cup i_l^+$ the *positive* partition defined by $\mathcal{I}$. Similarly, we call $i_1^- \cup i_2^- \cup \dots \cup i_l^-$ the *negative* partition defined by $\mathcal{I}$.

Notation 5.10. We partition the set of positive integers as $N_1 \cup N_2 \cup N_3$, where

$$N_1 = \{1\}, \qquad N_2 = \{n \geq 2 \,:\, n \text{ is even}\}, \qquad N_3 = \{n \geq 3 \,:\, n \text{ is odd}\}.$$

Notation 5.11. Given a sequence $\big((d_1,s_1,t_1),(d_2,s_2,t_2),\dots,(d_r,s_r,t_r)\big)$ of triples of non-negative integers and a subset I of $\{1,\dots,r\}$, we define the quantities

$$\begin{aligned} a(I) &= |\{i \in I \,:\, d_i \in N_3\}|, \\ b(I) &= |\{i \in I \,:\, d_i \in N_1\}| + \sum_{i \in I \,:\, d_i \in N_2} \frac{d_i}{2} + \sum_{i \in I \,:\, d_i \in N_3} \frac{d_i - 3}{2}, \\ s(I) &= |\{i \in I \,:\, d_i \in N_2\}| + \sum_{i \in I} s_i. \end{aligned}$$

Algorithm 5.12 (Good mutation class)**.** Given a non-empty sequence of triples of non-negative integers

$$\big((d_1,s_1,t_1),(d_2,s_2,t_2),\dots,(d_r,s_r,t_r)\big)$$

such that

- $d_i \geq 1$ and $s_i, t_i \geq 0$ for all $1 \leq i \leq r$,
- $d_1 + d_2 + \cdots + d_r \geq 2$ and $(d_1,\dots,d_r) \neq (2)$,

parameterizing a quiver of type III or IV (with spikes), we output its class (c), ($\mathrm{d}_{2,1}$), ($\mathrm{d}_{2,2}$), ($\mathrm{d}_{3,1}$) or ($\mathrm{d}_{3,2}$) and the parameters in that class as specified in Theorem 2.32 by performing the following operations.

1. Compute the subsets
$$\mathcal{I}_D = \{1 \leq i \leq r \,:\, d_i \in N_3\}, \qquad \mathcal{I}_T = \{1 \leq i \leq r \,:\, t_i > 0\}.$$
2. If $\mathcal{I}_D = \phi$ and $\mathcal{I}_T = \phi$, set b, S as
$$b = b(\{1,2,\dots,r\}), \qquad S = s(\{1,2,\dots,r\}).$$
If $b \geq 3$, we are in class ($\mathrm{d}_{2,1}$), otherwise we are in class (c) with parameters $(S,0,0,0)$
3. If $\mathcal{I}_D = \phi$ and $\mathcal{I}_T \neq \phi$, enumerate the elements of $\mathcal{I}_T$ in increasing order as $\mathcal{I}_T = \{i_1 < i_2 < \cdots < i_l\}$ with $l = |\mathcal{I}_T|$, and set b_j, T_j for $1 \leq j \leq l$ and S as
$$b_j = b(i_j^+), \qquad T_j = t_{i_j}, \qquad S = s(\{1,2,\dots,r\}).$$
If $b_1 + \cdots + b_l \geq 3$, we are in class ($\mathrm{d}_{2,2}$). Otherwise, we are in class (c) with parameters $(S+T_1,0,0,0)$ if $l=1$ or $(S+T_1,0,T_2,0)$ if $l=2$.
4. If $\mathcal{I}_D \neq \phi$ and $\mathcal{I}_T = \phi$, enumerate the elements of $\mathcal{I}_D$ in increasing order as $\mathcal{I}_D = \{i_{1,1} < i_{1,2} < \cdots < i_{1,a}\}$ and set a, b and $S_1,\dots,S_a$ as
$$a = a(\{1,2,\dots,r\}) = |\mathcal{I}_D|, \qquad b = b(\{1,2,\dots,r\}), \qquad S_j = s(i_{1,j}^-).$$
We are in class ($\mathrm{d}_{3,1}$).
5. If $\mathcal{I}_D \neq \phi$ and $\mathcal{I}_T \neq \phi$, enumerate the elements of $\mathcal{I}_T$ in increasing order as $\mathcal{I}_T = \{i_1 < i_2 < \cdots < i_l\}$ with $l = |\mathcal{I}_T|$. For any $1 \leq j \leq l$,
   - Enumerate the elements of $i_j^+ \cap \mathcal{I}_D$ (where the positive partition is taken with respect to the subset $\mathcal{I}_T$) in the order they appear within the interval i_j^+ as $i_{j,1} < i_{j,2} < \cdots < i_{j,a(i_j^+)}$.
   - Set a_j, b_j, T_j and $S_{j,1},\dots,S_{j,a_j}$ as
   $$a_j = a(i_j^+), \quad b_j = b(i_j^+), \quad T_j = t_{i_j}, \quad (S_{j,1},\dots,S_{j,a_j}) = \big(s(i_{j,1}^-), s(i_{j,2}^-), \dots, s(i_{j,a(i_j^+)}^-)\big)$$
   with the positive partition taken with respect to $\mathcal{I}_T$ and the negative one with respect to $\mathcal{I}_D$.

   We are in class ($\mathrm{d}_{3,2}$).

Notation 5.13. We call two sequences $(v_0,v_1,\dots,v_{m-1})$ and $(w_0,w_1,\dots,w_{m-1})$ *cyclic equivalent* if there is some $0 \leq j \leq m$ such that $w_i = v_{(i+j) \bmod m}$ for all $0 \leq i < m$.

Definition 5.14. We define the space $\mathcal{S}$ of *good mutation parameters* as a disjoint union of the following five sets. We also define an equivalence relation $\sim$ on $\mathcal{S}$ inside each set, and agree that elements from different sets are inequivalent.

(c) Triples (T_1,T_2,S) of non-negative integers. $(T_1,T_2,S) \sim (T_1',T_2',S')$ if and only if $S = S'$ and (T_1,T_2), (T_1',T_2') are cyclic equivalent.

($\mathrm{d}_{2,1}$) Pairs (b,S) with $b \geq 3$ and $S \geq 0$. $(b,S) \sim (b',S')$ if and only if $b = b'$ and $S = S'$.

($\mathrm{d}_{2,2}$) Pairs
$$\Big(\big((b_1,T_1),(b_2,T_2),\dots,(b_l,T_l)\big),S\Big)$$
for some $l \geq 1$, where the numbers b_j, T_j are positive, $b_1+\dots+b_l \geq 3$ and $S \geq 0$. Two such pairs are equivalent if and only if $S = S'$ and $\big((b_1,T_1),(b_2,T_2),\dots,(b_l,T_l)\big)$, $\big((b'_1,T'_1),(b'_2,T'_2),\dots,(b'_l,T'_l)\big)$ are cyclic equivalent.

($\mathrm{d}_{3,1}$) Pairs $\big(b,(S_1,\dots,S_a)\big)$ where $b \geq 0$ and $(S_1,\dots,S_a)$ is a sequence of $a > 0$ non-negative integers. Two such pairs are equivalent if and only if $b = b'$ and $(S_1,\dots,S_a)$, $(S'_1,\dots,S'_a)$ are cyclic equivalent.

($\mathrm{d}_{3,2}$) Sequences
$$\Big(\big(b_1,(S_{1,1},\dots,S_{1,a_1}),T_1\big),\big(b_2,(S_{2,1},\dots,S_{2,a_2}),T_2\big),\dots,\big(b_l,(S_{l,1},\dots,S_{l,a_l}),T_l\big)\Big),$$
of any length $l \geq 1$, where for any $1 \leq j \leq l$ the numbers a_j, b_j are non-negative integers not both zero, $(S_{j,1},\dots,S_{j,a_j})$ is a (possibly empty) sequence of a_j non-negative integers and T_j is a positive integer. The relation $\sim$ is just cyclic equivalence.

Remark 5.15. It is easy to decide whether two good mutation parameters are equivalent or not, because this involves only checking for cyclic equivalence.

Algorithm 5.12 in fact computes a map $\Sigma : \mathcal{Q} \to \mathcal{S}$ from the set $\mathcal{Q}$ of all quivers of types III or IV (with spikes) to the set $\mathcal{S}$ of good mutation parameters. On the other hand, the standard forms stated in Theorem 2.32 can be seen as a map $Q : \mathcal{S} \to \mathcal{Q}$. We also have two natural equivalence relations on these sets: the equivalence relation $\sim$ defined on $\mathcal{S}$ via cyclic equivalence, and the good mutation equivalence on $\mathcal{Q}$, which we also denote by $\sim$. The correctness of the algorithm is guaranteed by the following proposition whose proof is long but straightforward building on arguments similar to those presented in Section 5.2, and is therefore left to the interested reader.

Proposition 5.16. *Let $q, q' \in \mathcal{Q}$ and $\sigma, \sigma' \in \mathcal{S}$.*

(a) *If $q \sim q'$, then $\Sigma(q) \sim \Sigma(q')$.*
(b) *If $\sigma \sim \sigma'$ then $Q(\sigma) \sim Q(\sigma')$.*
(c) *If $\sigma \in \mathcal{S}$ then $\Sigma(Q(\sigma)) = \sigma$. In other words, applying Algorithm 5.12 to a standard form as in Theorem 2.32 recovers the parameters of that form.*
(d) *If $q \in \mathcal{Q}$ then there exists $\sigma \in \mathcal{S}$ such that $q \sim Q(\sigma)$. In other words, a quiver can be transformed by good mutations to a quiver in standard form as in Theorem 2.32.*

This completes the proof of Theorem 2.31 and Theorem 2.32.

Appendix A. Proofs of Cartan determinants

As a consequence of Proposition 1.4 the determinant of the Cartan matrix is invariant under derived equivalences. The aim of this section is to provide a proof of the formulae for the determinants of the Cartan matrices of all cluster-tilted algebras of Dynkin type D as given in Theorem 2.5. Recall that the quivers of the cluster-tilted algebras of type D are given by the quivers of types I, II, III and IV described in Section 1.6.

As these quivers are defined by gluing of rooted quivers of type A, it is useful to also have formulae for cluster-tilted algebras of Dynkin type A. Since cluster-tilted algebras of type A are gentle, the Cartan determinants can be obtained as a special case of [21] where formulae for the Cartan determinants of arbitrary gentle algebras are given; for a simplified proof for the special case of cluster-tilted algebras of type A see also [13].

Proposition A.1. *Let Q be a quiver mutation equivalent to a Dynkin quiver of type A. Then the Cartan matrix of the cluster-tilted algebra corresponding to Q has determinant* $\det C_Q = 2^{t(Q)}$.

For proving Theorem 2.5 we shall first show a useful reduction lemma. We need the following notation: if Q is a quiver and V a set of vertices in Q, then $Q \setminus V$ is the quiver obtained from Q by removing all vertices in V from Q and all arrows attached to them.

Lemma A.2. *Let Q be a quiver in the mutation class of a quiver of Dynkin type D, i.e. Q is of one of the types I,II,III,IV given in Section 1.6.*

(i) *Suppose Q contains a vertex a of valency 1. Then* $\det C_Q = \det C_{Q\setminus\{a\}}$.

(ii) *Suppose that Q contains an oriented triangle with vertices a, b, c (in this order, i.e. there is an arrow from b to c etc) where a and b have valency 2 in Q and where the quiver $Q' = Q \setminus \{a, b\}$ is mutation equivalent to a quiver of Dynkin type A or D. Then* $\det C_Q = 2 \cdot \det C_{Q\setminus\{a,b\}} = 2 \cdot \det C_{Q'}$.

Proof. (i) Since taking transposes does not change the determinant we can assume that a is a sink. Then the Cartan matrix of the cluster-tilted algebra corresponding to Q has the form

$$C_Q = \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline * & & & \\ \vdots & & C_{Q\setminus\{a\}} & \\ * & & & \end{array}\right)$$

from which the desired formula directly follows by Laplace expansion.

(ii) In the cluster-tilted algebra corresponding to Q, every product of two consecutive arrows in the triangle a, b, c is zero. Moreover, in the quiver $Q \setminus \{a\}$ there is a one-one correspondence between non-zero paths starting in c and non-zero paths starting in b by extending any of the former paths by the arrow from b to c (in fact, by the unique relations in the cluster-tilted algebra all these extensions remain non-zero). Therefore, the Cartan matrix of the cluster-tilted algebra corresponding to Q has the form

$$C_Q = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & *_1 & \dots & \dots & *_n \\ 1 & 0 & 1 & *_1 & \dots & \dots & *_n \\ * & 0 & * & & & & \\ \vdots & \vdots & \vdots & & C_{Q\setminus\{a,b,c\}} & & \\ \vdots & \vdots & \vdots & & & & \\ * & 0 & * & & & & \end{pmatrix}$$

where the first three rows are labelled by a, b and c, respectively. The entries marked by $*_i$ are really the same in the rows for b and c because of the one-one correspondence just mentioned. Moreover, note that in the (first) row for a and in the (second) column for b we have 0's except the two 1's indicated because of the zero-relations in the triangle with vertices a, b, c.

Denote by r_v the row in the above matrix corresponding to the vertex v. We now perform an elementary row operation, namely replace the first row r_a by $r_a - r_b + r_c$. Then we get

$$\det C_Q = \det \begin{pmatrix} 2 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & *_1 & \dots & \dots & *_n \\ 1 & 0 & 1 & *_1 & \dots & \dots & *_n \\ * & 0 & * & & & & \\ \vdots & \vdots & \vdots & & C_{Q\setminus\{a,b,c\}} & & \\ \vdots & \vdots & \vdots & & & & \\ * & 0 & * & & & & \end{pmatrix} = 2 \cdot \det C_{Q\setminus\{a,b\}}$$

where the last equality follows directly by Laplace expansion (with respect to the row of a and then the column of b).

□

We will also need the following three lemmas dealing with skeleta of type IV, i.e. the rooted quivers of type A consist of just one vertex.

Lemma A.3. *Let Q be a quiver of type IV which contains no spikes at all, i.e. it is just an oriented cycle of length $k \geq 3$. Then* $\det C_Q = k - 1$.

Lemma A.4. *Let Q be a quiver of type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$ of length $k \geq 3$, in other words, it is an oriented cycle of length k with all spikes present. Then* $\det C_Q = 2k - 1$.

Lemma A.5. *Let Q be a quiver of type IV with oriented cycle of length $k \geq 3$ and not all spikes are present. Let $c(Q)$ be the number of vertices on the oriented cycle which are part of two (consecutive) spikes. Then* $\det C_Q = k + c(Q) - 1$.

Proof of Lemma A.3. We have

$$\det C_Q = \det \begin{pmatrix} 1 & \dots & \dots & 1 & 0 \\ 0 & 1 & \dots & \dots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & 1 \end{pmatrix} = (-1)^{k-1} \det \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} = k-1,$$

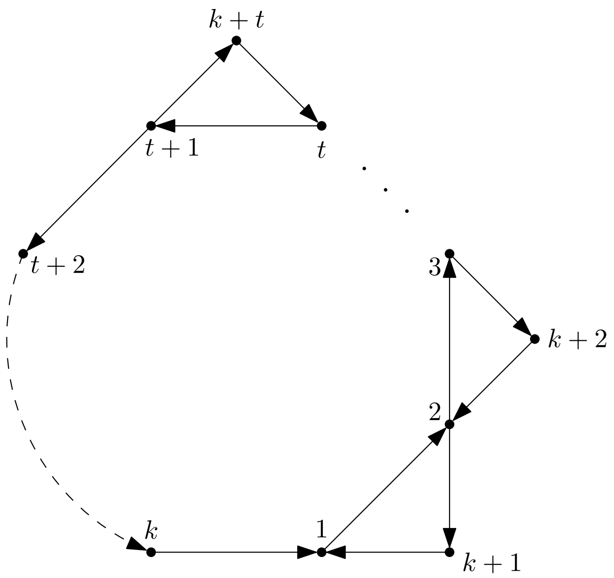
where for the last equality we have used the following formula whose verification is a standard exercise in linear algebra: for all $a, b \in \mathbb{R}$ we have

$$\det \begin{pmatrix} b & a & \dots & a \\ a & b & & \vdots \\ \vdots & & \ddots & a \\ a & \dots & a & b \end{pmatrix} = (b-a)^{k-1}(b+(k-1)a). \tag{A.1}$$

$\square$

Proof of Lemma A.4. By Lemma 4.5, the cluster-tilted algebra Λ_Q is derived equivalent to the one corresponding to the cycle of length $2k$. Since the determinant of the Cartan matrix is invariant under derived equivalence, the result now follows from Lemma A.3. $\square$

Proof of Lemma A.5. We shall closely look at one group of $t \geq 1$ consecutive spikes in Q and label the vertices as in the following figure



Then the Cartan matrix has the following shape

$$C_Q = \left(\begin{array}{cccc|cccc||cccc|cccc}
1 & 1 & \dots & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\
1 & 1 & \dots & 1 & 1 & \dots & \dots & 1 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & & \vdots & & & \vdots \\
1 & 1 & \dots & 1 & 1 & \dots & \dots & 1 & 0 & \dots & 1 & 0 & 0 & \dots & \dots & 0 \\
\hline
1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 1 & 0 & \dots & \dots & 0 \\
\hline
1 & 1 & \dots & 1 & & & & & 0 & 0 & \dots & 0 & & & & \\
1 & 1 & \dots & 1 & & ? & ? & & 0 & 0 & \dots & 0 & & ? & ? & \\
\vdots & \vdots & & \vdots & & ? & ? & & \vdots & \vdots & & \vdots & & ? & ? & \\
1 & 1 & \dots & 1 & & & & & 0 & 0 & \dots & 0 & & & & \\
\hline\hline
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 1 & & \vdots & \vdots & & & \vdots & 1 & \ddots & & & \vdots & & & \vdots \\
\vdots & & \ddots & 0 & \vdots & & & \vdots & 0 & \ddots & \ddots & 0 & \vdots & & & \vdots \\
\hline
0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 & 0 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\
\hline
0 & \dots & \dots & 0 & & & & & 0 & \dots & \dots & 0 & & & & \\
\vdots & & & \vdots & & ? & ? & & \vdots & & & \vdots & & ? & ? & \\
0 & \dots & \dots & 0 & & & & & 0 & \dots & \dots & 0 & & & &
\end{array}\right)$$

The two highlighted rows correspond to the vertices $t+1$ and $k+t$, respectively. For each vertex j let r_j be the row of C_Q corresponding to j.

Note that the column $k+t$ of C_Q has only two non-zero entries, namely in rows $t+1$ and $k+t$. We first replace row r_{t+1} by $r_{t+1} - r_{k+t} + r_t - r_1$ (in case $t = 1$ this indeed means just $r_{t+1} - r_{k+t}$). Then column $k+t$ has only one non-zero entry, namely on the diagonal; Laplace expansion along this column yields a new matrix $\tilde{C}$.

We consider the $(t+1)$-st row in this new matrix which has the form

$$\left(\begin{array}{ccccc|ccccc||cccc|cccc} 1 & 1 & \dots & 1 & 0 & 1 & 1 & \dots & 1 & N & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \end{array}\right)$$

where the number N at position $(t+1, k)$ is equal to 0 if $t = 1$ and equal to 2 if $t > 1$.

In case $t = 1$ we see that $\tilde{C}$ is equal to the Cartan matrix of the cluster-tilted algebra corresponding to the quiver $Q \setminus \{k+t\}$. This means that when computing the Cartan determinant we can remove isolated spikes, i.e. spikes which are not neighboring any other spike.

In this case $t = 1$ the statement of the theorem follows immediately by induction on the number of spikes (with the case of no spikes treated earlier as base of the induction). In fact, removing the isolated spike does not change the determinant (as we have just seen), and also the formula given in the theorem is not affected by removing an isolated spike.

Let us turn to the more complicated case $t > 1$ (which we shall also show by induction). If $t > 1$, the matrix $\tilde{C}$ is equal to the Cartan matrix of $Q \setminus \{k+t\}$, except for the $(t+1, k)$-entry which is 2 in $\tilde{C}$, but 1 in $C_{Q\setminus\{k+t\}}$.

To compare the determinants in this case we use the following easy observation. Let $C = (c_{ij})$ and $\tilde{C} = (\tilde{c}_{ij})$ be two matrices which only differ at the (m, n)-entry. Then

$$\det \tilde{C} - \det C = (-1)^{m+n}(\tilde{c}_{mn} - c_{mn})C_{mn}$$

where C_{mn} is the matrix obtained from C (or $\tilde{C}$) by removing row m and column n.

Applied to our situation we get

$$\det \tilde{C} - \det C_{Q\setminus\{k+t\}} = (-1)^{t+1+k}(2-1)\det C_{t+1,k} = (-1)^{t+1+k}\det C_{t+1,k}.$$

Since $\det \tilde{C} = \det C_Q$ we can rephrase this to get

$$\det C_Q = \det C_{Q\setminus\{k+t\}} + (-1)^{t+1+k}\det C_{t+1,k}. \tag{A.2}$$

By induction on the number of spikes of Q (with the case of no spikes treated earlier as base of the induction) we can deduce that $\det C_{Q\setminus\{k+t\}} = k + c(Q) - 2$ and hence

$$\det C_Q = k + c(Q) - 2 + (-1)^{t+1+k}\det C_{t+1,k}.$$

For proving the assertion of the theorem we therefore have to show that $(-1)^{t+1+k}\det C_{t+1,k} = 1$.

We keep the labelling of the rows and columns also for $C_{t+1,k}$ (i.e. there is no row with label $t+1$ or $k+t$, and no column with label k or $k+t$).

For convenience, the vertices $1,\dots,k$ on the cycle will be called cycle vertices and the remaining vertices will be called outer vertices in the sequel.

Note that in $C_{t+1,k}$ we have 0's on the diagonal in all rows indexed by cycle vertices which have no spike attached. More precisely, the rows corresponding to cycle vertices are of the form

$$\left(\begin{array}{ccc|ccc||ccccccc|ccc} 1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{array} \right)$$

if the vertex has a spike attached, and

$$\left(\begin{array}{cccc|ccc||ccc|ccc} 1 & \dots & 1 & 0 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \end{array} \right)$$

if there is no spike attached to the vertex.

For each cycle vertex $j \neq 1$ with no spike attached we perform the elementary row operation replacing r_j by $r_j - r_1$; this gives a unit vector with -1 on the diagonal. Laplace expansion along all these rows removes from $C_{t+1,k}$ all rows and columns corresponding to cycle vertices (not equal to vertex 1) with no spikes attached; for the determinant we thus get a sign $(-1)^{k-s(Q)-1}$ where $s(Q)$ is the total number of spikes of Q.

The matrix obtained after this removal process has rows indexed by vertex 1, the cycle vertices with spikes attached except vertex $t+1$, and the outer vertices except vertex $k+t$. Moreover it has the form

$$\left(\begin{array}{ccccccc|cccc}
1 & 1 & \dots & \dots & \dots & 1 & 1 & 0 & \dots & \dots & 0 \\
1 & 1 & \dots & \dots & \dots & 1 & 1 & 1 & 0 & \dots & 0 \\
\vdots & \vdots & & & & \vdots & \vdots & 0 & 1 & \ddots & \vdots \\
1 & 1 & \dots & \dots & \dots & 1 & 1 & \vdots & \ddots & \ddots & 0 \\
1 & 1 & \dots & \dots & \dots & 1 & 1 & 0 & \dots & 0 & 1 \\
\hline
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & ? & ? & \\
 & & & 0 & 1 & & & & ? & ? & \\
 & & & & \ddots & \ddots & & & & & \\
 & & & & & 0 & 1 & & & &
\end{array} \right)$$

where in the lower left block the crucial 0 on the main diagonal occurs in the column labelled by vertex t (because the row indexed by $k+t$ has been removed).

Now for each cycle vertex j with a spike attached we replace the row r_j by $r_j - r_1$, giving a unit vector. Consecutive Laplace expansion along these rows removes all columns corresponding to outer vertices (and all rows corresponding to cycle vertices with spikes attached). For each of these Laplace expansions we get a sign $(-1)^{s(Q)+1}$ and there are $s(Q)-1$ such expansions in total, giving an overall sign of $(-1)^{s(Q)^2-1} = (-1)^{s(Q)-1}$.

We are left with a $s(Q) \times s(Q)$-matrix of the form

$$\begin{pmatrix}
1 & 1 & \dots & \dots & \dots & 1 & 1 \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & & 0 & 1 & & \\
 & & & & \ddots & \ddots & \\
 & & & & & 0 & 1
\end{pmatrix}$$

whose determinant is easily seen to be $(-1)^{t+1}$.

Summarizing our arguments we get

$$\det C_{t+1,k} = (-1)^{k-s(Q)-1} \cdot (-1)^{s(Q)-1} \cdot (-1)^{t+1} = (-1)^{k+t-1}.$$

Substituting this into equation (A.2) we get for the Cartan determinant of Q the following

$$\begin{aligned}\det C_Q &= k + c(Q) - 2 + (-1)^{t+1+k} \det C_{t+1,k} \\ &= k + c(Q) - 2 + (-1)^{t+1+k}(-1)^{k+t-1} = k + c(Q) - 1\end{aligned}$$

which is exactly the formula claimed in Theorem 2.5. □

Proof of Theorem 2.5. (I) Applying part (i) of Lemma A.2 twice gives $\det C_Q = \det C_{Q\setminus\{a,b\}}$. By definition $Q' = Q \setminus \{a, b\}$ is a quiver of Dynkin type A, thus $\det C_{Q'} = 2^{t(Q')}$ by Proposition A.1. Clearly, $t(Q) = t(Q')$ for quivers of type I and hence

$$\det C_Q = \det C_{Q\setminus\{a,b\}} = \det C_{Q'} = 2^{t(Q')} = 2^{t(Q)}.$$

(II) Let Q be a quiver of type II. By applying Lemma A.2 inductively we can shrink each of the quivers Q' and Q'' (which are of Dynkin type A) to one vertex where for the Cartan determinant of the corresponding cluster-tilted algebra we get a factor 2 for each triangle we remove (see part (ii) of Lemma A.2). Thus we get

$$\det C_Q = 2^{t(Q')} \cdot 2^{t(Q'')} \cdot \det C_{\tilde{Q}} \tag{A.3}$$

where $\tilde{Q}$ is the quiver with vertices a, b, c, d obtained after shrinking each of Q' and Q'' to one vertex. Labelling the rows and columns in the order a, b, c, d the cluster-tilted algebra corresponding to $\tilde{Q}$ has Cartan matrix $C_{\tilde{Q}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ whose determinant is easily computed to be 2. This gives the desired formula as

$$\det C_Q = 2^{t(Q')} \cdot 2^{t(Q'')} \cdot \det C_{\tilde{Q}} = 2^{t(Q')+t(Q'')+1} = 2 \cdot \det C_{Q'} \det C_{Q''}.$$

(III) Completely analogous to the previous argument in type II we can shrink the subquivers Q' and Q'' of any quiver of type III to one vertex, ending up with an oriented 4-cycle $\tilde{Q}$. Labelling the rows and columns in the order a, c, b, d the cluster-tilted algebra corresponding to this 4-cycle has Cartan matrix $C_{\tilde{Q}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ which has determinant 3. As above we then get

$$\det C_Q = 2^{t(Q')} \cdot 2^{t(Q'')} \cdot \det C_{\tilde{Q}} = 3 \cdot 2^{t(Q')+t(Q'')} = 3 \cdot \det C_{Q'} \det C_{Q''}.$$

(IV) If there are no spikes at all, the result follows from Lemma A.3. Otherwise, by Lemma A.2 we can again assume that all the rooted quivers $Q^{(1)}, \ldots, Q^{(r)}$ of type A attached to the spikes have been shrinked to one vertex, yielding a factor

$$\prod_{j=1}^{r} 2^{t(Q^{(j)})} = \prod_{j=1}^{r} \det C_{Q^{(j)}}$$

for the Cartan determinant $\det C_Q$. The result then follows from Lemma A.4 and Lemma A.5.

□

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