Introduction to Higher Cubical Operads.
Second Part : The Functor of Fundamental Cubical Weak ∞-Groupoids for Spaces

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Abstract

In the second part of this article we use the cubical operad B^0_C of cubical weak ∞-categories (built in [10]) as a fundamental step to associate to any topological space X its fundamental cubical weak ∞-groupoids $\Pi_\infty(X)$, and this endows a functor $\mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}\mathbb{G}\mathrm{rp}$ which has a left adjoint functor CN_∞. This pair of adjunction $(CN_\infty, \Pi_\infty(-))$ should put an equivalence between the homotopy category of homotopy types and the homotopy category of ∞-$\mathbb{C}\mathbb{G}$rp of cubical weak ∞-groupoids with connections equipped with an adapted Quillen model structure.

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Contents

Introduction

20 years ago Michael Batanin in [1] had described the functor of fundamental globular weak ∞-groupoids for spaces in order to give a rigorous formulation of the Grothendieck conjecture on homotopy types [9] : in particular he built a functor from the category $\mathbb{T}\mathrm{op}$ of spaces to the category of globular weak ∞-groupoids. In order to do that he built an operadic approach of globular weak ∞-categories, that is his globular weak ∞-categories are algebra for a specific operad B^0_C. Two major steps for higher category theory were achieved in [1] :

- he builts a higher globular dimensional approach of non-symmetric operads à la Peter May;
- his definition of weak ∞-categories is more general than simplicial models of $(\infty,1)$-categories. For example it is proved in [12] that some algebraic models of $(\infty,1)$-categories are embedded in his weak ∞-categories.

In order to built the functor of fundamental globular weak ∞-groupoids for spaces he proved that the globular object $D^\bullet$ in $\mathbb{T}\mathrm{op}$ consisting of topological disks :

$$D^0 \underset{t^1_0}{\overset{s^1_0}{\rightrightarrows}} D^1 \underset{t^2_1}{\overset{s^2_1}{\rightrightarrows}} D^2 \cdots D^{n-1} \underset{t^n_{n-1}}{\overset{s^n_{n-1}}{\rightrightarrows}} D^n \cdots$$

is a B^0_C-coalgebra, which implication is the construction of the fundamental globular weak ∞-groupoid functor

$$\mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{G}\mathrm{rp}$$

In [17] Tom Leinster gave a simplification of the orginal definition of higher operads by Michael Batanin. However the very important examples of (co)endomorphism globular operads are built very naturally within the framework of globular monoidal categories, and this is not clear for us that the $\mathbb{T}$-categorial framework of Leinster can capture such natural point of view of (co)endomorphism globular operads. It seems that in [17], he succeeded to define such (co)endomorphism globular operads through $\mathbb{T}$-categories, but only in the context of locally cartesian closed categories. For example if C is a category with pullbacks and if E is a global object in the monoidal globular category $\mathbb{S}\mathrm{pan}(C)$ consisting of globular higher spans in C, it is possible to define its associated endomorphism operad $\mathbb{E}\mathrm{ND}(E)$ by using the theory of Batanin (see also [21]), but this is not clear for us how to get such operad $\mathbb{E}\mathrm{ND}(E)$ with $\mathbb{T}$-categories. Thus in order to write the first part of the article [10] we used the Leinster approach to build the operad which algebras are cubical weak ∞-categories, but to define cubical higher operads of endomorphism we found that the cubical analogue of the globular monoidal categories was much more natural.

In this article, which is the second part of [10], we use the cubical operad B^0_C of cubical weak ∞-categories (built in [10]) as a fundamental step to associate to any topological space X its fundamental cubical weak ∞-groupoids $\Pi_\infty(X)$, and this endows a functor $\mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}\mathbb{G}\mathrm{rp}$ which has a left adjoint functor CN_∞. This pair of adjunction $(CN_\infty, \Pi_\infty(-))$ should put an equivalence between the homotopy category of homotopy types and the homotopy category of $\infty\text{-}\mathbb{C}\mathbb{G}\mathrm{rp}$ of cubical weak ∞-groupoids with connections, through adapted Quillen model structures. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [18]). It is also important to know that non-operadical approach have been considered in [4, 8] to define other higher groupoid constructions for spaces.

Important tools to build this functor $\Pi_\infty(-)$ come from 2-category theory and especially thanks to the work of Mark Weber ([23, 24]) and Ross Street ([20, 21]) : pseudo-algebras for 2-monads and a generalization of the $\mathbb{S}\mathrm{pan}$ construction have been successfully considered for this interaction between elementary 2-topos and cubical geometry. An important feature of this article is also to show how the 2-categorical tools developed in [20, 21, 23, 24] can lead to generalization of the original theory of Michael Batanin's higher operads.

Plan of this paper :

- In the first section we define *monoidal cubical categories* as pseudo $\mathbb{S}$-algebras, where $\mathbb{S}$ is the 2-monad of free strict monoidal cubical categories on cubical categories.
- In the second section we state an important result of [11] which shows that for general situations the $\mathbb{S}\mathrm{pan}$-construction leads to pseudo algebraic structure. Then we give a nice combinatorial description of the cubical (co)spans taken from Marco Grandis ([7]). Then we define (co)endomorphisms operads by using the 2-categorical point of view of Ross Street and Mark Weber in [20, 21, 23, 24]. Our 2-categorical point of view of (co)endomorphisms operads can be adapted in the general context of pseudo algebras, and this is very important for a 2-categorical generalisation of the theory of Batanin.
- In the third section we proved that the cocubical object "box" (as defined in [5]) in $\mathbb{T}\mathrm{op}$:

$$
\begin{tikzcd}
I^0 \arrow[r, shift left, "s^0_1"] \arrow[r, shift right, "t^0_1"'] & I^1 \arrow[r, shift left=3, "s^2_{1,2}"] \arrow[r, shift left=1, "s^2_{1,1}"] \arrow[r, shift right=1, "t^2_{1,1}"'] \arrow[r, shift right=3, "t^2_{1,2}"'] & I^2 \arrow[r, shift left=5, "s^3_{2,3}"] \arrow[r, shift left=3, "s^3_{2,2}"] \arrow[r, shift left=1, "s^3_{2,1}"] \arrow[r, shift right=1, "t^3_{2,1}"'] \arrow[r, shift right=3, "t^3_{2,2}"'] \arrow[r, shift right=5, "t^3_{2,3}"'] & I^3 \arrow[r, shift left=7, "s^4_{3,1}"] \arrow[r, shift left=5, "s^4_{3,2}"] \arrow[r, shift left=3, "s^4_{3,3}"] \arrow[r, shift left=1, "s^4_{3,4}"] \arrow[r, shift right=1, "t^4_{3,4}"'] \arrow[r, shift right=3, "t^4_{3,3}"'] \arrow[r, shift right=5, "t^4_{3,2}"'] \arrow[r, shift right=7, "t^4_{3,1}"'] & I^4 \cdots I^{n-1} \arrow[r, shift left=9, "s^n_{n-1,1}"] \arrow[r, shift left=7, dotted] \arrow[r, shift left=5, "s^n_{n-1,i}"] \arrow[r, shift left=3, dotted] \arrow[r, shift left=1, "s^n_{n-1,n-1}"] \arrow[r, shift right=1, "t^n_{n-1,n-1}"'] \arrow[r, shift right=3, dotted] \arrow[r, shift right=5, "t^n_{n-1,i}"'] \arrow[r, shift right=7, dotted] \arrow[r, shift right=9, "t^n_{n-1,1}"'] & I^n \cdots
\end{tikzcd}
$$

is a B^0_C-coalgebra, where B^0_C is the $\mathbb{S}$-operad which algebras are cubical weak ∞-categories. Then we show how to "glue" the K_i-functors of Quillen in order to obtain a functor :

$$\mathbb{R}\text{ings} \xrightarrow{\mathrm{K}_\infty} \infty\text{-}\mathbb{C}\mathbb{G}\text{rp}$$

- The fourth and last section is a short "manifesto" for the following slogan : "*coalgebraic structures* govern different higher category theory". In particular we explain the main steps to get the cubical weak ∞-category of cubical weak ∞-categories, which is indeed of coalgebraic nature.

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1 Cubical monoidal categories as Pseudo-algebras

1.1 The cubical category

Consider the small category $\mathbb{C}$ with integers $\underline{n} \in \mathbb{N}$ as objects. Generators for $\mathbb{C}$ are, for all $\underline{n} \in \mathbb{N}$ given by *sources* $\underline{n} \xrightarrow{s^n_{n-1,j}} \underline{n-1}$ for each $j \in \{1,..,n\}$ and *targets* $\underline{n} \xrightarrow{t^n_{n-1,j}} \underline{n-1}$ for each $j \in \{1,..,n\}$ such that for $1 \leq i < j \leq n$ we have the following cubical relations

(i) $s^{n-1}_{n-2,i} \circ s^n_{n-1,j} = s^{n-1}_{n-2,j-1} \circ s^n_{n-1,i}$,

(ii) $s^{n-1}_{n-2,i} \circ t^n_{n-1,j} = t^{n-1}_{n-2,j-1} \circ s^n_{n-1,i}$,

(iii) $t^{n-1}_{n-2,i} \circ s^n_{n-1,j} = s^{n-1}_{n-2,j-1} \circ t^n_{n-1,i}$,

(iv) $t^{n-1}_{n-2,i} \circ t^n_{n-1,j} = t^{n-1}_{n-2,j-1} \circ t^n_{n-1,i}$

These generators plus these relations give the small category $\mathbb{C}$ called the *cubical category* that we may represent schematically with the low dimensional diagram :

$$
\begin{tikzcd}
\cdots C_4 \arrow[r, shift left=7, "s^4_{3,4}"] \arrow[r, shift left=5, "s^4_{3,3}"] \arrow[r, shift left=3, "s^4_{3,2}"] \arrow[r, shift left=1, "s^4_{3,1}"] \arrow[r, shift right=1, "t^4_{3,1}"'] \arrow[r, shift right=3, "t^4_{3,2}"'] \arrow[r, shift right=5, "t^4_{3,3}"'] \arrow[r, shift right=7, "t^4_{3,4}"'] & C_3 \arrow[r, shift left=5, "s^3_{2,3}"] \arrow[r, shift left=3, "s^3_{2,2}"] \arrow[r, shift left=1, "s^3_{2,1}"] \arrow[r, shift right=1, "t^3_{2,1}"'] \arrow[r, shift right=3, "t^3_{2,2}"'] \arrow[r, shift right=5, "t^3_{2,3}"'] & C_2 \arrow[r, shift left=3, "s^2_{1,2}"] \arrow[r, shift left=1, "s^2_{1,1}"] \arrow[r, shift right=1, "t^2_{1,1}"'] \arrow[r, shift right=3, "t^2_{1,2}"'] & C_1 \arrow[r, shift left, "s^1_0"] \arrow[r, shift right, "t^1_0"'] & C_0
\end{tikzcd}
$$

and this category $\mathbb{C}$ gives also the sketch $\mathcal{E}_\mathbb{S}$ of cubical sets used especially in [14] to produce the monads $\mathbb{S} = (S, \lambda, \mu)$, which algebras are cubical strict ∞-categories.

Definition 1 The category $\mathbb{C}\mathbb{S}$ets of cubical sets is the category of presheaves $[\mathbb{C}; \mathbb{S}\text{ets}]$. The terminal cubical set is denoted 1. $\square$

Definition 2 The 2-category $\mathbb{CCAT}$ of cubical categories is the 2-category of prestacks $[\mathbb{C};\mathbb{CAT}]$. The terminal cubical category is also denoted 1. $\square$

In particular it is shown in [14] that the category ∞-$\mathbb{CC}$at of strict cubical ∞-categories is sketchable by a projective sketch. Thus we put the following definition of *cubical strict monoidal categories* :

Definition 3 Strict monoidal cubical categories are internal cubical strict ∞-categories in $\mathbb{CAT}$. They form a strict 2-category $\mathbb{CM}_s\mathbb{C}$ where :

- 0-cells are internal cubical strict ∞-categories in $\mathbb{CAT}$;
- 1-cells are internal cubical strict ∞-functors in $\mathbb{CAT}$;
- 2-cells are internal globular[1] strict ∞-natural transformations in $\mathbb{CAT}$. $\square$

In [14] we denoted by $(\mathbb{S},\eta,\mu)$ the monad on $\mathbb{C}$Sets of cubical strict ∞-categories, and *cubical n-trees* are just n-cells of $\mathbb{S}(1)$. We shall prove in [11] that this monad is cartesian, and we denote again by $(\mathbb{S},\eta,\mu)$ its corresponding 2-monad on the 2-category $\mathbb{CCAT}$. Also the following 2-forgetful functor is 2-monadic : $\mathbb{CM}_s\mathbb{C} \longrightarrow \mathbb{CCAT}$, because the forgetful functor $\infty\text{-}\mathbb{CC}\mathrm{at} \longrightarrow \mathbb{C}\mathrm{Sets}$ is monadic and the 2-functor $\mathbb{CAT}_{\mathrm{pull}} \xrightarrow{\mathbb{CAT}(-)} 2\text{-}\mathbb{CAT}$, which takes a category X with pullbacks to the 2-category $\mathbb{CAT}(X)$ of internal categories preserves (finite) limits, thus preserves adjunctions and Eilenberg-Moore constructions. Thus we prefer to denote $\mathbb{S}$-$\mathbb{A}\mathrm{lg}_s$ this 2-category $\mathbb{CM}_s\mathbb{C}$ of strict monoidal cubical categories . This 2-monad $(\mathbb{S},\eta,\mu)$ gives weaker notions of algebras, and we recall it for any 2-monad $(\mathbb{S},\eta,\mu)$ on a 2-category $\mathcal{K}$ (see [2, 24]). In particular we shall need the notion of pseudo $\mathbb{S}$-algebra in order to define *monoidal cubical categories* below.

Definition 4 Let $(\mathbb{S},\eta,\mu)$ be a 2-monad on a 2-category $\mathcal{K}$. A pseudo-algebra structure (a,α_0,α) on an object $A\in\mathcal{K}$ is given by a 1-cell $\mathbb{S}(A) \xrightarrow{\quad a \quad} A$ and two invertible 2-cells in $\mathcal{K}$:

$$\begin{tikzcd}
\mathbb{S}^2(A) \arrow[r, "\mu(A)"] \arrow[d, "\mathbb{S}(a)"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathbb{S}(A) \arrow[d, "a"] \\
\mathbb{S}(A) \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr, "1_A"'] \arrow[r, phantom, shift right=4ex, "\overset{\alpha_0}{\Longrightarrow}"] & & \mathbb{S}(A) \arrow[dl, "a"] \\
& A &
\end{tikzcd}$$

such that the following equalities hold :

$$\begin{tikzcd}[column sep=small]
& \mathbb{S}^2(A) \arrow[rr, "\mu(A)"] & {=} & \mathbb{S}(A) \arrow[dr, "a"] & \\
\mathbb{S}^3(A) \arrow[ur, "\mu(\mathbb{S}(A))"] \arrow[rr] \arrow[dr, "\mathbb{S}^2(a)"'] & & \mathbb{S}^2(A) \arrow[ur] \arrow[dr, "\mathbb{S}(a)"] & \alpha\Uparrow & A \\
& \mathbb{S}^2(A) \arrow[rr, "\mathbb{S}(a)"'] \arrow[u, phantom, "\overset{\mathbb{S}(\alpha)}{\Longrightarrow}"] & & \mathbb{S}(A) \arrow[ur, "a"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& \mathbb{S}^2(A) \arrow[rr, "\mu(A)"] \arrow[dr] & & \mathbb{S}(A) \arrow[dr, "a"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\
\mathbb{S}^3(A) \arrow[ur, "\mu(\mathbb{S}(A))"] \arrow[dr, "\mathbb{S}^2(a)"'] \arrow[rr, phantom, "="] & & \mathbb{S}(A) \arrow[rr, "a"] & {} & A \\
& \mathbb{S}^2(A) \arrow[rr, "\mathbb{S}(a)"'] \arrow[ur] & & \mathbb{S}(A) \arrow[ur, "a"'] \arrow[u, phantom, "\overset{\alpha}{\Longleftarrow}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S}(A) \arrow[rr, "1_{\mathbb{S}(A)}"] \arrow[dr] \arrow[dd, "1_{\mathbb{S}(A)}"'] \arrow[d, phantom, shift left=6ex, "\overset{\mathbb{S}(\alpha_0)}{\Longrightarrow}"] & {=} & \mathbb{S}(A) \arrow[dd, "a"] \arrow[d, phantom, shift right=6ex, "\overset{\alpha}{\Longrightarrow}"] \\
{} & \mathbb{S}^2(A) \arrow[ur, "\mu(A)"'] \arrow[dl, "\mathbb{S}(a)"] & {} \\
\mathbb{S}(A) \arrow[rr, "a"'] & & A
\end{tikzcd}
= 1_a$$

The triple (A,α_0,α) is called a pseudo $\mathbb{S}$-algebra. If α_0 is an identity the pseudo algebra is said to be *normal.* If α_0 and α are identities then we recover the usual notion of $\mathbb{S}$-algebra, and in that case we say that A is equipped with a strict $\mathbb{S}$-algebra structure.

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[1] that is they are 2-globes between two cubical strict ∞-functors, whereas cubical strict ∞-natural transformations are 2-cubes with faces, four cubical strict ∞-functors. See [14]

Definition 5 Let (A, α_0, α) and (A', α'_0, α') two pseudo $\mathbb{S}$-algebras. A strong $\mathbb{S}$-morphism structure for a 1-cell $A \xrightarrow{f} A'$ is given by an invertible 2-cell :
$$\begin{tikzcd}
\mathbb{S}(A) \arrow[r, "a"] \arrow[d, "\mathbb{S}(f)"'] \arrow[dr, phantom, "\overset{\bar{f}}{\Longrightarrow}"] & A \arrow[d, "f"] \\
\mathbb{S}(A') \arrow[r, "a'"'] & A'
\end{tikzcd}$$
, such that we have the following equalities :

$$\begin{tikzcd}
 & \mathbb{S}(A) \arrow[rr, "a"] \arrow[d, phantom, "\overset{\alpha}{\Longleftarrow}"] & & A \arrow[dr, "f"] & \\
\mathbb{S}^2(A) \arrow[ur, "\mu(A)"] \arrow[rr, "\mathbb{S}(a)"] \arrow[dr, "\mathbb{S}^2(f)"'] & \arrow[d, phantom, "\overset{\mathbb{S}(\bar{f})}{\Longrightarrow}"] & \mathbb{S}(A) \arrow[ur, "a"] \arrow[dr, "\mathbb{S}(f)"] \arrow[r, phantom, "\bar{f}\Uparrow"] & & A' \\
 & \mathbb{S}^2(A') \arrow[rr, "\mathbb{S}(a')"'] & & \mathbb{S}(A') \arrow[ur, "a'"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbb{S}(A) \arrow[rr, "a"] \arrow[dr] & \arrow[d, phantom, "\overset{\bar{f}}{\Longrightarrow}"] & A \arrow[dr, "f"] & \\
\mathbb{S}^2(A) \arrow[ur, "\mu(A)"] \arrow[dr, "\mathbb{S}^2(f)"'] \arrow[r, phantom, "="] & & \mathbb{S}(A') \arrow[rr, "a'"] & \arrow[d, phantom, "\overset{\alpha'}{\Longleftarrow}"] & A' \\
 & \mathbb{S}^2(A') \arrow[ur, "\mu(A')"] \arrow[rr, "\mathbb{S}(a')"'] & & \mathbb{S}(A') \arrow[ur, "a'"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbb{S}(A) \arrow[dr, "a"] \arrow[dd] & \\
A \arrow[ur, "\eta(A)"] \arrow[ddd, "f"'] \arrow[r, phantom, "="] & \arrow[r, phantom, "\overset{\bar{f}}{\Longrightarrow}"] & A \arrow[ddd, "f"] \\
 & \mathbb{S}(A') \arrow[ddr] & \\
 & \alpha'_0\Uparrow & \\
A' \arrow[uur] \arrow[rr, "1_{A'}"'] & & A'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "\eta(A)"] \arrow[dr, "1_A"'] & \overset{\alpha_0}{\Longrightarrow} & \mathbb{S}(A) \arrow[dl, "a"] \\
 & A \arrow[d, "f"] & \\
 & A' &
\end{tikzcd}$$

Definition 6 Let f and f' be strong $\mathbb{S}$-morphisms :

$$\begin{tikzcd}
(a, \alpha_0, \alpha) \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & (a', \alpha'_0, \alpha')
\end{tikzcd} .$$

A 2-cell $f \overset{\psi}{\Longrightarrow} f'$ is an algebra 2-cell if the following equality holds :

$$\begin{tikzcd}
\mathbb{S}(A) \arrow[rr, "a"] \arrow[d, bend right, "\mathbb{S}(f)"'] \arrow[d, bend left, "\mathbb{S}(f')"] \arrow[d, phantom, "\overset{\mathbb{S}(\psi)}{\Longrightarrow}"] & \overset{\bar{f}'}{\Longrightarrow} & A \arrow[d, "f'"] \\
\mathbb{S}(A') \arrow[rr, "a'"'] & & A'
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{S}(A) \arrow[rr, "a"] \arrow[d, "\mathbb{S}(f)"'] & \overset{\bar{f}}{\Longrightarrow} & A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\overset{\psi}{\Longrightarrow}"] \\
\mathbb{S}(A') \arrow[rr, "a'"'] & & A'
\end{tikzcd}$$

Let us denote by Ps-$\mathbb{S}$-$\mathbb{A}$lg the 2-category which objects are pseudo $\mathbb{S}$-algebras, whose 1-cells are strong $\mathbb{S}$-morphisms and whose 2-cells are algebra 2-cells. The full sub-2-category of Ps-$\mathbb{S}$-$\mathbb{A}$lg consisting of the normal pseudo-algebras is denoted Ps$_0$-$\mathbb{S}$-$\mathbb{A}$lg, and the locally full sub-2-category of Ps-$\mathbb{S}$-$\mathbb{A}$lg consisting of the strict algebras and strict morphisms is denoted $\mathbb{S}$-$\mathbb{A}$lg$_s$.

Remark 1 We gave the description of Ps$_0$-$\mathbb{S}$-$\mathbb{A}$lg here as an indication. As a matter of fact for the globular setting it is possible to build a normal pseudo algebra for each globular monoidal categories in the sense of [1], but Mark Weber pointed out to me that Ps$_0$-$\mathbb{S}$-$\mathbb{A}$lg is 2-equivalent to Ps-$\mathbb{S}$-$\mathbb{A}$lg, and thus we prefer to use the context of the 2-category Ps-$\mathbb{S}$-$\mathbb{A}$lg to model monoidal cubical categories defined just below. □

Now let us comeback to the 2-monad $\mathbb{S} = (S, \lambda, \mu)$ on the 2-category of cubical categories $\mathbb{C}$CAT as described above, which strict 2-algebras are strict monoidal cubical categories .

Definition 7 The 2-category of monoidal cubical categories consists of the 2-category Ps-$\mathbb{S}$-$\mathbb{A}$lg of pseudo $\mathbb{S}$-algebras □

Also by using the theorem 5.1 and the theorem 5.12 of [2] we get the following biadjunction, similar to the one described in [21] :

Corollary 1 *The forgetful 2-functor U : $Ps\text{-}\mathbb{S}\text{-}\mathbb{A}lg \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbb{CCAT}$ such that :*

- *Ps-$\mathbb{S}$-$\mathbb{A}lg$ is the 2-category of pseudo $\mathbb{S}$-algebras;*
- *$\mathbb{CCAT}$ is the 2-category of cubical categories;*
- *F builds the free strict monoidal cubical categories functor.*

exhibits a biadjunction which restricts to a 2-adjunction on the strict monoidal cubical categories. □

Also we shall denote by $\mathbb{S}\text{-}\mathbb{A}\mathrm{lg_s} \underset{F}{\overset{V}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbb{CCAT}$ the underlying strict 2-adjunction of this biadjunction.

2 Cubical Higher Spans and Cubical Higher Cospans

2.1 The pseudo-algebraic structure of $Span(C)$

Let us first recall the $\mathbb{S}$pan construction ([20, 24]) : for any small category C there is a 2-adjunction :

$$\begin{tikzcd} \mathbb{CAT} \arrow[r, phantom, "\perp"] \arrow[r, shift right, "\mathbb{S}\mathrm{pan}_C"'] & {[C^{op}, \mathbb{CAT}]} \arrow[l, shift right, "\mathbb{EL}"'] \end{tikzcd}$$

where $\mathbb{S}\mathrm{pan}_C(\mathcal{E})(c) = [(C/c)^{op}, \mathcal{E}]$ and the category $\mathbb{EL}(X)$ has the following definition :

- objects are pairs (c, x) where $c \in C$ and $x \in X(c)$.
- morphisms : $(c, x) \longrightarrow (d, y)$, are pairs (f, α) where $d \xrightarrow{f} c$ is in C and $X(f)(x) \xrightarrow{\alpha} y$ is in $X(d)$.
- compositions and identities come from C and the categories $X(c)$.

Suppose now that $\mathbb{T} = (T, \eta, \mu)$ is a cartesian monad on $[C^{op}, \mathbb{S}\mathrm{ets}]$, and let us denote again by $\mathbb{T} = (T, \eta, \mu)$ its extension to a 2-monad on $[C^{op}, \mathbb{CAT}]$. In fact, for any category $\mathcal{E}$ with pullbacks it is proved in [11] that :

Theorem 1 (Kachour,Weber) *$\mathbb{S}pan_C(\mathcal{E})$ is a pseudo $\mathbb{T}$-algebra* □

In fact we can dualize such construction and produce a similar result which says that $\mathbb{C}\mathrm{ospan}_C(\mathcal{E})$ is a pseudo $\mathbb{T}$-algebra if $\mathcal{E}$ is a category with pushouts, and these produce the following diagram of functors :

$$\begin{tikzcd} \mathbb{CAT}_{\mathrm{push}} \arrow[rr, "j"] \arrow[dd, "(-)^{op}"'] \arrow[dr, "\mathbb{C}\mathrm{ospan}(-)"] & & \mathbb{CAT} \arrow[dd, "(-)^{op}"'] \arrow[dr, "\mathbb{C}\mathrm{ospan}(-)"] & \\ & \mathrm{Ps}\text{-}\mathbb{S}\text{-}\mathbb{A}\mathrm{lg} \arrow[rr, "i"] & & \mathbb{CCAT} \\ \mathbb{CAT}_{\mathrm{pull}} \arrow[rr, "k"'] \arrow[ur, "\mathbb{S}\mathrm{pan}(-)"'] & & \mathbb{CAT} \arrow[ur, "\mathbb{S}\mathrm{pan}(-)"'] & \end{tikzcd}$$

This result has two essential virtues : first it convince the reader that actually the structure behind the spans and the cospans construction are really of pseudo-algebraic nature; secondly it shows, and this is we believe the main fact, that probably not only globular and cubical higher category theory need such structures, but other useful higher category theory could need it.

However because of the "cubical scopes" of this article, we are going to describe cubical spans and cubical cospans in a more combinatorial way because this concrete point of view has the advantage to see it unpacked, and thus gives an accurate idea of what these cubical spans and cubical cospans looks like. This combinatorial description has been described first by Marco Grandis in [7], and it is instructive to compare it with the Batanin's combinatorial construction of globular spans and globular cospans [1]. The only new tools here are the connections on cubical (co)spans which are accurately describe.

In order to formalize cubical higher spans and cubical higher cospans we will use the *formal span category* V or the *formal cospan category* Λ used by Marco Grandis (see [7]). For simplicity we will explain only constructions for cubical higher spans, which use this small category V :

$$\begin{tikzcd} -1 & & 1 \\ & 0 \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

because for cubical higher cospans, constructions are duals, and use the small category Λ :

$$\begin{tikzcd} & 0 & \\ -1 \arrow[ur, "f"] & & 1 \arrow[ul, "g"'] \end{tikzcd}$$

Definition 8 Let C be a category. The category $\mathbb{S}\mathrm{pan}_n(C)$ of cubical n-spans in C is the category of functors $[V^n;C]$ and natural transformations between them. □

The combinatoric description of the category V^n shall be useful : each objects of V^n are n-uplets $(m_1,...,m_n)\in\{0,-1,1\}^n$. Also the category V^n underlies a n-cube structure, such that the object $(0,...,0)$ represents the n-face, and the n-uplets $(m_1,...,m_n)\in\{0,-1,1\}^n$ which countains exactly p integers m_j which are equal to zero, represent p-faces. Consider $(m_1,...,m_n)$ a $(p+1)$-face and suppose $m_{j_i}=0$ for $1\leq i\leq p+1$. Thus we get two morphisms in V^n :

$$\begin{tikzcd}[column sep=huge] (m_1,...,m_{j_i},...,m_n) \arrow[r, shift left, "{(m_1,...,m_{j_i-1},f,m_{j_i+1},...,m_n)}"] \arrow[r, shift right, "{(m_1,...,m_{j_i-1},g,m_{j_i+1},...,m_n)}"'] & (m_1,...,m_{j_i-1},\hat{m}_{j_i},m_{j_i+1},...,m_n) \end{tikzcd}$$

such that $(m_1,...,m_{j_i-1},f,m_{j_i+1},...,m_n)$ switch the value m_{j_i} to the value $\hat{m}_{j_i}=-1$ and $(m_1,...,m_{j_i-1},g,m_{j_i+1},...,m_n)$ switch the value m_{j_i} to the value $\hat{m}_{j_i}=1$.

Remark 2 Intuitively such map $(m_1,...,m_{j_i-1},f,m_{j_i+1},...,m_n)$ is a kind of s^{p+1}_{p,j_i} and the map $(m_1,...,m_{j_i-1},g,m_{j_i+1},...,m_n)$ is a kind of t^{p+1}_{p,j_i}. □

In particular the following arrows in V^n :

$$(0,...,0)\xrightarrow{(0,...,0,f,0,...,0)}(0,...,0,-1,0,...,0),\qquad (0,...,0)\xrightarrow{(0,...,0,g,0,...,0)}(0,...,0,1,0,...,0)$$

shall be important for an accurate description of the projective cone below, when we will describe the pseudo-algebraic structure produced by cubical higher spans in a category with pullbacks.

Now we want to put a cubical category structure on cubical spans. For that we just recall the constructions of Marco Grandis (see [7]).

- The *formal source functor* is given by $1\xrightarrow{\ s\ }V$, where $1=\{\star\}$ is the terminal category and s sends $\star$ to -1. Similarly the *formal target functor* is given by $1\xrightarrow{\ t\ }V$ where t sends $\star$ to 1. These give the *source functors* $V^{n-1}\xrightarrow{s^n_{n-1,i}}V^n$, given by $s^n_{n-1,i}:=V^{i-1}\times s\times V^{n-i}$ for $1\leq i\leq n$, and the *target functors* $V^{n-1}\xrightarrow{t^n_{n-1,i}}V^n$, given by $t^n_{n-1,i}:=V^{i-1}\times t\times V^{n-i}$ for $1\leq i\leq n$, and then we get the cubical category of spans in C :

$$\begin{tikzcd}[column sep=huge, row sep=huge] \cdots[V^4;C] \arrow[r, shift left=10, "s^4_{3,4}"] \arrow[r, shift left=6, "s^4_{3,3}"] \arrow[r, shift left=2, "s^4_{3,2}"] \arrow[r, shift left=-2, "s^4_{3,1}"] \arrow[r, shift right=2, "t^4_{3,1}"'] \arrow[r, shift right=6, "t^4_{3,2}"'] \arrow[r, shift right=10, "t^4_{3,3}"'] \arrow[r, shift right=14, "t^4_{3,4}"'] & {[V^3;C]} \arrow[r, shift left=6, "s^3_{2,3}"] \arrow[r, shift left=2, "s^3_{2,2}"] \arrow[r, shift left=-2, "s^3_{2,1}"] \arrow[r, shift right=2, "t^3_{2,1}"'] \arrow[r, shift right=6, "t^3_{2,2}"'] \arrow[r, shift right=10, "t^3_{2,3}"'] & {[V^2;C]} \arrow[r, shift left=2, "s^2_{1,2}"] \arrow[r, shift left=-2, "s^2_{1,1}"] \arrow[r, shift right=2, "t^2_{1,1}"'] \arrow[r, shift right=6, "t^2_{1,2}"'] & {[V;C]} \arrow[r, shift left, "s^1_0"] \arrow[r, shift right, "t^1_0"'] & C \end{tikzcd}$$

where for each $1\leq i\leq n$, $s^n_{n-1,i}$ and $t^n_{n-1,i}$ are functors :

$$\begin{tikzcd}[column sep=huge] {[V^n;C]} \arrow[r, shift left, "s^n_{n-1,i}"] \arrow[r, shift right, "t^n_{n-1,i}"'] & {[V^{n-1};C]} \end{tikzcd}$$

- The *formal reflexivity functor* is given by the unique functor $V \xrightarrow{\;!\;} 1$, and this gives for $1 \leq i \leq n$ the *reflexivity functors* $V^n \xrightarrow{1^{n-1}_{n,i}} V^{n-1}$, given by $1^{n-1}_{n,i} := V^{i-1} \times ! \times V^{n-i}$, and then we get a reflexivity structure on the cubical category of spans in C :

$$\begin{tikzcd} C \arrow[r, "1^0_1"] & {[V^1;C]} \arrow[r, shift left=2, "1^1_{2,1}"] \arrow[r, shift right=2, "1^1_{2,2}"'] & {[V^2;C]} \arrow[r, shift left=4, "1^2_{3,2}"] \arrow[r, "1^2_{3,1}"] \arrow[r, shift right=4, "1^2_{3,3}"'] & {[V^3;C]} \arrow[r, shift left=6, "1^3_{4,3}"] \arrow[r, shift left=2, "1^3_{4,1}"] \arrow[r, shift right=2, "1^3_{4,2}"] \arrow[r, shift right=6, "1^3_{4,4}"'] & {[V^4;C]\cdots} \end{tikzcd}$$

where for each $1 \leq i \leq n$, $1^{n-1}_{n,i}$ is a functor :

$$[V^{n-1};C] \xrightarrow{1^{n-1}_{n,i}} [V^n;C]$$

- Connections for cubical higher (co)spans are not defined in Grandis [7], thus we need to formalize it properly. V^2 may be seen as the following cubical 2-span :

$$\begin{tikzcd} & & & (0,0) \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\ (-1,0) \arrow[d] \arrow[drr] & & (0,1) \arrow[dll] \arrow[drr] & & (0,-1) \arrow[dll] \arrow[drr] & & (1,0) \arrow[dll] \arrow[d] \\ (-1,1) & & (-1,-1) & & (1,1) & & (1,-1) \end{tikzcd}$$

and if $A \xrightarrow{f} B$ is an 1-cell, then the 2-cell :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "1^{1,-}_{2,1}(f)"] & B \arrow[d, "1_B"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

is represented by the following 2-span :

$$\begin{tikzcd} & & & 1^{1,-}_{2,1}(f) \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\ f \arrow[d] \arrow[drr] & & 1_B \arrow[dll] \arrow[drr] & & f \arrow[dll] \arrow[drr] & & 1_B \arrow[dll] \arrow[d] \\ B & & A & & B & & B \end{tikzcd}$$

and the 2-cell

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] \arrow[dr, phantom, "1^{1,+}_{2,1}(f)"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is represented by the following 2-span :

$$\begin{tikzcd}[column sep=small]
 & & & 1^{1,+}_{2,1}(f) \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
1_A \arrow[d] \arrow[drr] & & f \arrow[dll] \arrow[drr] & & 1_A \arrow[dll] \arrow[drr] & & f \arrow[dll] \arrow[d] \\
A & & A & & B & & A
\end{tikzcd}$$

These show us how to formalise connections for cubical higher spans : the *formal connection functors* are thus given by : $V^2 \xrightarrow{1^-} V$, $\quad V^2 \xrightarrow{1^+} V$ defined on objects[2] of V^2 by

$$(0,0),(-1,0),(0,-1) \overset{1^-}{\longmapsto} 0, \quad (0,1),(1,0),(-1,1),(1,1),(1,-1) \overset{1^-}{\longmapsto} 1, \quad (-1,-1) \overset{1^-}{\longmapsto} -1,$$

and

$$(0,0),(0,1),(1,0) \overset{1^+}{\longmapsto} 0, \quad (-1,0),(0,-1),(-1,1),(-1,-1),(1,-1) \overset{1^+}{\longmapsto} -1, \quad (1,1) \overset{1^+}{\longmapsto} 1.$$

These give the *connection functors* : $V^{n+1} \xrightarrow{1^{n,-}_{n+1,i}} V^n$, $\quad V^{n+1} \xrightarrow{1^{n,+}_{n+1,i}} V^n$, given by $1^{n,-}_{n+1,i} := V^{i-1} \times 1^- \times V^{n-i}$ and $1^{n,+}_{n+1,i} := V^{i-1} \times 1^+ \times V^{n-i}$, and then we get the structure of connections on the cubical category of spans in C :

$$\begin{tikzcd}
{[V^1;C]} \arrow[r, shift left, "1^{1,-}_{2,1}"] \arrow[r, shift right, "1^{1,+}_{2,1}"'] & {[V^2;C]} \arrow[r, shift left=3, "1^{2,-}_{3,2}"] \arrow[r, shift left, "1^{2,-}_{3,1}"] \arrow[r, shift right, "1^{2,+}_{3,1}"'] \arrow[r, shift right=3, "1^{2,+}_{3,2}"'] & {[V^3;C]} \arrow[r, shift left=5, "1^{3,-}_{4,3}"] \arrow[r, shift left=3, "1^{3,-}_{4,2}"] \arrow[r, shift left, "1^{3,-}_{4,1}"] \arrow[r, shift right, "1^{3,+}_{4,1}"'] \arrow[r, shift right=3, "1^{3,+}_{4,2}"'] \arrow[r, shift right=5, "1^{3,+}_{4,3}"'] & {[V^4;C]} \arrow[r, shift left=7, "1^{4,-}_{5,4}"] \arrow[r, shift left=5, "1^{4,-}_{5,3}"] \arrow[r, shift left=3, "1^{4,-}_{5,2}"] \arrow[r, shift left, "1^{4,-}_{5,1}"] \arrow[r, shift right, "1^{4,+}_{5,1}"'] \arrow[r, shift right=3, "1^{4,+}_{5,2}"'] \arrow[r, shift right=5, "1^{4,+}_{5,3}"'] \arrow[r, shift right=7, "1^{4,+}_{5,4}"'] & {[V^5;C]\cdots}
\end{tikzcd}$$

where for each $1 \leq i \leq n$, $1^{n,-}_{n+1,i}$, $1^{n,+}_{n+1,i}$ are functors :

$$\begin{tikzcd}
{[V^n;C]} \arrow[r, shift left, "1^{n,-}_{n+1,i}"] \arrow[r, shift right, "1^{n,+}_{n+1,i}"'] & {[V^{n+1};C]}
\end{tikzcd}$$

Now suppose that C is a category equipped with pullbacks. This context allows to put a pseudo-algebra structure on cubical higher spans in C. In fact this cubical monoidal structure shall be given by these pullbacks. We will follow the definition of Grandis (see [7]) with a small variation on projective sketch. Our goal, for each $n \geq 1$ and each $1 \leq i \leq n$, is to build functors :

$$[V^n;C] \underset{[V^{n-1};C]}{\times} [V^n;C] \xrightarrow{\otimes^n_i} [V^n;C]$$

such that $[V^n;C] \underset{[V^{n-1};C]}{\times} [V^n;C]$ comes from the pullback :

$$\begin{tikzcd}
{[V^n;C] \underset{[V^{n-1};C]}{\times} [V^n;C]} \arrow[r] \arrow[d] & {[V^n;C]} \arrow[d, "s^n_{n-1,i}"] \\
{[V^n;C]} \arrow[r, "t^n_{n-1,i}"'] & {[V^{n-1};C]}
\end{tikzcd}$$

- First we consider the category V_2 given by the following pushout :

[2] Of course, these definition on objects give the one on arrows of V^2

$$\begin{tikzcd}
\{\star\} \arrow[r, "s"] \arrow[d, "t"'] & V \arrow[d, "k^+"] \\
V \arrow[r, "k^-"'] & V_2
\end{tikzcd}$$

Thus V_2 is given by the category :

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & & c \arrow[dl] \arrow[dr] & \\
-1 & & b & & 1
\end{tikzcd}$$

that we extend to the category $\hat{V}_2$:

$$\begin{tikzcd}
 & & 0 \arrow[dl, dotted] \arrow[dr, dotted] & & \\
 & a \arrow[dl] \arrow[dr] & & c \arrow[dl] \arrow[dr] & \\
-1 & & b & & 1
\end{tikzcd}$$

Also the following subcategory W of V_2 shall be considered :

$$\begin{tikzcd}
a \arrow[dr] & & c \arrow[dl] \\
 & b &
\end{tikzcd}$$

and the natural transformation $\Delta(0)$:

$$\begin{tikzcd}
W \arrow[r, bend left=60, "\Delta(0)"] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\tau\Downarrow"] & \hat{V}_2,
\end{tikzcd}$$

where $\Delta(0)$ is the constant functor with value 0. This allow to see the category $\hat{V}_2$ as the category V_2 equipped with a cone over W, that is $\hat{V}_2$ is a projective sketch equipped with the cone $\Delta(0) \xrightarrow{\tau} F$; also we have the concatenation functor : $V \xrightarrow{k} \hat{V}_2$ which sends 0 to 0, and -1 to -1, and finally 1 to 1, from the category V to the projective sketch $\hat{V}_2$. Now for each $n \geq 1$ and each $1 \leq i \leq n$, consider the pushout diagram :

$$\begin{tikzcd}
V^{n-1} \arrow[r, "s^n_{n-1,i}"] \arrow[d, "t^n_{n-1,i}"'] & V^n \arrow[d, "k^+_i"] \\
V^n \arrow[r, "k^-_i"'] & V^n_i
\end{tikzcd}$$

where $k^-_i = V^{i-1} \times k^- \times V^{n-i}$, $k^+_i = V^{i-1} \times k^+ \times V^{n-i}$ and $V^n_i = V^{i-1} \times V_2 \times V^{n-i}$. The category V^n_i may be thought as the gluing of itself along the functors $s^n_{n-1,i}$ and $t^n_{n-1,i}$, and also the category $\hat{V}^n_i := V^{i-1} \times \hat{V}_2 \times V^{n-i}$ may be thought as the category V^n_i equipped with a cone over its following subdiagram :

$$\begin{tikzcd}
(0,...,0) \arrow[dr, "{(0,...,0,f,0,...,0)}"'] & & (0,...,0) \arrow[dl, "{(0,...,0,g,0,...,0)}"] \\
 & (0,...,0,1,0,...,0) \sim (0,...,0,-1,0,...,0) &
\end{tikzcd}$$

Remark 3 In this subdiagram the symbol $\sim$ means the identification of $(0,...,0,1,0,...,0)$ and $(0,...,0,-1,0,...,0)$ under the pushout. $\square$

and the cone is formally described by the natural transformation :

$$\begin{tikzcd}
 & V^{i-1}\times\Delta(0)\times V^{n-i} \arrow[d, phantom, "\Downarrow \tau=V^{i-1}\times\tau\times V^{n-i}"] & \\
V^{i-1}\times W\times V^{n-i} \arrow[rr, "V^{i-1}\times F\times V^{n-i}"'] \arrow[rr, bend left] & & V^{i-1}\times\hat{V}_2\times V^{n-i}
\end{tikzcd}$$

- Now consider two cubical n-spans x and y such that $s^n_{n-1,i}(x)=t^n_{n-1,i}(y)$ in category C equipped with pullbacks :

$$\begin{tikzcd} V^n \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & C. \end{tikzcd}$$

We are in the following situation where we get the unique functor $[x,y]_i$:

$$\begin{tikzcd}
V^{n-1} \arrow[r, "s^n_{n-1,i}"] \arrow[d, "t^n_{n-1,i}"'] & V^n \arrow[d, "k_i^+"] \arrow[ddr, bend left, "y"] & \\
V^n \arrow[r, "k_i^-"'] \arrow[drr, bend right, "x"'] & V^n_i \arrow[dr, dotted, "{[x,y]_i}"] & \\
 & & C
\end{tikzcd}$$

thus we get the functor $\hat{[x,y]}_i$: $\hat{V}^n_i \xrightarrow{\hat{[x,y]}_i} C$ which is the extension of the functor $[x,y]_i$ on the category $\hat{V}^n_i$, which sends the cone $\tau_i=V^{i-1}\times\tau\times V^{n-i}$ to the following pullback in C :

$$\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[dr] & \\
x(0,...,0) \arrow[dr, "{x(0,...,0,f,0,...,0)}"'] & & y(0,...,0) \arrow[dl, "{y(0,...,0,g,0,...,0)}"] \\
 & x(0,...,0,1,0,...,0)=y(0,...,0,-1,0,...,0) &
\end{tikzcd}$$

- Thus we obtain the diagram : $V^n \xrightarrow{k_i} \hat{V}^n_i \xrightarrow{\hat{[x,y]}_i} C$, where $k_i=V^{i-1}\times k\times V^{n-i}$ comes from the concatenation functor : $V \xrightarrow{k} \hat{V}_2$, and we put : $y\otimes^n_i x=\hat{[x,y]}_i\circ k_i$. As for globular higher spans, these tensor products on arrows comes from universality of these pullbacks. Thus for each $n\geq 1$ and each $1\leq i\leq n$, we built functors :

$$[V^n;C]\underset{[V^{n-1};C]}{\times}[V^n;C]\xrightarrow{\otimes^n_i}[V^n;C]$$

which put on $\mathbb{S}\mathrm{pan}(C)$ a pseudo-algebra structure.

Of course the description of the pseudo-algebra $\mathbb{C}\mathrm{ospan}(C)$, where C is a category with pushouts, is obtained by dualizing these constructions.

2.2 B^0_C-algebras and B^0_C-coalgebras

Definition 9 If $\mathcal{C}$ is a monoidal cubical category then a *global object* of $\mathcal{C}$ is given by a morphism :

$$1 \xrightarrow{E} \mathcal{C}$$

in the category $\mathbb{C}\mathbb{C}\mathrm{AT}$ of cubical categories. $\square$

By the pseudo-universality of $1 \xrightarrow{\eta(1)} \mathbb{S}(1)$ we get the following morphism $[E]$ of monoidal cubical categories :

$$\begin{tikzcd}
\mathbb{S}(1) \arrow[rr, dotted, "{[E]}"] & & \mathcal{C} \\
1 \arrow[u, "\eta(1)"] \arrow[urr, "E"'] & &
\end{tikzcd}.$$

Now suppose : $\mathbb{S}(\mathcal{C}) \xrightarrow{v} \mathcal{C}$, is the structural map of the pseudo $\mathbb{S}$-algebra $\mathcal{C}$. It is important to notice that the freeness of $\mathbb{S}(1)$ describes this extension $[E]$ as the composition $v \circ \mathbb{S}(E)$:

$$\begin{tikzcd}
\mathbb{S}(1) \arrow[rr, "{[E]}"] \arrow[dr, "\mathbb{S}(E)"'] & & \mathcal{C} \\
& \mathbb{S}(\mathcal{C}) \arrow[ur, "v"'] &
\end{tikzcd}$$

This morphism $[E]$ is denoted $\mathbb{E}\mathrm{nd}(E)$ for the case of the monoidal cubical category $\mathcal{C} = \mathbb{S}\mathrm{pan}(C)$ where C is a category with pullbacks; thus a global object in it : $1 \xrightarrow{E} \mathbb{S}\mathrm{pan}(C)$ produces such extension $\mathbb{S}(1) \xrightarrow{\mathbb{E}\mathrm{nd}(E)} \mathbb{S}\mathrm{pan}(C)$, and furthermore this morphism $\mathbb{E}\mathrm{nd}(E)$ contains all informations we need to define the $\mathbb{S}$-operad of endomorphism $\mathbb{END}(E)$ associated to the global object E in $\mathbb{S}\mathrm{pan}(C)$:

Definition 10 For all $n \in \mathbb{N}$, n-cells of $\mathbb{END}(E)$ consist of elements of the set $hom_{\mathbb{S}\mathrm{pan}_n(C)}(\mathbb{E}\mathrm{nd}(E)(t), E(n))$, for each cubical n-tree $t \in \mathbb{S}(1)$. These n-cells form the set $\mathbb{END}(E)(n)$, and the corresponding cubical set $\mathbb{END}(E)$ underlies an $\mathbb{S}$-operad where the multiplication of it is defined as follow : if (x, y) is an n-cell of $\mathbb{S}(\mathbb{END}(E)) \underset{\mathbb{S}(1)}{\times} \mathbb{END}(E)$, and is such that[3] $\mu(1)(\mathbb{S}(a)(x)) = t'$ and $a(y) = t$:

$$\begin{tikzcd}
& & & \mathbb{S}(\mathbb{END}(E)) \underset{\mathbb{S}(1)}{\times} \mathbb{END}(E) \arrow[dl, "\pi_1"'] \arrow[drr, "\pi_2"] \arrow[dddd, dotted, "\gamma"] & & & \\
& & \mathbb{S}(\mathbb{END}(E)) \arrow[dl, "\mathbb{S}(a)"'] \arrow[dr, "\mathbb{S}(c)"] & & & \mathbb{END}(E) \arrow[dll, "a"'] \arrow[dr, "c"] & \\
& \mathbb{S}(1)^2 \arrow[dl, "\mu(1)"'] & & \mathbb{S}(1) & & & 1 \\
\mathbb{S}(1) & & & & & & \\
& & & \mathbb{END}(E) \arrow[ulll, "a"] \arrow[uurrr, "c"'] & & &
\end{tikzcd}$$

then $\gamma(x; y)$ is given by the composition $y \circ v(x)$ in $\mathbb{S}\mathrm{pan}_n(C)$:

$$\mathbb{E}\mathrm{nd}(E)(t') \xrightarrow{v(x)} \mathbb{E}\mathrm{nd}(E)(t) \xrightarrow{y} E(n)$$

where

$$\mathbb{S}(\mathbb{S}\mathrm{pan}(C)) \xrightarrow{v} \mathbb{S}\mathrm{pan}(C)$$

is the structural map of the pseudo $\mathbb{S}$-algebra $\mathbb{S}\mathrm{pan}(C)$; the unity of it is given, for each $n \in \mathbb{N}$, by the singleton $1_{E(n)} \in hom_{\mathbb{S}\mathrm{pan}_n(C)}(E(n), E(n))$. The axiom of associativity of the multiplication of $\mathbb{END}(E)$ comes from the associativity of compositions of each categories $\mathbb{S}\mathrm{pan}_n(C)$ $(n \in \mathbb{N})$, and we have the similar result for the axiom of unities. $\square$

[3]In this diagram $\mathbb{S}$ is seen as a monad on the category $\mathbb{C}\mathbb{S}\mathrm{ets}$ of cubical sets. See [10] for the definition of $\mathbb{S}$-operads.

Remark 4 It is important to notice that these definition of cubical higher operad of endomorphism associated to a global object can be generalized easily to any monoidal cubical categories (different to those of the form $\mathbb{S}\mathrm{pan}(C)$), and more, this could be done probably in the general setting of pseudo-algebras. But because the scope of this article is to have first an accurate description of the functor Π_∞ (3.2.3) we prefer to restrict ourself to this concrete description. □

Also we have the following easy result :

Corollary 2 *A global object* $1 \xrightarrow{\quad E \quad} \mathbb{S}pan(C)$ *is the same thing as to give a cubical object, still denoted by E, internal to the category C :* $\mathbb{C} \xrightarrow{\quad E \quad} C$ □

Now we are ready to define B^0_C-algebras :

Definition 11 Consider a category C with pullbacks, plus a cubical object E in it : $\mathbb{C} \xrightarrow{\quad E \quad} C$. E is equipped with a B^0_C-algebra structure if there is a morphism of $\mathbb{S}$-operads : $B^0_C \xrightarrow{\quad f \quad} \mathbb{END}(E)$. □

Operads of coendomorphisms and coalgebraic structures are defined similarly and dually, but because of their importance we prefer to give their precise dual definition : if C is a category with pushouts, thus $\mathcal{C} = \mathbb{C}\mathrm{ospan}(C)$ is a monoidal cubical category, and if : $1 \xrightarrow{\quad E \quad} \mathbb{C}\mathrm{ospan}(C)$ is a global object in it, then the corresponding extension $\mathbb{C}\mathrm{oend}(E)$ to $\mathbb{S}(1)$: $\mathbb{S}(1) \xrightarrow{\quad \mathbb{C}\mathrm{oend}(E) \quad} \mathbb{C}\mathrm{ospan}(C)$, contains all informations we need to define the $\mathbb{S}$-operad of coendomorphism $\mathbb{COEND}(E)$ associated to the global object E in $\mathbb{C}\mathrm{ospan}(C)$:

Definition 12 For all $n \in \mathbb{N}$, n-cells of $\mathbb{COEND}(E)$ consist of elements of the set $hom_{\mathbb{C}\mathrm{ospan}_n(C)}(E(n), \mathbb{C}\mathrm{oend}(E)(t))$, for each cubical n-tree $t \in \mathbb{S}(1)$. These n-cells form the set $\mathbb{COEND}(E)(n)$, and the corresponding cubical set $\mathbb{COEND}(E)$ underlies an $\mathbb{S}$-operad where the multiplication of it is defined as follow : if (x, y) is an n-cell of $\mathbb{S}(\mathbb{COEND}(E)) \underset{\mathbb{S}(1)}{\times} \mathbb{COEND}(E)$, and is such that $\mu(1)(\mathbb{S}(a)(x)) = t'$ and $a(y) = t$:

$$\begin{tikzcd}
 & & & \mathbb{S}(\mathbb{COEND}(E)) \underset{\mathbb{S}(1)}{\times} \mathbb{COEND}(E) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dddd, dotted, "\gamma"] & & \\
 & & \mathbb{S}(\mathbb{COEND}(E)) \arrow[dl, "\mathbb{S}(a)"'] \arrow[dr, "\mathbb{S}(c)"] & & \mathbb{COEND}(E) \arrow[dl, "a"'] \arrow[dr, "c"] & \\
 & \mathbb{S}(1)^2 \arrow[dl, "\mu(1)"'] & & \mathbb{S}(1) & & 1 \\
\mathbb{S}(1) & & & & & \\
 & & & \mathbb{COEND}(E) \arrow[ulll, "a"] \arrow[uurr, "c"'] & &
\end{tikzcd}$$

then $\gamma(x; y)$ is given by the composition $y \circ v(x)$ in $\mathbb{C}\mathrm{ospan}_n(C)$:

$$\mathbb{C}\mathrm{oend}(E)(t') \xrightarrow{\quad v(x) \quad} \mathbb{C}\mathrm{oend}(E)(t) \xrightarrow{\quad y \quad} E(n)$$

where

$$\mathbb{S}(\mathbb{C}\mathrm{ospan}(C)) \xrightarrow{\quad v \quad} \mathbb{C}\mathrm{ospan}(C)$$

is the structural map of the pseudo $\mathbb{S}$-algebra $\mathbb{C}\mathrm{ospan}(C)$; the unity of it is given, for each $n \in \mathbb{N}$, by the singleton $1_{E(n)} \in hom_{\mathbb{C}\mathrm{ospan}_n(C)}(E(n), E(n))$. The axiom of associativity of the multiplication of $\mathbb{COEND}(E)$ comes from the associativity of compositions of each categories $\mathbb{C}\mathrm{ospan}_n(C)$ ($n \in \mathbb{N}$), and we have the similar result for the axiom of unities. □

Also we have the following easy result :

Corollary 3 *A global object* $1 \xrightarrow{\quad E \quad} \mathbb{C}ospan(C)$ *is the same thing as to give a cocubical object, still denoted by E, internal to the category C :* $\mathbb{C}^{op} \xrightarrow{\quad E \quad} C$ □

Now we are ready to define B^0_C-coalgebras :

Definition 13 Consider a category C with pushouts, plus a cocubical object E in it : $\mathbb{C}^{op} \xrightarrow{E} C$. E is equipped with a B^0_C-coalgebra structure if there is a morphism of $\mathbb{S}$-operads : $B^0_C \xrightarrow{f} \mathbb{C}\mathrm{OEND}(E)$. □

Also it is easy to check that for each global object E of $\mathbb{S}\mathrm{pan}(C)$ where C has pullbacks, the construction of $\mathbb{E}\mathrm{ND}(E)$ endows a functor, and also for each global object E of $\mathbb{C}\mathrm{ospan}(C)$ where C has pushouts, the construction of $\mathbb{C}\mathrm{OEND}(E)$ is also functorial. Recall from 2.1 that we got the following diagram of functors :

$$\begin{tikzcd}
\mathbb{C}\mathrm{AT}_{\mathrm{push}} \arrow[rr, "j"] \arrow[dr, "\mathbb{C}\mathrm{ospan}(-)"] \arrow[dd, "(-)^{op}"'] & & \mathbb{C}\mathrm{AT} \arrow[dr, "\mathbb{C}\mathrm{ospan}(-)"] \arrow[dd, "(-)^{op}"'] & \\
& \mathrm{Ps}\text{-}\mathbb{S}\text{-}\mathbb{A}\mathrm{lg} \arrow[rr, "i"] & & \mathbb{C}\mathbb{C}\mathrm{AT} \\
\mathbb{C}\mathrm{AT}_{\mathrm{pull}} \arrow[rr, "k"'] \arrow[ur, "\mathbb{S}\mathrm{pan}(-)"'] & & \mathbb{C}\mathrm{AT} \arrow[ur, "\mathbb{S}\mathrm{pan}(-)"'] &
\end{tikzcd}$$

and we have the following result :

Corollary 4 • *If* $C \xrightarrow{F} D$ *is a morphism of the category* $\mathbb{C}AT_{pull}$, *and if*

$$\begin{tikzcd}
& 1 \arrow[dl, "E"'] \arrow[dr, "E'"] & \\
\mathbb{S}pan(C) \arrow[rr, "\mathbb{S}pan(F)"'] & & \mathbb{S}pan(D)
\end{tikzcd}$$

is a morphism of the category $(1 \downarrow i \circ \mathbb{S}\mathrm{pan}(-))$, *then it produces the morphism of* $\mathbb{S}$*-operads :*

$$\mathbb{E}ND(E) \xrightarrow{\mathbb{E}ND(\mathbb{S}pan(F))} \mathbb{E}ND(E')$$

Furthemore this construction is functorial and gives the functor

$$(1 \downarrow i \circ \mathbb{S}pan(-)) \xrightarrow{\mathbb{E}ND(-)} \mathbb{S}\text{-}\mathbb{O}per$$

• *If* $C \xrightarrow{F} D$ *is a morphism of the category* $\mathbb{C}AT_{push}$, *and if*

$$\begin{tikzcd}
& 1 \arrow[dl, "E"'] \arrow[dr, "E'"] & \\
\mathbb{C}ospan(C) \arrow[rr, "\mathbb{C}ospan(F)"'] & & \mathbb{C}ospan(D)
\end{tikzcd}$$

is a morphism of the category $(1 \downarrow i \circ \mathbb{C}ospan(-))$, *then it produces the morphism of* $\mathbb{S}$*-operads :*

$$\mathbb{C}OEND(E) \xrightarrow{\mathbb{C}OEND(\mathbb{C}ospan(F))} \mathbb{C}OEND(E')$$

Furthemore this construction is functorial and gives the functor

$$(1 \downarrow i \circ \mathbb{C}ospan(-)) \xrightarrow{\mathbb{C}OEND(-)} \mathbb{S}\text{-}\mathbb{O}per$$

3 Higher Cospans in $\mathbb{T}\mathrm{op}$

3.1 The global object *box* : $I^\bullet$

Here $I = [0,1]$ is the usual interval of $\mathbb{R}$. Consider the following internal cocubical object in $\mathbb{T}\mathrm{op}$:

$$\begin{tikzcd}
I^0 \arrow[r, shift left, "s_1^0"] \arrow[r, shift right, "t_1^0"'] & I^1 \arrow[r, shift left=3, "s_{1,2}^2"] \arrow[r, shift left=1, "s_{1,1}^2"] \arrow[r, shift right=1, "t_{1,1}^2"'] \arrow[r, shift right=3, "t_{1,2}^2"'] & I^2 \arrow[r, shift left=5, "s_{2,3}^3"] \arrow[r, shift left=3, "s_{2,2}^3"] \arrow[r, shift left=1, "s_{2,1}^3"] \arrow[r, shift right=1, "t_{2,1}^3"'] \arrow[r, shift right=3, "t_{2,2}^3"'] \arrow[r, shift right=5, "t_{2,3}^3"'] & I^3 \arrow[r, shift left=7, "s_{3,1}^4"] \arrow[r, shift left=5, "s_{3,2}^4"] \arrow[r, shift left=3, "s_{3,3}^4"] \arrow[r, shift left=1, "s_{3,4}^4"] \arrow[r, shift right=1, "t_{3,4}^4"'] \arrow[r, shift right=3, "t_{3,3}^4"'] \arrow[r, shift right=5, "t_{3,2}^4"'] \arrow[r, shift right=7, "t_{3,1}^4"'] & I^4 \cdots I^{n-1} \arrow[r, shift left=9, "s_{n-1,1}^n"] \arrow[r, shift left=7, dotted] \arrow[r, shift left=5, "s_{n-1,i}^n"] \arrow[r, shift left=3, dotted] \arrow[r, shift left=1, "s_{n-1,n-1}^n"] \arrow[r, shift right=1, "t_{n-1,n-1}^n"'] \arrow[r, shift right=3, dotted] \arrow[r, shift right=5, "t_{n-1,i}^n"'] \arrow[r, shift right=7, dotted] \arrow[r, shift right=9, "t_{n-1,1}^n"'] & I^n \cdots
\end{tikzcd}$$

defined by :

$$s^n_{n-1,i}(x_1, ..., x_{i-1}, x_i, ..., x_{n-1}) = (x_1, ..., x_{i-1}, 0, x_i, ..., x_{n-1})$$

$$t^n_{n-1,i}(x_1, ..., x_{i-1}, x_i, ..., x_{n-1}) = (x_1, ..., x_{i-1}, 1, x_i, ..., x_{n-1}).$$

This is a global object of the pseudo-algebra $\mathbb{C}\mathrm{ospan}(\mathbb{T}\mathrm{op})$. Following the notation in [5], this global object $I^\bullet$ shall be called the *box object*. Thanks to the pseudo-universality of $1 \xrightarrow{\eta(1)} \mathbb{S}(1)$ we get the following commutative (up to isomorphisms) diagram :

$$\begin{tikzcd}
\mathbb{S}(1) \arrow[rr, dotted, "\mathbb{C}\mathrm{oend}(I^\bullet)"] & & \mathbb{C}\mathrm{ospan}(\mathbb{T}\mathrm{op}) \\
1 \arrow[u, "\eta(1)"] \arrow[urr, "I^\bullet"'] & &
\end{tikzcd}$$

and from the cubical monoidal functor $\mathbb{C}\mathrm{oend}(I^\bullet)$ we get the $\mathbb{S}$-operad $\mathbb{COEND}(I^\bullet)$ (2.2). The next section is devoted to prove that $I^\bullet$ is a B^0_C-coalgebra, i.e that the $\mathbb{S}$-operad $\mathbb{COEND}(I^\bullet)$ is contractible and is equipped with a composition system in the sense of cubical higher operads [10].

3.2 $I^\bullet$ is a B^0_C-coalgebra

3.2.1 Composition systems on $\mathbb{COEND}(I^\bullet)$

The cubical $(n-1)$-sphere $\mathbb{S}^{n-1}_c$ is given by the sums :

$$\mathbb{S}^{n-1}_c := \coprod_{1 \leq i \leq n} (I^{i-1} \times \{0\} \times I^{n-i} \sqcup I^{i-1} \times \{1\} \times I^{n-i})$$

and we have the inclusion : $\mathbb{S}^{n-1}_c \hookrightarrow I^n$

For all $1 \leq i \leq n$ we are going to build by induction maps :

$$I^n \xrightarrow{\mu^n_i} I^n \underset{I^{n-1}}{\overset{i}{\sqcup}} I^n$$

such that $I^n \underset{I^{n-1}}{\overset{i}{\sqcup}} I^n$ is the following pushout :

$$\begin{tikzcd}
I^{n-1} \arrow[r, "s^n_{n-1,i}"] \arrow[d, "t^n_{n-1,i}"'] & I^n \arrow[d] \\
I^n \arrow[r] & I^n \underset{I^{n-1}}{\overset{i}{\sqcup}} I^n
\end{tikzcd}$$

that is, we start with $I^0 \xrightarrow{\ id\ } I^0$, and we suppose that the maps $I^{n-1} \xrightarrow{\mu_i^{n-1}} I^{n-1} \underset{I^{n-2}}{\overset{i}{\sqcup}} I^{n-1}$ are already defined for $1 \leq i \leq n-1$. We glue $\mathbb{S}_c^{n-1}$ with itself along the same face and we obtain the inclusion i :

$$
\begin{tikzcd}
I^{n-1} \arrow[r, "s^n_{n-1,i}"] \arrow[d, "t^n_{n-1,i}"] \arrow[rr, bend left, "s^n_{n-1,i}"] \arrow[dd, bend right, "t^n_{n-1,i}"'] & \mathbb{S}_c^{n-1} \arrow[r, hook] \arrow[d] & I^n \arrow[dd] \\
\mathbb{S}_c^{n-1} \arrow[r] \arrow[d, hook] & \mathbb{S}_c^{n-1} \underset{I^{n-1}}{\overset{i}{\sqcup}} \mathbb{S}_c^{n-1} \arrow[dr, hook, dotted, "i"] & \\
I^n \arrow[rr] & & I^n \underset{I^{n-1}}{\overset{i}{\sqcup}} I^n
\end{tikzcd}
$$

In order to build μ_i^n we are going to build first its interior $\mathring{\mu}_i^n$: $\mathbb{S}_c^{n-1} \xrightarrow{\mathring{\mu}_i^n} \mathbb{S}_c^{n-1} \underset{I^{n-1}}{\overset{i}{\sqcup}} \mathbb{S}_c^{n-1}$. It is defined by the following induction :

- If $i = j$ then we put $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{\ id\ } I^{i-1} \times \{0\} \times I^{n-i}$, where the identity map id sends the $(n-1)$-faces $I^{i-1} \times \{0\} \times I^{n-i}$ of the first copy $\mathbb{S}_c^{n-1}$ in $\mathbb{S}_c^{n-1} \underset{I^{n-1}}{\overset{i}{\sqcup}} \mathbb{S}_c^{n-1}$ to the $(n-1)$-faces $I^{i-1} \times \{0\} \times I^{n-i}$ of the second copy $\mathbb{S}_c^{n-1}$, and we put : $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{\ id\ } I^{i-1} \times \{1\} \times I^{n-i}$, where the identity map id sends the $(n-1)$-faces $I^{i-1} \times \{1\} \times I^{n-i}$ of the first copy $\mathbb{S}_c^{n-1}$ in $\mathbb{S}_c^{n-1} \underset{I^{n-1}}{\overset{i}{\sqcup}} \mathbb{S}_c^{n-1}$ to the $(n-1)$-faces $I^{i-1} \times \{1\} \times I^{n-i}$ of the second copy $\mathbb{S}_c^{n-1}$.

- If $1 \leq j < i \leq n$ then we put : $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{\mu_{i-1}^{n-1}} I^{j-1} \times \{0\} \times I^{n-j} \underset{I^{n-2}}{\overset{i-1}{\sqcup}} I^{j-1} \times \{0\} \times I^{n-j}$, and $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{\mu_{i-1}^{n-1}} I^{j-1} \times \{1\} \times I^{n-j} \underset{I^{n-2}}{\overset{i-1}{\sqcup}} I^{j-1} \times \{1\} \times I^{n-j}$, where the codomains are given by the following pushout :

$$
\begin{tikzcd}
I^{n-2} \arrow[r, "s^{n-1}_{n-2,i-1}"] \arrow[d, "t^{n-1}_{n-2,i-1}"'] & I^{n-1} \arrow[d] \\
I^n \arrow[r] & I^{n-1} \underset{I^{n-2}}{\overset{i-1}{\sqcup}} I^{n-1}
\end{tikzcd}
$$

- If $1 \leq i < j \leq n$ then we put : $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{\mu_{j-1}^{n-1}} I^{j-1} \times \{0\} \times I^{n-j} \underset{I^{n-2}}{\overset{j-1}{\sqcup}} I^{j-1} \times \{0\} \times I^{n-j}$, and $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{\mu_{j-1}^{n-1}} I^{j-1} \times \{1\} \times I^{n-j} \underset{I^{n-2}}{\overset{j-1}{\sqcup}} I^{j-1} \times \{1\} \times I^{n-j}$, where the codomains are given by the following pushout :

$$
\begin{tikzcd}
I^{n-2} \arrow[r, "s^{n-1}_{n-2,j-1}"] \arrow[d, "t^{n-1}_{n-2,j-1}"'] & I^{n-1} \arrow[d] \\
I^n \arrow[r] & I^{n-1} \underset{I^{n-2}}{\overset{j-1}{\sqcup}} I^{n-1}
\end{tikzcd}
$$

Thus we obtain the desired extension μ_i^n of $\overset{\circ}{\mu}{}_i^n$:

$$\begin{tikzcd}
I^n \arrow[rrd, dotted, "\mu_i^n"] & & \\
\mathbb{S}_c^{n-1} \arrow[u, hook] \arrow[r, "\overset{\circ}{\mu}{}_i^n"'] & \mathbb{S}_c^{n-1} \underset{I^{n-1}}{\overset{i}{\sqcup}} \mathbb{S}_c^{n-1} \arrow[r, hook, "i"'] & I^n \underset{I^{n-1}}{\overset{i}{\sqcup}} I^n
\end{tikzcd}$$

3.2.2 Contractibility of $\mathbb{C}$OEND$(I^\bullet)$

Consider two maps in $\mathbb{T}$op : $I^{n-1} \underset{g}{\overset{f}{\rightrightarrows}} X$, such that f and g are two $(n-1)$-cells of the operad $\mathbb{C}$OEND$(I^\bullet)$. Thus X is described as an iterated pushouts of the topological n-cubes I^n $(n \in \mathbb{N})$ given by the global object $I^\bullet$ in the pseudo-algebra $\mathbb{C}$ospan($\mathbb{T}$op); and in particular X is contractible. We are going to build the contraction $[f,g]_{n,j}^{n-1}$ by induction. Thus we suppose that for all $1 \leq j \leq n-1$ the maps $I^{n-1} \xrightarrow{[f,g]_{n-1,j}^{n-2}} X$ exist, and we start our induction with an easy choice of extension $[f,g]_1^0$, where f and g define here two points of X : $I \xrightarrow{[f,g]_1^0} X$. The contraction $[f,g]_{n,j}^{n-1}$ is given by a continuous map $I^n \xrightarrow{[f,g]_{n,j}^{n-1}} X$. In order to do that, for all $1 \leq j \leq n$, we need first to define the map : $\mathbb{S}_c^{n-1} \xrightarrow{\langle f,g\rangle_{n,j}^{n-1}} X$. This map $\langle f,g\rangle_{n,j}^{n-1}$ has the following definition :

- for $i = j$ we put : $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{f} X$, and $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{g} X$
- If $1 \leq i < j \leq n$ then $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i}^{n-1}(f),s_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2}} X$, and $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i}^{n-1}(f),t_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2}} X$
- If $1 \leq j < i \leq n$ then $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i-1}^{n-1}(f),s_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2}} X$, and $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i-1}^{n-1}(f),t_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2}} X$

then we obtain the desired extension :

$$\begin{tikzcd}
I^n \arrow[rd, dotted, "{[f,g]_{n,j}^{n-1}}"] & \\
\mathbb{S}_c^{n-1} \arrow[u, hook] \arrow[r, "\langle f,g\rangle_{n,j}^{n-1}"'] & X
\end{tikzcd}$$

Now consider two $(n-1)$-cells of $\mathbb{C}$OEND$(I^\bullet)$: $I^{n-1} \underset{g}{\overset{f}{\rightrightarrows}} X$, such that for $1 \leq j \leq n-1$ we have

$$f \circ s_{n-2,j}^{n-1} = g \circ s_{n-2,j}^{n-1} : \ I^{n-2} \xrightarrow{s_{n-2,j}^{n-1}} I^{n-1} \underset{g}{\overset{f}{\rightrightarrows}} X,$$

that is $s_{n-2,j}^{n-1}(f) = s_{n-2,j}^{n-1}(g)$. We are going to build the contraction $[f,g]_{n,j}^{n-1,-}$ by induction. Thus we suppose that for all $1 \leq j \leq n-2$ the maps $I^{n-1} \xrightarrow{[f,g]_{n-1,j}^{n-2,-}} X$ exist, and we start our induction with an easy choice of extension $[f,g]_{2,1}^{1,-}$, where f and g define here two paths in X : $I^2 \xrightarrow{[f,g]_{2,1}^{1,-}} X$ The map $[f,g]_{n,j}^{n-1,-}$ is given by a continuous map $I^n \xrightarrow{[f,g]_{n,j}^{n-1,-}} X$. In order to do that, for all $1 \leq j \leq n-1$, we need first to define the map : $\mathbb{S}_c^{n-1} \xrightarrow{\langle f,g\rangle_{n,j}^{n-1,-}} X$. This map $\langle f,g\rangle_{n,j}^{n-1,-}$ has the following definition :

- if $i = j$ we put : $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{f} X$, and $I^{j} \times \{0\} \times I^{n-j-1} \xrightarrow{g} X$,

  and $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{[t_{n-2,j}^{n-1}(f),t_{n-2,j}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$, and $I^{j} \times \{1\} \times I^{n-j-1} \xrightarrow{[t_{n-2,j}^{n-1}(f),t_{n-2,j}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$.

- If $1 \leq i, j \leq n$ then we put :
  - if $1 \leq i < j \leq n-1$ then $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s^{n-1}_{n-2,i}(f), s^{n-1}_{n-2,i}(g)]^{n-2,-}_{n-1,j-1}} X$,

    and $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t^{n-1}_{n-2,i}(f), t^{n-1}_{n-2,i}(g)]^{n-2,-}_{n-1,j-1}} X$.
  - if $2 \leq j+1 < i \leq n$ then $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s^{n-1}_{n-2,i-1}(f), s^{n-1}_{n-2,i-1}(g)]^{n-2,-}_{n-1,j}} X$,

    and $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t^{n-1}_{n-2,i-1}(f), t^{n-1}_{n-2,i-1}(g)]^{n-2,-}_{n-1,j}} X$

  then we obtain the desired extension :

$$\begin{tikzcd} I^n \arrow[drr, dotted, "{[f,g]^{n-1,-}_{n,j}}"] & & \\ \mathbb{S}^{n-1}_c \arrow[u] \arrow[rr, "{\langle f,g\rangle^{n-1,-}_{n,j}}"'] & & X \end{tikzcd}$$

Now consider two $(n-1)$-cells of $\mathbb{C}\mathrm{OEND}(I^\bullet)$: $I^{n-1} \overset{f}{\underset{g}{\rightrightarrows}} X$, such that X is contractible, and such that for $1 \leq j \leq n-1$ we have

$$f \circ t^{n-1}_{n-2,j} = g \circ t^{n-1}_{n-2,j} : \; I^{n-2} \xrightarrow{t^{n-1}_{n-2,j}} I^{n-1} \overset{f}{\underset{g}{\rightrightarrows}} X,$$

that is $t^{n-1}_{n-2,j}(f) = t^{n-1}_{n-2,j}(g)$. We are going to build the contraction $[f,g]^{n-1,+}_{n,j}$ by induction. Thus we suppose that for all $1 \leq j \leq n-2$ the maps $I^{n-1} \xrightarrow{[f,g]^{n-2,+}_{n-1,j}} X$ exist, and we start our induction with an easy choice of extension $[f,g]^{1,+}_{2,1}$, where f and g define here two paths in X : $I^2 \xrightarrow{[f,g]^{1,+}_{2,1}} X$. The map $[f,g]^{n-1,-}_{n,j}$ is given by a continuous map $I^n \xrightarrow{[f,g]^{n-1,+}_{n,j}} X$. In order to do that, for all $1 \leq j \leq n-1$, we need first to define the map : $\mathbb{S}^{n-1}_c \xrightarrow{\langle f,g\rangle^{n-1,+}_{n,j}} X$. This map $\langle f,g\rangle^{n-1,+}_{n,j}$ has the following definition :

- if $i = j$ we put : $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{f} X$, and $I^{j} \times \{1\} \times I^{n-j-1} \xrightarrow{g} X$,

  and $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{[s^{n-1}_{n-2,j}(f), s^{n-1}_{n-2,j}(g)]^{n-2,-}_{n-1,j}} X$, and $I^{j} \times \{0\} \times I^{n-j-1} \xrightarrow{[s^{n-1}_{n-2,j}(f), s^{n-1}_{n-2,j}(g)]^{n-2,-}_{n-1,j}} X$,
- If $1 \leq i, j \leq n$ then we put :
  - if $1 \leq i < j \leq n-1$ then $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s^{n-1}_{n-2,i}(f), s^{n-1}_{n-2,i}(g)]^{n-2,+}_{n-1,j-1}} X$,

    and $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t^{n-1}_{n-2,i}(f), t^{n-1}_{n-2,i}(g)]^{n-2,+}_{n-1,j-1}} X$.
  - if $2 \leq j+1 < i \leq n$ then $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s^{n-1}_{n-2,i-1}(f), s^{n-1}_{n-2,i-1}(g)]^{n-2,+}_{n-1,j}} X$,

    and $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t^{n-1}_{n-2,i-1}(f), t^{n-1}_{n-2,i-1}(g)]^{n-2,+}_{n-1,j}} X$.

  Then we obtain the desired extension :

$$\begin{tikzcd} I^n \arrow[drr, dotted, "{[f,g]^{n-1,+}_{n,j}}"] & & \\ \mathbb{S}^{n-1}_c \arrow[u] \arrow[rr, "{\langle f,g\rangle^{n-1,+}_{n,j}}"'] & & X \end{tikzcd}$$

and it is then straightforward to check the different axioms of contractions for such extensions $[f,g]^{n-1}_{n,j}$, $[f,g]^{n-1,-}_{n,j}$ and $[f,g]^{n-1,+}_{n,j}$. With 3.2.1 and 3.2.2 we thus have proved the :

Theorem 2 *$I^\bullet$ is a B^0_C-coalgebra.* □

3.2.3 The fundamental morphism of operads

Let us fix a topological space $X \in \mathbb{T}\mathrm{op}$. From it we get a functor $\mathbb{T}\mathrm{op} \xrightarrow{\mathbb{M}\mathrm{ap}(-,X)} \mathbb{S}\mathrm{ET}^{op}$ in $\mathbb{C}\mathrm{AT}_{\mathrm{push}}$, thus from the functor : $\mathbb{C}\mathrm{AT}_{\mathrm{push}} \xrightarrow{\mathbb{C}\mathrm{ospan}(-)} \mathrm{Ps}\text{-}\mathbb{S}\text{-}\mathbb{A}\mathrm{lg}$ we get the following morphism in $\mathrm{Ps}\text{-}\mathbb{S}\text{-}\mathbb{A}\mathrm{lg}$:

$$\mathbb{C}\mathrm{ospan}(\mathbb{T}\mathrm{op}) \xrightarrow{\mathbb{C}\mathrm{ospan}(\mathbb{M}\mathrm{ap}(-,X))} \mathbb{C}\mathrm{ospan}(\mathbb{S}\mathrm{ET}^{op})$$

Also we have the functor :

$$(1 \downarrow i \circ \mathbb{C}\mathrm{ospan}(-)) \xrightarrow{\mathbb{C}\mathrm{OEND}(-)} \mathbb{S}\text{-}\mathbb{O}\mathrm{per}$$

which sends the following morphism $\mathbb{M}\mathrm{ap}(-,X)$ of $(1 \downarrow i \circ \mathbb{C}\mathrm{ospan}(-))$:

$$\begin{tikzcd} & 1 \arrow[dl, "I^{\bullet}"'] \arrow[dr, "\mathbb{M}\mathrm{ap}(I^{\bullet},X)^{op}"] & \\ \mathbb{C}\mathrm{ospan}(\mathbb{T}\mathrm{op}) \arrow[rr, "\mathbb{C}\mathrm{ospan}(\mathbb{M}\mathrm{ap}(-,X))"'] & & \mathbb{C}\mathrm{ospan}(\mathbb{S}\mathrm{ET}^{op}) \end{tikzcd}$$

to the morphism of operads :

$$\mathbb{C}\mathrm{OEND}(I^{\bullet}) \xrightarrow{\mathbb{C}\mathrm{OEND}(\mathbb{C}\mathrm{ospan}(\mathbb{M}\mathrm{ap}(-,X)))} \mathbb{C}\mathrm{OEND}(\mathbb{M}\mathrm{ap}(I^{\bullet},X)^{op}) \simeq \mathbb{E}\mathrm{ND}(\mathbb{M}\mathrm{ap}(I^{\bullet},X))$$

this shows that $\mathbb{M}\mathrm{ap}(I^{\bullet},X)$ is an algebra for $\mathbb{C}\mathrm{OEND}(I^{\bullet})$. But we proved in 3.2 that $I^{\bullet}$ is also a B^0_C-coalgebra, which means that we have a morphism of operads :

$$B^0_C \xrightarrow{\ !\ } \mathbb{C}\mathrm{OEND}(I^{\bullet})$$

which shows that we have a morphism of operads :

$$B^0_C \xrightarrow{\mathbb{C}\mathrm{OEND}(\mathbb{C}\mathrm{ospan}(\mathbb{M}\mathrm{ap}(-,X)))\circ !} \mathbb{E}\mathrm{ND}(\mathbb{M}\mathrm{ap}(I^{\bullet},X))$$

that is the cubical set $\mathbb{M}\mathrm{ap}(I^{\bullet},X)$:

$$\begin{tikzcd}[column sep=large] \cdots \mathbb{M}\mathrm{ap}(I^n,X) \arrow[r, shift left=5, "s^n_{n-1,n}"] \arrow[r, dotted, shift left=4] \arrow[r, shift left=3, "s^n_{n-1,i}"] \arrow[r, dotted, shift left=2] \arrow[r, shift left=1, "s^n_{n-1,1}"] \arrow[r, shift right=1, "t^n_{n-1,1}"'] \arrow[r, dotted, shift right=2] \arrow[r, shift right=3, "t^n_{n-1,i}"'] \arrow[r, dotted, shift right=4] \arrow[r, shift right=5, "t^n_{n-1,n}"'] & \mathbb{M}\mathrm{ap}(I^{n-1},X) \cdots \mathbb{M}\mathrm{ap}(I^4,X) \arrow[r, shift left=4, "s^4_{3,4}"] \arrow[r, shift left=3, "s^4_{3,3}"] \arrow[r, shift left=2, "s^4_{3,2}"] \arrow[r, shift left=1, "s^4_{3,1}"] \arrow[r, shift right=1, "t^4_{3,1}"'] \arrow[r, shift right=2, "t^4_{3,2}"'] \arrow[r, shift right=3, "t^4_{3,3}"'] \arrow[r, shift right=4, "t^4_{3,4}"'] & \mathbb{M}\mathrm{ap}(I^3,X) \arrow[r, shift left=3, "s^3_{2,3}"] \arrow[r, shift left=2, "s^3_{2,2}"] \arrow[r, shift left=1, "s^3_{2,1}"] \arrow[r, shift right=1, "t^3_{2,1}"'] \arrow[r, shift right=2, "t^3_{2,2}"'] \arrow[r, shift right=3, "t^3_{2,3}"'] & \mathbb{M}\mathrm{ap}(I^2,X) \arrow[r, shift left=2, "s^2_{1,2}"] \arrow[r, shift left=1, "s^2_{1,1}"] \arrow[r, shift right=1, "t^2_{1,1}"'] \arrow[r, shift right=2, "t^2_{1,2}"'] & \mathbb{M}\mathrm{ap}(I,X) \arrow[r, shift left=1, "s^1_0"] \arrow[r, shift right=1, "t^1_0"'] & \mathbb{M}\mathrm{ap}(I^0,X) \end{tikzcd}$$

is equipped with a structure of weak cubical ∞-category. This weak cubical ∞-category $\Pi_{\infty}(X)$ is in fact a weak cubical ∞-groupoid (see [14] for the definition of cubical weak ∞-groupoids), called the *fundamental cubical weak ∞-groupoid* of X. Also if $X \xrightarrow{f} Y$ is a continuous map between X and Y, then from our functorial constructions we get the following commutative diagram :

$$\begin{tikzcd} & & \mathbb{E}\mathrm{ND}(\mathbb{M}\mathrm{ap}(I^{\bullet},X)) \arrow[dd, "\mathbb{E}\mathrm{ND}(\mathbb{M}\mathrm{ap}(I^{\bullet},f))"] \\ B^0_C \arrow[urr, "\mathbb{C}\mathrm{OEND}(\mathbb{C}\mathrm{ospan}(\mathbb{M}\mathrm{ap}(-,X)))\circ !"] \arrow[r, "!"] \arrow[drr, "\mathbb{C}\mathrm{OEND}(\mathbb{C}\mathrm{ospan}(\mathbb{M}\mathrm{ap}(-,Y)))\circ !"'] & \mathbb{C}\mathrm{OEND}(I^{\bullet}) \arrow[ur] \arrow[dr] & \\ & & \mathbb{E}\mathrm{ND}(\mathbb{M}\mathrm{ap}(I^{\bullet},Y)) \end{tikzcd}$$

which exhibits the *fundamental cubical weak ∞-groupoid functor* :

$$\mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}\mathbb{G}\mathrm{rp}$$

which has a left adjoint functor CN_∞. This pair of adjunction $(CN_\infty, \Pi_\infty(-))$ should put an equivalence between the homotopy category of homotopy types and the homotopy category of ∞-$\mathbb{C}\mathbb{G}$rp of cubical weak ∞-groupoids with connections equipped with an adapted Quillen model structure. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [18]).

The Grothendieck conjecture on homotopy types [9] predicts that the category $\mathbb{T}$op of topological spaces is Quillen equivalent to the category of globular weak ∞-groupoids. An accurate formulation of this conjecture is in [1] where Michael Batanin has built a *fundamental globular weak ∞-groupoid functor* :

$$\mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty-\mathbb{G}\mathrm{rp}$$

by using the coalgebraic feature of the coglobular object of $\mathbb{T}$op consisting of topological disks. Also it was proved in [3] that the category of globular strict ∞-categories is equivalent to the category of cubical strict ∞-categories. This work [3] shows that technics to compare globular higher category and cubical higher category exist. A natural question is to ask if such technics can be generalized for weak structures. Also it is important to notice an other analogy between globular higher category and cubical higher category : in the work [12] we built algebraic models of globular weak ∞-groupoids, and in an other work [14] we also built algebraic models of cubical weak ∞-groupoids, which are similar to their globular analogue : they are similar in the sense that they are both defined as algebras for specific monads. Indeed our globular weak ∞-groupoids are algebras for a monad on the category of globular sets, and our cubical weak ∞-groupoids are also algebras for a monad on the category of cubical sets. All these material putting together give a real perspectives to solve the the Grothendieck conjecture on homotopy types[4].

3.2.4 Application for higher K-theory

The functor $\Pi_\infty(-)$ could be intuitively thought as the gluing of all the homotopy groups functor π_i together, and because the π_i are cohomologies, $\Pi_\infty(-)$ could be thought as a *higher dimensional cohomology*, that is a functor between ∞-categories, or an ∞-functor which behave like cohomologies. It seems that such objects are of interest for the *Stolz-Teichner* program[5] [22] who try to investigate ideas from physic (TQFT=Topological Quantum Field Theory) through cohomologies, and also ETQFT (Extended TQFT) through higher dimensional cohomologies and vice-versa.

In this section we explain how to "glue" algebraic K_i-functors ($i \in \mathbb{N}$) of Quillen :

$$\mathbb{R}\mathrm{ings} \xrightarrow{\mathrm{K}_i} \mathbb{T}\mathrm{op}$$

into a single functor K_∞, where here $\mathbb{R}$ings is the category of rings with unit.

But first let us recall some basic facts which are defined more accurately in [19] : the functors K_i are defined by the compostion :

$$\mathbb{R}\mathrm{ings} \xrightarrow{\mathrm{BGL}(-)^+ \times \mathrm{K}_0(-)} \mathbb{T}\mathrm{op} \xrightarrow{\pi_i(-)} \mathbb{G}\mathrm{rp}$$

where for any rings R with unit :

- $\mathrm{GL(R)} = \bigcup\limits_{n=1}^{\infty} \mathrm{GL(n,R)}$
- BGL(R) is the classifying space of the group GL(R)
- the +-construction on BGL(R) is taken relative to the perfect subgroup $E(R)$ (elementary matrices) of GL(R)
- $K_0(R)$ is given the discrete topology

Thus we get the functor K_∞ which is given by the composition :

$$\mathbb{R}\mathrm{ings} \xrightarrow{\mathrm{BGL}(-)^+ \times \mathrm{K}_0(-)} \mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}\mathbb{G}\mathrm{rp}$$

[4]It is important to aware that the author suffer by a lack of financial support. Between 2013 and 2017, only three months have been financially supported. The author believes that with decent financial support he will be in a better condition to attack this conjecture

[5]These ideas take their roots in the work of Graham Segal on Conformal Field Theory.

4 Importance of Coalgebraic structures for Globular and Cubical Higher Category Theory

4.1 The Batanin's construction and the author's construction

In the article [1], Michael Batanin has built the contractible operad B^0_C which algebras are globular weak ∞-categories. He also proved that the globular object $D^\bullet$ in $\mathbb{T}\mathrm{op}$ consisting of topological disks :

$$D^0 \overset{s^1_0}{\underset{t^1_0}{\rightrightarrows}} D^1 \overset{s^2_1}{\underset{t^2_1}{\rightrightarrows}} D^2 \cdots D^{n-1} \overset{s^n_{n-1}}{\underset{t^n_{n-1}}{\rightrightarrows}} D^n \cdots$$

is a B^0_C-coalgebra, which implication is the construction of the fundamental globular weak ∞-groupoid functor

$$\mathbb{T}\mathrm{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{G}\mathrm{rp}$$

In the other hand, in the article [13] the author built a coglobular object $B^\bullet_C$

$$B^0_C \overset{s^1_0}{\underset{t^1_0}{\rightrightarrows}} B^1_C \overset{s^2_1}{\underset{t^2_1}{\rightrightarrows}} B^2_C \cdots B^{n-1}_C \overset{s^n_{n-1}}{\underset{t^n_{n-1}}{\rightrightarrows}} B^n_C \cdots$$

such that B^0_C is the contractible operad just above of Michael Batanin, B^1_C is the contractible operad which algebras are globular weak ∞-functors, B^2_C is the contractible operad which algebras are globular weak ∞-natural transformations, etc. Also we have the surprising fact : if $B^\bullet_C$ is a B^0_C-coalgebra then it implies that the globular weak ∞-category of globular weak ∞-categories exists. We didn't prove yet this fact[6], however this is an important improvement for globular higher category theory for two main reasons :

- in the beginning it was non-trivial to know why globular weak ∞-categories, globular weak ∞-functors, globular weak ∞-natural transformations, etc. organize in a globular weak ∞-category. Now we have replaced this very complex combinatorial question by a precise statement : the coendomorphism operad $\mathbb{C}\mathrm{OEND}(B^\bullet_C)$ should be contractible[7], like its topological little son[8] $\mathbb{C}\mathrm{OEND}(D^\bullet)$.
- it brings a spectacular analogy between topological spaces and globular higher categories, which was hope by Grothendieck and Thomason. Let us gives a first smell of such analogy :
  - Consider the following 1-cell in the operad $\mathbb{C}\mathrm{OEND}(D^\bullet)$ of topological disks :

    $$D^1 \xrightarrow{\mu^1_0} D^1 \underset{D^0}{\sqcup} D^1$$

    and consider a topological space X. With these we get the following 1-cell of the fundamental weak ∞-groupoid $\Pi_\infty(X)$:

    $$\mathbb{M}\mathrm{ap}(D^1, X) \underset{\mathbb{M}\mathrm{ap}(D^0,X)}{\times} \mathbb{M}\mathrm{ap}(D^1, X) \xrightarrow{\quad \circ^1_0 \quad} \mathbb{M}\mathrm{ap}(D^1, X)$$

  - Suppose that $\mathbb{C}\mathrm{OEND}(B^\bullet_C)$ is contractible. It is then possible to consider the following 1-cell in the operad $\mathbb{C}\mathrm{OEND}(B^\bullet_C)$ of operadical disks :

    $$B^1_C \xrightarrow{\mu^1_0} B^1_C \underset{B^0_C}{\sqcup} B^1_C$$

    and with this 1-cell of $\mathbb{C}\mathrm{OEND}(B^\bullet_C)$, we get the following 1-cell in the suspected globular weak ∞-category of globular weak ∞-categories :

    $$\mathbb{A}\mathrm{lg}(B^1_C)(0) \underset{\mathbb{A}\mathrm{lg}(B^0_C)(0)}{\times} \mathbb{A}\mathrm{lg}(B^1_C)(0) \xrightarrow{\quad \circ^1_0 \quad} \mathbb{A}\mathrm{lg}(B^1_C)(0)$$

    which is the composition of globular weak ∞-functors !

[6] Also because the author suffer of lack of financial support.
[7] which imply that it is equipped with a composition system. See [13]
[8] or little brother ...

4.2 Steps toward the cubical weak ∞-category of cubical weak ∞-categories

Consider the following internal cocubical object in a subcategory[9] $\mathcal{C}$ of the category $\mathbb{M}nd$ of monads, such that $\mathcal{C}$ has pushouts.

$$\begin{tikzcd}[column sep=large]
B^0_C \arrow[r, shift left=1, "s^0_1"] \arrow[r, shift right=1, "t^0_1"'] & B^1_C \arrow[r, shift left=3, "s^2_{1,2}"] \arrow[r, shift left=1, "s^2_{1,1}"] \arrow[r, shift right=1, "t^2_{1,1}"'] \arrow[r, shift right=3, "t^2_{1,2}"'] & B^2_C \arrow[r, shift left=5, "s^3_{2,3}"] \arrow[r, shift left=3, "s^3_{2,2}"] \arrow[r, shift left=1, "s^3_{2,1}"] \arrow[r, shift right=1, "t^3_{2,1}"'] \arrow[r, shift right=3, "t^3_{2,2}"'] \arrow[r, shift right=5, "t^3_{2,3}"'] & B^3_C \arrow[r, shift left=7, "s^4_{3,1}"] \arrow[r, shift left=5, "s^4_{3,2}"] \arrow[r, shift left=3, "s^4_{3,3}"] \arrow[r, shift left=1, "s^4_{3,4}"] \arrow[r, shift right=1, "t^4_{3,4}"'] \arrow[r, shift right=3, "t^4_{3,3}"'] \arrow[r, shift right=5, "t^4_{3,2}"'] \arrow[r, shift right=7, "t^4_{3,1}"'] & B^4_C \cdots B^{n-1}_C \arrow[r, shift left=9, "s^n_{n-1,1}"] \arrow[r, shift left=7, dotted] \arrow[r, shift left=5, "s^n_{n-1,i}"] \arrow[r, shift left=3, dotted] \arrow[r, shift left=1, "s^n_{n-1,n-1}"] \arrow[r, shift right=1] \arrow[r, shift right=3, dotted, "t^n_{n-1,n-1}"] \arrow[r, shift right=5] \arrow[r, shift right=7, dotted, "t^n_{n-1,i}"] \arrow[r, shift right=9, "t^n_{n-1,1}"'] & B^n_C \cdots
\end{tikzcd}$$

such that B^0_C is the $\mathbb{S}^0$-operad which algebras are cubical weak ∞-categories. Also the $\mathbb{S}^1$-operad B^1_C which algebras are cubical weak ∞-functors, the $\mathbb{S}^2$-operad B^2_C which algebras are cubical weak ∞-natural transformations, etc. where $\mathbb{S}^1$ is the cartesian monad which algebras are cubical strict ∞-functors, $\mathbb{S}^2$ is the cartesian monad which algebras are cubical strict ∞-natural transformations, etc. are not difficult to be built. For example the underlying combinatorics of the $\mathbb{S}^1$-collection of B^1_C comes easily from the monad of cubical weak ∞-functors as defined in [14] and the underlying combinatorics of the $\mathbb{S}^2$-collection of B^2_C comes easily from the monad of cubical weak ∞-natural transformations as defined in [14]. Also according to the cubical combinatorics it is straightforward to see that the cartesian monad $\mathbb{S}^n$ of cubical strict n-transformations act on the category $\mathbb{C}\mathbb{S}\mathrm{ets}^{2^n}$, the cartesian product 2^n times in $\mathbb{C}\mathrm{AT}$ of the category of cubical sets with itself. In order to build these contractible $\mathbb{S}^n$-operads B^n_C we have different technics to do it. We can use for example the formalism of the $\mathbb{T}$-*categorical stretchings* as developed in [16], or we can use the theory of Garner [6] to build a fibrant replacement of the $\mathbb{S}^n$, or we can more classically just use the technology developed in [1, 10].

This internal cocubical object $B^\bullet$ of $\mathcal{C}$ is a global object of the pseudo-algebra $\mathbb{C}\mathrm{ospan}(\mathcal{C})$, where here we deal with cubical higher cospans. Thanks to the functor defined in 2.2

$$(1 \downarrow i \circ \mathbb{C}\mathrm{ospan}(-)) \xrightarrow{\mathbb{C}\mathrm{OEND}(-)} \mathbb{S}\text{-}\mathbb{O}\mathrm{per}$$

the following morphism $\mathbb{A}\mathrm{lg}(-)$ of $(1 \downarrow i \circ \mathbb{C}\mathrm{ospan}(-))$:

$$\begin{tikzcd}
 & 1 \arrow[dl, "I^\bullet"'] \arrow[dr, "\mathbb{A}\mathrm{lg}(B^\bullet_C)^{op}"] & \\
\mathbb{C}\mathrm{ospan}(\mathcal{C}) \arrow[rr, "\mathbb{C}\mathrm{ospan}(\mathbb{A}\mathrm{lg}(-))"'] & & \mathbb{C}\mathrm{ospan}(\mathbb{S}\mathrm{ET}^{op})
\end{tikzcd}$$

is sent to the morphism of operads :

$$\mathbb{C}\mathrm{OEND}(B^\bullet_C) \xrightarrow{\mathbb{C}\mathrm{OEND}(\mathbb{C}\mathrm{ospan}(\mathbb{A}\mathrm{lg}(-)))} \mathbb{C}\mathrm{OEND}(\mathbb{A}\mathrm{lg}(B^\bullet_C)^{op}) \simeq \mathbb{E}\mathrm{ND}(\mathbb{A}\mathrm{lg}(B^\bullet_C)) \ .$$

This shows that $\mathbb{A}\mathrm{lg}(B^\bullet_C)$ is an algebra for $\mathbb{C}\mathrm{OEND}(B^\bullet_C)$. Now suppose $B^\bullet_C$ is also a B^0_C-coalgebra. In fact we put the following conjecture :

Conjecture *The operad of coendomorphism $\mathbb{C}OEND(B^\bullet_C)$ is contractible.*

Contractibility here means the cubical one, as developed in [10], where we consider contractions similar to their globular analogue, plus the "connections-contractions" which are for contractions what connections are for cubical ∞-categories.

If we accept this conjecture then it means that we have a morphism of operads :

$$B^0_C \xrightarrow{\ !\ } \mathbb{C}\mathrm{OEND}(B^\bullet_C)$$

which means that we have a morphism of operads :

$$B^0_C \xrightarrow{\mathbb{C}\mathrm{OEND}(\mathbb{C}\mathrm{ospan}(\mathbb{A}\mathrm{lg}(-)))\circ !} \mathbb{E}\mathrm{ND}(\mathbb{A}\mathrm{lg}(B^\bullet_C))$$

which shows that the cubical set $\mathbb{A}\mathrm{lg}(B^\bullet_C)(0)$[10] :

[9] Such subcategory exists according to a private communication with Ross Street and John Bourke who give me such accurate construction of it. We won't describe it here because of lack of time.

[10] For each $n \in \mathbb{N}$, $\mathbb{A}\mathrm{lg}(B^n_C)(0)$ means the class of objects of the category $\mathbb{A}\mathrm{lg}(B^n_C)$.

$$\cdots \mathbb{A}\mathrm{lg}(B^n_C)(0) \underset{\substack{t^n_{n-1,1} \\ \cdots \\ t^n_{n-1,i} \\ \cdots \\ t^n_{n-1,n}}}{\overset{\substack{s^n_{n-1,n} \\ \cdots \\ s^n_{n-1,i} \\ \cdots \\ s^n_{n-1,1}}}{\rightrightarrows}} \mathbb{A}\mathrm{lg}(B^{n-1}_C)(0) \cdots \mathbb{A}\mathrm{lg}(B^4_C)(0) \underset{\substack{t^4_{3,1} \\ t^4_{3,2} \\ t^4_{3,3} \\ t^4_{3,4}}}{\overset{\substack{s^4_{3,4} \\ s^4_{3,3} \\ s^4_{3,2} \\ s^4_{3,1}}}{\rightrightarrows}} \mathbb{A}\mathrm{lg}(B^3_C)(0) \underset{\substack{t^3_{2,1} \\ t^3_{2,2} \\ t^3_{2,3}}}{\overset{\substack{s^3_{2,3} \\ s^3_{2,2} \\ s^3_{2,1}}}{\rightrightarrows}} \mathbb{A}\mathrm{lg}(B^2_C)(0) \underset{\substack{t^2_{1,1} \\ t^2_{1,2}}}{\overset{\substack{s^2_{1,2} \\ s^2_{1,1}}}{\rightrightarrows}} \mathbb{A}\mathrm{lg}(B^1_C)(0) \underset{t^1_0}{\overset{s^1_0}{\rightrightarrows}} \mathbb{A}\mathrm{lg}(B^0_C)(0)$$

is equipped with a structure of weak cubical ∞-category. This is the cubical weak ∞-category of cubical weak ∞-categories. Like for globular higher category theory, we thus have an amazing analogy between topological spaces and cubical higher categories, up to these conjectures related to coalgebraicity. This is our operadical point of view which allows such analogies. Thanks to it we can mimic the globular approach of weak Grothendieck ∞-topos as described in [15] to have a real smell of what is a cubical weak Grothendieck ∞-topos. We would like to insist that this article overall shows how 2-categorical materials developed in [20, 21, 23, 24] and recently in [11], can provide some good generalizations, where different higher category theory with different shapes, could be developed within this framework, and where we can imagine for example that other geometries for higher groupoids associated to topological spaces are possible.

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