DESY 06–233
SFB/CPP–06–58

THE NEXT-TO-LADDER APPROXIMATION FOR DYSON-SCHWINGER EQUATIONS

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1. Introduction

Ladder approximations have been one of the most basic attempts to simplify and truncate Dyson–Schwinger equations in field theory in a still meaningful way. From a mathematical viewpoint they simplify the combinatorics of the forest formula considerably, and are solvable by a scaling Ansatz for sufficiently simple kinematics.

Here, we discuss such a scenario, but iterate one- and two-loop skeletons jointly, combining some analytic progress with a thorough discussion of the underlying algebraic properties.

1.1. **Purpose of this paper.** The main purpose is to sum an infinite series of graphs based on the iteration of two underlying skeleton graphs. We progress in a manner such that our methods can be generalized to any countable number of skeletons. We restrict to linear Dyson Schwinger equations, a case relevant for theories at a fixpoint of the renormalization group. We proceed using one-dimensional Mellin transforms, a privilege of linearity of which we make full use. See [1, 2, 3, 4] for the general approach.

2. The Dyson–Schwinger Equation

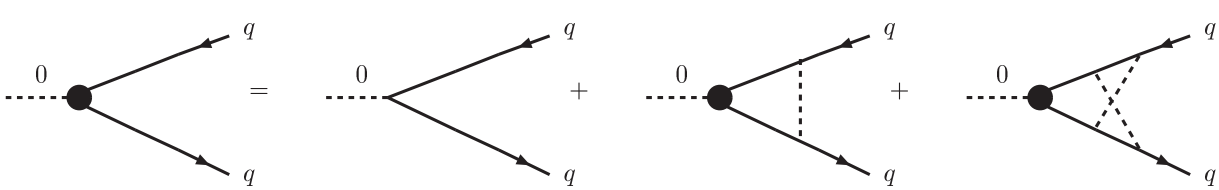
2.1. **The integral equation.** The equation which we consider is in massless Yukawa theory in four-dimensional Minkowski space $\mathbb{M}$, for pedagogical purposes. We define a renormalized Green function describing the coupling of a scalar particle to a fermion line by

$$
\begin{aligned}
G_R(a,\ln(-q^2/\mu^2)) \quad &= \quad 1 - a\int_{\mathbb{M}} \frac{d^4k}{i\pi^2}\left\{\frac{1}{\not{k}} G_R(a,\ln(-k^2/\mu^2))\frac{1}{\not{k}}\frac{1}{(k-q)^2}\right\}_- \\
&\quad + a^2\int_{\mathbb{M}}\frac{d^4k}{i\pi^2}\int_{\mathbb{M}}\frac{d^4l}{i\pi^2}\left\{\frac{\not{l}(\not{l}+\not{k})G_R(a,\ln(-(k+l)^2/\mu^2))(\not{l}+\not{k})(\not{k}+\not{q})}{[(k+l)^2]^2 l^2 (k+q)^2 k^2 (l-q)^2}\right\}_-, \qquad (1)
\end{aligned}
$$

where $\{\}_-$ indicates subtraction at $\mu^2 = -q^2$, so that $G_R(a,0) = 1$:

$$
\left\{G_R(a,\ln(-k^2/\mu^2))\right\}_- = G_R(a,\ln(-k^2/\mu^2)) - G_R(a,\ln(-k^2/-q^2)). \qquad (2)
$$

The kinematics are such that the fermion has momentum q and the external scalar particle carries zero momentum. The equation can graphically be represented as



[1]supported by DFG Sonderforschungsbereich Transregio 9, Computergestützte Theoretische Physik.
[2]supported by CNRS and in parts by grant nsf-dms/0603781.

where the blob represents the unknown Green function $G_R(a, \ln(-q^2/\mu^2))$. This linear Dyson–Schwinger equation can be solved by a scaling solution, $L = \ln(-q^2/\mu^2)$,

$$G_R(a, L) = \exp\{-\gamma_G(a)L\}. \tag{3}$$

Indeed, this satisfies the desired normalization and leads to the equation

$$\exp\{-\gamma_G(a)L\} = 1 + (\exp\{-\gamma_G(a)L\} - 1)\left[aF_1(\gamma_G) + a^2F_2(\gamma_G)\right], \tag{4}$$

where the two Mellin transforms are the functions determined by

$$F_1 : \rho \to -\int \frac{d^4k}{i\pi^2} \frac{1}{k^2(k-q)^2} \left(\frac{k^2}{q^2}\right)^{-\rho} \tag{5}$$

and similarly

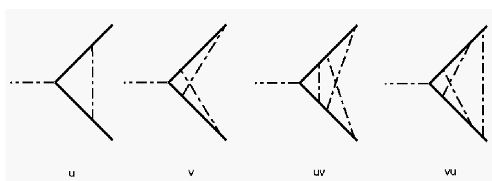
$$F_2 : \rho \to \int_{\mathbb{M}} \frac{d^4k}{i\pi^2} \int_{\mathbb{M}} \frac{d^4l}{i\pi^2} \frac{\not{l}(\not{k} + \not{q})}{(k+l)^2 l^2 (k+q)^2 k^2 (l-q)^2} \left[\frac{(l+k)^2}{q^2}\right]^{-\rho}. \tag{6}$$

Clearing the factor $[\exp\{-\gamma_G(a)L\} - 1]$ in this equation gives

$$1 = aF_1(\gamma_G) + a^2F_2(\gamma_G). \tag{7}$$

It remains to determine F_1, F_2 explicitely and solve this implicit equation for γ_G in terms of a. We do so in the next sections but first discuss the perturbative structure behind this solution.

2.2. **The algebraic structure.** We can identify any graph in this resummation with a word in two letters u, v say, for example:

(8)


We have renormalized Feynman rules ϕ_R such that

$$\phi_R(u)(L) = -L \lim_{\rho \to 0} \rho F_1(\rho), \tag{9}$$

and

$$\phi_R(v)(L) = -L \lim_{\rho \to 0} \rho F_2(\rho). \tag{10}$$

The Green function $G_R(a, L)$ is obtained as the evaluation by ϕ_R of the fixpoint of the combinatorial Dyson Schwinger equation

$$X(a) = \mathbb{I} + aB_+^u(X(a)) + a^2B_+^v(X(a)). \tag{11}$$

We have

$$X(a) = 1 + au + a^2(uu + v) + \ldots = \exp_\curlyvee [au + a^2v], \tag{12}$$

where $\curlyvee$ is the shuffle

$$H_{\text{lin}} \times H_{\text{lin}} \to H_{\text{lin}}, \tag{13}$$

$$B_+^i(w_1) \curlyvee B_+^j(w_2) = B_+^i(w_1 \curlyvee B_+^j(w_2)) + B_+^j(w_2 \curlyvee B_+^i(w_1)), \tag{14}$$

$\forall i, j \in \{u, v\}$. Note that for example $u \curlyvee u = 2uu$.

The two maps B_+^i are Hochschild one-cocycles, and $X(a)$ is group-like:

$$\Delta X(a) = X(a) \otimes X(a). \tag{15}$$

Correspondingly, decomposing $X(a) = 1 + \sum_{k\geq 1} a^k c_k$, we have

$$\Delta c_k = \sum_{j=0}^{k} c_j \otimes c_{k-j}, \tag{16}$$

with $c_0 = 1$. This is a decorated version of the Hopf algebra of undecorated ladder trees t_n with coproduct $\Delta t_n = \sum_{j=0}^{n} t_j \otimes t_{n-j}$. Feynman rules become iterated integrals as

$$\phi_R(B_+^i(w))(L) = \int \left\{\phi(w)(\ln k^2/\mu^2) d\mu_i(k)\right\}_- , \tag{17}$$

where $d\mu_i$ is the obvious integral kernel for $i \in u, v$, cf. (1). Apart from the shuffle product, we have disjoint union as a product which makes the Feynman rules into a character

$$\phi(w_1 \cdot w_2) = \phi(w_1)\phi(w_2). \tag{18}$$

These two commutative products $\curlyvee, \cdot$ allow to express the primitive elements associated with shuffles of letters u, v, see for example [6]:

Theorem 1. *The primitive elements are given by polarization of the primitive elements p_n of the undecorated ladder trees t_n. These are given by $p_n = \frac{1}{n}[S \star Y](t_n)$.*

Here, Y denotes the grading operator, defined by $Y(t_k) = kt_k$ and the star product is defined as usual by $O_1 \star O_2 = \cdot \circ (O_1 \otimes O_2) \circ \Delta$. Polarization of the undecorated primitive elements p_n means that we decorate each vertex of p_n with $u + v$.

The set $\mathcal{P}(u, v)$ of primitive elements is hence spanned by elements p_{i_u,i_v}, where the integeres i_u, i_v count the number of letters u and v in the polarization of $t_{i_u+i_v}$. For example the primitive element corresponding to the undecorated ladder tree t_2 is $p_2 = t_2 - \frac{1}{2}t_1t_1$. Polarization yields

$$\begin{aligned} &p_{2,0} = \frac{1}{2}(u \curlyvee u - u \cdot u) = uu - \frac{1}{2}u \cdot u, \quad p_{0,2} = \frac{1}{2}(v \curlyvee v - v \cdot v) = vv - \frac{1}{2}v \cdot v, \\ &p_{1,1} = u \curlyvee v - u \cdot v = uv + vu - u \cdot v. \end{aligned} \tag{19}$$

3. The Mellin transforms

The general structure of the Mellin transform can be obtained from quite general considerations. The crucial input comes from powercounting and conformal symmetry.

Theorem 2. *The Mellin transforms above are invariant under the transformation $\rho \to 1 - \rho$.*

Proof: Explicit computation. We give it here for F_1. We assume $\Re\rho > 0$ so that F_i is well defined as a function. Then, the conformal inversion $k_\mu \to k'_\mu = k_\mu/k^2$ gives explicitly

$$-\int \frac{d^4k'}{i\pi^2} \frac{1}{k'^2(k'-q)^2} \left(\frac{k'^2}{q^2}\right)^{-1+\rho} \tag{20}$$

for F_1. F_2 can be treated similarly by conformal inversion in both Minkowski spaces. □

3.1. The Mellin transform of the one-loop kernel.

This Mellin transform is readily integrated to deliver

$$F_1(\rho) = -\frac{1}{\rho(1-\rho)}, \tag{21}$$

exhibiting the expected conformal symmetry.

3.2. **The Mellin transform of the two-loop kernel.** Determining this Mellin transform is the core part of this paper. We proceed by making use of the advantage that we remain in four dimensions, and use results of [5]. We are interested in the integral

$$F_2(\rho) = (-q^2)^\rho \int_{\mathbb{M}} \frac{d^4k}{i\pi^2} \int_{\mathbb{M}} \frac{d^4l}{i\pi^2} \frac{\not{l}(\not{k}+\not{q})[-(l+k)^2]^{-\rho}}{(k+l)^2 l^2 (k+q)^2 k^2 (l-q)^2}. \tag{22}$$

Integration is over the eight dimensional cartesian product of two Minkowski spaces furnished with a quadratic form

$$a^2 = a_0^2 - a_1^2 - a_2^2 - a_3^2. \tag{23}$$

A simple tensor calculus delivers

$$F_2(\rho) = \frac{1}{2}\left\{-2G_4(1,1+\rho)G_4(1,1+\rho) + I_6(1,1,\rho,1,1,2-\rho) + I_6(1,1,1+\rho,1,1,1-\rho)\right\}, \tag{24}$$

where

$$G_D(a,b) = \frac{\Gamma(a+b-D/2)\Gamma(D/2-a)\Gamma(D/2-b)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)}, \tag{25}$$

so that $G_4(1,1+\rho) = \frac{1}{\rho(1-\rho)}$. We use the notation of [5] for I_6. In this notation, we have $I_6 = \overline{I_6}$. Setting $u \to 0$ and $v = -\rho$ or $v = 1-\rho$ we can determine the two I_6 integrals as a limit $u \to 0$ in Eq.(19)(op.cit.) as

$$I_6(1,1,1-v,1,1,1+v) = 8\sum_{n=1}^{\infty} n\zeta_{2n+1}(1-2^{-2n})v^{2n-2}, \tag{26}$$

and similarly for $v = 1-\rho$. We hence find the above DSE in the form

$$1 = -a\frac{1}{\gamma_G(1-\gamma_G)} - a^2\left\{\frac{1}{\gamma_G^2(1-\gamma_G)^2} - 4\sum_{n=1}^{\infty} n\zeta_{2n+1}(1-2^{-2n})\left[\gamma_G^{2n-2} + (1-\gamma_G)^{2n-2}\right]\right\}. \tag{27}$$

4. The solution

We can solve for γ_G in the above in two different ways, expressing the solution as an infinite product or via the logarithmic derivative of the Γ function.

4.1. **Solution as an infinite product.** We have:

$$G_R(a,L) = \exp\left\{\sum_{p\in\mathcal{P}(u,v)} a^{|p|}\phi_R(p)(L)\right\}. \tag{28}$$

Here, the sum is over all primitives $p \in \mathcal{P}(u,v)$, where $\mathcal{P}$ is the set of primitives assigned to any tree t_n decorated arbitrarily by letters in the alphabet u, v, as described above. The proof is an elementary exercise in the Taylor expansion of the two Mellin transforms. Note that $\phi_R(p)(L)$ is linear in L for primitive p, $\partial_L^2\phi_R(L) = 0$. We hence find for $\gamma_G(a)$

$$\gamma_G(a) = -\frac{\partial \ln G}{\partial L}|_{L=0} = -\sum_p a^{|p|}\phi_R(p)/L. \tag{29}$$

Convergence of the sum is covered by the implicit function theorem, which provides for γ_G through the two Mellin transforms in the DSE. We hence proceed to the second way of expressing the solution.

4.2. **Solution via the ψ-function.** We can express the DSE using the logarithmic derivative of the Γ function and we obtain

$$
\begin{aligned}
(30) \qquad 1 \quad = \quad & -a\frac{1}{\gamma_G(1-\gamma_G)} - a^2\left\{\frac{1}{\gamma_G^2(1-\gamma_G)^2}\right. \\
& +\frac{1}{\gamma_G}\left[\psi'\left(1+\gamma_G\right)-\psi'\left(1-\gamma_G\right)\right]+\frac{1}{1-\gamma_G}\left[\psi'\left(2-\gamma_G\right)-\psi'\left(\gamma_G\right)\right] \\
& \left.-\frac{1}{2\gamma_G}\left[\psi'\left(1+\frac{\gamma_G}{2}\right)-\psi'\left(1-\frac{\gamma_G}{2}\right)\right]-\frac{1}{2(1-\gamma_G)}\left[\psi'\left(\frac{3-\gamma_G}{2}\right)-\psi'\left(\frac{1+\gamma_G}{2}\right)\right]\right\}.
\end{aligned}
$$

Here

$$
(31) \qquad \psi'(x) = \frac{d^2}{dx^2}\ln\Gamma(x).
$$

Again, the two-loop solution shows explicitly the conformal symmetry $\gamma_G \to 1-\gamma_G$. Note that the apparent second order poles at $\gamma_G = 0$ and $\gamma_G = 1$ on the rhs are only first order poles upon using standard properties of the logarithmic derivative of the Γ function, as it has to be. This provides an implicit equation for γ_G, which can be solved numerically.

5. Conclusions

We determined the Mellin transform of the two-loop massless vertex in Yukawa theory. We used it to resum a linear Dyson–Schwinger equation. Following [1, 2, 3], more complete applications will be provided elsewhere. The same two-loop Mellin transform also appears in setting up the full DSE in other renormalizable theories [4].

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