

ON THE DIMENSION GROWTH OF GROUPS

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ABSTRACT. The (asymptotic) dimension growth functions of groups were introduced by Gromov in 1999. In this paper, we show connections between dimension growth and expansion properties of graphs, Ramsey theory and the Kolmogorov-Ostrand dimension of groups and prove that all solvable subgroups of the R.Thompson group F have polynomial dimension growth. We introduce controlled dimension growth function and prove that the exponentially controlled dimension growth is exponential for the Thompson group F and some solvable of class 3 groups. The paper contains many open questions.

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1. INTRODUCTION

Gromov introduced the notion of asymptotic dimension [Gromov1] to study finitely generated groups. It turns out that groups with finite asymptotic dimension satisfy many famous conjectures like the Novikov Higher Signature Conjecture [Yu1], [Bartels], [Carlsson-Goldfarb], [Dr1], [DFW], [Bartels-Rosenthal]. Many of the popular types of groups have finite asymptotic dimension. Such are hyperbolic groups [Gromov1] virtually polycyclic groups (hence nilpotent groups), and solvable groups with finite rational Hirsch length [BD], [DS], Coxeter groups [DJ], arithmetic subgroups of algebraic groups over $\mathbb{Q}$ [Ji], any finitely generated linear group over a field of positive characteristic [GTy], relatively hyperbolic groups with

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parabolic subgroup of finite asymptotic dimension [Osin], mapping class groups [BBF], group acting “nicely” on finite dimensional CAT(0) cubical complexes [Wright], etc. Examples of asymptotically infinite dimensional groups include wreath product $\mathbb{Z} \wr \mathbb{Z}$ (and every other wreath product $A \wr B$ where A is not torsion and B is infinite), Thompson group F , Grigorchuk’s groups, Gromov’s group containing an expander. The infinite dimensionality of the wreath products and F easily follows from the fact that for every n , these groups contain $\mathbb{Z}^n$ as a subgroup. Grigorchuk group does not contain $\mathbb{Z}$ since it is a torsion group. It is infinite dimensional because for every n it coarsely contains $\mathbb{R}_+^n$ [Smith]. This argument does not apply to Gromov’s groups containing expanders, since they can have finite cohomological dimension [Gromov3]; the infinite dimensionality of them follows from the fact that these groups do not satisfy Yu’s property A defined below (some of them do not even coarsely embed into Hilbert spaces) while all groups of finite asymptotic dimension satisfy property A [Higson-Roe], see also Corollary 2.10 below.

The dimension theoretic approach still could be useful in the case of asymptotically infinite dimensional groups. Thus, in [Gromov2] the notion of asymptotic dimension growth was introduced. It was shown in [Dr2] that the groups with polynomial asymptotic dimension growth have property A and some examples of groups with infinite asymptotic dimension and polynomial dimension growth were constructed. In particular, the Novikov conjecture is true for them. Recently, answering our question, Ozawa extended this result to all groups with subexponential dimension growth [Ozawa].

Property A (already mentioned) was introduced by Guoliang Yu [Yu2]. It can be viewed as non-equivariant version of amenability. Recall that a group (or a metric space) X satisfies property A if for every ε and R there exists a function ξ from X to $\ell_1(X)$ with $\xi(x) \geq 0$, $\|\xi(x)\| = 1$, $x \in \text{supp}\xi(x) \subseteq B(x, R)$ for some $S = S(\varepsilon, R)$ for every $x \in X$, so that for every two points $x, y \in X$ at distance at most R , $\|\xi(x) - \xi(y)\| \leq \varepsilon$. A group X is amenable if one can choose the functions $\xi(x) \in \ell_1(X)$ with the above property to be shifts of the properly rescaled indicator function of a bounded set of elements containing the identity (a Følner set) by elements of the group X .

Although most known groups do have property A, for some groups like the R. Thompson group F property A is still not known and is considered almost as hard as amenability. Property A implies coarse embeddability of a group into the Hilbert space. In fact, for finitely presented groups, it is still an open question whether it is equivalent to coarse embeddability into a Hilbert space. In view of D. Farley’s result [Farley], the R. Thompson group F is coarsely (in fact even equivariantly) embeddable into the Hilbert space. The compression number of such embeddings was computed in [AGS], and unfortunately the answer lies exactly on the border ($=1/2$) where it does not allow to derive property A [GK]. Note that low compression number of a group does not imply high dimension growth. For example, groups constructed in [ADS] have finite asymptotic dimension and compression number 0.

Thus the question about dimension growth of the R. Thompson group is related to the famous amenability problem of F .

Definition 1.1. Let λ be a positive number. We say that two points x, y in a metric space X are λ -connected if there exists a sequence of points $x = z_0, z_1, \dots, z_n = y$ such that the distance between any two consecutive points is at most λ (that sequence will be called a λ -path). We call a set of points λ -cluster if every two points in that set are λ -connected.

Definition 1.2. Let λ and D be positive numbers, X be a metric space. We say that $(\lambda, D)\text{-dim}(X) \leq n$ if X can be colored in at most $n + 1$ colors so that every monochromatic λ -cluster has diameter at most D .

A coloring satisfying the above condition will be called a (λ, D) -coloring of X in $n + 1$ colors. The maximal monochromatic λ -clusters of the coloring will be simply called its λ -clusters.

Remark 1.3. Note that we do not insist that every point is colored in only one color. Nevertheless if a point is colored in several colors, we can pick any of them, making the point colored in only one color. As a result, the sizes of clusters can get only smaller. So we can always assume, when dealing with the dimension growth that we color the metric space X in such a way that each point is colored in one color. In Section 6, we shall consider a version of this function for which it will be essential that points are colored in several colors.

As usual, we say that two non-decreasing functions $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ have the same *growth* if there are positive constants a, c such that $f(at) \geq g(t) - c$ and $g(at) \geq f(t) - c$ for all t . Clearly, this is an equivalence relation on the set of all monotone functions. The equivalence class of a function f is called the *growth* of that function.

We denote by $d_X(\lambda) = \inf\{(\lambda, D)\text{-dim}(X) \mid D \in \mathbb{R}_+\}$.

Definition 1.4. The growth of the function $d_X(\lambda)$ is called *the dimension growth* of X .

For a function $D = D(\lambda)$ the growth of the function

$$d_{D,X}(\lambda) = (\lambda, D(\lambda))\text{-dim}(X)$$

is called the *D-controlled dimension growth* of X .

The upper limit $\limsup_{\lambda \rightarrow \infty} d_X(\lambda)$ is called *the asymptotic dimension* of X and is denoted $asdim(X)$. Note that $asdim(X)$ takes values in $\mathbb{N}_+ \cup \{\infty\}$.

The upper limit $\limsup_{\lambda \rightarrow \infty} (\lambda, D(\lambda))\text{-dim}(X)$ is called the *D(λ)-controlled asymptotic dimension* of X . In particular, when D is linear it is called the linearly controlled dimension or the *Assouad-Nagata dimension* [Assouad]. Note that the study of asymptotic dimension with control functions was proposed by Gromov in [Gromov1]. Some research in this direction mostly motivated by the Assouad-Nagata dimension was done in [BDLM].

Note that the definition of dimension function in [Dr2] is similar but different: the *asymptotic dimension growth* $\text{ad}_X(\lambda)$ from [Dr2] is the minimal dimension of the nerve of a uniformly bounded cover of X with the Lebesgue number $\geq \lambda$. By taking a $\lambda/2$ -enlargement of a colored cover with λ -disjoint colors (i.e. the diameters of all λ -clusters are uniformly bounded) one can construct a cover of the same multiplicity with the Lebesgue number $\geq \lambda/2$. This yields the inequality $\text{ad}_X(\lambda/2) \leq d_X(\lambda)$. Therefore, the growth of ad_X does not exceed the growth of d_X .

Question 1.5. Is the opposite inequality true, i.e. is the dimension growth equal to the asymptotic dimension growth (for groups or general metric spaces)?

Certainly the answer is “yes” when ad_X is a constant. In that case both functions give alternative definitions of the asymptotic dimension. Thus, the functions d_X and ad_X generalize two definitions of asymptotic dimension to metric spaces with infinite asymptotic dimension.

Remark 1.6. Note that the function $(\lambda, D)\text{-dim}(X)$ substantially depends on the control function D . For example, the asymptotic dimension of $\mathbb{Z}$ with control $\lambda + 1$ is equal to 1, but $(\lambda, \sqrt{\lambda})\text{-dim}(\mathbb{Z}) \approx \sqrt{\lambda}$ and $(\lambda, 0)\text{-dim}(\mathbb{Z}) = \lambda$.

Thus, it makes little sense to consider control $D \leq \lambda$. Note also that if the control function is constant the dimension growth is equivalent to the volume growth (see Lemma 2.3 below).

Question 1.7. Is exponential control sufficient for detection the asymptotic dimension of a finitely generated group? In more formal way, is it true that for every finitely generated group G

$$\limsup_{\lambda \rightarrow \infty} (\lambda, D)\text{-dim}(G) = \text{asdim}(G)$$

for some exponential function D ?

Another dimension function was defined in [CFY]: the asymptotic dimension growth $f(\lambda)$ of a metric space X is the infimum over all n for which there is a uniform bounded cover $\mathcal{U}$ such that, for every $x \in X$, the ball $B_r(x)$ intersects at most $n+1$ members of $\mathcal{U}$. It is easy to see that $f(\lambda/2) \leq \text{ad}_X(\lambda) \leq f(2\lambda)$ and hence f and ad_X have the same growth.

The dimension growth, the controlled dimension growth, and asymptotic dimension growth (d_X , $d_{D,X}$ for all D , and ad_X) are quasi-isometry invariants (if we identify equivalent control functions D) and therefore they are invariants of finitely generated groups.

In this paper we study the dimension growth of wreath products and the R.Thompson group F . The dimension growth of F turns out to be exponential for an exponential control. Note that this does not imply the exponential asymptotic dimension growth of F although the answer to the following question (related to Question 1.7) is unknown.

Question 1.8. Are there finitely generated groups (metric spaces) X with exponentially controlled exponential dimension growth and subexponential (uncontrolled) dimension growth d_X ? In particular, are there finitely generated groups with finite asymptotic dimension and super-linearly (or even exponentially) controlled exponential dimension growth.

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2. PRELIMINARIES

2.1. Volume growth and dimension growth. We recall that the *chromatic number* of a graph is the minimal number of colors (if exists) such that the vertices of the graph can be colored in a way that adjacent vertices have different colors.

Lemma 2.1. *Let K be a possibly infinite graph of valency $\leq c$. Then its chromatic number $\leq c+1$.*

Proof. Using the Zorn lemma, take a maximal $(c+1)$ -colorable induced subgraph K' of K . Suppose that $K' \neq K$. Any vertex v of K that is not in K' has at most c colored neighbors and hence it can be colored and added to K' . That contradicts the maximality of K' . $\square$

Remark 2.2. We note that for $D \geq D'$, $d_{D,X} \leq d_{D',X}$, also $d_X \leq d_{D,X}$ for all D .

A version of the next proposition for the function ad is proved in [Dr2].

Lemma 2.3. *The dimension growth of a finitely generated group G with any control function does not exceed its volume growth.*

Proof. Let f be the volume growth function of G (relative to some finite generating set). We consider a graph $R_\lambda(G)$ whose vertices are all elements of G where every two vertices at distance $\leq \lambda$ are joined by an edge (this is of course the 1-skeleton of the Rips complex of G). Then the valency of every vertex of $R_\lambda(G)$ is $\leq f(\lambda)$. By Lemma 2.1 the graph has chromatic number $\leq f(\lambda) + 1$. Thus, a coloring of the graph in $f(\lambda) + 1$ colors is a coloring of the Cayley graph of G with monochromatic λ -clusters of diameter 0. Note that $f(\lambda) + 1$ is equivalent to $f(\lambda)$. $\square$

The following corollary was mentioned in [Gromov2] as obvious.

Corollary 2.4. *The dimension growth of any finitely generated group with any control function is at most exponential.*

2.2. Dimension growth and quasi-isometries. We recall that a map of metric spaces $\phi : X \rightarrow Y$ is called a *coarse embedding* if there are strictly monotone tending to infinity functions $\rho_1, \rho_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a number $r > 0$ such that

$$\rho_1(d_X(x, x')) \leq d_Y(\phi(x), \phi(x')) \leq \rho_2(d_X(x, x'))$$

for all $x, x' \in X$ with $d(x, x') \geq r$. A typical example of a coarse embedding is an inclusion of a finitely generated subgroup in a finitely generated group both supplied with the word metrics. In that case $r = 0$, ρ_2 is linear, and ρ_1 could with any greater than linear growth of ρ_1^{-1} .

Lemma 2.5. *Let $\phi : G' \rightarrow G$ be a 1-Lipschitz isomorphism of finitely generated groups. Then $\lambda\text{-dim}(G') \leq \lambda\text{-dim}(G)$.*

Proof. It is enough to note that the ϕ -preimage of every uniformly bounded λ -disjoint family in G is a uniformly bounded and λ -disjoint family in G' . $\square$

Lemma 2.6. *Let $\phi : X \rightarrow Y$ be a coarse embedding with functions ρ_1, ρ_2 : Then*

$$(\lambda, D)\text{-dim}(Y) \geq (\rho_2^{-1}(\lambda), \rho_1^{-1}(D))\text{-dim}(X).$$

Proof. We assume that $\lambda, D > r$. Suppose that $(\lambda, D)\text{-dim}(Y) \leq m$. For any (λ, D) -coloring of Y in m colors $c : Y \rightarrow \{1, \dots, m\}$ the composition $c \circ \phi$ defines a coloring of X in m colors. Note that for any two clusters C and C' in Y of the same color

$$\lambda < d_Y(C, C') = \inf_{y \in C, y' \in C'} d_Y(y, y') \leq \rho_2(d_X(\phi^{-1}(C), \phi^{-1}(C')))$$

and

$$\rho_1(\text{diam}_X(\phi^{-1}(C))) \leq \text{diam}_Y(C) \leq D.$$

Therefore,

$$\rho_2^{-1}(\lambda) < d_X(\phi^{-1}(C), \phi^{-1}(C')) \text{ and } \text{diam}_X(\phi^{-1}(C)) \leq \rho_1^{-1}(D).$$

Thus, $c \circ \phi$ is a $(\rho_2^{-1}(\lambda), \rho_1^{-1}(D))$ -coloring of X in m colors. So,

$$(\rho_2^{-1}(\lambda), \rho_1^{-1}(D))\text{-dim}(X) \leq m.$$

This completes the proof. $\square$

Applying Lemma 2.6 to quasi-isometries gives

Lemma 2.7. *Let $\phi : X \rightarrow Y$ be a quasi-isometric embedding:*

$$\frac{1}{c_1} d_X(x, x') - r_1 \leq d_Y(\phi(x), \phi(x')) \leq c d_X(x, x') + r$$

for all $x, x' \in X$ where $c, c_1 \geq 1, r, r_1 \geq 0$. Then for $\lambda > r$,

$$(\lambda, D)\text{-dim}(Y) \geq \left(\frac{\lambda-r}{c}, c_1(D+r_1)\right)\text{-dim}(X).$$

Thus if for some function D , the dimension growth of Y with control $D(\lambda)$ is $d(\lambda)$, then the dimension growth of X with control $c_1(D(c\lambda+r)+r_1)$ is at most $d(c\lambda+r)$.

For a metric d on a discrete space X and $r > 0$ we denote by $d+r$ the new metric $\bar{d}$ defined as $\bar{d}(x, y) = d(x, y) + r$ provided $x \neq y$ and $\bar{d}(x, x) = 0$. We call it *the metric d shifted by a constant r* .

Lemma 2.7 immediately implies

Lemma 2.8. *For every r, λ, D such that $0 \leq r < \lambda, r < D$,*

$$(\lambda, D)\text{-dim}(X, d+r) = (\lambda-r, D-r)\text{-dim}(X, d).$$

2.3. Dimension growth and expansion in graphs. Let G be a graph such that there exists a number $\varepsilon > 0$ such that for every r there exists a finite subgraph $G_r \subseteq G$ with the following property

$(P_r(\varepsilon))$ For every subset A of vertices of G_r of diameter (in G) $\leq r$, $|\partial_{G_r}(A)| \geq \varepsilon|A|$ where $\partial_{G_r} = \{v \in G_r \mid \text{dist}(v, A) = 1\}$ denotes the boundary in G_r .

Theorem 2.9. *Suppose that there exists $\varepsilon > 0$ such that a graph $G = (V_G, E_G)$ (where V_G is the set of vertices and E_G is the set of edges) satisfies $(P_r(\varepsilon))$ for all r . Then the dimension growth of G is exponential.*

Proof. Consider any even number $\lambda > 0$. Let $V_G = \cup_{i=1}^{k+1} U_i$ be a coloring of the vertices of G in $k+1$ colors such that all λ -clusters U_i^j have diameters at most d . Take $r > d + \lambda$ and consider the graph $G_r = (V_r, E_r)$. Let $W_i^j = U_i^j \cap G_r$. We have $\cup W_i^j$ equal to the set V_r of all vertices of G_r . Note that by the choice of r and by property $(P_r(\varepsilon))$, the $\lambda/2$ -neighborhood $N_{\lambda/2}(W_i^j)$ of W_i^j in G_r has at least $(1+\varepsilon)^{\lambda/2}$ elements. Since different λ -clusters of the same color are λ -disjoint, we have that the sum of cardinalities $|N_{\lambda/2}(W_i^j)|$ is at most $|V_r|(k+1)$. On the other hand, that sum is at least $(1+\varepsilon)^{\lambda/2}$ times the sum of cardinalities $|W_i^j|$, i.e. at least $|V_r|(1+\varepsilon)^{\lambda/2}$. Hence $k+1 \geq (1+\varepsilon)^{\lambda/2}$ which implies the statement of the theorem. $\square$

The following two corollaries of Theorem 2.9 follow from Ozawa's result [Ozawa] that metric spaces of subexponential dimension growth satisfy property A, and the result of Willett [Willett] that certain graphs do not satisfy property A. Theorem 2.9 gives direct proofs of these corollaries. See also the recent paper [BNSWW] where it is proved that in the case of graphs of bounded degree the condition of Theorem 2.9 implies that G does not satisfy property A.

Corollary 2.10. *Let G be a graph containing an expander. Then the dimension growth of G is exponential.*

Proof. Indeed, by definition, every sufficiently large (finite) graph in an expander sequence satisfies property $(P_r(\varepsilon))$ for some ε (which is related to the spectral gap of the expander) and all r . $\square$

Corollary 2.11. *Let G be a graph such that for every $r \geq 0$ there exists a subgraph G_r of G with degree of each vertex at least 3 and with girth $> r$. Then the dimension growth of G is exponential. In particular, the dimension growth of the Gromov “random monster” [Gromov3] constructed using any sequence of finite regular graphs with increasing girth (not necessarily an expander sequence of finite graphs) is exponential.*

Proof. Indeed, if a graph G_r has girth $\geq r$ and minimal degree of a vertex $k \geq 3$, then every subgraph A of G_r of diameter $\leq r$ is a forest and there are at least $|A|/2$ vertices in A that are connected to at least $(k-2)$ vertices in G_r outside A , and each of these vertices is connected to at most k vertices in A . Thus one can take $\varepsilon = \frac{k-2}{2k}$. $\square$

Remark 2.12. By [Sa] there exists a 5-dimensional aspherical Riemannian manifold whose fundamental group contains expander and hence has exponential dimension growth. That answers a question from [Gromov2, Page 158].

In view of Theorem 2.9 and [BNSWW], the following question becomes very interesting.

Question 2.13. Is it true that a finitely generated group (or a graph with bounded degrees of vertices) satisfies property A if and only if it has subexponential dimension growth?

3. DIMENSION GROWTH OF DIRECT SUMS OF $\mathbb{Z}$

3.1. Coloring unit cubes. The following two examples are well known.

Example 3.1. For every $\lambda \geq 1$, color a number $z \in \mathbb{Z}$ in black if $\lfloor \frac{z}{2\lambda} \rfloor$ is even, and in white otherwise. This is a $(\lambda, 2\lambda)$ -coloring of $\mathbb{Z}$. Hence

$$(\lambda, 2\lambda)\text{-dim}(\mathbb{Z}) \leq 1.$$

Example 3.2. Using the checker coloring of vertices of $\mathbb{Z}^n$ (the color of the point $(x_1, \dots, x_n)$ of $\mathbb{Z}^n$ is the sum $\sum x_i$ modulo 2) one gets

$$(1, 0)\text{-dim}(\mathbb{Z}^n, \ell_1) = 1.$$

It will be clear later that expansion of “small” subsets in unit cubes plays important role in computing dimension growth of groups. See also Gromov’s papers [Gromov4] and [Gromov5] where a connection between this question and Shannon inequalities is considered.

Lemma 3.3. *Let G be the binary cube $\{0, 1\}^n$ with the ℓ_1 -metric. Then for every $r > 0$, such that $\varepsilon = \frac{n}{r+1} - 2 > 0$, G satisfies property $(P_r(\varepsilon))$.*

Proof. Let $A \subseteq G$. Induction on $d = \text{diam}(A)$.

If $d = 0$, then A is a point and the inequality $|\partial A| = n > \frac{n}{r+1} - 2 = \left(\frac{n}{r+1} - 2\right) |A|$ is true.

Assume that $(P_r(\varepsilon))$ holds for $d-1$ and let $\text{diam}(A) = d \leq r$. We may assume that $\vec{0} \in A$ where $\vec{0} = (0, 0, \dots, 0)$. Let $F = A \cap S(\vec{0}, d)$, where $S(x, t)$ denotes the sphere in $G = \{0, 1\}^n$ of radius t centered in x . Since the diameter of A is d , we can assume that F is not empty. By the induction assumption,

$$|\partial(A \setminus F)| > \varepsilon |A \setminus F|.$$

Let $B = \partial A \cap S(\vec{0}, d+1)$. Since the diameter of A is d , we have $\partial(A \setminus F) \subset \partial A \cup F$, we obtain $\partial(A \setminus F) \cup B \subset \partial A \cup F$. Note that $B \cap \partial(A \setminus F) = \emptyset$. Also $\partial A \cap F = \emptyset$. Thus, the above unions are disjoint

$$\partial(A \setminus F) \sqcup B \subset \partial A \sqcup F$$

where $\sqcup$ denotes disjoint union. Hence

$$|\partial(A \setminus F)| + |B| \leq |\partial A| + |F|.$$

Note that every point $y \in B$ is obtained from some $x \in F$ by replacing one 0 coordinate by 1. Since $\|x\| = d$, there are exactly $n - d$ coordinates of x that are equal to 0. Thus each $x \in F$ corresponds to $n - d$ points in B . Each $y \in G$ can be obtained from a point from F at most in $d + 1$ ways. Thus,

$$|G| \geq \frac{(n - d)|F|}{d + 1}.$$

Therefore

$$\begin{aligned} |\partial A| &\geq |\partial(A \setminus F)| + |B| - |F| \geq \left(\frac{n}{r+1} - 2\right) |A \setminus F| + \frac{(n - d)|F|}{d + 1} - |F| = \\ &\left(\frac{n}{r+1} - 2\right) |A \setminus F| + \left(\frac{n}{d+1} - \frac{d}{d+1} - 1\right) |F| \geq \left(\frac{n}{r+1} - 2\right) |A \setminus F| + \left(\frac{n}{d+1} - 2\right) |F| \\ &\text{since } d \leq r, \\ &\geq \left(\frac{n}{r+1} - 2\right) |A \setminus F| + \left(\frac{n}{r+1} - 2\right) |F| = \left(\frac{n}{r+1} - 2\right) |A| = \varepsilon |A|. \end{aligned}$$

□

For every subset A of vertices of a graph X and every $r \geq 1$ we use the notation

$$\partial_r A = \{x \in X \mid 0 < d(x, A) \leq r\}$$

for the r -boundary of a set A . Then $\partial A = \partial_1 A$.

Corollary 3.4. *For any subset $A \subset \{0, 1\}^n$ of diameter $\leq r \leq n/4$ and $n > 16$*

$$|\partial_2 A| > \frac{n^2}{4(r+2)^2} |A|.$$

Proof. Note that $\partial_2 A = \partial A \sqcup \partial(A \cup \partial A)$. Therefore,

$$\begin{aligned} |\partial_2 A| &= |\partial A| + |\partial(A \cup \partial A)| \geq \left(\frac{n}{r+1} - 2\right) |A| + \left(\frac{n}{r+2} - 2\right) (|A| + |\partial A|) \\ &\geq \left(\left(\frac{n}{r+2} - 2\right) \left(\frac{n}{r+1} - 1\right) + \left(\frac{n}{r+2} - 2\right)\right) |A| \geq \\ &\left(\left(\frac{n}{r+2} - 2\right) \left(\frac{n}{r+2} - 1\right) + \left(\frac{n}{r+2} - 2\right)\right) |A| = \left(\frac{n}{r+2} - 2\right) \frac{n}{r+2} |A| \geq \frac{n^2}{4(r+2)^2} |A|. \end{aligned}$$

Here we used the inequality $\frac{n}{(r+2)} \geq \frac{8}{3}$ for $r < \frac{n}{4}$ and $n \geq 16$. □

We recall that a λ -cluster, $\lambda \in \mathbb{N}$, of $A \subset \{0, 1\}^n$ is a maximal subset $C \subset A$ such that for every pair of points $x, x' \in C$ there is a sequence $x_i \in C$ with $x_0 = x$, $x_k = x'$, and $d(x_i, x_{i+1}) \leq \lambda$. Thus, every two λ -clusters are at distance $\geq \lambda + 1$. The proof of the following lemma is similar to the proof of Theorem 2.9.

Lemma 3.5. *The binary n -cube $\{0, 1\}^n$, $n > 64$, cannot be colored by n colors such that each 4-cluster of every color has diameter less than $\leq \sqrt{n}/4$.*

Proof. Assume that there is such a coloring:

$$\{0, 1\}^n = A_1 \cup \dots \cup A_n$$

with $A_i = \cup_j A_i^j$, $\text{diam}(A_i^j) \leq r < \sqrt{n}/4$, each A_i^j is a 4-cluster of color i , and $d(A_i^j, A_i^k) \geq 5$ for all i, j , and $k \neq j$. By Corollary 3.4,

$$|\partial_2 A_i^j| > \frac{n^2}{4(r+2)^2} |A_i^j|$$

for all i and j where $r < \sqrt{n}/4$. Since $d(A_i^j, A_i^k) \geq 5$, we have $\partial_2 A_i = \sqcup_j \partial_2 A_i^j$ (disjoint union). Therefore,

$$|\partial_2 A_i| > \frac{n^2}{4(r+2)^2} |A_i|.$$

Since $\sum_{i=1}^n |A_i| = 2^n$ (the number of vertices in the binary cube) we obtain,

$$\sum_{i=1}^n |\partial_2 A_i| > \frac{n^2}{4(r+2)^2} 2^n.$$

Since $2^n > |A_i|$, we have $n2^n > \frac{n^2}{4(r+2)^2} 2^n$ or equivalently, $r+2 > \sqrt{n}/2$. Then $\sqrt{n}/4 > \sqrt{n}/2 - 2$ which is equivalent to $8 > \sqrt{n}$. This contradicts the assumption that $n > 64$. $\square$

Corollary 3.6. $(4, \sqrt{n}/4)\text{-dim}(\{0, 1\}^n, \ell_1) \geq n$ for $n > 64$.

Corollary 3.7. If a metric space X contains isometric copies of binary cubes $\{0, 1\}^n$ for all n , then $4\text{-dim}(X) = \infty$. In particular, $4\text{-dim}(\bigoplus^\infty \mathbb{Z}) = \infty$.

Proof. Assume that $4\text{-dim}(X) = k < \infty$. This means that for some r , $(4, r)\text{-dim}(X) = k$. By Corollary 3.6, $(4, r)\text{-dim}(\{0, 1\}^l) = l$ for all $l \geq \max\{4r^2, 64\}$, a contradiction. $\square$

3.2. Dimension growth and the Ramsey theory. Answering our question Dmitri Panov and Justin Moore [Panov-Moore] gave two proofs that $2\text{-dim}(\bigoplus^\infty \mathbb{Z}) = \infty$. Here we include a proof by Justin Moore. It shows a connection between dimension growth and the Ramsey theory. The proof can be easily adapted to any metric space that contains isometric copies of arbitrary large binary cubes (as in Corollary 3.7).

Theorem 3.8. $2\text{-dim}(\bigoplus^\infty \mathbb{Z}) = \infty$.

Proof. Every finite subset M of $\mathbb{N}$ corresponds to a vector $v(M)$ from $\mathbb{Z}^\infty$ with coordinates 0, 1 in the natural way ($v(M)$ is the indicator function of M). Choose any $k \geq 1$. Let $P_k(\mathbb{N})$ denote the set of all k -element subsets of $\mathbb{N}$. Every finite coloring of $\mathbb{Z}^\infty$ induces a finite coloring of $P_k(\mathbb{N})$. By the classic result of Ramsey [GRS] there exists a subset $M \subseteq \mathbb{N}$ of size $2k$ such that all k -element subsets of M have the same color. Therefore we can find subsets $T_1, T_2, \dots, T_k$ of size k from M such that the symmetric distance between T_i and T_{i+1} is 2, $i = 1, \dots, k-1$, and T_1, T_k are disjoint. Then the vectors $v(T_1), \dots, v(T_k)$ belong to the same 2-cluster of the coloring and the diameter of that cluster is $\geq 2k$. Thus for every coloring of $\mathbb{Z}^\infty$ in finite number of colors and every k there exists a 2-cluster of diameter $\geq k$, hence 2-clusters must have arbitrary large diameters. This immediately implies the statement of the theorem. $\square$

3.3. Dimension growth and the game of Hex. The following questions seem to be interesting and non-trivial (see Remark 5.7 below).

Question 3.9. For every $k \geq 1$ let $f(k) = 2\text{-dim}(\mathbb{Z}^k)$. What is the rate of growth of f ? Is this function bounded? Is $f(k) = k$ for every $k \geq 1$? Is $f(k) \geq k^\alpha$ for some $\alpha > 0$?

Note that this question is similar in spirit to the famous game of Hex [Gale]. Recall that the n -dimensional Hex board of size k consists of all vertices $z = (x_1, \dots, x_n) \in \mathbb{Z}^n$ such that $1 \leq z_i \leq k, i = 1, \dots, n$ that is all vertices of an n -dimensional cube I_k^n of size k . A pair of vertices $(z_1, \dots, z_n), (z'_1, \dots, z'_n)$ is called *adjacent* if $\max_i(|z_i - z'_i|, 1 \leq i \leq n) = 1$ and all differences $z_i - z'_i$ are of the same sign. For every $i = 1, \dots, n$ let $H_i^- = \{(z_1, \dots, z_n) \mid z_i = 1\}$, $H_i^+ = \{(z_1, \dots, z_n) \mid z_i = k\}$. The following theorem can be found, for example, in [Gale].

Theorem 3.10. *For every k, n and every coloring of I_k^n in n colors there exists a monochromatic path of color i (for some $i = 1, \dots, n$) connecting H_i^- and H_i^+ .*

In order to answer Question 3.9 one needs to consider the following modified game Hex_1 with the same board but calling two vertices adjacent if the ℓ_1 -distance between them is 1 (in the standard Hex game the distance is ℓ_∞). It is easy to see that the function $f(k)$ from Question 3.9 would be equal to $k + 1$ if we had a statement similar to Theorem 3.10 for the game Hex_1 .

It is also known [Gale] that Theorem 3.10 is equivalent to the Brouwer fixed point theorem (in the sense that both theorems easily follow from each other). It would be interesting to find a fixed point-type statement which implies an answer to Question 3.9.

4. WREATH PRODUCTS

Let B and A be finitely generated groups with finite generating sets S and T and word metrics $|\cdot|_B$ and $|\cdot|_A$ respectively. Then the (reduced) wreath product $B \wr A$ is the semidirect product of $B^{(A)}$ (the group of all functions $C_0(A, B) = \{A \rightarrow B\}$ with finite support) and A for the natural action of A on $B^{(A)}$. We define the ℓ_1 -metric on $B^{(A)}$ as follows:

$$d_{\ell_1}(f, g) = \sum_{a \in A} d_B(f(a), g(a)).$$

For every $a \in A$ we denote by B_a the group of functions $A \rightarrow B$ with the support $\{a\}$. We shall always identify B with the group B_e . Note that $B_a = a^{-1}Ba$. If the operation in B is written additively, we shall write $\bigoplus_A B$ instead of $B^{(A)}$.

If S is a generating set for B , and T is a generating set for A , then $S \subset B$ together with $T \subset A$ generate $B \wr A$. An explicit formula for the word metric $d_{B \wr A}$ on $B \wr A$ was found by Parry.

Theorem 4.1 ([Parry]). *Let $g = ba \in B \wr A$, where $a \in A, b \in B$. Let $b = b_1^{a_1} \dots b_n^{a_n}$ for some $b_i \in B$ and $a_i \in A$ where B is identified with the subgroup of $B^{(A)}$ consisting of all functions $A \rightarrow B$ with support $\{1\}$. Let p be the shortest path in the Cayley graph of A that starts at 1, visits all vertices a_i and ends at a . Then the word length of $|g|_{B \wr A}$ of g in $B \wr A$ is the length of p plus $\sum |b_i|_B$.*

The following statement immediately follows from Theorem 4.1.

Corollary 4.2. (1) *For every $a \in A$ the metric on $B_a \cong B$ induced from $B \wr A$ is the metric on B shifted by $2|a|$: $d_{B \wr A}(x^a, y^a) = d_B(x, y) + 2|a|$ for $x \neq y \in B$.*

(2) *$d_{B \wr A}(f, g) \geq d_{\ell_1}(f, g)$ for all $f, g \in B^{(A)}$. Where d_{ℓ_1} is the ℓ_1 -metric on $B^{(A)}$.*

We define B_k to be the k th iterated wreath product of $\mathbb{Z}$. Formally, $B_0 = \mathbb{Z} = \langle b_0 \rangle$ and if $B_k = \langle b_0, \dots, b_k \rangle$ is already constructed, then

$$B_{k+1} = B_k \wr \mathbb{Z}$$

where the “top” $\mathbb{Z}$ is generated by b_{k+1} .

By induction we define a canonical subgroup $D_k \cong \bigoplus \mathbb{Z} \subset B_k$: $D_0 = B_0 = \mathbb{Z}$ and $D_{k+1} = \bigoplus_{i \in \mathbb{Z}} D_k$. Note that D_k is a sum of copies $\mathbb{Z}_{\vec{i}}$ of $\mathbb{Z}$ indexed by vectors $\vec{i} \in \mathbb{Z}^k$ with $\|\vec{i}\|_1 = |i_1| + \dots + |i_k|$. Let d_k be the metric on D_k induced from B_k .

Lemma 4.3. *The metric d_k restricted to the summand $\mathbb{Z} \subset D_k$ indexed by $\vec{i}$ is the standard metric shifted by $\|\vec{i}\|$:*

$$|x|_{d_k} = |x| + \|\vec{i}\|_1$$

for $x \in \mathbb{Z}$, $x \neq 0$.

Proof. Induction on k . Let $\mathbb{Z}$ be indexed by $\vec{i} \in \mathbb{Z}^{k+1}$. By Corollary 4.2(1) $|x|_{d_{k+1}} = |x|_{d_k} + |i_{k+1}|$ for $x \in \mathbb{Z}_{\vec{i}} \subset (B_k)_{i_{k+1}} \subset B_k \wr \mathbb{Z}$. By induction assumption, $|x|_{d_k} = |x| + |i_1| + \dots + |i_k|$. Then $|x|_{d_{k+1}} = |x| + \|\vec{i}\|$. $\square$

Lemma 4.4. *For any $r > 0$ the identity map*

$$id : \left(\bigoplus_{\|\vec{i}\|_1 \leq r} \mathbb{Z}, \ell_1 \right) \rightarrow \left(\bigoplus_{\|\vec{i}\|_1 \leq r} \mathbb{Z}, d_k \right) \subset D_k$$

is $(1, r)$ -bi-Lipschitz:

$$\|x - y\|_{\ell_1} \leq d_k(x, y) \leq r \|x - y\|_{\ell_1}.$$

Proof. In view of Lemma 4.3 the d_k -norm of every vector from the standard basis of $\bigoplus_{\|\vec{i}\| \leq r} \mathbb{Z}$ does not exceed r . This implies that the above identity map is r -Lipschitz.

The inequality $\|x\|_{\ell_1} \leq \|x\|_{d_k}$ will be proven by induction on k . Let

$$x \in \bigoplus_{\vec{i} \in \mathbb{Z}^{k+1}, \|\vec{i}\| \leq r} \mathbb{Z} \subset D_{k+1} = \bigoplus_{j \in \mathbb{Z}} D_k.$$

Thus, $x = (x_s)$ with $x_s \in (D_k)_s$. By Corollary 4.2(2), $\|x\|_{d_{k+1}} \geq \sum_{s \in \mathbb{Z}} \|x_s\|_{d_k}$. By induction assumption, $\|x_s\|_{d_k} \geq \|x_s\|_{\ell_1}$. Therefore,

$$\|x\|_{d_{k+1}} \geq \sum_{s \in \mathbb{Z}} \|x_s\|_{\ell_1} = \|x\|_{\ell_1}.$$

$\square$

Let $P \subset \mathbb{R}^k$ be a polytope with integral vertices. The Ehrhart polynomial $L(P, t)$ of P is defined as

$$L(P, t) = |tP \cap \mathbb{Z}^k|$$

where tP is dilation of P and $|\cdot|$ denotes the cardinality. Thus $L(P, t)$ is the number of integer points in the polytope P dilated by the factor t . It is known that $L(P, t)$ is a polynomial of degree k with positive coefficients [Beck-Robbins].

The *regular cross-polytope* in $\mathbb{R}^k$ is the polytope spanned by the vertices $\{\pm e_i \mid i = 1, \dots, k\}$ where $\{e_i\}$ is the orthonormal basis of $\mathbb{R}^k$. The Ehrhart polynomial for the regular cross-polytope $P_k \subset \mathbb{R}^k$ is known [Beck-Robbins]:

$$L(P_k, x) = \sum_{i=0}^k \frac{2^i x(x-1) \dots (x-i+1)}{i!}.$$

Theorem 4.5. *For each k there are α and β such that*

$$(\lambda, R)\text{-dim}(B_k) \geq \beta \lambda^k$$

for $R < \alpha \lambda^{\frac{k}{2}}$ for sufficiently large λ .

Proof. Pick numbers r and R such that $\frac{\lambda}{2r+1} \geq 5$ and $R < \sqrt{L(P_k, r)}/4$. By Lemma 4.4 the identity map

$$id : \left(\bigoplus_{\|\vec{i}\|_1 \leq r} \mathbb{Z}, \ell_1 \right) \rightarrow \left(\bigoplus_{\|\vec{i}\|_1 \leq r} \mathbb{Z}, d_k \right) \subset D_k$$

is $(1, r)$ -bilipschitz.

Considering the subspaces and applying Lemma 2.7 we obtain

$$\begin{aligned} (\lambda, R)\text{-dim}(B_k, d_k) &\geq (\lambda, R)\text{-dim}(D_k, d_k) \geq (\lambda, R)\text{-dim}\left(\bigoplus_{\|\vec{i}\|_1 \leq r} \mathbb{Z}, d_k \right) \\ &\geq \left(\frac{\lambda}{r}, R \right)\text{-dim} \bigoplus_{\|\vec{i}\|_1 \leq r} \mathbb{Z} \geq |\{ \vec{i} \in \mathbb{Z}^k \mid \|\vec{i}\|_1 \leq r \}| = L(P_k, r). \end{aligned}$$

Here the last equality is by definition. The preceding inequality holds true by Corollary 3.6 provided $r < \lambda/4$ and $R < \alpha \lambda^{\frac{k}{2}}$ for some α which depends on k only. In that case, $(\lambda, R)\text{-dim}(B_k) \geq \beta \lambda^k$ for some β which depends on k only. $\square$

The proof of the following Lemma is similar to the proof of Lemma 4.4.

Lemma 4.6. *For any $r > 0$ the identity map*

$$id : \left(\bigoplus_{\|g\|_G \leq r} \mathbb{Z}, \ell_1 \right) \rightarrow \left(\bigoplus_{\|g\|_G \leq r} \mathbb{Z}, d_{\mathbb{Z} \wr G} \right) \subset Z \wr G$$

is $(1, r)$ -bi-Lipschitz.

Theorem 4.7. *Let G be a group of exponential growth. Then for some exponential function D the dimension growth of the group $\mathbb{Z} \wr G$ with control D is exponential.*

Proof. Let B_r be the ball of radius r in G .

In view of Lemma 4.6 the identity map

$$Z = \left(\left(\bigoplus_{g \in B_r} \mathbb{Z} \right), \ell_1 \right) \xrightarrow{i} \left(\left(\bigoplus_{g \in B_r} \mathbb{Z} \right), d \right) = X$$

satisfies

$$d_Z(z, z') \leq d_X(z, z') \leq r d_Z(z, z').$$

Application of Lemma 2.7 to the above quasi-isometries and Corollary 3.6 gives us the chain of inequalities

$$(\lambda, R)\text{-dim}\left(\left(\bigoplus_{g \in G} \mathbb{Z} \right), d \right) \geq (\lambda, R)\text{-dim}\left(\left(\bigoplus_{g \in B_r} \mathbb{Z} \right), d \right) \geq \left(\frac{\lambda}{r}, R \right)\text{-dim}\left(\bigoplus_{g \in B_r} \mathbb{Z} \right) \geq |B_r|$$

whenever

$$R \leq \frac{\sqrt{|B_r|}}{4r}$$

and $\frac{\lambda}{r} > 4$ where $|B_r|$ denotes the cardinality of the ball. Note that

$$\frac{\sqrt{|B_r|}}{4r} \geq \frac{e^{\beta r}}{4r}$$

for some $\beta > 0$. We take $r = \lambda/5$ to satisfy the second condition and take α such that $e^{\alpha\lambda} < \frac{e^{\beta\lambda/5}}{4\lambda/5}$. Then for $R(\lambda) < e^{\alpha\lambda}$ we have the required condition. $\square$

Corollary 4.8. *For some exponential function D the dimension growth of the solvable of class 3 group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ with control D is exponential.*

Proof. Indeed, the volume growth function of $\mathbb{Z} \wr \mathbb{Z}$ is exponential. $\square$

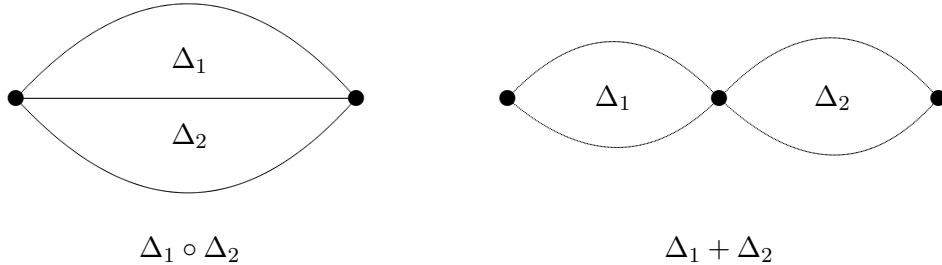
5. LOWER BOUND FOR DIMENSION GROWTH OF THE R. THOMPSON GROUP

In this section, it will be convenient to view the R. Thompson group as a diagram group over the semigroup presentation $\langle x \mid x^2 = x \rangle$.

Let us recall the definition of a *diagram group* (see [GS1, GS3] for more formal definitions). A (semigroup) *diagram* is a planar directed labeled graph tessellated into cells, defined up to an isotopy of the plane. Each diagram Δ has the top path $\mathbf{top}(\Delta)$, the bottom path $\mathbf{bot}(\Delta)$, the initial and terminal vertices $\iota(\Delta)$ and $\tau(\Delta)$. These are common vertices of $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$. The whole diagram is situated between the top and the bottom paths, and every edge of Δ belongs to a (directed) path in Δ between $\iota(\Delta)$ and $\tau(\Delta)$. More formally, let X be an alphabet. For every $x \in X$ we define the *trivial diagram* $\varepsilon(x)$ which is just an edge labeled by x . The top and bottom paths of $\varepsilon(x)$ are equal to $\varepsilon(x)$, $\iota(\varepsilon(x))$ and $\tau(\varepsilon(x))$ are the initial and terminal vertices of the edge. If u and v are words in X , a *cell* $(u \rightarrow v)$ is a planar graph consisting of two directed labeled paths, the top path labeled by u and the bottom path labeled by v , connecting the same points $\iota(u \rightarrow v)$ and $\tau(u \rightarrow v)$. There are three operations that can be applied to diagrams in order to obtain new diagrams.

1. **Addition.** Given two diagrams Δ_1 and Δ_2 , one can identify $\tau(\Delta_1)$ with $\iota(\Delta_2)$. The resulting planar graph is again a diagram denoted by $\Delta_1 + \Delta_2$, whose top (bottom) path is the concatenation of the top (bottom) paths of Δ_1 and Δ_2 . If $u = x_1x_2 \dots x_n$ is a word in X , then we denote $\varepsilon(x_1) + \varepsilon(x_2) + \dots + \varepsilon(x_n)$ (i.e. a simple path labeled by u) by $\varepsilon(u)$ and call this diagram also *trivial*.

2. **Multiplication.** If the label of the bottom path of Δ_2 coincides with the label of the top path of Δ_1 , then we can *multiply* Δ_1 and Δ_2 , identifying $\mathbf{bot}(\Delta_1)$ with $\mathbf{top}(\Delta_2)$. The new diagram is denoted by $\Delta_1 \circ \Delta_2$. The vertices $\iota(\Delta_1 \circ \Delta_2)$ and $\tau(\Delta_1 \circ \Delta_2)$ coincide with the corresponding vertices of Δ_1, Δ_2 , $\mathbf{top}(\Delta_1 \circ \Delta_2) = \mathbf{top}(\Delta_1)$, $\mathbf{bot}(\Delta_1 \circ \Delta_2) = \mathbf{bot}(\Delta_2)$.



3. **Inversion.** Given a diagram Δ , we can flip it about a horizontal line obtaining a new diagram Δ^{-1} whose top (bottom) path coincides with the bottom (top) path of Δ .

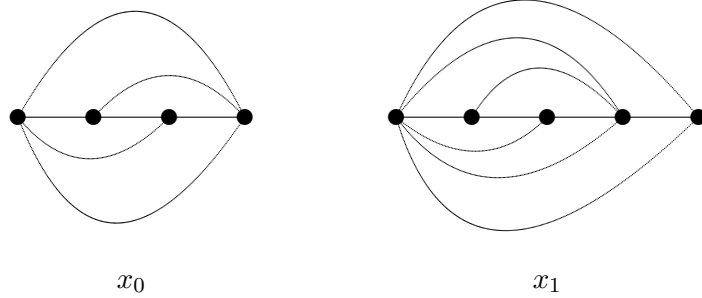
Definition 5.1. A diagram over a collection of cells P is any planar graph obtained from the trivial diagrams and cells of P by the operations of addition, multiplication and inversion. If the top path of a diagram Δ is labeled by a word u and the bottom path is labeled by a word v , then we call Δ a (u, v) -diagram over P .

Two cells in a diagram form a *dipole* if the bottom part of the first cell coincides with the top part of the second cell, and the cells are inverses of each other. In this case, we can obtain a new diagram removing the two cells and replacing them by the top path of the first cell. This operation is called *elimination of dipoles*. The new diagram is called *equivalent* to the initial one. A diagram is called *reduced* if it does not contain dipoles. It is proved in [GS1, Theorem 3.17] that every diagram is equivalent to a unique reduced diagram.

If the top and the bottom paths of a diagram are labeled by the same word u , we call it a *spherical* (u, u) -diagram. Now let $P = \{c_1, c_2, \dots\}$ be a collection of cells. The diagram group $\mathcal{D}(P, u)$ corresponding to the collection of cells P and a word u consists of all reduced spherical (u, u) -diagrams obtained from the cells of P and trivial diagrams by using the three operations mentioned above. The product $\Delta_1 \Delta_2$ of two diagrams Δ_1 and Δ_2 is the reduced diagram obtained by removing all dipoles from $\Delta_1 \circ \Delta_2$. The fact that $\mathcal{D}(P, u)$ is a group is proved in [GS1].

Example 5.2. If X consists of one letter x and P consists of one cell $x \rightarrow x^2$, then the group $\mathcal{D}(P, x)$ is the R. Thompson group F [GS1].

Here are the diagrams representing the two standard generators x_0, x_1 of the R. Thompson group F . All edges are labeled by x and oriented from left to right, so we omit the labels and orientation of edges.



It is easy to represent, say, x_0 as a product of sums of cells and trivial diagrams:

$$x_0 = (x \rightarrow x^2) \circ (\varepsilon(x) + (x \rightarrow x^2)) \circ \left((x \rightarrow x^2)^{-1} + \varepsilon(x) \right) \circ \left((x \rightarrow x^2)^{-1} \right).$$

There is a natural *diagram metric* on every diagram group $\mathcal{D}(P, u)$: $\text{dist}(\Delta, \Delta')$ is the number of cells in the diagram $\Delta^{-1} \Delta'$.

Lemma 5.3 ([Burillo, AGS]). *For the R. Thompson group F , the diagram metric is $(6, 2)$ -quasi-isometric to the word metric corresponding to the standard generating set $\{x_0, x_1\}$.*

Lemma 5.4. *There are constants $C_1, C_2 > 0$ such that for every n there is a group embedding of $\xi_n : \mathbb{Z}^{2^n} \rightarrow F$ into the Thompson group F such that ξ_n is a $(C_1 n, C_2)$ -quasi-isometric*

embedding:

$$\frac{1}{C_1 n} |x - x'|_1 - C_2 \leq d_F(\xi_n(x), \xi_n(x')) \leq C_1 n \|x - x'\|_1 + C_2$$

where $\|\cdot\|_1$ is the standard l_1 -metric on $\mathbb{Z}^{2^n}$.

Proof. We are going to use the following construction from [AGS]. For any $n \geq 0$, let us define 2^n elements of F that commute pairwise. All these elements will be reduced (x, x) -diagrams over $\mathcal{P} = \langle x \mid x^2 = x \rangle$. For $n = 0$, let Δ be the diagram that corresponds to the generator x_0 (see above). It has 4 cells.

Suppose that $n \geq 1$ and we have already constructed diagrams Δ_i ($1 \leq i \leq 2^{n-1}$) that commute pairwise. For every i we consider two (x^2, x^2) -diagrams: $\varepsilon(x) + \Delta_i$ and $\Delta_i + \varepsilon(x)$. We get 2^n spherical diagrams with base x^2 that obviously commute pairwise. It remains to conjugate them to obtain 2^n spherical diagrams with base x having the same property. Namely, we take $\pi \circ (\varepsilon(x) + \Delta_i) \circ \pi^{-1}$ and $\pi \circ (\Delta_i + \varepsilon(x)) \circ \pi^{-1}$.

Let us denote the elements of F obtained in this way by g_i ($1 \leq i \leq 2^n$). It is easily proved, say, by induction on n , that there exists a (x^{2^n}, x) -diagram u_n with n cells and (x^{2^n}, x^{2^n}) -diagrams $v_{n,i} = \varepsilon(x^i) + \Delta + \varepsilon(x^{2^n-i-1})$, $i = 0, \dots, 2^n - 1$, such that each g_i is equal to $u_n^{-1} v_{n,i} u_n$. Hence each g_i has $2n + 4$ cells and its word length in F is bounded between n/C and Cn where C is a constant. Hence the subgroup A_n generated by $g_1, \dots, g_{2^n}$ is isomorphic to $\mathbb{Z}^{2^n}$.

Now if we consider the diagram $g_1^{k_1} \dots g_{2^n}^{k_{2^n}}$ for any integers $k_1, \dots, k_{2^n}$, the number of cells in that diagram is between $4(|k_1| + \dots + |k_{2^n}|)$ and $2n + 4(|k_1| + \dots + |k_{2^n}|)$. It follows from Lemma 5.3 that the restriction of the word metric of F on the subgroup A_n is between $\frac{1}{C_1} |\cdot| - C_2 n$ and $C_1 |\cdot| + C_2 n$ where $|\cdot|$ is the standard l_1 -metric on $\mathbb{Z}^{2^n}$, C_1, C_2 are constants > 1 . $\square$

Remark 5.5. Note that the constants C_1 and C_2 in Lemma 5.4 do not exceed 25 and do not depend on n .

Theorem 5.6. *There exists an exponential function D such that the dimension growth of the Thompson group F with control D is exponential.*

Proof. Let $A_n = \xi_n(\mathbb{Z}^{2^n})$. In view of Lemma 5.4, Lemma 2.7, and Corollary 3.6 we obtain

$$(\lambda, e^{\alpha\lambda})\text{-dim}(F) \geq (\lambda, e^{\alpha\lambda})\text{-dim}(A_n) \geq \left(\frac{\lambda - C_2}{C_1 n}, C_1 n (e^{\alpha\lambda} + C_2) \right) \text{-dim} \bigoplus_{i=1}^{2^n} \mathbb{Z} = 2^n$$

provided $\frac{\lambda - C_2}{C_1 n} \geq 5$ and $C_1 n (e^{\alpha\lambda} + C_2) < 2^{\frac{n}{2}-2}$. This holds for $n = \frac{\lambda - C_2}{5C_1}$ and some α . $\square$

Remark 5.7. It is not known whether the dimension growth of F (with no control) is exponential. It seems that the embedding of $\mathbb{Z}^{2^n}$ into F described here has almost the smallest possible quasi-isometry constants. It does not look like similarly distorted embeddings of F^{2^n} or even $(F \wr \mathbb{Z})^{2^n}$ into F (which can be defined as above) help proving that the dimension growth of F is exponential. Whether there are less distorted copies of $\mathbb{Z}^k$ or F^k inside F is an open problem. On the other hand, if F has in fact a subexponential dimension growth, then from Ozawa [Ozawa], it would follow that F has Guoliang Yu's property A. This would solve a very difficult open problem (see the Introduction).

Note that if for some $\lambda_0 > 0$ and $\alpha > 0$, $\lambda_0\text{-dim}(\mathbb{Z}^k) \geq k^\alpha$ for every k (see Question 3.9), then the dimension growth of F is exponential. Indeed, in that case the (C_1n, C_2) -quasi-isometric embedding of $\mathbb{Z}^{2^n}$ into F described above gives by Lemmas 2.8 and 2.7 the following inequalities (for some $C_3 > 0$):

$$(\lambda_0 C_1 n + C_2)\text{-dim}(F) \geq \lambda_0\text{-dim}(\mathbb{Z}^{2^n}) \geq C_3 (2^n)^\alpha = C_3 2^{\alpha n}$$

for every $n \geq 1$. Hence

$$n\text{-dim}(F) \geq 2^{C_4 n}$$

for some $C_4 > 0$ and all $n \geq 1$.

6. UPPER BOUNDS. THE KOLMOGOROV-OSTRAND DIMENSION GROWTH

In order to estimate dimension growth from above, we will use another function which can be traced back to the work of Kolmogorov and Ostrand on Hilbert's 13-th problem [Kolmogorov, Ostrand]. Let X be a metric space. Consider colorings of X in where every point can be colored in several colors. Let $\lambda > 0$. The definition of monochromatic λ -clusters remains the same.

Definition 6.1. Let $D: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a function which is non-decreasing in each variable. For every $\lambda > 0$, we say that the Kolmogorov-Ostrand (KO) λ -dimension of X does not exceed n with control D if for every $m \geq 0$ there exists a coloring of X in $m + n$ colors such that every λ -cluster has diameter at most $D(m)$ and every point is colored in at least $m + 1$ colors. The smallest such n is called the KO λ -dimension of X with control $D(m, \lambda)$ (written as $(\lambda, D)\text{-KODim}(X)$). If the diameters of clusters are uniformly bounded and a control function is not specified, one obtain the notion of dimension $\lambda\text{-KODim}(X)$.

Example 6.2. Let $D(m, \lambda) = 2(m + 1)(\lambda + 1)$. Then

$$(\lambda, D)\text{-KODim}(\mathbb{R}) \leq 2.$$

Proof. Indeed, let $m \geq 0$. We color every $x \in \mathbb{R}$ in color $i \in \{0, \dots, m + 1\}$ if

$$\left\lfloor \frac{x}{\lambda + 1} \right\rfloor \not\equiv 2i + 1 \pmod{2m + 2}.$$

Then every point x is colored in $m + 1$ colors if $\lfloor \frac{x}{\lambda + 1} \rfloor$ is odd and is colored in all $m + 2$ colors otherwise. Every λ -cluster of every color is an interval of size $(2m + 1)(\lambda + 1) < 2(m + 1)(\lambda + 1)$. $\square$

Lemma 6.3. For every metric space X ,

$$\lambda\text{-dim}(X) \leq \lambda\text{-KODim}(X) - 1.$$

For every control function $D: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ which is non-decreasing in each variable

$$(\lambda, D(0, \lambda))\text{-dim}(X) \leq (\lambda, D)\text{-KODim}(X) - 1.$$

Proof. Indeed, let $(\lambda, D)\text{-KODim}(X) \leq n(\lambda)$ for some fixed λ . Take $m = 0$. Then there exists a coloring of X in $n(\lambda)$ colors such that every λ -cluster has diameter at most $D(0, \lambda)$. Then $(\lambda, D(0, \lambda))\text{-dim}(X) \leq n(\lambda) - 1$. $\square$

We always assume that the product of two metric spaces $X \times Y$ is supplied with the ℓ_1 -metric.

Lemma 6.4. *Suppose that*

$$(\lambda, D_X)\text{-}K\text{Odim}(X) \leq n_X(\lambda) \quad \text{and} \quad (\lambda, D_Y)\text{-}K\text{Odim}(Y) \leq n_Y(\lambda).$$

Then

$$(\lambda, D)\text{-}K\text{Odim}(X \times Y) \leq n(\lambda) = n_X(\lambda) + n_Y(\lambda) - 1$$

$$\text{with } D(m, \lambda) = D_X(m + n_Y(\lambda), \lambda) + D_Y(m + n_X(\lambda), \lambda).$$

Proof. Fix $\lambda > 0$ and $m \in \mathbb{N}$. By assumption there exists a coloring $X = \cup_{i=1}^{m+n(\lambda)} U^{(i)}$ (resp. $Y = \cup_{i=1}^{m+n(\lambda)} V^{(i)}$) where the diameter of every λ -cluster does not exceed $D_X(m + n_Y(\lambda), \lambda)$ (resp. $D_Y(m + n_X(\lambda), \lambda)$) and every point is colored in $m + n_Y(\lambda) + 1$ (resp. $m + n_X(\lambda) + 1$) colors. Consider the following coloring of $X \times Y$ in $m + n(\lambda)$ colors:

$$X \times Y = \cup \left(U^{(i)} \times V^{(i)} \right)$$

(i.e. we color each $U^{(i)} \times V^{(i)}$ in color i). Note that every λ -cluster of that coloring is the direct product of a cluster U of $U^{(i)}$ and a cluster V of $V^{(i)}$ for some i . Since $\text{diam}(U \times V) = \text{diam}U + \text{diam}V$, the diameter of every λ -cluster is at most

$$D_X(m + n_Y(\lambda), \lambda) + D_Y(m + n_X(\lambda), \lambda).$$

Now pick any point $(x, y) \in X \times Y$. Note that by our assumption, x is colored by some colors $i \in I$, $|I| \geq m + n_Y(\lambda) + 1$ and y is colored by some colors $j \in J$, $|J| \geq m + n_X(\lambda) + 1$. Since $(m + n_X(\lambda) + 1) + (m + n_Y(\lambda) + 1) = m + 1 + (m + n(\lambda))$, the intersection of I and J has at least $m + 1$ numbers. Hence (x, y) is colored in at least $m + 1$ colors as required. $\square$

Lemma 6.5. *Suppose that $(\lambda, D)\text{-}K\text{Odim}(X) \leq n(\lambda)$. Then for every $k \in \mathbb{N}$,*

$$(\lambda, D_k)\text{-}K\text{Odim}(X^k) \leq kn(\lambda) - k + 1$$

where $D_k(m, \lambda) = kD(m + (k - 1)n(\lambda), \lambda)$.

Proof. Induction on k . It is true formula for $k = 1$.

Assume that

$$(\lambda, D_{k-1})\text{-}K\text{Odim}(X^{k-1}) \leq (k - 1)n(\lambda)$$

for $D_{k-1}(m, \lambda) = (k - 1)D(m + (k - 2)n(\lambda), \lambda)$. We apply Lemma 6.4 to $X^{k-1} \times X$ to obtain

$$(\lambda, D)\text{-}K\text{Odim}(X^k) \leq n(\lambda) = (k - 1)n(\lambda) - (k - 1) + 1 + n(\lambda) - 1 = kn(\lambda) - k + 1$$

with

$$\begin{aligned} D(m, \lambda) &= D_{k-1}(m + n(\lambda), \lambda) + D(m + (k - 1)n(\lambda) - k + 1, \lambda) \\ &= (k - 1)D(m + n(\lambda) + (k - 2)n(\lambda), \lambda) + D(m + (k - 1)n(\lambda) - k + 1, \lambda) \\ &= (k - 1)D(m + (k - 1)n(\lambda), \lambda) + D(m + (k - 1)n(\lambda) - k + 1, \lambda) \\ &\leq kD(m + (k - 1)n(\lambda), \lambda) = D_k(m, \lambda). \end{aligned}$$

$\square$

Corollary 6.6. $(\lambda, (4n^2 - 2n)(\lambda + 1))\text{-dim}(\mathbb{Z}^n) \leq n$.

Proof. We apply this proposition to Example 6.2 where $n(\lambda) = 2$ and

$$D(m, \lambda) = 2(m+1)(\lambda+1)$$

to obtain $(\lambda, D_n)\text{-KODim}(\mathbb{Z}^n) \leq 2n - n + 1 = n + 1$ for $D_n(m, \lambda)$. By Lemma 6.3

$$(\lambda, D_n(0, \lambda))\text{-dim}(\mathbb{Z}^n) \leq n.$$

Note that

$$D_n(0, \lambda) = nD(2(n-1), \lambda) = 2n(2n-1)(\lambda+1) = (4n^2 - 2n)(\lambda+1).$$

□

This Corollary shows that the constant c in the definition of the Assouad-Nagata dimension of $\mathbb{Z}^n$ has a quadratic in n upper bound. Will Sawin [Sawin] improved this constant to $\frac{(n-1)(n+2)}{2}$, still quadratic.

We recall that the wreath product $B \wr A$ is a semi-direct product of the restricted power $B^{(A)}$ and A . In case of fixed generating sets for A and B the group $B \wr A$ is equipped with a natural metric $d_{B \wr A}$ described in Theorem 4.1 Let ρ denote the restriction of this metric to $B^{(A)}$.

We take the ℓ_1 metric on the product of metric spaces.

Lemma 6.7. *The metric spaces $(B \wr A, d_{B \wr A})$ and $(B^{(A)}, \rho) \times A$ are quasi-isometric.*

Proof. Given $a \in A$ and a finite set $a_1, \dots, a_n \in A$, we consider a shortest path p in A (to be precise - in the Cayley graph of A) from e to a that visits all a_i s and a shortest loop ω in A based at e that visits all a_i s. Let $|p|$ and $|\omega|$ denote the lengths of p and ω respectively. We note that $|p| \leq |\omega| + \|a\|_A$, $|\omega| \leq 2|p|$, and $\|a\|_A \leq |p|$. Therefore, $|\omega| + \|a\|_A \leq 3|p|$.

Then by Parry's theorem for $\bar{b}a \in B^{(A)}A = B \wr A$, $\bar{b} = b_1^{a_1} \dots b_n^{a_n} \in B^{(A)}$, $a, a_i \in A$, $i = 1, \dots, n$

$$\|\bar{b}a\|_{B \wr A} = |p| + \sum_{i=1}^n \|b_i\|_B \quad \text{and} \quad \|\bar{b}\|_{B \wr A} = |\omega| + \sum_{i=1}^n \|b_i\|_B.$$

Thus,

$$\|\bar{b}a\|_{B \wr A} \leq \|\bar{b}\|_{B \wr A} + \|a\|_A \leq 3\|\bar{b}a\|_{B \wr A}.$$

□

The proof of the following statement is the same as that for Lemma 2.7.

Lemma 6.8. *Let $\phi : X \rightarrow Y$ be a quasi-isometric embedding:*

$$\frac{1}{c_1} d_X(x, x') - r_1 \leq d_Y(\phi(x), \phi(x')) \leq c d_X(x, x') + r$$

for all $x, x' \in X$ where $c, c_1 \geq 1, r, r_1 \geq 0$. Then for $\lambda > r$,

$$(\lambda, D)\text{-KODim}(Y) \geq \left(\frac{\lambda-r}{c}, c_1(D+r_1)\right)\text{-KODim}(X).$$

Lemma 6.9. *Let $\phi : G' \rightarrow G$ be a 1-Lipschitz homomorphism of finitely generated groups. Then $\lambda\text{-KODim}(G') \leq \lambda\text{-KODim}(G)$.*

If additionally for all $x \in G'$, ϕ satisfies the condition $\|x\|_{G'} \leq \|\phi(x)\|_G + r$. Then $(\lambda, D)\text{-KODim}(G') \leq (\lambda, D+r)\text{-KODim}(G)$.

Proof. The first statement is proved the same way as Lemma 2.5. The second statement follows from Lemma 6.8. □

Lemma 6.10. *Let G be a (not necessarily finitely generated) group with a proper left invariant metric $\text{dist}(\cdot, \cdot)$. Let U be the ball in G of radius λ centered at $e \in G$ and let H be a subgroup containing U . Then left cosets of H are λ -separated.*

Proof. Indeed, if $xh, x'h'$ are in two different left cosets ($h, h' \in H, x, x' \in G$) but

$$\text{dist}(xh, x'h') \leq \lambda,$$

then $\text{dist}(1, h^{-1}x^{-1}x'h') \leq \lambda$, so $h^{-1}x^{-1}x'h' \in U \subseteq H$. Hence $x^{-1}x' \in H$, and $xH = x'H$, a contradiction. $\square$

We denote by ρ the word metric on $B \wr \mathbb{Z}$ restricted to $B^{(\mathbb{Z})}$.

Lemma 6.11. *Suppose that $(\lambda, D)\text{-KODim}(B) \leq n(\lambda)$. Then*

$$(\lambda, D')\text{-KODim}(B^{(\mathbb{Z})}, \rho) \leq (2\lambda + 1)(n(\lambda) - 1) + 1$$

with $D'(m, \lambda) = (2\lambda + 1)D(m + 2\lambda n(\lambda), \lambda) + 4\lambda + 1$.

Proof. Note that the λ -ball U in $(B^{(\mathbb{Z})}, \rho)$ centered at e generates a subgroup of $B^{2\lambda+1} = B^{[-\lambda, \lambda] \cap \mathbb{Z}} \subset B^{(\mathbb{Z})}$. By Lemma 6.10 every D' -controlled λ -coloring of $(B^{[-\lambda, \lambda] \cap \mathbb{Z}}, \bar{d})$ defines by means of translations by elements of $B^{(\mathbb{Z})}$ a D' -controlled λ -coloring of $B^{(\mathbb{Z})}$ with the same set of colors. Additionally, if every element of $x \in B_\lambda$ is painted by at least $m + 1$ color, the same holds true for all $y \in B^{(\mathbb{Z})}$. Thus,

$$(\lambda, D')\text{-KODim}(B^{(\mathbb{Z})}, \rho) = (\lambda, D')\text{-KODim}(B^{2\lambda+1}, \rho).$$

Since the identity map $i : (B^{2\lambda+1}, \rho) \rightarrow (B^{s\lambda+1}, \ell_1)$ is a 1-Lipschitz isomorphism satisfying the condition $\|x\|_\rho \leq \|x\|_{\ell_1} + 4\lambda + 1$ by Lemma 6.9 we obtain

$$(\lambda, \bar{D} + 4\lambda + 1)\text{-KODim}(B^{2\lambda+1}, \rho) \leq (\lambda, \bar{D})\text{-KODim}(B^{2\lambda+1}, \ell_1).$$

By Lemma 6.5

$$(\lambda, \bar{D})\text{-KODim}(B^{2\lambda+1}, \ell_1) \leq (2\lambda + 1)(n(\lambda) - 1) + 1$$

with $\bar{D}(m, \lambda) = (2\lambda + 1)D(m + 2\lambda n(\lambda), \lambda)$. $\square$

Theorem 6.12. *For every $k \in \mathbb{N}$ there are constants a_k and b_k such that*

$$(1) \quad (\lambda, D_k)\text{-KODim}(B_k) \leq a_k \lambda^k$$

with $D_k(m, \lambda) = b_k(m + \lambda^k) \lambda^{k+1}$.

Proof. Induction on k . For $k = 0$ this is Example 6.2.

Assume that $(\lambda, D_k)\text{-KODim}(B_k) \leq a_k \lambda^k$ with $D_k(m, \lambda) = b_k(m + \lambda^k) \lambda^{k+1}$. By Lemma 6.11 and induction assumption

$$(\lambda, D')\text{-KODim}(B_k^{(\mathbb{Z})}, \rho) \leq (2\lambda + 1)(a_k \lambda^k - 1) + 1$$

for $D'(m, \lambda) = (2\lambda + 1)D_k(m + 2\lambda a_k \lambda^k, \lambda) + 4\lambda + 1$. By Lemma 6.4

$$(\lambda, \bar{D})\text{-KODim}\left(\left(B_k^{(\mathbb{Z})}, \rho\right) \times \mathbb{Z}\right) \leq (2\lambda + 1)(a_k \lambda^k - 1) + 2 \leq 4a_k \lambda^{k+1}$$

for

$$\bar{D}(m, \lambda) = D'(m + 2, \lambda) + 2(m + 4a_k \lambda^{k+1} + 1)(\lambda + 1) =$$

$$(2\lambda + 1) D_k \left(m + 2 + 2\lambda a_k \lambda^k, \lambda \right) + 4\lambda + 1 + 2 \left(m + 4a_k \lambda^{k+1} + 1 \right) (\lambda + 1) \\ \leq 3b_k \lambda \left(m + 2 + 2\lambda a_k \lambda^k + \lambda^k \right) \lambda^{k+1} + 4a_k \lambda \left(m + \lambda^{k+1} \right) \leq 4a_k b_k \left(m + \lambda^{k+1} \right) \lambda^{k+2}.$$

By Lemma 6.7 $(B_k^{(\mathbb{Z})}, \rho) \times \mathbb{Z}$ is quasi-isometric to $B_k \wr \mathbb{Z} = B_{k+1}$. By Lemma 6.8, there are constants a_{k+1} and b_{k+1} such that

$$(\lambda, D_{k+1})\text{-KDim}(B_k \wr \mathbb{Z}) \leq a_{k+1} \lambda^{k+1}$$

with $D_{k+1}(m, \lambda) = b_{k+1} (m + \lambda^{k+1}) \lambda^{k+2}$. $\square$

Corollary 6.13. *The dimension growth of B_k is at most λ^k .*

Proof. Take $m = 0$ in formula (1) of Theorem 6.12 and apply Lemma 6.3. $\square$

Question 6.14. What is the dimension growth of B_k ? The answer is not known even for $k = 1$, i.e. for the group $\mathbb{Z} \wr \mathbb{Z}$.

Corollary 6.15. *For every solvable subgroup G of the R. Thompson group F , there exist polynomials $\Delta_G(\lambda)$ and $C_G(\lambda)$ such that $(\lambda, \Delta_G(\lambda))\text{-dim}(G)$ is at most $C_G(\lambda)$.*

Proof. Indeed, by the result of Collin Bleak [Bleak], every solvable subgroup of F embeds into the direct power $(B_k)^m$ for some k, m . $\square$

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