EIGENVALUES OF LAPLACIAN AND MULTI-WAY ISOPERIMETRIC CONSTANTS ON WEIGHTED RIEMANNIAN MANIFOLDS

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Abstract. We investigate the distribution of eigenvalues of the weighted Laplacian on closed weighted Riemannian manifolds of nonnegative Bakry-Émery Ricci curvature. We derive some universal inequalities among eigenvalues of the weighted Laplacian on such manifolds. These inequalities are quantitative versions of the previous theorem by the author with Shioya. We also study some geometric quantity, called multi-way isoperimetric constants, on such manifolds and obtain similar universal inequalities among them. Multi-way isoperimetric constants are generalizations of the Cheeger constant. Extending and following the heat semigroup argument by Ledoux and E. Milman, we extend the Buser-Ledoux result to the k-th eigenvalue and the k-way isoperimetric constant. As a consequence the k-th eigenvalue of the weighted Laplacian and the k-way isoperimetric constant are equivalent up to polynomials of k on closed weighted manifolds of nonnegative Bakry-Émery Ricci curvature.

1. Introduction

1.1. **Eigenvalues of the weighted Laplacian.** Let (M, μ) be a pair of a Riemmanian manifold M and a Borel probability measure μ on M of the form $d\mu = \exp(-\psi) d\,vol_M$, $\psi \in C^2(M)$. We call such a pair (M, μ) an *weighted Riemannian manifold*. We define the *weighted Laplacian* (also called the *Witten Laplacian*) $\triangle_\mu$ by

$$\triangle_\mu := \triangle - \nabla\psi \cdot \nabla,$$

where $\triangle$ is the usual positive Laplacian on M. If M is closed, then the spectrum of the weighted Laplacian $\triangle_\mu$ is discrete, where $\triangle_\mu$ is considered as a self-adjoint operator on $L^2(M, \mu)$. We denote its eigenvalues

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with multiplicity by

$$0 = \lambda_0(M,\mu) < \lambda_1(M,\mu) \leq \lambda_2(M,\mu) \leq \cdots \leq \lambda_k(M,\mu) \leq \cdots .$$

Our purpose in this paper is to understand the relationships between the eigenvalues $\lambda_k(M,\mu)$ for different k. For that purpose let us focus on diameter estimates in terms of eigenvalues of the weighted Laplacian due to Li and Yau [30, Theorem 10] and Cheng [9, Corollary 2.2] (see also [46]). Combining their results one could obtain that $\lambda_k(M,\mu) \leq c(k,n)\lambda_1(M,\mu)$ for any natural number k and any closed weighted Riemannian manifold (M,μ) of nonnegative Bakry-Émery Ricci curvature, here $c(k,n)$ is a constant depending only on k and the dimension n of M. The dependence of the constant $c(k,n)$ on n comes from Cheng's result. In order to bypass the dimension dependence of the inequality by Cheng, we consider the *observable diameter* $\mathrm{ObsDiam}((M,\mu);-\kappa)$, $\kappa > 0$, introduced by Gromov in [22]. The observable diameter comes from the study of 'concentration of measure phenomenon' and it might be interpreted as a substitute of the usual diameter. See Definition 5.5. The observable diameter is closely related with the first nontrivial eigenvalue of the weighted Laplacian as was firstly observed by Gromov and V. Milman in [21]:

$$\mathrm{ObsDiam}_{\mathbb{R}}((M,\mu);-\kappa) \leq \frac{6}{\sqrt{\lambda_1(M,\mu)}} \log \frac{1}{\kappa}. \tag{1.1}$$

Under assuming the nonnegativity of Bakry-Émery Ricci curvature, E. Milman obtained the opposite inequality ([39, 40, 41]):

$$\mathrm{ObsDiam}_{\mathbb{R}}((M,\mu);-\kappa) \geq \frac{1-2\kappa}{2\sqrt{\lambda_1(M,\mu)}}.$$

See (2.3), Proposition 2.12, and Lemma 5.6 for the proof of the above two inequalities. Observe that these two inequalities are independent of the dimension of the underlying manifold. One might regard the Gromov-V. Milman inequality as a dimension-free Cheng's inequality for $k = 1$ and also the E. Milman inequality as a dimension-free Li-Yau's inequality.

One of the main results in this paper is the following:

Theorem 1.1. *There exists a universal numeric constant $c > 0$ such that if (M,μ) is a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature and k is a natural number, then we have*

$$\lambda_k(M,\mu) \leq \exp(ck)\lambda_1(M,\mu).$$

Theorem 1.1 also holds for a convex domain with C^2 boundary in a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature and with the Neumann boundary condition, the proof of which is identical.

The crucial point of Theorem 1.1 is that the constant $\exp(ck)$ is independent of the dimension of the underlying manifold and quantitative. In [18, Theorem 1.1] the author proved with Shioya that the fraction $\lambda_k(M,\mu)/\lambda_1(M,\mu)$ is bounded from above by some universal constant depending only on k. However the estimate was not quantitative since the proof in [18] relies on some compactness argument.

In Theorem 1.1, the nonnegativity of Bakry-Émery Ricci curvature is necessary as was remarked in [18]. In fact for any $\varepsilon > 0$ there exists a closed Riemannian manifold M of Ricci curvature $\geq -\varepsilon$ such that $\lambda_2(M)/\lambda_1(M) \geq 1/\varepsilon$. Taking an appropriate scaling, some 'dumbbell space' becomes such an example, see [18, Example 4.9] for details.

The following corollary corresponds to Cheng's inequality for general k:

Corollary 1.2. *There exists a universal numeric constant $c > 0$ such that if (M,μ) is a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature and k is a natural number, then we have*

$$\mathrm{ObsDiam}_{\mathbb{R}}((M,\mu);-\kappa) \leq \frac{\exp(ck)}{\sqrt{\lambda_k(M,\mu)}} \log \frac{1}{\kappa}.$$

Corollary 1.2 follows from Theorem 1.1 together with the Gromov-V. Milman inequality (1.1).

In order to treat the k-th eigenvalue we will work on the notion of *'separation'*, which is regarded as a generalization of the concentration of measure phenomenon (see Subsection 2.2). It tells the information whether or not there exists a pair which are not separated in some sense among any $k+1$-tuple subsets with a fixed volume. According to the work of Chung, Grigor'yan, and Yau [10, 11], it is related with the information of general eigenvalues of the weighted Laplacian (see Theorem 2.10). Under assuming the nonnegativity of Bakry-Émery Ricci curvature, we prove that if a nonseparated pair always exists among any $k+1$-tuple of subsets, then it also holds among any k-tuple of subsets. In order to prove it, we use the *curvature-dimension condition* $\mathrm{CD}(0,\infty)$ in the sense of Lott-Villani [32] and Sturm [49, 50], which is equivalent to the nonnegativity of Bakry-Émery Ricci curvature. Idea of the proof of Theorem 1.1 will be discussed in Section 3 in more detail.

1.2. **Multi-way isoperimetric constants.** Let (M, μ) be a closed weighted Riemannian manifold. Recall that *Minkowski's (exterior) boundary measure* of a Borel subset A of M, which we denote by $\mu^+(A)$, is defined as

$$\mu^+(A) := \liminf_{r \to 0} \frac{\mu(O_r(A)) - \mu(A)}{r},$$

where $O_r(A)$ denotes the open r-neighborhood of A. We consider the following geometric quantity:

Definition 1.3 (Multi-way isoperimetric constants)**.** For a natural number k, we define the *k-way isoperimetric constant* as

$$h_k(M, \mu) := \inf_{A_0, A_1, \cdots, A_k} \max_{0 \leq i \leq k} \frac{\mu^+(A_i)}{\mu(A_i)},$$

where the infimum runs over all collections of $k+1$ non-empty, disjoint Borel subsets $A_0, A_1, \cdots, A_k$ of M. $h_1(M, \mu)$ is also called the *Cheeger constant*.

Note that $h_k(M, \mu) \leq h_{k+1}(M, \mu)$ by the definition. We are interested in the distribution of $h_1(M, \mu), h_2(M, \mu), \cdots, h_k(M, \mu), \cdots$ on the real line and the relation between $\lambda_k(M, \mu)$ and $h_k(M, \mu)$.

The Cheeger-Maz'ja inequality ([33, 34, 35] and [7], see [39, Theorem 1.1]) states that

$$h_1(M, \mu) \leq 2\sqrt{\lambda_1(M, \mu)}. \tag{1.2}$$

In [28], resolving a conjecture by Miclo [37] (see also [14]), Lee, Gharan, and Trevisan obtained a higher order Cheeger-Maz'ja inequality for general graphs. Although they proved it for graphs, by an appropriate modification of their proof (e.g., by replacing sums with integrals), it is also valid for weighted Riemannian manifolds. In Appendix, we will discuss a point that we have to be care when we treat their argument for the smooth setting.

Theorem 1.4 (Lee et al. [28, Theorem 3.8])**.** *There exists a universal numerical constant $c > 0$ such that for all closed weighted Riemannian manifold (M, μ) and a natural number k we have*

$$h_k(M, \mu) \leq ck^3\sqrt{\lambda_k(M, \mu)}.$$

Lee et al. proved the above order k^3 can be improved to k^2 for graphs ([28, Theorem 1.1]). Since it is uncertain that Lemma 4.7 in [28] holds or does not hold for the case of weighted Riemannian manifolds, the author does not know k^3 order in Theorem 1.4 can be improved to k^2 order.

The opposite inequality of the Cheeger-Maz'ja inequality (1.2) was shown by Buser [6] and Ledoux [26]. They proved the existence of a universal numeric constant $c > 0$ such that

$$c\sqrt{\lambda_1(M,\mu)} \leq h_1(M,\mu) \tag{1.3}$$

for any closed weighted Riemannian manifold (M,μ) of nonnegative Bakry-Émery Ricci curvature. Combining Theorems 1.1 and 1.4 with (1.3) we obtain

$$h_k(M,\mu) \lesssim k^3\sqrt{\lambda_k(M,\mu)} \lesssim k^3\exp(ck)\sqrt{\lambda_1(M,\mu)} \lesssim k^3\exp(ck)h_1(M,\mu),$$

where $A \lesssim B$ denotes $A \leq CB$ for some universal numeric constant $C > 0$. Consequently we have the following:

Theorem 1.5. *There exists a universal numeric constant $c > 0$ such that if (M,μ) is a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature and k is a natural number, then we have*

$$h_k(M,\mu) \leq k^3\exp(ck)h_1(M,\mu).$$

M. Mimura obtained similar universal inequalities among multi-way isoperimetric constants for Cayley graphs.

Following and extending the heat semigroup argument by Ledoux [26] and E. Milman [41], we obtain the extension of the Buser-Ledoux Theorem:

Theorem 1.6. *Assume that a closed weighted Riemannian manifold (M,μ) has nonnegative Bakry-Émery Ricci curvature. Then for any natural number k we have*

$$(80k^3)^{-1}\sqrt{\lambda_k(M,\mu)} \leq h_k(M,\mu).$$

As a consequence $h_k(M,\mu)$ and $\sqrt{\lambda_k(M,\mu)}$ are equivalent up to polynomials of k under assuming the nonnegativity of Bakry-Émery Ricci curvature.

1.3. **Applications to the stability of eigenvalues of the weighted Laplacian and multi-way isoperimetric constants.** In [41, Section 5] and [42, Section 5.2], E. Milman obtained several stability results of the Cheeger constants. Although one could apply Theorems 1.1 and 1.5 to several stability results by E. Milman, we apply our results to only one of his theorems in [41]. The interested reader is referred to [41, 42] for further information

For a domain Ω with C^2 boundary in a complete Riemannian manifold, we denote by $\eta_k(\Omega)$ the k-th eigenvalue of Laplacian with Neumann condition.

Corollary 1.7. *Let K, L be two bounded convex domains in $\mathbb{R}^n$ and assume that both K and L have C^2 boundary. If*

$$vol(K \cap L) \geq v_K \, vol(K) \text{ and } vol(K \cap L) \geq v_L \, vol(L),$$

then

$$\eta_k(K) \geq \frac{\exp(-ck) v_K^4}{\{\log(1 + 1/v_L)\}^2} \eta_k(L),$$

and

$$h_k(K) \geq \frac{\exp(-ck) v_K^2}{k^3 \log(1 + 1/v_L)} h_k(L).$$

where $c > 0$ is a universal numeric constant.

In particular, if $vol(K) \simeq vol(L) \simeq vol(K \cap L)$ then $\eta_k(K) \simeq_k \eta_k(L)$ and $h_k(K) \simeq_k h_k(L)$. Here $A \simeq B$ (resp., $A \simeq_k B$) stands for A and B are equivalent up to universal numeric constants (resp., constants depending only on k).

E. Milman obtained the above corollary for $k = 1$ ([41, Theorem 1.7]). The above corollary follows from his theorem together with Theorems 1.1 and 1.5.

In the same spirit we investigate the (rough) stability property of eigenvalues of the weighted Laplacian and multi-way isoperimetric constants with respect to perturbation of spaces (Section 5). We discuss the case where two weighted manifolds M and N of nonnegative Bakry-Émery Ricci curvature are close with respect to the concentration topology introduced by Gromov in [22]. Roughly speaking, the two spaces M and N are close with respect to the concentration topology if 1-Lipschitz functions on M are close to those on N in some sense.

1.4. **Organization of the paper.** Section 2 collects some back ground material. In Section 3, after explaining some basics of the theory of optimal transportation, we prove Theorem 1.1. In Section 4, we prove Theorem 1.6. In Section 5 we study the (rough) stability property of eigenvalues of the weighted Laplacian and multi-way isoperimetric constants with respect to the concentration topology. In Section 6 we discuss several questions concerning this paper and some conjecture raised in [18].

2. Preliminaries

We review some basics needed in this paper.

2.1. **Concentration of measure.** In this subsection we explain the known relation among the 1st eigenvalue of the weighted Laplacian, the Cheeger constant, and the concentration of measure in the sense of Lévy and V. Milman ([29], [43]).

Let X be an *mm-space*, i.e., a complete separable metric space with a Borel probability measure μ_X.

Definition 2.1 (Concentration function, [1])**.** For $r > 0$ we define the real number $\alpha_X(r)$ as the supremum of $\mu_X(X \setminus O_r(A))$, where A runs over all Borel subsets of X such that $\mu_X(A) \geq 1/2$. The function $\alpha_X : (\,0, +\infty\,) \to \mathbb{R}$ is called the *concentration function*.

Lemma 2.2 ([1], [27, Lemma 1.1])**.** *If $\mu_X(A) \geq \kappa > 0$, then*

$$\mu_X(X \setminus O_{r+r_0}(A)) \leq \alpha_X(r)$$

for any $r, r_0 > 0$ such that $\alpha_X(r_0) < \kappa$.

The following Gromov and V. Milman's theorem asserts that Poincaré inequalities imply appropriate exponential concentration inequalities ([21], [27, Theorem 3.1]).

Theorem 2.3 ([21])**.** *Let (M, μ) be a closed weighted Riemannian manifold. Then we have*

$$\alpha_{(M,\mu)}(r) \leq \exp(-\sqrt{\lambda_1(M,\mu)}r/3)$$

for any $r > 0$. In particular, we have

$$\alpha_{(M,\mu)}(r) \leq \exp(-h_1(M,\mu)r/6) \tag{2.1}$$

for any $r > 0$.

The second statement (2.1) follows from the first statement together with the Cheeger-Maz'ja inequality (1.2).

Remark 2.4. Integrating Cheeger's linear isoperimetric inequality also implies the second inequality (2.1) (see [38, Proposition 1.7]).

In the series of works [39, 40, 41], E. Milman obtained the converse of Theorem 2.3 under assuming the nonnegativity of Bakry-Émery Ricci curvature. He proved that a uniform tail-decay of the concentration function implies the linear isoperimetric inequality (Cheeger's isoperimetric inequality) under assuming the nonnegativity of Bakry-Émery Ricci curvature. E. Milman's theorem plays a key role in the proof of Theorem **??**.

For an weighted Riemannian manifold (M, μ), we define the (*infinite-dimensional*) *Bakry-Émery Ricci curvature tensor* as

$$Ric_\mu := Ric_M + \operatorname{Hess}\psi.$$

Theorem 2.5 (E. Milman, [40, Theorem 2.1])**.** *Let (M,μ) be a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature. If $\alpha_{(M,\mu)}(r) \leq \kappa$ for some $r > 0$ and $\kappa \in (0, 1/2)$, then*

$$h_1(M,\mu) \geq \frac{1-2\kappa}{r}.$$

In particular, we have

$$\lambda_1(M,\mu) \geq \Big(\frac{1-2\kappa}{2r}\Big)^2.$$

The key ingredient of E. Milman's approach to the above result is the concavity of isoperimetric profile under the assumption of the nonnegativity of Bakry-Émery Ricci curvature, the fact based on the regularity theory of isoperimetric minimizers (see [39, Appendix]). See also [27] for the heat semigroup approach to Theorem 2.5.

2.2. **Separation distance.** We define the separation distance which plays an important role when treating eigenvalues of the weighted Laplacian. The separation distance was introduced by Gromov in [22].

Definition 2.6 (Separation distance)**.** For any $\kappa_0, \kappa_1, \cdots, \kappa_k \geq 0$ with $k \geq 1$, we define the (k-)*separation distance* $\mathrm{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_k)$ of X as the supremum of $\min_{i\neq j} d_X(A_i, A_j)$, where $A_0, A_1, \cdots, A_k$ are any Borel subsets of X satisfying that $\mu_X(A_i) \geq \kappa_i$ for all $i = 0, 1, \cdots, k$.

It is immediate from the definition that if $\kappa_i \geq \tilde{\kappa}_i$ for each $i = 0, 1, \cdots, k$, then

$$\mathrm{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_k) \leq \mathrm{Sep}(X; \tilde{\kappa}_0, \tilde{\kappa}_1, \cdots, \tilde{\kappa}_k).$$

Note that if the support of μ_X is connected, then

$$\mathrm{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_k) = 0$$

for any $\kappa_0, \kappa_1, \cdots, \kappa_k > 0$ such that $\sum_{i=0}^k \kappa_i > 1$.

For a Borel subset A of an mm-space X we put

$$\mu_A := \frac{\mu_X|_A}{\mu_X(A)}$$

Lemma 2.7. *If A satisfies $\mu_X(A) \geq \kappa$, then*

$$\mathrm{Sep}((A, \mu_A); \kappa_0, \kappa_1, \cdots, \kappa_k) \leq \mathrm{Sep}(X; \kappa\kappa_0, \kappa\kappa_1, \cdots, \kappa\kappa_k)$$

for any $\kappa_0, \kappa_1, \cdots, \kappa_k > 0$.

Proof. Take $k+1$ Borel subsets $A_0, A_1, \cdots, A_k$ of A such that $\mu_A(A_i) \geq \kappa_i$ for any i. The lemma immediately follows from that $\mu_X(A_i) \geq \mu_X(A)\kappa_i \geq \kappa\kappa_i$. □

We denote the closed r-neighborhood of a subset A in a metric space by $C_r(A)$.

Lemma 2.8. *Let X be an mm-space and put $r := \mathrm{Sep}(X, \kappa_0, \kappa_1, \cdots, \kappa_k)$. Assume that k Borel subsets $A_0, A_1, \cdots, A_{k-1}$ of X satisfy $\mu_X(A_i) \geq \kappa_i$ for every $i = 0, 1, \cdots, k-1$ and $d_X(A_i, A_j) > r$ for every $i \neq j$. Then we have*

$$\mu_X\Big(\bigcup_{i=0}^{k-1} C_r(A_i)\Big) \geq 1 - \kappa_k.$$

Proof. Suppose that for some $\varepsilon_0 > 0$,

$$\mu_X\Big(\bigcup_{i=0}^{k-1} C_{r+\varepsilon_0}(A_i)\Big) \leq 1 - \kappa_k.$$

Putting $A_k := X \setminus \bigcup_{i=0}^{k-1} C_{r+\varepsilon_0}(A_i)$ we have $\mu_X(A_k) \geq \kappa_k$ and $d_X(A_k, A_i) \geq r + \varepsilon_0$ for any $i = 0, 1, \cdots, k-1$. Thus we get

$$r < \min_{i \neq j} d_X(A_i, A_j) \leq \mathrm{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_k) = r,$$

which is a contradiction. Hence $\mu_X(\bigcup_{i=0}^{k-1} C_{r+\varepsilon}(A_i)) > 1 - \kappa_k$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ we obtain the conclusion. □

The following lemma asserts that exponential concentration inequalities and logarithmic 2-separation inequalities are equivalent:

Lemma 2.9. *Let X be an mm-space.*

(1) *If X satisfies*

$$\mathrm{Sep}(X; \kappa, \kappa) \leq \frac{1}{C} \log \frac{c}{\kappa} \tag{2.2}$$

for any $\kappa > 0$, then we have $\alpha_X(r) \leq c \exp(-Cr)$ for any $r > 0$.

(2) *Conversely, if X satisfies $\alpha_X(r) \leq c' \exp(-C'r)$ for any $r > 0$, then we have*

$$\mathrm{Sep}(X; \kappa, \kappa) \leq \frac{2}{C'} \log \frac{c'}{\kappa}$$

for any $\kappa > 0$.

Proof. (1) Assume that X satisfies (2.2) and let $A \subseteq X$ be a Borel subset such that $\mu_X(A) \geq 1/2$. For $r > 0$ we put $\kappa := \mu_X(X \setminus O_r(A))$. Since

$$r \leq d_X(X \setminus O_r(A), A) \leq \mathrm{Sep}(X; \kappa, 1/2) \leq \mathrm{Sep}(X; \kappa, \kappa) \leq \frac{1}{C} \log \frac{c}{\kappa},$$

we have $\kappa \leq c \log(-Cr)$, which gives the conclusion of (1).

(2) Assuming that $\alpha_X(r) \leq c' \exp(-C'r)$, we take two Borel subsets $A, B \subseteq X$ such that $\mu_X(A) \geq \kappa$, $\mu_X(B) \geq \kappa$, and $d_X(A,B) = \mathrm{Sep}(X;\kappa,\kappa)$. Let $\tilde{r}$ be any positive number satisfying

$$\alpha_X(\tilde{r}) \leq c' \exp(-C'\tilde{r}) < \kappa,$$

i.e.,

$$\tilde{r} > \frac{1}{C'} \log \frac{c'}{\kappa}.$$

Since $\mu_X(A) \geq \kappa$, by Lemma 2.2, we have

$$1 - \mu_X(O_{2\tilde{r}}(A)) \leq \alpha_X(\tilde{r}) < \kappa.$$

Hence we have

$$\mu_X(O_{2\tilde{r}}(A) \cap B) > (1-\kappa) + \kappa - 1 = 0,$$

which yields $\mathrm{Sep}(X;\kappa,\kappa) = d_X(A,B) \leq 2\tilde{r}$. Letting $\tilde{r} \to C'^{-1} \log(c'/\kappa)$ we obtain (2). □

Theorem 2.3 together with Lemma 2.9 (2) implies that for any closed weighted Riemannian manifold (M,μ) we have

$$\mathrm{Sep}((M,\mu);\kappa,\kappa) \leq \frac{6}{\sqrt{\lambda_1(M,\mu)}} \log \frac{1}{\kappa}. \tag{2.3}$$

Chung, Grigor'yan, and Yau generalized the above inequality in the following form:

Theorem 2.10 (Chung et al. [11, Theorem 3.1])**.** *Let (M,μ) be a closed weighted Riemannian manifold. Then, for any $k \in \mathbb{N}$ and any $\kappa_0, \kappa_1, \cdots, \kappa_k > 0$, we have*

$$\mathrm{Sep}((M,\mu);\kappa_0,\kappa_1,\cdots,\kappa_k) \leq \frac{1}{\sqrt{\lambda_k(M,\mu)}} \max_{i \neq j} \log\Big(\frac{e}{\kappa_i \kappa_j}\Big).$$

Combining Theorems 1.4 and 2.10 we obtain the following proposition:

Proposition 2.11. *There exists a universal numeric constant $c > 0$ such that for all closed weighted Riemannian manifold (M,μ), a natural number k, and $\kappa_0, \kappa_1, \cdots, \kappa_k > 0$, we have*

$$\mathrm{Sep}((M,\mu);\kappa_0,\kappa_1,\cdots,\kappa_k) \leq \frac{ck^3}{h_k(M,\mu)} \max_{i \neq j} \log \frac{e}{\kappa_i \kappa_j}.$$

We end this subsection reformulating Theorem 2.5 in terms of the separation distance for later use.

Proposition 2.12. *Let (M,μ) be a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature. Then, for any $\kappa > 0$, we have*

$$\mathrm{Sep}((M,\mu);\kappa,\kappa) \geq \frac{1-2\kappa}{h_1(M,\mu)}$$

and

$$\mathrm{Sep}((M,\mu);\kappa,\kappa) \geq \frac{1-2\kappa}{2\sqrt{\lambda_1(M,\mu)}}.$$

Proof. We prove only the first assertion. The proof of the second assertion is identical to the first one. According to Theorem 2.5, if $r \in (0,\infty)$ and $\kappa \in (0,1/2)$ satisfy

$$r < \frac{1-2\kappa}{h_1(M,\mu)}, \tag{2.4}$$

then we have $\alpha_{(M,\mu)}(r) > \kappa$. There exists $A \subseteq M$ such that $\mu(A) \geq 1/2$ and $\mu(M \setminus O_r(A)) > \kappa$. Hence we have

$$r = d_M(A, M \setminus O_r(A)) \leq \mathrm{Sep}((M,\mu);1/2,\kappa) \leq \mathrm{Sep}((M,\mu);\kappa,\kappa).$$

Combining the above inequality with (2.4) gives the conclusion. □

Since $\mathrm{Sep}((M,\mu);\kappa,\kappa) \leq \operatorname{diam} M$, the second inequality of Theorem 2.12 recovers the Li-Yau inequality [30].

2.3. **Three distances between probability measures.** Let X be a complete separable metric space. We denote by $\mathcal{P}(X)$ the set of Borel probability measures on X.

Definition 2.13 (Prohorov distance)**.** Given two measures $\mu,\nu \in \mathcal{P}(X)$ and $\lambda \geq 0$, we define the *Prohorov distance* $\mathrm{di}_\lambda(\mu,\nu)$ as the infimum of $\varepsilon > 0$ such that

$$\mu(C_\varepsilon(A)) \geq \nu(A) - \lambda\varepsilon \text{ and } \nu(C_\varepsilon(A)) \geq \mu(A) - \lambda\varepsilon \tag{2.5}$$

for any Borel subsets $A \subseteq X$.

For any $\lambda \geq 0$, the function di_λ is a complete separable distance function on $\mathcal{P}(X)$. If $\lambda > 0$, then the topology on $\mathcal{P}(X)$ determined by the Prohorov distance function di_λ coincides with that of the weak convergence (see [4, Section 6]). The distance functions di_λ for all $\lambda > 0$ are equivalent to each other. Also it is known that if $\mu(C_\varepsilon(A)) \geq \nu(A) - \lambda\varepsilon$ for any Borel subsets A of X, then $\mathrm{di}_\lambda(\mu,\nu) \leq \varepsilon$. In other words, the second inequality in (2.5) follows from the first one (see [4, Section 6]).

For $(x,y) \in X \times X$, we put $\mathrm{proj}_1(x,y) := x$ and $\mathrm{proj}_2(x,y) := y$. For two finite Borel measures μ and ν on X, we write $\mu \leq \nu$ if

$\mu(A) \leq \nu(A)$ for any Borel subset $A \subseteq X$. A finite Borel measure π on $X \times X$ is called a *partial transportation* from $\mu \in \mathcal{P}(X)$ to $\nu \in \mathcal{P}(X)$ if $(\mathrm{proj}_1)_*(\pi) \leq \mu$ and $(\mathrm{proj}_2)_*(\pi) \leq \nu$. Note that we do not assume π to be a probability measure. For a partial transportation π from μ to ν, we define its *deficiency* $\operatorname{def} \pi$ by $\operatorname{def} \pi := 1 - \pi(X \times X)$. Given $\varepsilon > 0$, the partial transportation π is called an *ε-transportation* from μ to ν if it is supported in the subset

$$\{(x, y) \in X \times X \mid d_X(x, y) \leq \varepsilon\}.$$

Definition 2.14 (Transportation distance)**.** Let $\lambda \geq 0$. For two probability measures $\mu, \nu \in \mathcal{P}(X)$, we define the *transportation distance* $\mathrm{Tra}_\lambda(\mu, \nu)$ between μ and ν as the infimum of $\varepsilon > 0$ such that there exists an ε-transportation π from μ to ν satisfying $\operatorname{def} \pi \leq \lambda\varepsilon$.

The following theorem is due to V. Strassen.

Theorem 2.15 ([51, Corollary 1.28], [22, Section $3\frac{1}{2}$.10])**.** *For any $\lambda > 0$, we have*

$$\mathrm{Tra}_\lambda = \mathrm{di}_\lambda .$$

Let (X, d_X) be a complete metric space. We indicate by $\mathcal{P}^2(X)$ the set of all Borel probability measures $\nu \in \mathcal{P}(X)$ such that

$$\int_X d_X(x, y)^2 d\nu(y) < +\infty$$

for some $x \in X$.

Definition 2.16 ($(L^2$-)Wasserstein distance)**.** For two probability measures $\mu, \nu \in \mathcal{P}^2(X)$, we define the *$L^2$-Wasserstein distance* $d_2^W(\mu, \nu)$ between μ and ν as the infimum of

$$\Big(\int_{X \times X} d_X(x, y)^2 d\pi(x, y) \Big)^{1/2},$$

where $\pi \in \mathcal{P}^2(X \times X)$ runs over all *couplings* of μ and ν, i.e., probability measures π with the property that $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for any Borel subset $A \subseteq X$. It is known that this infimum is achieved by some transport plan, which we call an *optimal transport plan* for $d_2^W(\mu, \nu)$.

If the underlying space X is compact, then the topology on $\mathcal{P}(X)$ induced from the L^2-Wasserstein distance function coincides with that of the weak convergence (see [51, Theorem 7.12]).

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need to explain some useful tools from the theory of optimal transportation. Refer to [51, 52] for more details.

Let (X, d_X) be a metric space. A rectifiable curve $\gamma : [0,1] \to X$ is called a *geodesic* if its arclength coincides with the distance $d_X(\gamma(0), \gamma(1))$ and it has a constant speed, i.e., parameterized proportionally to the arclength. We say that a metric space is a *geodesic space* if any two points are joined by a geodesic between them. It is known that $(\mathcal{P}^2(X), d_2^W)$ is compact geodesic space as soon as X is ([50, Proposition 2.10]).

Let M be a close Riemannian manifold. For two probability measures $\mu_0, \mu_1 \in \mathcal{P}^2(M)$ which are absolutely continuous with respect to $d\,vol_M$, there is a unique geodesic $(\mu_t)_{t\in[0,1]}$ between them with respect to the L^2-Wasserstein distance function d_2^W ([36, Theorem 9]).

For an mm-space X let us denote by Γ the set of minimal geodesics $\gamma : [0,1] \to X$ endowed with the distance

$$d_\Gamma(\gamma_1, \gamma_2) := \sup_{t\in[0,1]} d_X(\gamma_1(t), \gamma_2(t)).$$

Define the *evaluation map* $e_t : \Gamma \to X$ for $t \in [0,1]$ as $e_t(\gamma) := \gamma(t)$. A probability measure $\Pi \in \mathcal{P}(\Gamma)$ is called a *dynamical optimal transference plan* if the curve $\mu_t := (e_t)_*\Pi$, $t \in [0,1]$, is a minimal geodesic in $(\mathcal{P}^2(X), d_2^W)$. Then $\pi := (e_0 \times e_1)_*\Pi$ is an optimal coupling of μ_0 and μ_1, where $e_0 \times e_1 : \Gamma \to X \times X$ is the "endpoints" map, i.e., $(e_0 \times e_1)(\gamma) := (e_0(\gamma), e_1(\gamma))$.

Lemma 3.1 ([32, Proposition 2.10])**.** *If (X, d_X) is locally compact, then any minimal geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}^2(X), d_2^W)$ is associated with a dynamical optimal transference plan Π, i.e., $\mu_t = (e_t)_*\Pi$.*

Let μ and ν be two probability measures on a set X. We define the *relative entropy* $\mathrm{Ent}_\mu(\nu)$ of ν with respect to μ as follows. If ν is absolutely continuous with respect to μ, writing $d\nu = \rho d\mu$, then

$$\mathrm{Ent}_\mu(\nu) := \int_M \rho \log \rho d\mu,$$

otherwise $\mathrm{Ent}_\mu(\nu) := \infty$.

Definition 3.2 (Curvature-dimension condition, [32], [49, 50])**.** Let K be a real number. We say that an mm-space satisfies the *curvature-dimension condition* $CD(K, \infty)$ if for any $\nu_0, \nu_1 \in \mathcal{P}^2(X)$ there exists a minimal geodesic $(\nu_t)_{t\in[0,1]}$ in $(\mathcal{P}^2(X), d_2^W)$ from ν_0 to ν_1 such that

$$\mathrm{Ent}_{\mu_X}(\nu_t) \le (1-t)\,\mathrm{Ent}_{\mu_X}(\nu_0) + t\,\mathrm{Ent}_{\mu_X}(\nu_1) - \frac{K}{2}(1-t)t\; d_2^W(\nu_0, \nu_1)^2$$

for any $t \in [0,1]$.

In the above definition, assume that both ν_0 and ν_1 are absolutely continuous with respect to μ_X. Then Jensen's inequality applied to the convex function $r \mapsto r \log r$ gives

(3.1)

$$\begin{aligned}&\log \mu_X(\mathrm{Supp}\, \nu_t)\\ \geq\ &-(1-t)\int_M \rho_0 \log \rho_0 d\mu_X - t\int_M \rho_1 \log \rho_1 d\mu_X + \frac{Kt(1-t)}{2}\, d_2^W(\mu_0,\mu_1)^2,\end{aligned}$$

where ρ_0 and ρ_1 are densities of ν_0 and ν_1 with respect to μ_X respectively. In particular, for two Borel subsets $A, B \subseteq X$ with $\mu_X(A), \mu_X(B) > 0$, we have

(3.2)

$$\begin{aligned}&\log \mu_X(\mathrm{Supp}\, \nu_t)\\ \geq\ &(1-t)\log \mu_X(A) + t \log \mu_X(B) + \frac{Kt(1-t)}{2}\, d_2^W\left(\frac{\mu_X|_A}{\mu_X(A)}, \frac{\mu_X|_B}{\mu_X(B)}\right)^2\end{aligned}$$

([50]).

Theorem 3.3 ([12, 13], [44], [48])**.** *For a complete weighted Riemannian manifold (M,μ), we have $Ric_\mu \geq K$ for some $K \in \mathbb{R}$ if and only if (M,μ) satisfies $CD(K,\infty)$.*

Theorem 1.1 follows from the following key theorem together with Theorem 2.10 and Proposition 2.12.

Theorem 3.4. *Let (M,μ) be a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature. If (M,μ) satisfies*

$$\mathrm{Sep}((M,\mu); \underbrace{\kappa, \kappa, \cdots, \kappa}_{k+1\ times}) \leq \frac{1}{D}\log\frac{1}{\kappa^2} \tag{3.3}$$

for any $\kappa > 0$, then we have

$$\mathrm{Sep}((M,\mu); \underbrace{\kappa, \kappa, \cdots, \kappa}_{k\ times}) \leq \frac{c}{D}\log\frac{1}{\kappa^2} \tag{3.4}$$

for any $\kappa > 0$ and for some universal numeric constant $c > 0$.

The idea of the proof of Theorem 3.4 is the following. It turns out that it is enough to prove (3.4) for sufficiently small $\kappa > 0$ and

sufficiently large $c > 0$. We suppose the converse of this, i.e.,

$$\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k\text{ times}}) > \frac{c}{D}\log\frac{1}{\kappa^2}$$

for sufficiently small $\kappa > 0$ and sufficiently large $c > 0$. Put $\alpha := (c/D)\log(1/\kappa)$. By the definition of the separation distance there exists k Borel subsets $A_0, A_1.\cdots, A_{k-1} \subseteq M$ such that $\min_{i\neq j} d(A_i, A_j) > \alpha$ and $\mu(A_i) \geq \kappa$ for any i. If we choose the constant c large enough so that

$$\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k\text{ times}},\kappa^{100}) \leq \mathrm{Sep}((M,\mu);\underbrace{\kappa^{100},\kappa^{100},\cdots,\kappa^{100}}_{k+1\text{ times}}) \leq \alpha/100,$$

then by Lemma 2.8 we have

$$\mu\Big(\bigcup_{i=0}^{k-1} C_{\alpha/100}(A_i)\Big) \geq 1-\kappa^{100}.$$

It means that if $\kappa > 0$ is sufficiently small, the measure of the set $\bigcup_{i=0}^{k-1} C_{\alpha/100}(A_i)$ is nearly 1. Although it is not true, we assume that

$$\mu\Big(\bigcup_{i=0}^{k-1} C_{\alpha/100}(A_i)\Big) = 1 \tag{3.5}$$

in order to tell the idea of the proof. Putting $A := C_{\alpha/100}(A_0)$ and $B := \bigcup_{i=1}^{k-1} C_{\alpha/100}(A_i)$, we have $M = A\cup B$, $A\cap B = \emptyset$, $\mu(A) \geq \kappa, \mu(B) \geq \kappa$, and $d(A,B) \geq \alpha/2$.

Let $(\mu_t)_{t\in[0,1]}$ be a geodesic from $\mu_A := (1/\mu(A))\mu|_A$ to μ with respect to d_2^W. For sufficiently small $t > 0$ we have $d(x,A) < \alpha/2 \leq d(A,B)$ for any $x \in \mathrm{Supp}\,\mu_t$, which gives $\mathrm{Supp}\,\mu_t \subseteq A$. This leads a contradiction since by (3.2) we have

$$\log\mu(A) \geq \log\mu(\mathrm{Supp}\,\mu_t) \geq (1-t)\log\mu(A) + t\log\mu(M),$$

which implies $\log\mu(A) \geq 0$. Although (3.5) is always not true, we show below that the above idea can be accomplished by controlling separated subsets and estimating average distances between them.

Proof of Theorem 3.4. It suffices to prove that there exist two universal numeric constants $c_0, \kappa_0 > 0$ such that

$$\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k\text{ times}}) \leq \frac{c_0}{D}\log\frac{1}{\kappa^2} \tag{3.6}$$

for any $\kappa \leq \kappa_0$. In fact, if $\kappa \geq 1/2$, then the left-hand side of the above inequality is zero and there is nothing to prove. In the case where $\kappa_0 < \kappa \leq 1/2$, by (3.6) we have

$$\begin{aligned}\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k \text{ times}}) &\leq \mathrm{Sep}((M,\mu);\underbrace{\kappa_0,\kappa_0,\cdots,\kappa_0}_{k \text{ times}}) \\ &\leq \frac{c_0 \log \frac{1}{\kappa_0^2}}{D \log \frac{1}{\kappa^2}} \log \frac{1}{\kappa^2} \\ &\leq \frac{c_0 \log \frac{1}{\kappa_0^2}}{D \log 4} \log \frac{1}{\kappa^2},\end{aligned}$$

which implies the conclusion of the theorem.

Suppose the contrary to (3.6), i.e.,

$$\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k \text{ times}}) > \frac{c_1}{D} \log \frac{1}{\kappa^2}, \tag{3.7}$$

where $c_1 > 0$ is a sufficiently large universal numeric constant and $\kappa > 0$ is a sufficiently small number. Both the largeness of c_1 and the smallness of κ will be specified later. Note that the assumption (3.7) immediately gives $k\kappa < 1$ (otherwise, the left-hand side of (3.7) is zero). We denote the right-hand side of (3.7) by α, i.e.,

$$\alpha := \frac{c_1}{D} \log \frac{1}{\kappa^2}.$$

Claim 3.5. *If $c_1 > 0$ (resp., $\kappa > 0$) in (3.7) is large enough (resp., small enough), then there exist two closed subsets $B_0, B_1 \subseteq M$ such that $B_0 \subseteq B_1$, $\kappa/4 \leq \mu(B_0) \leq 1/2$, $\mu(B_1) \geq 1 - \kappa^6$, and*

$$d_M(B_0, B_1 \setminus B_0) \geq c_2 \max\left\{\alpha, \frac{\kappa}{\sqrt{\lambda_1(M,\mu)}}\right\}$$

for some universal numeric constant $c_2 > 0$.

Proof. The assumption (3.7) implies the existence of k Borel subsets $A_0, A_1, \cdots, A_{k-1} \subseteq M$ such that $\mu(A_i) \geq \kappa$ for any i and

$$d_M(A_i, A_j) \geq \alpha \text{ for any } i \neq j.$$

If $\kappa < 1/8$ and $c_1 \geq 8$, then by (3.3) we have

$$\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k \text{ times}},1/8) \leq \mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k+1 \text{ times}}) \leq \frac{1}{D} \log \frac{1}{\kappa^2} \leq \frac{\alpha}{8}.$$

Hence Lemma 2.8 yields

$$\mu\Big(\bigcup_{i=0}^{k-1} C_{\alpha/8}(A_i)\Big) \geq \frac{7}{8}. \tag{3.8}$$

Note that

$$d_M(C_{\alpha/8}(A_i), C_{\alpha/8}(A_j)) \geq \alpha/4 \tag{3.9}$$

for any $i \neq j$. According to Proposition 2.12 we take $X_0, X_1 \subseteq M$ such that

$$\mu(X_i) \geq \frac{1}{2} - \frac{\kappa}{4} \quad (i = 0, 1) \tag{3.10}$$

and

$$d_M(X_0, X_1) \geq \frac{\kappa}{8\sqrt{\lambda_1(M, \mu)}}. \tag{3.11}$$

Set $Y := X_0 \cup X_1$. By (3.10) we have $\mu(Y) \geq 1 - \frac{\kappa}{2}$ and thus

$$\mu(Y \cap C_{\alpha/8}(A_i)) \geq \mu(Y \cap A_i) \geq \Big(1 - \frac{\kappa}{2}\Big) + \kappa - 1 \geq \frac{\kappa}{2}$$

for each $i = 0, 1, \cdots, k-1$. Suppose that $\mu(X_i \cap C_{\alpha/8}(A_l)) < \kappa/4$ for some $i \in \{0, 1\}$ and for any $l = 0, 1, \cdots, k-1$. Then we have

$$\mu\Big(X_i \cap \bigcup_{l=0}^{k-1} C_{\alpha/8}(A_l)\Big) \leq \frac{k\kappa}{4} < \frac{1}{4}.$$

Combining (3.8) with (3.10) we also get

$$\mu\Big(X_i \cap \bigcup_{l=0}^{k-1} C_{\alpha/8}(A_l)\Big) \geq \Big(\frac{1}{2} - \frac{\kappa}{4}\Big) + \frac{7}{8} - 1 \geq \frac{1}{4},$$

which is a contradiction (we have used $\kappa < 1/8$). Therefore for each $l = 0, 1, \cdots, k-1$ we may choose $n_l \in \{0, 1\}$ so that

$$\mu(X_{n_l} \cap C_{\alpha/8}(A_l)) \geq \kappa/4 \tag{3.12}$$

and

$$I_i := \{l \in \{0, 1, \cdots, k-1\} \mid n_l = i\} \neq \emptyset \quad (i = 0, 1).$$

For each $i = 0, 1$, we set

$$A_i' := X_i \cap \bigcup_{l \in I_i} C_{\alpha/8}(A_l).$$

Combining (3.9) with (3.11) yields

$$d_M(A_0', A_1') \geq \max\Big\{\frac{\kappa}{8\sqrt{\lambda_1(M, \mu)}}, \frac{\alpha}{4}\Big\} =: \beta. \tag{3.13}$$

Since

$$\mathrm{Sep}\Big((M,\mu);\underbrace{\frac{\kappa}{2},\frac{\kappa}{2},\cdots,\frac{\kappa}{2}}_{k\text{ times}},\kappa^6\Big)\le \mathrm{Sep}((M,\mu);\underbrace{\kappa^6,\kappa^6,\cdots,\kappa^6}_{k+1\text{ times}})\le \frac{c_3}{D}\log\frac{1}{\kappa^2}$$

for some universal numeric constant $c_3>0$, we get

$$\mathrm{Sep}\Big((M,\mu);\underbrace{\frac{\kappa}{2},\frac{\kappa}{2},\cdots,\frac{\kappa}{2}}_{k\text{ times}},\kappa^6\Big)\le \frac{\alpha}{16}\Big(\le\frac{\beta}{4}\Big)$$

provided that c_1 in (3.7) is large enough. Put $B_0:=C_{\alpha/16}(A_0')$ and $B_1:=C_{\alpha/16}(A_0')\cup C_{\alpha/16}(A_1')$. We may assume that $\mu(C_{\alpha/16}(A_0'))\le 1/2$. Thanks to Lemma 2.8 it is easy to check that $\mu(B_1)\ge 1-\kappa^6$. By (3.12) and (3.13), we see that B_0 and B_1 possess the other desired properties. □

We consider two Borel probability measures μ_{B_i}, $i=0,1$, defined by

$$\mu_{B_i}:=\frac{\mu|_{B_i}}{\mu(B_i)}.$$

The following claim is essentially due to Gromov [22] (see also [18, Claim 5.10]). He used it in the context of the convergence theory of mm-spaces without detailed proof. Since our context is different from his one, we include the proof for the concreteness of this paper. The proof below is shorter than the one in [18, Claim 5.10].

Claim 3.6 ([22, Section $3\frac{1}{2}$.47])**.** *There exist a universal numeric constant $c_4>0$ and a coupling π of μ_{B_0} and μ_{B_1} such that*

$$\pi\Big(\Big\{(x,y)\in M\times M\mid d_M(x,y)>\frac{c_4\log\frac{1}{\kappa^2}}{\sqrt{\lambda_1(M,\mu)}}\Big\}\Big)\le\kappa^6.$$

Proof. We use the identity $\mathrm{di}_\lambda(\mu_{B_0},\mu_{B_1})=\mathrm{Tra}_\lambda(\mu_{B_0},\mu_{B_1})$ (Theorem 2.15). Put $\delta:=\frac{c_4}{\sqrt{\lambda_1(M,\mu)}}\log\frac{1}{\kappa^2}$, where $c_4>0$ is a numeric universal constant which will be determined later. We shall prove that

$$\mu_{B_1}(C_\delta(A))\ge\mu_{B_0}(A)-\kappa^6 \tag{3.14}$$

for any Borel subset $A\subseteq B_1$, which implies the claim. In fact, applying (3.14) to Theorem 2.15 gives that there exists a δ-transportation π_0 from μ_{B_0} to μ_{B_1} such that $\mathrm{def}\,\pi_0\le\kappa^6$. If $\mathrm{def}\,\pi_0=0$, then we set $\pi:=\pi_0$. If $\mathrm{def}\,\pi_0>0$, then set

$$\pi:=\pi_0+\frac{1}{\mathrm{def}\,\pi_0}(\mu_{B_0}-(\mathrm{proj}_1)_*\pi_0)\times(\mu_{B_1}-(\mathrm{proj}_2)_*\pi_0).$$

It is easy to check that π fulfills the desired property.

To prove (3.14) we may assume that $\mu_{B_0}(A) \geq \kappa^6$, which yields that

$$\mu_{B_1}(A) \geq \mu(A) \geq \kappa^6 \mu(B_0) \geq \kappa^7/4.$$

Using Lemma 2.7 and (2.3) we choose $c_4 > 0$ so that

$$\begin{aligned}\mathrm{Sep}\left((B_1, \mu_{B_1}); \frac{\kappa^7}{4}, \frac{\kappa^7}{4}\right) &\leq \mathrm{Sep}\left((M,\mu); (1-\kappa^6)\frac{\kappa^7}{4}, (1-\kappa^6)\frac{\kappa^7}{4}\right) \\ &\leq \frac{c_4}{\sqrt{\lambda_1(M,\mu)}} \log \frac{1}{\kappa^2} \ (= \delta).\end{aligned}$$

Lemma 2.8 implies that

$$\mu_{B_1}(C_\delta(A)) \geq 1 - \frac{\kappa^7}{4} \geq 1 - \kappa^6 \geq \mu_{B_0}(A) - \kappa^6,$$

which is (3.14). □

We set

$$\Delta := \Big\{(x,y) \in M \times M \mid d(x,y) \leq \frac{c_4 \log \frac{1}{\kappa^2}}{\sqrt{\lambda_1(M,\mu)}}\Big\}.$$

We consider two Borel probability measures $\mu_0 := a(\mathrm{proj}_1)_*(\pi|_\Delta)$ and $\mu_1 := a(\mathrm{proj}_2)_*(\pi|_\Delta)$, where $a := \pi(\Delta)^{-1}$. By Claim 3.6 we have

$$1 \leq a \leq \frac{1}{1-\kappa^6} \tag{3.15}$$

and

$$d_2^W(\mu_0,\mu_1)^2 \leq a \int_{M\times M} d(x,y)^2 d\pi|_\Delta(x,y) \leq \Big\{\frac{c_4 \log \frac{1}{\kappa^2}}{\sqrt{\lambda_1(M,\mu)}}\Big\}^2. \tag{3.16}$$

Take an optimal dynamical transference plan Π such that $(e_i)_*\Pi = \mu_i$ for each $i = 0, 1$. Putting $r := d_M(B_0, B_1 \setminus B_0)$, we consider

$$\Gamma_t := \{\gamma \in \mathrm{Supp}\,\Pi \mid d_M(e_0(\gamma), e_t(\gamma)) \leq r/2\}.$$

By (3.16) we have

$$\frac{r^2}{4}\Pi(\Gamma \setminus \Gamma_t) \leq d_2^W((e_0)_*\Pi, (e_t)_*\Pi)^2 = t^2\, d_2^W(\mu_0,\mu_1)^2 \leq \Big\{\frac{c_4 t \log \frac{1}{\kappa^2}}{\sqrt{\lambda_1(M,\mu)}}\Big\}^2.$$

According to Claim 3.5 we thus get

$$\Pi(\Gamma_t) \geq 1 - \frac{c_5 t^2 \Big(\log \frac{1}{\kappa^2}\Big)^2}{\kappa^2} \tag{3.17}$$

for some universal numeric constant $c_5 > 0$. For $s \in [0,1]$ we put $\nu_s := (e_s)_* \frac{\Pi|_{\Gamma_t}}{\Pi(\Gamma_t)}$. By the definition of ν_s we obtain the following.

Claim 3.7. $\mathrm{Supp}\,\nu_t \cap B_1 \subseteq B_0$.

By using Claim 3.7, we get

$$\begin{aligned}(3.18)\quad \log\mu(B_0)+\frac{\kappa^6}{\mu(B_0)} &\geq \log\mu(B_0)+\log\left(1+\frac{\kappa^6}{\mu(B_0)}\right)\\ &= \log(\mu(B_0)+\kappa^6)\\ &\geq \log\{\mu(\mathrm{Supp}\,\nu_t\cap B_1)+\mu(\mathrm{Supp}\,\nu_t\setminus B_1)\}\\ &= \log\mu(\mathrm{Supp}\,\nu_t)\end{aligned}$$

Note that $(\nu_s)_{s\in[0,1]}$ is a geodesic between ν_0 and ν_1. Since

$$(3.19)\quad \nu_i=\frac{(e_i)_*\Pi|_{\Gamma_t}}{\Pi(\Gamma_t)}\leq\frac{(e_i)_*\Pi}{\Pi(\Gamma_t)}=\frac{\mu_i}{\Pi(\Gamma_t)}\leq\frac{a}{\Pi(\Gamma_t)}(\mathrm{proj}_{i+1})_*\pi=\frac{a}{\Pi(\Gamma_t)}\mu_{B_i}$$

for $i=0,1$, each ν_i is absolutely continuous with respect to μ, and especially the above geodesic $(\nu_s)_{s\in[0,1]}$ is unique. For each $i=0,1$, we write $d\nu_i=\rho_i d\mu$. By (3.1), we get

$$(3.20)\quad \log\mu(\mathrm{Supp}\,\nu_t)\geq-(1-t)\int_M\rho_0\log\rho_0 d\mu-t\int_M\rho_1\log\rho_1 d\mu.$$

For a subset $A\subseteq M$ we denote by 1_A the characteristic function of A, i.e., $1_A(x):=1$ if $x\in A$ and $1_A(x):=0$ if $x\in M\setminus A$.

Claim 3.8. *We have*

$$\rho_i\log\rho_i\leq\frac{c_t1_{B_i}}{\mu(B_i)}\log\frac{c_t1_{B_i}}{\mu(B_i)}\quad(i=0,1),$$

where $c_t:=a/\Pi(\Gamma_t)$.

Proof. By (3.19) we have $\rho_i\leq(c_t/\mu(B_i))1_{B_i}$. Since $c_t\geq1$ and $u\log u\leq v\log v$ for any two positive numbers u,v such that $u\leq v$ and $v\geq1$, we obtain the claim. □

Combining Claim 3.8 with (3.18) and (3.20) we have

$$\begin{aligned}&\log\mu(B_0)+\frac{\kappa^6}{\mu(B_0)}\\ \geq\ &-(1-t)\int_M\frac{c_t1_{B_0}}{\mu(B_0)}\log\frac{c_t1_{B_0}}{\mu(B_0)}d\mu-t\int_M\frac{c_t1_{B_1}}{\mu(B_1)}\log\frac{c_t1_{B_1}}{\mu(B_1)}d\mu\\ =\ &-c_t\log c_t+c_t(1-t)\log\mu(B_0)+c_tt\log\mu(B_1).\end{aligned}$$

Substituting $t := \kappa^3$, we thereby obtain

$$\begin{aligned}(3.21)\qquad &\log(1/2) + 4\kappa^2 \\ &\geq \log\mu(B_0) + \frac{\kappa^6}{\kappa^3\mu(B_0)} \\ &\geq -\frac{c_t}{\kappa^3}\log c_t + \frac{c_t-1}{\kappa^3}(1-\kappa^3)\log\mu(B_0) + c_t\log\mu(B_1).\end{aligned}$$

Using (3.15) and (3.17) we estimate each term on the right-side of the above inequalities as

$$\begin{aligned}&\frac{c_t\log c_t}{\kappa^3} \\ &= \frac{a}{\Pi(\Gamma_t)}\cdot\frac{\log a - \log\Pi(\Gamma_t)}{\kappa^3} \\ &\leq \frac{1}{1-\kappa^6}\Big(1 - \frac{c_5\kappa^6\Big(\log\frac{1}{\kappa^2}\Big)^2}{\kappa^2}\Big)^{-1} \\ &\qquad\times\frac{1}{\kappa^3}\Big(\log\frac{1}{1-\kappa^6} - \log\Big(1 - \frac{c_5\kappa^6\Big(\log\frac{1}{\kappa^2}\Big)^2}{\kappa^2}\Big)\Big) \\ &\leq \frac{1}{1-\kappa^6}\Big(1 - c_5\kappa^4\Big(\log\frac{1}{\kappa^2}\Big)^2\Big)^{-1}\cdot 2\Big(\kappa^3 + c_5\kappa\Big(\log\frac{1}{\kappa^2}\Big)^2\Big),\end{aligned}$$

$$\begin{aligned}\Big|\frac{c_t-1}{\kappa^3}\log\mu(B_0)\Big| &\leq \frac{a-\Pi(\Gamma_t)}{\kappa^3\Pi(\Gamma_t)}\log\frac{2}{\kappa} \\ &\leq \frac{\frac{1}{1-\kappa^4} - 1 + c_5\kappa^4\Big(\log\frac{1}{\kappa^2}\Big)^2}{\kappa^3\Pi(\Gamma_t)}\log\frac{2}{\kappa} \\ &\leq \kappa\frac{1 + c_5(1-\kappa^4)\Big(\log\frac{1}{\kappa^2}\Big)^2}{(1-\kappa^4)\Big(1 - c_5\kappa^4\Big(\log\frac{1}{\kappa^2}\Big)^2\Big)}\log\frac{2}{\kappa},\end{aligned}$$

and

$$|c_t\log\mu(B_1)| \leq \frac{a}{\Pi(\Gamma_t)}\log\frac{1}{1-\kappa^6} \leq \frac{2\kappa^6}{(1-\kappa^6)\Big(1 - c_5\kappa^4\Big(\log\frac{1}{\kappa^2}\Big)\Big)}.$$

These estimates imply the right-side of the inequalities (3.21) is close to zero for sufficiently small $\kappa > 0$. Since the left-side of the inequality (3.21) is about $\log(1/2) < 0$ for sufficiently small $\kappa > 0$, this is a contradiction. This completes the proof of the theorem. □

4. Proof of Theorem 1.6

On a closed weighted Riemannian manifold (M, μ), denote by $(P_t)_{t\geq 0}$ the semigroup associated with the infinitesimal generator Δ_μ. For each $t \geq 0$, $P_t : C^\infty(M) \to C^\infty(M)$ is a bounded linear operator and we extend the action of P_t to $L^p(\mu)$ $(p \geq 1)$.

The following gradient estimate of the heat semigroup is due to Bakry and Ledoux [3]. One might regard it as a dimension-free Li-Yau parabolic gradient inequality [31].

Lemma 4.1 (Bakry-Ledoux, [3, Lemma 4.2])**.** *Let (M, μ) be a closed weighted Riemannian manifold of Bakry-Émery Ricci curvature bounded from below by a nonpositive real number K. Then for any $t \geq 0$ and $f \in C^\infty(M)$ we have*

$$c(t)|\nabla P_t(f)|^2 \leq P_t(f^2) - (P_t(f))^2,$$

where

$$c(t) := \frac{1 - \exp(2Kt)}{-K} \ (= 2t \text{ if } K = 0).$$

Corollary 4.2. *If (M, μ) has nonnegative Bakry-Émery Ricci curvature, then for any $t \geq 0$, $p \geq 2$, and $f \in C^\infty(M)$, we have*

$$\||\nabla P_t(f)|\|_{L^p(\mu)} \leq \frac{1}{\sqrt{2t}}\|f\|_{L^p(\mu)}.$$

From Corollary 4.2 Ledoux obtained the following lemma:

Lemma 4.3 (Ledoux, [26, (5.5)])**.** *Assume that (M, μ) has nonnegative Bakry-Émery Ricci curvature. Then for any $f \in C^\infty(M)$, we have*

$$\|f - P_t(f)\|_{L^1(\mu)} \leq \sqrt{2t}\||\nabla f|\|_{L^1(\mu)}.$$

Proof of Theorem 1.6. Take any $k+1$ non-empty, disjoint Borel subsets $A_0, A_1, \cdots, A_k \subseteq M$. We may assume that $\mu(A_0) \leq \mu(A_1) \leq \cdots \leq \mu(A_k)$, and thus

$$\sum_{i=0}^{k-1} \mu(A_i) \leq 1 - \frac{1}{k+1} \text{ and } \mu(A_i) \leq 1/2 \text{ for any } i = 0, 1, \cdots, k-1.$$

We put $t := 4k(k+1)/\lambda_k(M, \mu)$. We shall prove that there exists i_0, $0 \leq i_0 \leq k-1$, such that

$$\mu^+(A_{i_0}) \geq (80k^3)^{-1}\sqrt{\lambda_k(M, \mu)}\mu(A_{i_0}). \tag{4.1}$$

For each $i = 0, 1, \cdots, k-1$, let $1_{A_i,\varepsilon}(x) := \min\{0, 1 - \frac{1}{\varepsilon}\, d(x, A_i)\}$ denote a Lipschitz approximation of 1_{A_i}. Note that

$$\frac{\mu(C_\varepsilon(A_i)) - \mu(A_i)}{\varepsilon} \geq \int_M |\nabla 1_{A_i,\varepsilon}| d\mu,$$

where for a Lipschitz function $f : M \to \mathbb{R}$ and $x \in M$, we put

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d_M(y, x)}.$$

Letting $\varepsilon \to 0$, by Lemma 4.3 we have

$$\sqrt{2t}\mu^+(A_i) \geq \|1_{A_i} - P_t(1_{A_i})\|_{L^1(\mu)}.$$

Since the right-side of the above inequality can be written as

$$\begin{aligned}
&\int_{A_i} (1 - P_t(1_{A_i}))d\mu + \int_{M\setminus A_i} P_t(1_{A_i}) d\mu \\
&= 2\Big(\mu(A_i) - \int_{A_i} P_t(1_{A_i}) d\mu\Big) \\
&= 2\Big(\mu(A_i)(1 - \mu(A_i)) - \int_M (P_t(1_{A_i}) - \mu(A_i))(1_{A_i} - \mu(A_i)) d\mu\Big),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sqrt{2t}\mu^+(A_i) \\
&\quad \geq 2\Big(\mu(A_i)(1 - \mu(A_i)) - \int_M (P_t(1_{A_i}) - \mu(A_i))(1_{A_i} - \mu(A_i)) d\mu\Big).
\end{aligned} \tag{4.2}$$

Observe that $P_t(1_{A_i}) - \mu(A_i)$, $i = 0, 1, \cdots, k-1$, are linearly independent and orthogonal to constant functions on M. Thus the Rayleigh quotient representation of $\lambda_k(M, \mu)$ yields that there exist $a_0, a_1, \cdots, a_{k-1} \in \mathbb{R}$ such that

$$\lambda_k(M, \mu) \leq \frac{\| |\nabla(\sum_{i=0}^{k-1} a_i (P_t(1_{A_i}) - \mu(A_i)))| \|^2_{L^2(\mu)}}{\| \sum_{i=0}^{k-1} a_i (P_t(1_{A_i}) - \mu(A_i)) \|^2_{L^2(\mu)}}. \tag{4.3}$$

Put $f_0 := \sum_{i=0}^{k-1} a_i 1_{A_i}$. We consider the following two cases: (I) $\|f_0 - \int_M f_0 d\mu\|_{L^2(\mu)} \geq 2\|f_0 - P_t(f_0)\|_{L^2(\mu)}$, (II) $\|f_0 - \int_M f_0 d\mu\|_{L^2(\mu)} \leq 2\|f_0 - P_t(f_0)\|_{L^2(\mu)}$.

We prove that the case (I) cannot happen from the our choice of t. Suppose that (I) holds. In this case we get

$$\|\sum_{i=0}^{k-1} a_i(P_t(1_{A_i})-\mu(A_i))\|_{L^2(\mu)} = \Big\|P_t(f_0)-\int_M f_0 d\mu\Big\|_{L^2(\mu)} \geq \frac{1}{2}\Big\|f_0-\int_M f_0 d\mu\Big\|_{L^2(\mu)}. \tag{4.4}$$

We estimate the right-side of the above inequality from below:

Claim 4.4. *We have*

$$\int_M \Big(\sum_{i=0}^{k-1} a_i(1_{A_i}-\mu(A_i))\Big)^2 d\mu \geq \frac{1}{k+1}\sum_{i=0}^{k-1} a_i^2 \int_M (1_{A_i}-\mu(A_i))^2 d\mu.$$

Proof. Since

$$\int_M (1_{A_i}-\mu(A_i))^2 d\mu = \mu(A_i)(1-\mu(A_i)) \tag{4.5}$$

and

$$\int_M \Big(\sum_{i=0}^{k-1} a_i(1_{A_i}-\mu(A_i))\Big)^2 d\mu = \sum_{i=0}^{k-1} a_i^2\mu(A_i) - \Big(\sum_{i=0}^{k-1} a_i\mu(A_i)\Big)^2,$$

it suffices to prove

$$\Big(1-\frac{1}{k+1}\Big)\sum_{i=0}^{k-1} a_i^2\mu(A_i) + \frac{1}{k+1}\sum_{i=0}^{k-1} a_i^2\mu(A_i)^2 \geq \Big(\sum_{i=0}^{k-1} a_i\mu(A_i)\Big)^2. \tag{4.6}$$

Since

$$\begin{aligned}\Big(\sum_{i=0}^{k-1} a_i\mu(A_i)\Big)^2 &= \Big(\sum_{j=0}^{k-1}\mu(A_j)\Big)^2 \cdot \Big(\sum_{i=0}^{k-1}\frac{\mu(A_i)}{\sum_{j=0}^{k-1}\mu(A_j)}a_i\Big)^2 \\ &\leq \Big(\sum_{j=0}^{k-1}\mu(A_j)\Big)^2 \sum_{i=0}^{k-1}\frac{\mu(A_i)}{\sum_{j=0}^{k-1}\mu(A_j)}a_i^2 \\ &\leq \Big(1-\frac{1}{k+1}\Big)\sum_{i=0}^{k-1} a_i^2\mu(A_i),\end{aligned}$$

we have (4.6). This completes the proof of the claim. □

Claim 4.4 together with (4.3) and (4.4) implies the existence of i_0, $0 \leq i_0 \leq k-1$, such that

$$\lambda_k(M,\mu)\|1_{A_{i_0}}-\mu(A_{i_0})\|^2_{L^2(\mu)} \leq 4k(k+1)\||\nabla P_t(1_{A_{i_0}})|\|^2_{L^2(\mu)}.$$

Using Corollary 4.2 and $t = 4k(k+1)/\lambda_k(M,\mu)$ we obtain

$$\begin{aligned}\lambda_k(M,\mu)\|1_{A_{i_0}} - \mu(A_{i_0})\|^2_{L^2(\mu)} &\le \frac{2k(k+1)}{t}\|1_{A_{i_0}} - \mu(A_{i_0})\|^2_{L^2(\mu)} \\ &= 2^{-1}\lambda_k(M,\mu)\|1_{A_{i_0}} - \mu(A_{i_0})\|^2_{L^2(\mu)},\end{aligned}$$

which is a contradiction.

Since (II) holds, Lemma 4.3 yields

$$\begin{aligned}\frac{1}{4}\Big\|f_0 - \int_M f_0 d\mu\Big\|^2_{L^2(\mu)} &\le \|P_t(f_0) - f_0\|^2_{L^2(\mu)} \\ &\le k\sum_{i=0}^{k-1} a_i^2\|P_t(1_{A_i}) - 1_{A_i}\|^2_{L^2(\mu)} \\ &\le k\sum_{i=0}^{k-1} a_i^2\|P_t(1_{A_i}) - 1_{A_i}\|_{L^1(\mu)} \\ &\le k\sqrt{2t}\sum_{i=0}^{k-1} a_i^2\||\nabla 1_{A_i}|\|_{L^1(\mu)} \\ &= k\sqrt{2t}\sum_{i=0}^{k-1} a_i^2\mu^+(A_i).\end{aligned} \tag{4.7}$$

According to Claim 4.4 and (4.7), there exists i_0, $0 \le i_0 \le k-1$, such that

$$\|1_{A_{i_0}} - \mu(A_{i_0})\|^2_{L^2(\mu)} \le 4k(k+1)\sqrt{2t}\mu^+(A_{i_0}).$$

Thus we get

$$\begin{aligned}\int_M (P_t(1_{A_{i_0}}) - \mu(A_{i_0}))(1_{A_{i_0}} - \mu(A_{i_0}))d\mu &\le \|1_{A_{i_0}} - \mu(A_{i_0})\|^2_{L^2(\mu)} \\ &\le 4k(k+1)\sqrt{2t}\mu^+(A_{i_0}).\end{aligned}$$

Since $\mu(A_{i_0}) \le 1/2$, it follows from (4.2) that

$$(8k^2+8k+1)\sqrt{2t}\mu^+(A_{i_0}) \ge 2\mu(A_{i_0})(1-\mu(A_{i_0})) \ge \mu(A_{i_0}).$$

Recalling that $t = 4k(k+1)/\lambda_k(M,\mu)$, we finally obtain

$$\mu^+(A_{i_0}) \ge \frac{\sqrt{\lambda_k(M,\mu)}}{(16k(k+1)+2)\sqrt{2k(k+1)}}\mu(A_{i_0}) \ge \frac{\sqrt{\lambda_k(M,\mu)}}{80k^3}\mu(A_{i_0}),$$

which implies (4.1). This completes the proof of the theorem. □

Remark 4.5. From the proof of [3] Bakry-Ledoux's lemma (Lemma 4.1) follows from the following Bakry-Émery type L^2-gradient estimate:

$$|\nabla P_t(f)|^2(x) \leq e^{-2Kt} P_t(|\nabla f|^2)(x) \tag{4.8}$$

for any Lipschitz function f and any $x \in X$. Gigli, Kuwada, and Ohta proved the gradient estimate (4.8) for compact finite-dimensional Alexandrov spaces satisfying $\mathrm{CD}(K, \infty)$ ([19, Theorem 4.3]). Here Alexandrov spaces are metric spaces whose 'sectional curvature' is bounded from below in the sense of the triangle comparison property. In particular the same argument in this section implies that Theorem 1.6 holds for compact finite-dimensional Alexandrov spaces satisfying $\mathrm{CD}(0, \infty)$. Refer to [24] for the Laplacian on Alexandrov spaces. We remark that Theorem 1.4 holds for compact finite-dimensional Alexandrov spaces from the proof of [28]. Consequently the k-th eigenvalue of Laplacian and the k-way isoperimetric constant are equivalent up to polynomials of k for compact finite-dimensional Alexandrov spaces satisfying $\mathrm{CD}(0, \infty)$. In particular it is also valid for compact finite-dimensional Alexandrov spaces of nonnegative curvature, since such spaces satisfy $\mathrm{CD}(0, \infty)$ ([45], [53]).

5. Rough stability of eigenvalues of the weighted Laplacian and multi-way isoperimetric constants

We first review the concentration topology. Recall that the *Hausdorff distance* between two closed subsets A and B in a metric space X is defined by

$$d_H(A, B) := \inf\{\, \varepsilon > 0 \mid A \subseteq C_\varepsilon(B),\ B \subseteq C_\varepsilon(A) \,\}.$$

Let (I, μ) be a probability space. We denote by $\mathcal{F}(I, \mathbb{R})$ the space of all μ-measurable functions on I. Given $\lambda \geq 0$ and $f, g \in \mathcal{F}(I, \mathbb{R})$, we put

$$\mathrm{me}_\lambda(f, g) := \inf\{\, \varepsilon > 0 \mid \mu(|f - g| > \varepsilon) \leq \lambda\varepsilon \,\},$$

where $\mu(|f - g| > \varepsilon) := \mu(\{x \in I \mid |f(x) - g(x)| > \varepsilon\})$. Note that, if any two functions $f, g \in \mathcal{F}(I, \mathbb{R})$ with $f = g$ a.e. are identified to each other, then me_λ is a distance function on $\mathcal{F}(I, \mathbb{R})$ for any $\lambda \geq 0$ and its topology on $\mathcal{F}(I, \mathbb{R})$ coincides with the topology of the convergence in measure for any $\lambda > 0$. The distance functions me_λ for all $\lambda > 0$ are mutually equivalent.

Let d be a *semi-distance function on* I, i.e., a nonnegative symmetric function on $I \times I$ satisfying the triangle inequality. We indicate by $\mathcal{L}ip_1(d)$ the space of all 1-Lipschitz functions on I with respect to d.

Note that $\mathcal{L}ip_1(d)$ is a closed subset in $(\mathcal{F}(I,\mathbb{R}),\mathrm{me}_\lambda)$ for any $\lambda \geq 0$. For $\lambda \geq 0$ and two semi-distance functions d and d' on I, we define

$$H_\lambda\mathcal{L}\iota_1(d,d') := d_H(\mathcal{L}ip_1(d),\mathcal{L}ip_1(d')),$$

where d_H is the Hausdorff distance function in $(\mathcal{F}(X,\mathbb{R}),\mathrm{me}_\lambda)$. $H_\lambda\mathcal{L}\iota_1$ is a distance function on the space of all semi-distance functions on X for all $\lambda \geq 0$, and the two distance functions $H_\lambda\mathcal{L}\iota_1$ and $H_{\lambda'}\mathcal{L}\iota_1$ are equivalent to each other for any $\lambda,\lambda' > 0$. We denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$.

For any mm-space X there exists a Borel measurable map $\varphi : [\,0,1\,) \to X$ with $\varphi_*\mathcal{L} = \mu_X$ (see [23, Theorem 17.41]). We call such a map φ a *parameter of* X. Note that a parameter of X is not unique in general. For a parameter φ of X, we define a function $\varphi^*\, d_X : [\,0,1\,)\times[\,0,1\,) \to \mathbb{R}$ by $\varphi^*\, d_X(s,t) := d_X(\varphi(s),\varphi(t))$ for any $s,t \in [\,0,1\,)$.

Definition 5.1 (Observable distance function)**.** For two mm-spaces X and Y we define

$$\underline{H}_\lambda\mathcal{L}\iota_1(X,Y) := \inf H_\lambda\mathcal{L}\iota_1(\varphi_X^*\, d_X, \varphi_Y^*\, d_Y),$$

where the infimum is taken over all parameters $\varphi_X : [\,0,1\,) \to X$ and $\varphi_Y : [\,0,1\,) \to Y$.

We say that two mm-spaces are *isomorphic* to each other if there is a measure preserving isometry between the spaces. Denote by $\mathcal{X}$ the space of isomorphic classes of mm-spaces. The function $\underline{H}_\lambda\mathcal{L}\iota_1$ is a distance function on $\mathcal{X}$ for any $\lambda \geq 0$. Note that $\underline{H}_\lambda\mathcal{L}\iota_1$ and $\underline{H}_{\lambda'}\mathcal{L}\iota_1$ are equivalent to each other for any $\lambda,\lambda' > 0$.

Definition 5.2 (Concentration topology)**.** We say that a sequence of mm-spaces X_n, $n = 1,2,\cdots$, *concentrates to* an mm-space Y if X_n converges to Y as $n \to \infty$ with respect to $\underline{H}_1\mathcal{L}\iota_1$. The topology on the set $\mathcal{X}$ induced by the observable distance function is called the *concentration topology.*

The term 'concentration topology' comes from the following: We say that a sequence of mm-spaces $\{X_n\}$ is a *Lévy family* if $\lim_{n\to\infty}\alpha_{X_n}(r) = 0$ for any $r > 0$. Due to Lévy's lemma ([29], [27, Proposition 1.3]) we obtain the following:

Proposition 5.3 ([22])**.** *A sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces is a Lévy family if and only if it concentrates to the one-point mm-space.*

For example, the sequence of n-dimensional unit spheres in $\mathbb{R}^{n+1}$, $n = 1,2,\cdots$, concentrates to the one-point space by Lévy's result([29]).

The concentration topology is strictly weaker than the measured Gromov-Hausdorff topology on the space of mm-spaces ([16]). We

mention that the concentration topology coincides with the measured Gromov-Hausdorff topology on the set of mm-spaces satisfying $\mathrm{CD}(K,N)$ for fixed K and $N < +\infty$. In fact, the set becomes compact with respect to the measured Gromov-Hausdorff topology because we have the doubling condition with a uniform doubling constant under the condition $\mathrm{CD}(K,N)$.

Answering a conjecture by Fukaya in [15], Cheeger and Colding proved the continuity of eigenvalues of Laplacian on Riemmanian manifolds with respect to the measured Gromov-Hausdorff topology under the condition $\mathrm{CD}(K,N)$ for fixed $K, N \in \mathbb{R}$ ([8]). We consider an analogy of the above Cheeger-Colding result with respect to the concentration topology:

Corollary 5.4. *There exists a universal numeric constant $c > 0$ satisfying the following. Let $\{(M_n, \mu_n)\}$ be a sequence of closed weighted Riemannian manifolds of nonnegative Bakry-Émery Ricci curvature and assume that the sequence concentrates to a closed weighted Riemannian manifold (M_∞, μ_∞). Then for any natural number k we have*

$$\limsup_{n\to\infty} \max\left\{\frac{\lambda_k(M_n,\mu_n)}{\lambda_k(M_\infty,\mu_\infty)}, \frac{\lambda_k(M_\infty,\mu_\infty)}{\lambda_k(M_n,\mu_n)}\right\} \le \exp(ck) \tag{5.1}$$

and

$$\limsup_{n\to\infty} \max\left\{\frac{h_k(M_n,\mu_n)}{h_k(M_\infty,\mu_\infty)}, \frac{h_k(M_\infty,\mu_\infty)}{h_k(M_n,\mu_n)}\right\} \le k^3 \exp(ck). \tag{5.2}$$

Note that dimension of M_n may diverge to infinity as $n \to \infty$.

The rest of this subsection is devoted to prove Corollary 5.4. For the proof we first recall the definition of observable diameter introduced by Gromov in [22]:

Definition 5.5 (Observable diameter)**.** Let $\kappa > 0$. We define the *partial diameter*

$$\mathrm{diam}(\mu_X, 1-\kappa)$$

of μ_X as the infimum of $\mathrm{diam}\, A$ over all Borel subsets $A \subseteq X$ with $\nu(A) \ge 1-\kappa$. Define the *observable diameter*

$$\mathrm{ObsDiam}_{\mathbb{R}}(X; -\kappa)$$

of X as the supremum of $\mathrm{diam}(f_*\mu_X, 1-\kappa)$ over all 1-Lipschitz functions $f : X \to \mathbb{R}$.

The idea of the observable diameter comes from the quantum and statistical mechanics, i.e., we think of μ_X as a state on a configuration space X and f is interpreted as an observable.

The next lemma expresses the relation between the observable diameter and the separation distance. The proof of the lemma is found in [17, Subsection 2.2]

Lemma 5.6 ([22])**.** *Let X be an mm-space. For any $\kappa, \kappa' > 0$ with $\kappa > \kappa'$, we have*

(1) $\mathrm{Sep}(X; \kappa, \kappa) \leq \mathrm{ObsDiam}_{\mathbb{R}}(X; -\kappa')$,
(2) $\mathrm{ObsDiam}_{\mathbb{R}}(X; -2\kappa) \leq \mathrm{Sep}(X; \kappa, \kappa)$.

Lemma 5.7. *Let X, Y be two mm-spaces and assume that $\underline{H}_1\mathcal{L}\iota_1(X, Y) < \varepsilon < 1$. Then for any $\kappa \in (\,\varepsilon, 1\,)$ we have*

$$\mathrm{ObsDiam}_{\mathbb{R}}(Y; -\kappa) \leq \mathrm{ObsDiam}_{\mathbb{R}}(X; -(\kappa - \varepsilon)) + 2\,\varepsilon\,.$$

Proof. The condition $\underline{H}_1\mathcal{L}\iota_1(X, Y) < \varepsilon$ implies the existence of two parameters $\varphi_X : [\,0, 1\,) \to X$ and $\varphi_Y : [\,0, 1\,) \to Y$ such that

$$d_H(\varphi_X^*\,\mathcal{L}ip_1(X), \varphi_Y^*\,\mathcal{L}ip_1(Y)) < \varepsilon\,.$$

Hence, for any $f \in \mathcal{L}ip_1(Y)$, there exists $g \in \mathcal{L}ip_1(X)$ such that

$$\mathcal{L}(|f \circ \varphi_Y - g \circ \varphi_X| > \varepsilon) < \varepsilon\,.$$

Take a Borel subset $A \subseteq \mathbb{R}$ such that $g_*\mu_X(A) \geq 1 - \kappa + \varepsilon$ and $\mathrm{diam}(g_*\mu_X, 1 - (\kappa - \varepsilon)) = \mathrm{diam}\, A$. Putting

$$B := f \circ \varphi_Y(\{|f \circ \varphi_Y - g \circ \varphi_X| \leq \varepsilon\} \cap (g \circ \varphi_Y)^{-1}(A)),$$

we find

$$f_*\mu_Y(B) \geq (1 - \varepsilon) + (1 - \kappa + \varepsilon) - 1 = 1 - \kappa.$$

Given $s, t \in \{|f \circ \varphi_Y - g \circ \varphi_X| \leq \varepsilon\} \cap (g \circ \varphi_Y)^{-1}(A)$ we have

$$\begin{aligned} &|f \circ \varphi_Y(s) - f \circ \varphi_Y(t)| \\ \leq\ &|f \circ \varphi_Y(s) - g \circ \varphi_X(s)| + |g \circ \varphi_X(s) - g \circ \varphi_X(t)| \\ &+ |g \circ \varphi_X(t) - f \circ \varphi_Y(t)| \\ \leq\ &\mathrm{diam}\, A + 2\,\varepsilon, \end{aligned}$$

which implies $\mathrm{diam}(f_*\mu_Y, 1 - \kappa) \leq \mathrm{diam}\, A + 2\,\varepsilon$. This completes the proof. □

Lemma 5.8. *Let (M, μ_M) and (N, μ_N) be two closed weighted Riemannian manifolds of nonnegative Bakry-Émery Ricci curvature such that $\underline{H}_1\mathcal{L}\iota_1((M, \mu_M), (N, \mu_N)) < 1/2$. Assume that two positive numbers ε, δ satisfies $\underline{H}_1\mathcal{L}\iota_1((M, \mu_M), (N, \mu_N)) < \varepsilon < 1/2$ and $\varepsilon + \delta < 1/2$. Then we have*

$$\lambda_1(N, \mu_N) \geq \lambda_1(M, \mu_M)\Big\{\frac{\delta}{2\,\varepsilon\,\sqrt{\lambda_1(M, \mu_M)} - 6\log(\frac{1}{4} - \frac{\varepsilon}{2} - \frac{\delta}{2})}\Big\}^2 \tag{5.3}$$

and

$$h_1(N,\mu_N) \geq h_1(M,\mu_M) \cdot \frac{\delta}{\varepsilon\, h_1(M,\mu_M) - 6\log(\frac{1}{4}-\frac{\varepsilon}{2}-\frac{\delta}{2})}. \tag{5.4}$$

Proof. Combining (2.3), Lemmas 5.6 and 5.7 gives that for any $\kappa > \varepsilon$ we have

$$\begin{aligned}\mathrm{ObsDiam}_{\mathbb{R}}((N,\mu_N);-\kappa) &\leq \mathrm{ObsDiam}_{\mathbb{R}}((M,\mu_M);-(\kappa-\varepsilon)) + 2\,\varepsilon \\ &\leq \mathrm{Sep}\left((M,\mu_M);\frac{\kappa-\varepsilon}{2},\frac{\kappa-\varepsilon}{2}\right) + 2\,\varepsilon \\ &\leq \frac{6}{\sqrt{\lambda_1(M,\mu_M)}}\log\frac{2}{\kappa-\varepsilon} + 2\,\varepsilon\,.\end{aligned}$$

Lemma 5.6 again yields

$$\mathrm{Sep}((N,\mu_N);\kappa,\kappa) \leq \frac{6}{\sqrt{\lambda_1(M,\mu_M)}}\log\frac{2}{\kappa-\varepsilon} + 2\,\varepsilon\,.$$

As in the proof of Lemma 2.9 (1) we obtain

$$\alpha_{(N,\mu_N)}(r) \leq \varepsilon + 2\exp(-6^{-1}\sqrt{\lambda_1(M,\mu_M)}(r-2\,\varepsilon))$$

for any $r > 2\,\varepsilon$. By subsutituting

$$r := 2\,\varepsilon - \frac{6\log(\frac{1}{4}-\frac{\varepsilon}{2}-\frac{\delta}{2})}{\sqrt{\lambda_1(M,\mu_M)}}$$

we obtain $\alpha_{(N,\mu_N)}(r) \leq 2^{-1} - \delta$. Applying Theorem 2.5 then implies the inequality (5.3). The proof of (5.4) is similar and we omit it. □

Proof of Corollary 5.4. Due to Theorem 2.5 we have $\sup_{n\in\mathbb{N}} \lambda_1(M_n,\mu_n) < +\infty$ unless $\{(M_n,\mu_n)\}$ concentrates to the one point space. Since the condition $CD(0,\infty)$ is preserved under the concentration topology ([18, Theorem 1.2]), the limit weighted manifold (M_∞,μ_∞) has nonnegative Bakry-Émery Ricci curvature. Combining Lemma 5.8 with Theorem 1.1 we obtain the corollary. □

The proof of Corollary 5.4 also follows from the following lemma and corollary together with Theorems 1.1 and 2.5:

Lemma 5.9. *Let X, Y be two mm-spaces such that $\underline{H}_1\mathcal{L}\iota_1(X,Y) < \varepsilon < 1/(k+1)$. Then for any $\kappa_0,\kappa_1,\cdots,\kappa_k,\kappa_0',\kappa_1',\cdots,\kappa_k' > 0$ such that $\kappa_i - (k+1)\,\varepsilon \geq \kappa_i'$ for any i, we have*

$$\mathrm{Sep}(Y;\kappa_0,\kappa_1,\cdots,\kappa_k) \leq \mathrm{Sep}(X;\kappa_0',\kappa_1',\cdots,\kappa_k') + 2\,\varepsilon\,.$$

Proof. Take $k+1$ Borel subsets $A_0, A_1, \cdots, A_k \subseteq Y$ such that $\mu_Y(A_i) \geq \kappa_i$ for any i and $\min_{i\neq j} d_Y(A_i, A_j) = \mathrm{Sep}(Y; \kappa_0, \kappa_1, \cdots, \kappa_k)$. Since $\underline{H}_1\mathcal{L}\iota_1(X,Y) < \varepsilon$ there exist two parameters $\varphi_X : [0,1) \to X$ and $\varphi_Y : [0,1) \to Y$ such that $H_1\mathcal{L}\iota_1(\varphi_X^* d_X, \varphi_Y^* d_Y) < \varepsilon$. For each $i = 0, 1, \cdots, k$, we put $f_i(x) := d_Y(x, A_i)$. Since each f_i is 1-Lipschitz, the condition $H_1\mathcal{L}\iota_1(\varphi_X^* d_X, \varphi_Y^* d_Y) < \varepsilon$ implies the existence of $k+1$ 1-Lipschitz functions $g_i : X \to \mathbb{R}$, $i = 0, 1, \cdots, k$, such that $\mathrm{me}_1(f_i \circ \varphi_Y, g_i \circ \varphi_X) < \varepsilon$. Putting

$$\tilde{I} := \bigcap_{i=0}^{k} \{|f_i \circ \varphi_Y - g_i \circ \varphi_X| \leq \varepsilon\}$$

we have $\mathcal{L}(\tilde{I}) \geq 1 - (k+1)\,\varepsilon$. For each $i = 0, 1, \cdots, k$ we define $B_i \subseteq X$ as $B_i := \varphi_X(\varphi_Y^{-1}(A_i) \cap \tilde{I})$. Note that $\mu_X(B_i) \geq \mathcal{L}(\varphi_Y^{-1}(A_i) \cap \tilde{I}) \geq \kappa_i - (k+1)\,\varepsilon$. For any $a_i \in \varphi_Y^{-1}(A_i) \cap \tilde{I}, a_j \in \varphi_Y^{-1}(A_j) \cap \tilde{I}$, $i \neq j$, we get

$$\begin{aligned} d_X(\varphi_X(a_i), \varphi_X(a_j)) &\geq |g_i(\varphi_X(a_i)) - g_j(\varphi_X(a_j))| \\ &\geq |f_i(\varphi_Y(a_i)) - f_j(\varphi_Y(a_j))| - 2\,\varepsilon \\ &\geq\ d_Y(A_i, A_j) - 2\,\varepsilon, \end{aligned}$$

which implies that

$$\min_{i\neq j} d_X(B_i, B_j) \geq \min_{i\neq j} d_Y(A_i, A_j) - 2\,\varepsilon = \mathrm{Sep}(Y; \kappa_0, \kappa_1, \cdots, \kappa_k) - 2\,\varepsilon\,.$$

This completes the proof. □

Corollary 5.10. *Assume that a sequence $\{X_n\}$ of mm-spaces concentrate to an mm-space Y. Then we have*

$$\liminf_{n\to\infty} \mathrm{Sep}(X_n; \kappa_0', \kappa_1', \cdots, \kappa_k') \geq \mathrm{Sep}(Y; \kappa_0, \kappa_1, \cdots, \kappa_k) \tag{5.5}$$

and

$$\limsup_{n\to\infty} \mathrm{Sep}(X_n; \kappa_0, \kappa_1, \cdots, \kappa_k) \leq \mathrm{Sep}(Y; \kappa_0', \kappa_1', \cdots, \kappa_k') \tag{5.6}$$

for any $\kappa_0, \kappa_1, \cdots, \kappa_k, \kappa_0', \kappa_1', \cdots, \kappa_k' > 0$ such that $\kappa_i > \kappa_i'$.

6. Questions

In this section we raise several questions which are concerned with this paper. We also discuss the conjecture which was posed in [18]. Throughout this section, unless otherwise stated, we will always assume that (M, μ) is a closed weighted Riemannian manifold of nonnegative Bakry-Émery Ricci curvature.

Question 6.1. *Independent of k, is it possible to bound $\lambda_{k+1}(M,\mu)/\lambda_k(M,\mu)$ or $h_{k+1}(M,\mu)/h_k(M,\mu)$ from above by a universal numeric constant ?*

Masato Mimura asked me about the fraction of $\lambda_{k+1}(M,\mu)/\lambda_k(M,\mu)$. Theorem 1.1 leads to the above question for eigenvalues of the weighted Laplacian. Due to Theorems 2.10 and 3.4, in order to give an affirmative answer to Question 6.1 for eigenvalues it suffices to extend E. Milman's theorem (Theorem 2.5) in terms of $\lambda_k(M,\mu)$ and the k-separation distance, i.e., any k-separation inequalities imply appropriate lower bounds of the k-th eigenvalue $\lambda_k(M,\mu)$. Or more weakly, it suffices to prove that any logarithmic k-separation inequalities of the form (3.3) give appropriate estimates of the k-th eigenvalue $\lambda_k(M,\mu)$ from below. This can also be considered as an extension of [20, Theorem 1.14]. In [20] Gozlan, Roberto, and Samson proved that any exponential concentration inequalities imply appropriate Poincaré inequalities under assuming $\mathrm{CD}(0,\infty)$. Notice that by Lemma 2.9 exponential concentration inequalities are nothing but logarithmic 2-separation inequalities.

For multi-way isoperimetric constants, we also need to improve k^3 order in Proposition 2.11 to some universal numeric constant. The following integration argument makes possible to improve k^3 order but it is not logarithmic separation inequalities:

Proposition 6.2. *Let (M,μ) be a closed weighted Riemannian manifold and k a natural number. Then for any $\kappa > 0$ we have*

$$\mathrm{Sep}((M,\mu);\underbrace{\kappa,\kappa,\cdots,\kappa}_{k+1\ times}) \leq \frac{2}{\log 2}\cdot\frac{\log(2/\kappa)}{h_k(M,\mu)\kappa}.$$

Proof. Let $A_0, A_1, \cdots, A_k$ be $k+1$ Borel subsets of M such that $\mu(A_i) \geq \kappa$ for any $0 \leq i \leq k$. Our goal is to prove the following inequality:

$$D := \min_{i\neq j} d_M(A_i,A_j) \leq \frac{2}{\log 2}\cdot\frac{\log(2/\kappa)}{h_k(M,\mu)\kappa}. \tag{6.1}$$

In order to prove (6.1) we may assume that each A_i is given by a finite union of open balls. For $r \in [0, D/2)$ we put $B_i := O_r(A_i)$, $i = 0, 1, \cdots, k-1$, and $B_k := M \setminus \bigcup_{i=0}^{k-1} B_i$. By the definition of $h_k(M,\mu)$, we have $\mu^+(B_{i_0}) \geq h_k(M,\mu)\mu(B_{i_0})$ for some i_0. Assume first that $i_0 = k$. Since each B_i consists of a finite union of open balls we obtain

$$\sum_{i=0}^{k-1}\mu^+(B_i) = \mu^+(B_k) \geq h_k(M,\mu)\mu(B_k) \geq h_k(M,\mu)\kappa.$$

In the case where $i_0 \leq k-1$, we get

$$\sum_{i=0}^{k-1}\mu^+(B_i) \geq \mu^+(B_{i_0}) \geq h_k(M,\mu)\mu(B_{i_0}) \geq h_k(M,\mu)\kappa$$

Combining the above two inequalities implies that

$$\mu\Big(\bigcup_{i=0}^{k-1} O_r(A_i)\Big) - \mu\Big(\bigcup_{i=0}^{k-1} A_i\Big) = \int_0^r \sum_{i=0}^{k-1} \mu^+(O_s(A_i))ds \geq h_k(M,\mu)\kappa r,$$

which yields

$$\mu\Big(M \setminus \bigcup_{i=0}^{k-1} O_r(A_i)\Big) \leq (1 - h_k(M,\mu)\kappa r)\mu\Big(M \setminus \bigcup_{i=0}^{k-1} A_i\Big). \tag{6.2}$$

What follows is a straightforward adaption of Gromov-V. Milman's argument in [21, Theorem 4.1]. Put $\varepsilon := (2\kappa h_k(M,\mu))^{-1}$. If $\varepsilon \leq r$, then there exists a natural number j such that $j\,\varepsilon \leq r < (j+1)\,\varepsilon$. Iterating (6.2) k times shows

$$\begin{aligned}
\mu\Big(M \setminus O_r\Big(\bigcup_{i=0}^{k-1} A_i\Big)\Big) &\leq \mu\Big(M \setminus O_{j\,\varepsilon}\Big(\bigcup_{i=0}^{k-1} A_i\Big)\Big) \\
&\leq (1 - h_k(M,\mu)\kappa\,\varepsilon)\mu\Big(M \setminus O_{(j-1)\,\varepsilon}\Big(\bigcup_{i=0}^{k-1} A_i\Big)\Big) \\
&\quad \cdots \\
&\leq (1 - h_k(M,\mu)\kappa\,\varepsilon)^j \mu\Big(M \setminus \bigcup_{i=0}^{k-1} A_i\Big) \\
&\leq (1 - h_k(M,\mu)\kappa\,\varepsilon)^j \\
&= \exp(-j\log 2) \\
&\leq \exp(-(r/\,\varepsilon)\log 2) \\
&= \exp(-h_k(M,\mu)r\kappa 2\log 2).
\end{aligned}$$

If $r < \varepsilon$, then we have

$$\mu\Big(M \setminus O_r\Big(\bigcup_{i=0}^{k-1} A_i\Big)\Big) \leq 1 \leq 2 \cdot 2^{-\,\varepsilon^{-1}\,r} \leq 2\exp(-h_k(M,\mu)r\kappa 2\log 2)$$

Put $r := D/2$. Combining the above two inequalities we obtain

$$\kappa \leq \mu(A_k) \leq \mu\Big(M \setminus O_{\frac{D}{2}}\Big(\bigcup_{i=0}^{k-1} A_i\Big)\Big) \leq 2\exp(-h_k(M,\mu)D\kappa\log 2),$$

which implies (6.1). This completes the proof. □

Question 6.3. *What is the right order of* $\sqrt{\lambda_k(M,\mu)}/h_k(M,\mu)$, $\lambda_k(M,\mu)/\lambda_1(M,\mu)$, *and* $h_k(M,\mu)/h_1(M,\mu)$ *in* k*? Especially can we bound* $\lambda_k(M,\mu)/\lambda_1(M,\mu)$ *and* $h_k(M,\mu)/h_1(M,\mu)$ *from above by some polynomial function of* k *?*

The following two questions are concerned with the stability of eigenvalues of the weighted Laplacian and multi-way isoperimetric constants.

Question 6.4. *Is it true that if two convex domains $K, L \subseteq \mathbb{R}^n$ satisfy $vol(K) \simeq vol(L)$, then $\eta_k(K) \simeq \eta_k(L)$ or $h_k(K) \simeq h_k(L)$?*

Question 6.5. *Can we get the stability of eigenvalues of the weighted Laplacian and multi-way isoperimetric constants with respect to the concentration topology ? Or more weakly can we replace* $\exp(ck)$ *and* $k^3 \exp(ck)$ *in Corollary 5.4 with some universal numeric constant ?*

In view of Corollary 5.10 an extension of E. Milman's theorem for the k-separation distance and the k-th eigenvalue would imply the latter question in Question 6.5.

In [18, Conjecture 6.11] we raised the following conjecture.

Conjecture 6.6. *For any natural number k there exists a positive constant C_k depending only on k such that if X is a compact finite-dimensional Alexandrov space of nonnegative curvature, then we have*

$$\lambda_k(X) \leq C_k \lambda_1(X).$$

Since Theorems 1.4 and 1.6 hold for compact finite-dimensional Alexandrov spaces of nonnegative curvature, the above question amounts to saying the existence of C_k such that $h_k(X) \leq C_k h_1(X)$.

We remark that Theorem 2.10 holds for compact finite-dimensional Alexandrov spaces. In fact, the only we need in the proof is the Davies-Gaffney heat kernel estimate

$$\int_A \int_B p_t(x,y) d\mu(x) d\mu(y) \leq \sqrt{\mu(A)\mu(B)} \exp\Big(- \frac{d^2(A,B)}{4t} \Big)$$

for any Borel subsets A, B and asymptotic expansion of heat kernel by eigenvalues and eigenfunctions of Laplacian ([10]). These are true for compact finite-dimensional Alexandrov spaces ([47], [24]). However it is not known the corresponding theorem of E. Milman's theorem (Theorem 2.5) for Alexandrov spaces. Note that we used Theorem 2.5 in the proof of Theorem 3.4. In order to give an affirmative answer to Conjecture 6.6, it suffices to prove that any concentration inequalities imply appropriate exponential concentration inequalities under assuming $\mathrm{CD}(0,\infty)$ or Theorem 3.4 holds for general $\mathrm{CD}(0,\infty)$ spaces by Gozlan-Roberto-Samson's theorem [20, Theorem 1.14].

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Appendix

The only point we need to be care when we prove Lee-Gharan-Trevisan's theorem (Theorem 1.4) for the smooth setting is the following lemma:

Lemma 6.7 ([28, Lemma 2.1])**.** *Let X be an mm-space and $f : X \to \mathbb{R}^n$ a Lipschitz map. Then there exists a closed subset A of X such that $A \subseteq \mathrm{Supp}\, f$ and*

$$\frac{\mu_X^+(A)}{\mu_X(A)} \leq 2\frac{\||\nabla f|\|_{L^2(\mu_X)}}{\|f\|_{L^2(\mu_X)}}.$$

Proof. For any positive real number t we put

$$A_t := \{x \in X \mid |f(x)|^2 \geq t\}.$$

Note that $A_t \subseteq \mathrm{Supp}\, f$ for any $t > 0$ and

$$\int_0^\infty \mu_X(A_t)dt = \|f\|^2_{L^2(\mu_X)} \tag{6.3}$$

The co-area inequality ([5, Lemma 3.2]) implies that

$$\begin{aligned} \int_0^\infty \mu_X^+(A_t)dt &\leq \int_M |\nabla(|f|^2)|(x)d\mu_X(x) \\ &\leq 2\int_M |f(x)||\nabla f|(x)d\mu_X(x) \\ &\leq 2\|f\|_{L^2(\mu_X)}\||\nabla f|\|_{L^2(\mu_X)}. \end{aligned} \tag{6.4}$$

Combining (6.3) with (6.4) gives

$$\frac{\int_0^\infty \mu_X^+(A_t)dt}{\int_0^\infty \mu_X(A_t)dt} \leq 2\frac{\||\nabla f|\|_{L^2(\mu_X)}}{\|f\|_{L^2(\mu_X)}},$$

which implies the conclusion of the lemma. □