# Ergostructures, Ergologic and the Universal Learning Problem: Chapters 1, 2, 3.

Misha Gromov

October 1, 2013

## Contents



# 1 Structures and Metaphors.

*Every sentence I utter must be understood not as an affirmation, but as a question.* NIELS BOHR.[1]

[1] I could not find on the web when and where Bohr said/wrote it. Maybe he never did. But here and everywhere, a quote is not an appeal to an authority but an acknowledgment of something having been already said.

Our ultimate aim is to develop *mathematical* means for *designing learning systems* $\mathcal{L}$ that would be similar in their essential properties to the minds of humans and certain animals but we limit ourselves at this point to *mathematically describing* expected *properties* of such systems.

LEARNING AND STRUCTURES. We want to understand the process(es) of learning, e.g. of *mother tongue* or of a *mathematical theory*, in the context of what we call *ergostructures*. Such structures, as we see them, are present in the depth of the human (some animal?) minds, in natural languages, in logical/combinatorial organizations of branches of mathematics and, in a less mature form, in biological systems – from metabolic and regulatory networks in living cells up to, possibly, ecological networks.

Learning from this perspective is

a dynamical process of building the *internal ergostructure* of an $\mathcal{L}$
from the *raw structures* in the incoming flows $\mathcal{FS}$ of signals, where
$\mathcal{FS}$ may or may not itself contain an ergostructure or its ingredients.

Such an $\mathcal{L}$ interacting with a flow of signals is similar to a a photosynthesizing plant growing in a stream of photons of light or to an amoeba navigating in a see of chemical nutrients and/or of smaller microbes: $\mathcal{L}$ recognizes and selects what is *interesting* for itself in such a flow and uses it for *building* its own structure.

This analogy is not fully far fetched. There is no *significant* difference between human activities and those by amoebas and even by bacteria, well,... on the GRAND SCALE. Say, the probability of finding first $10^9$ digit of $e = 2.718....$ "written" at some location $u$ of a universe $\mathcal{U}$ increases by a factor $> 10^{100}$, if you find a bacterium kind machine feeding on a source of almost amorphous *free energy* at a point $u'$ within a few billion light years from $u$.

Besides, what enters the brain mainly comes from plants, animals, humans and human artifacts – it would be little to learn for our nose, ears and even your eyes if not for Life around us.

This, however, does not apply to your *somatosensory* input: much of it comes from non-biological external sources. The somatosensory system is also exceptional in several other respects: it is short range and, most significantly, *proprioception* – your "body/muscle sense" is almost fully *interactive*. This is because the brain's output is mostly directed toward the muscles in the body and to feel your body you have to move it. (Besides muscles, the brain sends signals to the endocrine system and also it "talks to itself" but the conscious control over these brain activities is limited.)

And the integrated picture of the world that remains stable under transformation by Euclidean isometries of space and scaling transformations – the crowning creation of your visual+somatorsensory system[2], has little to do with surrounding life, but nearly all interesting visual sensations are of the biological origin, except, maybe, for the hypnotizing charm of wandering water streams and the irresistible beauty of cloud shapes sliding overhead.

It is virtually impossible to say something of substance on *mind, thinking, intellegence,* without a resort to metaphors that serve to create an illusion of

[2]An essential part of "somatorsensory" in understanding geometric space is *proprioception* – the sense of your body that is coupled with the *motor system*; the tactile/touch feeling is also involved but to a lesser(?) extent.

understanding and to hide inconsistency of the ideas they purport to convey.[3]

To make any sense of *learning, understanding*, etc., one has to to find mathematical counterparts of these concepts. These for us are certain unknowns $x$ the essential properties $P$ of which we try to guess with $x$ satisfying $P$ thought of as equations $P(x)$. Only when such an equation is "written down" one may proceed to search for its solutions, that is designing a leaning system $\mathcal{L}$ implementing $x$.

Eventually, we have to express this in truly mathematical terms, but we resort to a metaphoric language for a while, since we do not want to narrow our field of vision and miss the target with prematurely precise definitions.

(Being non-precise is tolerable in mathematics, since the ambience of mathematical *structures*, allows, within certain limits, a non-rigorous yet productive discourse with concepts that are not immediately clearly defined.)

Understanding and modeling human thinking processes is dissimilar to other problems in science and engineering but the following example may be instructive.

*How to get to the Moon.*

People might have been wondering about this for millennia but the following *formulation(s)* of the question became possible only with development of physical science and mathematics in the last three of centuries.

What are *all conceivably possible orbits/trajectories* for bodies traveling between the Earth and the Moon?
What kind of *propulsion mechanism(s)* can bring you to such an orbit?

It is unthinkable to plan a trip to the Moon prior to forming adequate concepts of *equations* mathematically describing *properties* of these "orbits"[4] and of "mechanisms".[5]

Similarly, without a proper *reformulation* of the problem and *a mathematical description* of essential, partly conjectural, properties of

*thinking/learning/understanding systems,*

it is unimaginable to make a radical advance in the study of "thinking mechanisms"; also a "blind design" of a functional model of a "thinking system" appears unrealistic.

## 1.1 Universality, Freedom and Curiosity

*Out of chaos God made a world,*
*and out of high passions comes a people.*
BYRON.

Our inspiration for design of learning systems comes from what may seem as an almost godlike ability of a human (and some animal) infant's brain of building a *consistent* model of *external world* from an *apparent* chaos of electric/chemical signals that come into it.

[3]In poetry, unlike how it is in science, we joyfully welcome illusions created by the beauty of metaphors.

[4]Orbits are described by *differential equations* of motion in a graviton field.

[5]A basic aspect of a relevant "mechanism" is described by the *Tsiolkovsky ideal rocket equation.*

We conjecture that an infant's learning process follows an *universal* set of *simple* rules for extracting structural information from these *not* truly "chaotic" signals, where

these rules must *indiscriminately* apply to *diverse classes* of incoming signals.

Universality is the most essential property we require from our learning systems – this is the key that opens the door for a non-cosmetic use of mathematics;[6] reciprocally, if successful, mathematics will furnish universality in learning.

At the moment, one may only speculate in favor of universality by appealing to "evolutionary thrift of Nature" and to "brain plasticity". Ultimately, we want to write down a *short* list of *universal* rules for "extracting" *mathematical structures* from *general* "flows of signals". And these flows may come in many different flavors – well organized and structured as mathematical deductions processes, or as unorderly as *"a shower of little electrical leaks"* depicted by Charles Sherrington in his description of the brain..

Of course, nontrivial structures can be found by a learning system, (be it universal or specialized) only in "interesting" flows of signals. For instance, nothing can be extracted from fully random or from constant flows. But if signals are modulated by *something* from "real world" we want to reconstruct as much of the *mathematical structure* of this *something* with these rules as the brain of a human infant can do.

Universality necessitates *non-pragmatic* character of learning. Indeed, *formulating* each utilitarian goal is *specific* for this goal – there is no universal structure on the "set of goals". Thus,

*the essential mechanism of learning is goal free*
*and independent of an external reinforcement*[7]

.

Georg Cantor's words

*The essence of mathematics lies in its freedom*

equally apply to learning in place of mathematics. Freedom for us means *freedom to learn.*

Universal learning systems, must be designed as self propelled learners that need no purpose, no instruction, no reinforcement for learning.

This, a priori, is no more paradoxical than, say, your digestive system functioning with no teacher instruction or a mechanical system moving by *inertia* with no external forces. External constrains and forces change the behaviour of such systems, but they hardly can be regarded as the source of motion.

It would be unrealistic making any conjecture on how such rules could be implemented by the neurophysiology of the human brain, although it seems plausible that they are incorporated into the "architecture of pathways" of signal processing's in the brain. But we shall try to guess as much as possible about these rules by looking at the universal learning problem from a mathematical

[6]This is meaningless unless you say what kind of mathematics you have in mind. Mathematical creatures, such, for example, as *Turing machine* and *Pythagorean theorem* differ one from another as much as a single-stranded RNA virus form a human embryo.

[7]Feeling of pain when you fall down or bump into something may be helpful in learning to run – this is debatable; but contrary to what a behavioristically minded educator would think, *reward/punishment reinforcement* does not channel the learning process by *reinforcing* it, but rather by *curtailing and constraining* it. Compare [13] [11].

perspective.

*Cuiriosity as Intrinsic Motivation.* The idea of what we call *ergosystems* is close to what was earlier proposed by Oudeyer, Kaplan and Hafner,[8] in the context of robotics under the name of *Intrinsically Motivated Curiosity Driven Robots.*

This "motivation" is implemented by a class of *predictor programs*, that depend on a parameter $B$ which is coupled with (e.g. by being a function of) the behaviour of robots. These programs $Pred = Pred(H, B)$ "predict" in a certain specified way incoming signals on the basis of the history $H$, while the robots (are also programed to) optimize (in a specific formally defined way) the quality of this prediction by varying $B$. *Curiousity driven robots* are being designed and build in Ouduer's lab.

### Information/Prediction Profile.

*The Maximal Prediction* idea is also central in our thinking on ergosystems but we emphasize "structure" instead of "behaviour", with *degree of predictability* being seen as a part of the structure of flows of signals within and without an ergosystem.

This "degree" is defined as a function in three (groups of) variables: that are

*the system $\mathcal{L}$ itself and two fragment $F_1$ and $F_2$ in the flow of signals $\mathcal{FS}$,* where

*$\mathcal{L}$ predicts "something of $F_2$" on the basis of its knowledge of $F_1$.*

This "something" refers to the result of some *reduction* procedure applied to $F_2$, where such a reduction may be suggested by $\mathcal{L}$ itself or by another ergosystem, e.g. by a human ergobrain.

An instance of that would be predicting a *class of a word* $W$ in a text $S$ on the basis of several preceding words or classes of such words. Such a class may be either syntactic, such as *part of speech*: verb/noun/..., or semantic, e.g. referring to vision, hearing, motion, an animal, an inanimate object or something else.

And "degree of predictability" of a class of a word $W$ derived from correlations of this class with words that *follow as well as precede* $W$ is also structurally informative.

In fact, the *proper direction*, that is "follow" versus "proceed" relation, is *not intrinsic* for (a record of) a flow of speech. But, possibly, it can be reconstructed via some *universal* feature of the "predictability (information) profile" of such a flow *common to all* languages[9], similarly (but not quite) to how the arrow of time is derived from evolution of macroscopic observables of large physical ensembles

Besides the structure in $\mathcal{S}$, this degree also tells you how competent the ergo learner $\mathcal{L}$ is, where one can also judge the ability of $\mathcal{L}$ to learn by evaluating

[8]See [OKH] – (Oudeyer, P., Kaplan, F., Hafner, V.V.: Intrinsic Motivation Systems for Autonomous Mental Development. IEEE Transactions on Evolutionary Computation 11:1, (2007) and [www.pyoudeyer.com].

[9]Phonetics of a recorded speech suggests an easy solution but it would be more interesting to do it with a deeper levels of the linguistic structures. In English, for instantce, the correlation of a *short* word $W_1$ with a neighboring $W_2$ is stronger if $W_2$ follows $W_1$ rather than proceeds it; but this may be not so in other languages.

how much this competence increases with extra information about $\mathcal{S}$ getting available to $\mathcal{L}$.

The idea of "*interesting*", that is the feature of a structure that excites "curiosity" of an $\mathcal{L}$, can be best grasped by looking at the extreme instances of *uninteresting* flows of signals – the constant ones:

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○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

There is (almost) nothing to predict here, nothing to learn, there is no substance is this flow for building your internal ergostructure. (If you were deprived of freedom to learn by being confined to an infinite flat plane with no single distinguished feature on it, you will be soon mentally dead; boredom cripples and kills – literally, not metaphorically.)

And random *stochastically constant* sequences do not look significantly more interesting.

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This appears "non-interesting" because one loses control over incoming signals: ours ergo idea of "interesting" is suspended in balance between *maximal novelty* of what comes and *being in control* of what happens.

(Pure randomness looks boringly uneventful to your eye but your *vestibular* and the *proprioceptive/somatosensory systems*[10] would enjoy propelling your body through a rugged terrain with occasional random jumps from one rocky stone to another making the trip dangerous, exciting and interesting[11].)

*On (Im)Practicality of Universality.* Multi-purpose gadgets are not among Greatest Engineering Achievements of the Twentieth Century: *flying submarines*, if they were a success, then only in James Bond films.[12] On the other hand, the 20th century machine computation has converged to universality; the basic machine learning will, most probably, follow this path in the 21st century.

## 1.2 Throwing Stones and Learning to Speak.

Mastering accurate throwing, a uniquely human[13] capacity, could have been, conceivably, a key factor in the early hominid brain evolution.[14] According to *the unitary hypothesis*, the same neural circuitry may be responsible for other sequential motor activities, including those involved into the speech production and language. [1], [12]

Let us draw a parallel between several aspects of mechanics of throwing and different perspectives on learning.

From a thrower point of view the most important is his/her *aim*, that must be achieved with a correct *initial condition* – the velocity vector of a stone –

[10] These sensory systems tell you what the current (absolute and relative) positions, velocities and accelerations of your body and of its parts are, with most accelerations being perceived via stresses in your skeletal muscles.

[11] Only rarely, grown-up *non-human* animals are able to derive pleasure from doing something irrational.

[12] There are sea birds, e.g. *pelagic cormorants* and *common murres* who are (reasonably) good flyers and who also can dive, some up to more than 50(150?)m. The technology for building comparably universal/adaptable machines may be waiting ahead of us.

[13] Elephants may be better than humans at precision throwing.

[14] 500 000-year-old hafted stone projectile points, 4-9cm long, were found in the deposits at Kathu Pan in South Africa, http://www.newscientist.com/article/dn22508-first-stonetipped-spear-thrown-earlier-than-thought.html

that then will follow the trajectory toward a desired target. You may (and you better do) fully forget the laws of Newtonian mechanics for this purpose.

But from a physicist's point of view, it is the *second law + the force field* (graviton and the air resistance) that determine the motion – the initial condition is a secondary matter.

A mathematician goes a step further away from the ancient hunter and emphasizes the general idea of time dependent processes being described/modeled by *differential equations.*

We – physicists and mathematicians with all our science would not stand a chance against *Homo heidelbergensis*[15] in a stone or spear throwing contest; however, we, at least some of us, shall do better in mathematically designing *gravity-assist trajectories* from Earth to other Solar system bodies.

(This would not convince Homo heidelbergensis in our intellectual superiority but rather would make him/her lough at the fools engaged in the useless activity of aiming at inedible targets.

> *It* [science] *triumphantly tells him*[/her]*: how many million miles it is from the earth to the sun; at what rate light travels through space; how many million vibrations of ether per second are caused by light, and how many vibrations of air by sound; it tells of the chemical components of the Milky Way, of a new element – Helium – of micro-organisms and their excrements, of the points on the hand at which electricity collects, of X-rays, and similar things.*
> *"But I don't want any of those things," says a plain and reasonable* [Heidelberg] *man*[/woman] *–"I want to know how to live".*
>
> LEV TOLSTOY, 1903.)

Formally, the concept of *goal free learning* is analogous to a mathematical physicist's view on mechanical motion: there is nothing special, nothing *intrinsically* interesting *neither in the hunter's aim* $A$, no matter how hungry he/she is, *nor in the initial condition* $I$, although much skill is needed to achieve it. But the transformation $T = T_L$ from $I$ to $A$, that incorporates the laws of motion expressed by differential equations, is regarded as something *universal* and the most essential from our point of view.

There are many possible aims $A$ and initial conditions $I$ but not so many fundamental laws $L$ and of transformations $T = T_L : I \mapsto A$ associated to them.[16] This what makes these laws so precious in our eyes.

Similarly, one may think of learning as of a transformation of an initial input and/or of a learning instruction $I$ to the final aim $A$ of learning.

Here we are even in a poorer position than the ancient hunter: we have hardly an inkling of what the corresponding "transformation by learning" $T_L$ does as it brings you from the initial input/instruction $I$ to our your aim $A$:

What is the "space" where all this happens?

What is the structure of the trajectory that leads from $I$ to $A$?

[15] *Homo heidelbergensis*, a probable ancestor of Homo sapiens as well as of Neanderthals and of Denisovans, lived in Africa, Europe and western Asia between 1 000 000 and 200 000 years ago.

[16] This stands in a sharp contradiction with *Cantor's theorem*: there are more *logically conceivable* functions $f : x \mapsto y = f(x)$ than arguments $x$. But logic should not be taken literally when it comes to the "real life mathematics".

And, unlike a teaching instructor, we are not concerned with *observable* $I$ and $A$, but with mathematical models of *invisible* intrinsic structures of transformations $T$ that are built according to "universal laws of learning".

It is not that we deny importance of goals, instructions and external stimuli for learning, but we relegate them to the secondary roles in the formula $T_L : I \mapsto A$. We try to understand learning processes regardless of their specific aims, or, rather, we want to see general aim generating mechanisms within the "universal laws" of learning.

*Why Universal?* It would be probably unrealistic (?) to expect that the evolution had time enough to select for many long sequential learning programs specific to different goals. On the other hand we do see manifestation of universality even for rather simple programs, e.g. the one that underlies what is called *Hawk/Goose effect*: a baby animal who learns to distinguish frequently observed shapes sliding overhead from those that appear rarely. The former eventually stops soliciting HIDE! response, while every unusual kind of a shape, e.g. that of a hawk, makes an animal run for cover.

This and more sophisticated learning programs develop in the environment of evolutionary older *general "ideas"* that are kind of *tugs*, such as dangerous/harmless, edible/useless, etc., selectively associated in an animal/human brain with a variety of *particular* "real somethings" in the world. But such programs themselves, if they are complicated, must be rather *universal*, since it is improbable(?) to have *many different long* programs, each being *individually* evolutionary selected for a *specific* task.

Well... nothing in biology, no universality is ever mathematically perfect – "laws of biology" are no brothers of laws of physics. A mathematician would hardly call a correspondence between the set of 64 quadruples of four units and the set of twenty other units, "universal", while such a correspondence is, probably, the most fundamental general feature of Life on Earth. (The primal universality in biology from a mathematician's point of view is *one-dimensional* organization of polynucleotides and polypeptides along with the "information transfer" implemented by 3D-folding of heteropolimers.)

## 1.3 Facets of Learning.

There are, in a rough outline, two actively pursued directions in the study of learning:

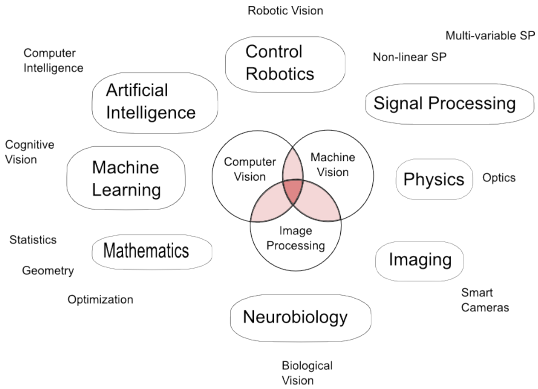
(1) The study of the neuro-physiological *mechanisms* of learning and computer design of (conjecturally) similar mechanisms.

(2) Description of various *specific* learning *problems*, a study of how are these are *solved* by humans and/or animals and an algorithmic approach to their *solution*.

To have just a glimpse of an idea of what goes on look at the corresponding pages in Wikipedia:

"learning", "memory", "motor learning", "language acquisition",
"mental representation" "outline of artificial intelligence",
"machine learning", "computer vision".

We do not attempt to contribute anything either to (1) or to (2) but we shall try to look at *learning* from a different perspective.




Our first objective is to identify a maximally general (quasi)mathematical concept reflecting essential features of learning processes.

Such a process must apply to an abstractly defined class of "incoming flows of signals", denote these by $[In]$, like those entering the brain via sensory receptor cells; then learning is seen as some kind of transformation applied to these flows. The results of such a transformation are twofold:

(A) The "visible" part of such a transformation is an outgoing flow of signals, we denote this flow by $[Out]$. The basic example of it is what goes from the brain to the muscles of the body.[17] This flow modulates an interaction – think of this as a "conversation" or a "game" – of the brain with the external world.

(B) What is invisible is incorporation of results of the $[In] \to [Out]$ transformation in the internal structure of the learning system. Building this (ergo) structure, call it $[ErgSt]$, constitutes the major part of the activity of the "brain who learns", but this process is invisible[18]; hopefully, we may guess how this $[ErgSt]$ looks like by imagining how a mathematician would proceed in making such a structure.

However, even the apparently easy problem of developing a language for speaking of incoming flows $[In]$ is by no means trivial since

on the one hand, we want to describe the incoming signals in mathematical terms with no reference to the external world where they originate from;

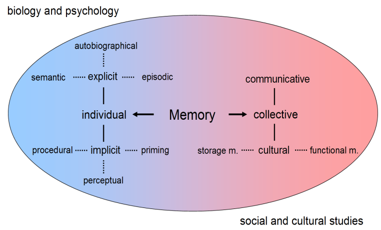
on the other hand, we want to keep track of the "real world meaning" of these flows.

We shall follow the usual recipe for solving this dilemma by resorting to *doublespeak*: we shall manipulate with $[In]$ as with an abstract mathematical entity but shall speak of it in metaphorical terms as if it were still residing in the "real world".

An essential issue in artificial learning, as we see it, is finding mathematical means for description of $[ErgSt]$. Without understanding this structure, attempts to reconstruct the transformation $[In] \to [Out]$ are like planning a trip to the Moon with no idea of rocket propulsion in your mind.

[17]What we know of the structure/message carried by $[Out]$ is mainly manifested by the (broadly understood) *behavior* of an organism.

[18]This is similar to *metabolism* versus *digestion*: the product of the latter is visible without being especially interesting. But the energy transfers and biochemical building processes in the cells are not discernible to the causal eye, but this is what we find fascinatingly interesting.




*If you have succeeded in explaining your new theory to an Australian penguin you ought to have devised a cleverer theory.*

Ernest Rutherford (misquote)

*Two (Seemingly Far-fetched) Examples.* (i) When he was 16, Ramanujan read a book by G. S. Carr. "A Synopsis of Elementary Results in Pure and Applied Mathematics" that collected 5000 theorems and formulas. Then in the course of his short life, Ramanujan has written down about 4000 new formulas, where one of the first was

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}=3.$$

What kind of mathematical structure could adequately describe "mysterious something" in the human brain/mind that caused the transformation from the flow of written symbols from Carr's book to the formulas written by Ramanujan?

Unless we develop a fair idea of what such a structure can be, we would not accept any speculation on the nature of mathematics, be it suggested by a psychologist or by a mathematician.

(ii) If a 4 years old child sees somebody balancing a stick on the tip of the finger, the child will try to imitate this; eventually, without any help or approval by adults, he/she is likely to master the trick.

What is the mathematics behind this?

A naive/trivial solution would be reformulating the problem in terms of classical mechanics and control theory. The balancing problem is easily solvable in these terms but this solution has several shortcomings:

• It does not apply where the external forces are unknown.

• It does not scale up: no such robot came anywhere close do a healthy human in its agility.

• It does not account for what make a child to be persistent in learning the trick.

• And, worst of all, the control theory and similar mathematical theories suggest no hint at a universal mechanism behind balancing sticks and inventing mathematical formulas.

(We share our inborn ability to count as well as the intuitive perception of the Euclidean geometry with many animals, pigeons, for example.[19] But enchantment of "useless formulas" and fascination by "meaningless tricks" can

[19]Human thinking, unlike(?) that of pigeons, is intertwined with the language learned in the cradle. Separating *inborn* from *acquired* needs collecting and filtering data on several culturally isolated ethnicities. The extreme difficulty of this is witnessed by the experience of Keren and Daniel Everetts who lived for many years with *Pirahã people* of Amazonia.

not be seriously studied on an animal model. The essence of being human is our inexplicable attraction to "meaningless and useless" things.)

## 1.4 Ego and Ergo.

> *Man is so complicated a machine that it is impossible*
> *to get a clear idea of the machine beforehand,*
> *and hence impossible to define it.*
> JULIEN OFFRAY DE LA METTRIE, MAN A MACHINE. 1748.

> *Nature shows us only the tail of the lion.*
> *But... the lion belongs with it even if he*
> *cannot reveal himself to the eye all at once*
> *because of his huge dimension.*
> ALBERT EINSTEIN. LETTER TO H. ZANGGER 1914.

In our [SLE] paper[20] we collect evidence for the basic learning mechanisms in humans (and some animals) being *universal, logically simple and goal free.* An organized totality of these mechanisms is what we call *ergobrain* – the essential, albeit nearly invisible, "part" of human mind – an elaborate mental machine that serves as an *interface* between the neuro-physiological brain and the *(ego)*mind.

We bring this "invisible" into focus by rewriting the Cartesian

**I** THINK therefore **I** AM

as

cogito ERGO sum.

*"I think"* and "*I am*" are what we call *ego-concepts* – structurally shallow products of *common sense.* But ERGO – a mental transformation of the seemingly *chaotic flow* of electric/chemical signals the brain receives into a coherent picture of *a world* that defines your personal idea of existence has a beautifully organized *mathematical structure.*

Apparently, MIND contains two quite different separate entities, that we call *egomind* and *ergobrain.*

*Egomind* is what you see as your personality. It includes all what you perceive as your conscious self – all your thoughts, feelings and passions, with subconscious as a byproduct of this *ego.* Most (all) of what we know of egomind is expressible in the common sense language – this language, call it *ego-reasoning,* that is a reflection of egomind, is perfectly adapted to to our every day life as well as to the needs of a practicing psychologist.

An essential (but not the only one) difficulty in accounting for the passage

$$\textit{brain} \rightsquigarrow \textit{(ego)mind}$$

is incompatibility of the languages used for description of the electrochemical processes in the brain and of the mental processes in the (ego)mind.[21]

*Ergobrain*, that is supposed to serve as an interface between the brain and the mind, mediates this transformation from electrochemical dynamics of neuronal networks to what we perceive as our "thinking".

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[20] *Structures, Learning and Ergosystems,* [www.ihes.fr/~gromov/PDF/ergobrain.pdf].

[21] This is similar to the incompatibility of the classical and quantum languages in physics called *collapse of quantum states.*

Ergobrain is something abstract and barely existing from ego's point of view. Ultimately, ergobrain is describable in the language of what we call (mathematical universal learning) *ergosystems* but it is hard to say at the present point what ergobrain truly is, since almost all of it is invisible to the conscious (ego)mind. (An instance of such an "invisible" is the mechanism of *conditional reflexes* that is conventionally regarded as belonging with the brain rather than with the mind.)

Certain aspects of ergo may be seen experimentally, e.g. by following *saccadic eye movements*, but a direct access to ergo-processes is limited.[22]

But there are properties of the working ergo in our brain/mind that are, however, apparent. For example, the *maximal number* $N_\circ$ of concepts our ergobrain can manipulate with *without structurally organizing them* ("chunking" in the parlance of psychologists) equals three or four.[23] This is seen on the conscious level but such a bound is likely to apply to all signal processing by the ergobrain.

For instance, this $N_\circ$ for (the rules of) chess is between three and four: the three unorganized concepts are those of "rook", "bishop" and "knight", with a weak structure distinguishing king/queen.

Contrary to what many linguists say, similar constrains are present in the structures of natural language where they bound the number of times operations allowed by a generative grammar may be implemented in a single sentence.

About Emotions.

Animal (including human) emotional responses to external stimuli seem rather straightforward with no structurally elaborate ergo mediating between neuronal and endocrine systems.

We think of emotions as colors or typefaces – a couple of dozen of different kinds of them, which the brain may choose for writing a particular message, such as

run! run! RUN! RUN!

## 1.5 Ergo-Ideas and Common Sense.

*Common sense is the collection of prejudices acquired by age eighteen.*
Einstein.

This saying by Einstein is not intentionally paradoxical. There is a long list of human conceptual advances based on *non-trivial* refutations of the *old way which is also the common-sense way.*[24] The first entry on this list – *heliocentrism* – was envisioned by by Philolaus, albeit not quite as we see it today, twenty four centuries ago. The age of enlightenment was marked by the counterintuitive idea of Galileo's *inertia*, while the 20th century contributed *quantum physics – absurd from the point of view of common sense* – in Richard Feynman's words. (Amusingly, Einstein sided with common sense on the issue of *quantum*.)

[22]This is similar to how it is with the cellular/molecular structures and functions, where the "ergo of the cell", one might say, is the machinery controlled by the *housekeeping genes that is not directly involved in any kind of production by the cell.*

[23]Some people claim their $N_\circ$ is as large as (Miller's) "magical seven" but this seems unlikely from our mathematical perspective; also some psychologists also find the number four more realistic.

[24]This is the way of thinking by *a plain, reasonable working man* as Lev Tolstoy tells his readers.

Your egomind with its *pragmatic ego-reasoning* – common sense as much as your emotional self, is a product of evolutionary selection. The two "selves" stay on guard of your survival and passing on your genes.

But *ergo*, unlike ego, was not specifically targeted by selection – it was adopted by evolution out of sheer logical necessity as, for example, the *1-dimensionality* of DNA molecules.

A pragmatically teleological ego-centered mode of thinking that was installed by evolution into our conscious mind along with the caldron of *high passions* seems to us intuitively natural and logically inescapable. But this mode was selected by Nature for[25] our social/sexual success and personal survival, not at all for a structural modeling of the world including the mind itself.

The self-gratifying ego-vocabulary of

*intuitive, intelligent, rational, serious, objective,*
*important, productive, efficient, successful, useful.*

will lead you astray in any attempt of a rational description of processes of learning; these words may be used only metaphorically. We *can not*, as Lavoisier says,

*to improve a science without improving the language or nomenclature which belongs to it.*

The intuitive common sense concept of *human intellegence* – an idea insulated in the multilayered cocoon of *teleology* –purpose, function, usefulness, survival, is a persistent human illusion. If we want to to understand the *structural essence* of the mind, we need to to break out of this cocoon, wake up from this illusion and pursue a different path of thought.

It is hard, even for a mathematician, to accept that your conscious mind, including the basic (but not all) mathematical/logical intuition, is run by a blind evolutionary program resulting from "ego-conditioning" of your animal/human ancestor's minds by million years of "selection by survival" and admit that mathematics is the only valid alternative to common sense.

Yet, we do not fully banish common sense but rather limit its use to concepts and ideas *within* mathematics. To keep on the right track we use a semi-mathematical reasoning – we call it *ergologic* – something we need to build along the way. We use, as a guide, the following

Ergolist of Ideas.

*interesting, meaningful, informative, funny, beautiful,*
*curious, amusing, amazing, surprising,*
*confusing, perplexing, predictable, nonsensical, boring.*

These concepts, are neither "objective" nor "serious" in the eyes of the ego-mind, but they are *universal*, unlike say "useful" that depend on what, *specifically*, "useful" refers to. These ergo-ideas will direct us toward understanding of how the ergobrain works, and especially will help us to model processes running in a child's mind, that hardly can be called serious, rational or objective, the processes that build a coherent world out of chaos of signals that enter child's brain.

*Those who dance are often thought mad*

[25] This embarrassing "for" is a fossilized imprint of the teleological bent in our language.

*by those who hear no music.*
TAO TE CHING.

What we wrote on the above few pages can hardly convince anybody in the credibility of the idea of some invisible mathematical ergobrain running along with your pragmatic (ego)mind and in the existence of *performant* (universal goal free learning) ergosystems.

("Performant" is, in truth, no more applicable to an ergosystem than to a child at play: both do not follow your instructions and do not get engaged into solving your problems. From the ego perspective what ergo does, e.g. composing a most beautiful but utterly useless chess problem, appears plain stupid and meaningless. Reciprocatory, utilitarian ego's activity, e.g. laboriously filing tax returns, is dead boring for ergo.)

An evidence in favor of ergobrain – a powerful mathematically elaborate machine hidden in *everybody's* head that is responsible for non-pragmatic mechanism(s) of learning can be seen in

• mastering bipedal locomotion in a heterogeneous environment by human infants – walking, running and not bumping into things, as well as learning to speak, to read and to write, including learning languages and writing poetry by deaf-blind people;[26]

• possession of almost supernatural *artistic or mathematical abilities* by exceptionally rare people, e.g. by the mathematician *Srinivasa Ramanujan*;[27]

• non-pragmatic *playful* nature of learning in animal (including human) infants during the periods of their lives when the responsibility for their survival resides in the paws of their parents;

• attraction to *useless* (from survival perspective) activities by human and by certain animals;

• creating and communicating *mathematics*, e.g. the potential ability by many (probably, by several hundred, if not thousands or even millions, people on Earth) to understand *Fermat's last theorem*[28] by reading a thousand-page *written proof* of it.

We bring forth specific example on the above points in our [SLE]-paper but a justification of the concept of ergobrain, is not, formally speaking, needed for a mathematical theory of ergosystems. However, it provides you with a psychological support in groping toward such a theory. And imagining how ergobrain thinks and trying to understand what ergo-logic can conceivably be, will not only help you in developing mathematics of ergosystems but also may guide you in a search for other mathematical structures.

## ERGO IN SCIENCE

The mental set-up that makes the very existence of science possible is of ergo. Here is how Poincaré puts it:

*The scientist does not study nature because it is useful to do so.*
*He studies it ... because it is beautiful.*

[26]We make no conjecture on whether the logical architecture of "the art of walking" is similar or dissimilar to that of "the art of speaking".

[27]Writing off these rare events to "mere accidents", is like judging *explosions of supernovae* as "random and nothing else", just because only a dozen of supernovae were recorded in our galaxy with more than hundred billion stars (none since October 9, 1604);

[28]No integers $x > 0, y > 0, z > 0$ and $n > 2$ satisfy $x^n + y^n = z^n$.

*... intimate beauty which comes from the harmonious order of its parts, and which a pure intelligence can grasp.*

But grasping, embracing this beauty in your mind may be prohibitively difficult. Nothing in Nature that is worth understanding comes in "a few simple words". If you happen to learn something novel for you in science without much intellectual effort and hard work – this is either not especially novel or it is not science.

Even most familiar and apparently simple things in science are intuitively hard to accept, such as the second law of Newton that presents a manifestly mathematical (ergo)way of thinking about motion:

*Lex II: Mutationem motus proportionalem esse vi motrici impressae,*
*et fieri secundum lineam rectam qua vis illa imprimitur.*

This law, even more so than the first law, runs against how our visual and somatosensory (mainly *proprioception*– the body sense) systems represent properties of motion in our mind.[29] Yet, some people find themselves comfortable with Newtonian laws; analyzing how they learn to understand them may shade some light on general mechanisms of ergo learning.[30]

Science is a child of the art of *not* understanding. If we want to approach the problem of *thinking machines* we must visualize the extent and the source of our non-understanding *thinking*. The key question is not "can a machine think?" but:

*Is there enough structural universality in the process of "thinking" in order to allow a mathematical modeling of this process?*

Formulating questions about "thinking" without making guesses on the nature of the underlying mathematical structure(s) is like talking about LAWS OF PHYSICS with no ideas of *number* and *space* in your head.

There is no visible *non-trivial* mathematical structure in what we conciously perceive in our (ego)mind and it is unlikely that there is a *realistically describable* structure (mathematical model) of human (neuro)brain capable to account for such mental processes as learning a language, for example. But we conjecture that such structure(s) does reside in the *ergobrain*.

ON BEING TRIVIAL.

Triviality is a mathematician's scarecrow but no-trivial constructions are often made with a few trivial constituents.

For instance, certain structures *are* "trivial" when taken in isolation, such as highly disconnected (often bipartite) graphs that represent the [object]–[name] relations where the edges join words with the corresponding visual images or [question]–[answer] graphs of human dialogs – the brains of the stupidest animals depend on such graphs. Yet, mathematical derivations issuing from *several* trivial graphs make *non-trivial* structures in the human ergobrain.

[29]The essential logic of this reconstruction is of ergo but it serves the survival of our ego and serves it well, better than mathematical Newtonian model would do.

[30]Majority of us, even if we can correctly recite the three laws of motion, do not believe in these laws. We intuitively reject them in view of the apparent inconsistency of these laws with much of what we see with our own eyes, such as the motion of a pendulum that visibly contradicts to the *conservation of momentum law*.

WHAT IS ABSTRACT AND WHAT IS OBVIOUS?

Explaining "simple and apparent things" by means of something "abstract and difficult", that may not a priori even exist, is against common sense, but this is how it is in science and in mathematics.

For example, the "obvious" properties of light and matter we see everywhere around us make sense *only* in the context of *quantum theory of electromagnetic fields*, with the energy source of the sunlight being inconceivable without the theory of *strong interactions* in *atomic nuclei*.

Something as simple as the air we breath is the product of unbelievably complicated quantum-chemical process of *photosynthesis* and the whole edifice of Life on Earth is based on statistical mechanics of large *heteropolymeric molecules* .

## 1.6 Ergo Perspective on Chess

> *Arithmetical or algebraical calculations are, from their very nature, fixed and determinate....* [But] *no one move in chess necessarily follows upon any one other.*
>
> EDGAR ALLAN POE, "MAELZEL'S CHESS-PLAYER", APRIL 1836.

In the early 19th century, when Poe was writing his article on *Maelzel's Chess-Player Automaton*, an ability to play chess was seen by many (all?) as a quintessential instance/measure of the intellectual power of the human mind. But the *mere existence* of *chess algorithms* is obvious.[31]

You play white. Let $\mathbf{eva}_0(P^*)$, where $* = \circ$ or $* = \bullet$, depending on whether a move by white($\circ$) or by black($\bullet$) is pending, be a "natural" numerical[32] *evaluation function* of a position $P^*$, e.g. the sum of judicially assigned weights to the pieces – positive weights to the white pieces and negative to the black ones.

Inspect possible *white* moves $wh$ for all $P^\circ$, denote by $P^\circ + wh$ the resulting new positions, and similarly consider all $P^\bullet + bl$ . Define new evaluation function $\mathbf{eva}_1(P)$ by

$$\mathbf{eva}_1(P^\circ) = \max_{wh} \mathbf{eva}_0(P^\circ + wh)$$
$$\mathbf{eva}_1(P^\bullet) = \min_{bl} \mathbf{eva}_0(P^\bullet + bl).$$

Keep doing this and get

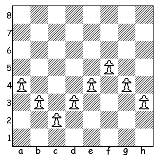
$$\mathbf{eva}_0 \Rightarrow \mathbf{eva}_1 \Rightarrow \mathbf{eva}_2 \Rightarrow \mathbf{eva}_3 \Rightarrow ... \Rightarrow \mathbf{eva}_N \Rightarrow ...$$

Stop, say, at $N = 20$ and let your computer (that plays white) maximize $\mathbf{eva}_{20}(P + wh)$ for all its moves $wh$. Such a program, probably, will beat anybody, but... no computer can inspect twenty half-moves in realistic time.

As recently as in 1950's, Hubert Dreyfus, a critic of artificial intelligence believed that a child would beat any chess program.

[31] This, must(?) have been understood by Wolfgang von Kempelen, the creator of Chess-Player, and by contemporary mathematicians and scientists, e.g. by Benjamin Franklin who played with this "automaton". But I could not find a reference.

[32] This is *unnatural*, there is nothing intrinsically *numerical* in chess. Logically, what we need for an evaluation is (somewhat less than) an *order relation* on positions. But "ergo-evaluation" is more subtle and less logical.



In 1957, Dreyfus was defeated by a **eva**$_2$ chess program that was *implemented on a computer* by Alex Bernstein and his collaborators.[33]

Fourty years later in 1997, Deep Blue (non-impressively) defeated the wold champion Kasparov, 3.5-2.5. The computer could evaluate 200 million positions per second; it inspected, depending on the complexity a position, from $N = 6$ to $N \approx 20$ half-moves. The program contained a list of endgames and it adjusted the evaluation function by analyzing thousands of master games.

So, when Poe insists that

> *... no analogy whatever between the operations of the Chess-Player, and those of the calculating machine of Mr. Babbage,*

one might judge him as mathematically naive; yet, Poe's conclusion was fully correct.

> *It is quite certain that the operations of the Automaton*
> *are regulated by mind, and by nothing else.*
> *... this matter is susceptible of a mathematical demonstration, a priori.*

The idea behind what Poe says is valid: Turing(Babbage) machines and **eva**-algorithms make poor chess players – they can match ergobrain programs *only* if granted superhuman resources of computational power.

This does not preclude, however, an approach with a quite different, possibly yet unknown to us, (ergo?)mathematics, but some people conjecture that the human (ego?)mind is "fundamentally non-algorithmic".

In his book *Shadows of the Mind*, Roger Penrose, who opposes the idea of *thinking machines,*[34] presents a chess position where

> *black has eight pawns, while white, in addition to eight pawns, has two rooks (and the white squared bishop, if you wish). The black pawns stay on black squares in an unbroken chain (as in the above drawing) that separates the black king from the white pieces. White pawns are positioned in front of the black ones and fully shield them from the rooks.*

Thus, the black king is safe in-so-far as black pawns do not change positions. But if a black pawn captures a white rook, then the chain of the black pawns will be disrupted and the black king *eventually* mated.

Any current computer-chess program would accept a sacrifice of a white rook, since "eventually" shows only in another twenty-thirty half-moves, while no human player will make such a silly mistake.

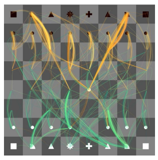
But Doron Zeilberger, who fights against the *Human Supremacy* idea, insists[35] that

> *symbol-crunching* [program], *valid for an $m \times n$ board rather than only*

[33] In 1945, the *first*(?) chess program was *written* by Konrad Zuse in his *Plankalkül* – a high-level programming language.

[34] See more on www.calculemus.org/MathUniversalis/NS/10/01penrose.html

[35] www.math.rutgers.edu/~zeilberg/Opinion100.html.



*an* $8 \times 8$ *board, will perform as good as a human player.*

Also, Zeilberger is critical of Penrose's use of *Gödel's incompleteness theorem* (see 2.1) as an argument against *thinking machines.*

Chess has been supplying an experimental playground to all kind of people pondering over the enigma of the human mind.

Logicians-philosophers marvel at how *formal rules* but not, say, the shape, color or texture of the pieces, determine what players do with them.

For example, Wittgenstein instructs (mocks?)[36] the reader:

*The meaning of calling a piece "the king"*
*is solely defined by its role in the game.*

He continues –

*imagine alien anthropologists landing on planet earth ...*
*discover ... a chess king...*
[It] *remains an enigma to their understanding.*
*Without ... other artifacts, the chess king is*
*only a chunk of wood* (or plastic, or... ).

(It is hard to resist continuing with *...or chocolate...* . But what the philosopher had in mind was not as trivial as it looks.)

Unlike logicians, students of the *ego*mind search for the meaning of chess in apparent or hidden urges of players to compete, to win, to grab, where making checkmate for a male chess player substitutes for killing his father in accordance with *Oedipus complex*.[37]

From the *human ergobrain perspective*, the relevance of chess is seen in its *intrinsic attractiveness* to certain people[38] and the central problem of chess in designing a (relatively) simple *learning (ergo)program* $\mathcal{L}$ that would find chess *interesting* and/or will be able (and "willing") to learn playing chess by itself whenever it is given an access to the records of sufficiently many (how many?) chess positions, chess problems and/or (fragments of) chess games.

And since chess, as most (all?) ergo-activities, is *interactive*, a learning will go faster if $\mathcal{L}$ is allowed an access to computer chess programs.

We conjecture the existence of such an $\mathcal{L}$, that is, moreover will be (rather) *universal* – not chess-special in a remotest way. It may come from somebody who has never heard of chess or of any other human game. However, such a program $\mathcal{L}$, being a pure ergo, may behave differently from a human player, e.g. it will not necessarily strive to win.

[36] Wittgenstein is often quoted with *A serious and good philosophical work could be written consisting entirely of jokes.*

[37] Was it intended by Freud as a macabre joke? Sphinx might have accepted this solution to the riddle of chess, but we feel more at ease with *Flatulus complex* see [SLE] §6.7 [3].

[38] If you remold the piece "king" into "sphinx", the game will not loose its attractiveness to more than half of chess perceptive people.

Such self-taught ergolearner program implemented on a modern computer will play chess better than any human or any existing specialized computer chess program, but this is *not* the main ergo issue. And it is nether the power of logical formality – something trivial from the ergo (and from general mathematical) point of view, what makes chess attractive and interesting.

An ergolearner delights in the beauty of the *structure* of $\mathcal{CHESS}_{ergo}$, some kind of combinatorial arrangement of "all" interesting games and/or positions. An ergolearner tries to understand *(ergo)principles of chess* that transcend the formal rules of the game

These "principles" enable one, for instance, to distinguish positions arising in (interesting) games from *meaningless* positions, as it is seen in how chess masters memorize *meaningful* positions that come from actual games but they are as bad as all of us when it comes to random arrangements[39] of chess pieces on the board. In its own way, chess tells us us something interesting about *meaning*.

## 1.7 Meaning of Meaning.

*Meanings of words are determined to a large extent by their distributional patterns.*

ZELIG HARRIS.

This "*meaning*" of Harris is quite different from the common usage of the word "*meaning*" that invariably refers to "the real world" with "meaningful" being almost synonymous to what is advantages for preservation and propagation of (observable features encoded by) your genes. (The speakers of the word are usually blissfully unaware of this and they are getting unhappy with such interpretation of meanings of their actions.)

The former is a *structural meaning* the full extent of which may be discerned only in the dynamics of the learning processes in humans, while the latter – the concept *pragmatic meaning*, is shared by all living organisms, at least by all animals from insects on. This idea of meaning – the commandment to survive – was firmly installed in our brain hardware by the evolutionary selection several hundred million years before anything resembling humans came to existence.

A possible way to look beyond the survival oriented mode of thinking is to turn your mind toward something like chess, something that does not (contrary to what Freudists say) carry a significantly pronounced imprint of the evolutionary success of your forefathers.

But even if you manage to switch your mind from ego- to ergo-mode, you may remain skeptical about (ergo)chess telling you something nontrivial about learning languages and understanding their meanings.

Superficially (this is similar but different to what was was suggested by Wittgenstein), one may approach a dialog in a natural language as a chess-like game that suggests an idea of (ergo) meaning: the *meaning* of an uttering $U$ is derived similarly to that of the meaning of a position $P$ in chess: the latter is determined by the combinatorial arrangement of $P$ within the ergostructure $\mathcal{CHESS}_{ergo}$ of "all" ergo-interesting chess positions/games while the former is

[39]The number of possible chess positions is estimated around $10^{45}$. Probably, $10^{12}$-$10^{18}$ among them are "meaningful".

similarly determined by its location in the architecture of $\mathcal{TONGUE}_{ergo}$ of a language.

More generally, we want to entertain the following idea.

*The meanings assigned by ergostructures (e.g. by our ergobrains) to signals are* ***entirely*** *established by patterns of combinatorial arrangements and of statistical distributions of "units of signals", be they words, tunes, shapes or other kinds of "units".*
*Understanding is a* ***structurally organized*** *ensemble of these patterns in a human/animal ergobrain or in a more general ergosystem.*

But even leaving aside the lack of precision in all these "pattern", "arrangement", etc. one may put forward several objections to this idea.

The most obvious one is that words, and signals in general, are "just names" for objects in the "real world"; the "true meaning" resides in this world. But from the brain perspective, the only "reality" is the interaction and/or communication of the brain with incoming flows of signals. The "real word" is an abstraction, a model invented by the brain, a conjectural "external invisible something" that is responsible for these flows. Only this "brain's reality" and its meaning may admit a mathematical description and be eventually tested on a computer.[40]

(There are many different answers to the questions "What is meaning?", "What is understanding?" offered by linguists, psychologists and philosophers.[41] We, on the other hand, do not suggest such an answer, since we judge our understanding of the relevant ergo-structures as immature. The expression "*structurally organized ensemble*" is not intended as a definition, but rather as an indication of a possible language where the concept of *understanding* can be productively discussed.)

Another objection may be that learning chess and understanding its meaning, unlike learning native languages by children, depends on specific verbal instructions by a teacher.

However, certain children, albeit rarely – this was said about Paul Morphy, Jose Raul Capablanca, Mikhail Tal and Joshua Waitzkindo – learn chess by observing how adults play. And as for supernovas, it would be foolish to rejects this evidence as "statistically insignificant".

More serious problems that are harder to dismiss and that we shall address later on are the following.

($\circ$) The structures $\mathcal{TONGUE}_{ergo}$ of natural languages are *qualitatively* different from $\mathcal{CHESS}_{ergo}$ in several respects.

Unlike how it is with chess, the rules of languages are non-deterministic, they are not explicitly given to us and many of them remain unknown. Languages are bent under the load of (ego)pragmatics and distorted by how their syntactic tree-like structures are packed into 1-dimensional strings.

Self and Time. The most interesting feature of natural languages – *self-referentiality* of their (ergo)syntax (e.g. expressed by *pronouns* and/or by certain *subordinate clauses)* that allows languages to *meaningfully* "speak" about themselves.

[40] We do not want to break free from the *real world*, but from the hypnosis of the words EXISTENCE/NON-EXISTENCE coming along with it.

[41] References can be found on the corresponding pages of Wikipedia.

This is present in most condensed form in *anaphoras* such as in

*X thinks he is a good chess player,*

and related features common to all human languages are seen in *deixis*, such as in

*but I am afraid you may be disappointed by the naivety of his moves,*

along with various forms of *grammatical aspects* linked to the *idea of time*.

(It is hard to say how much of *time in the mind* is necessitated by the time dynamics of the neurobrain, what had been installed by the evolution and what comes with flows of incoming signals. And it is unclear if *time* is an essential structural component of the ergobrain and if should it be a necessary ingredient of universal learning programs.)

None of these have counterparts in chess[42] or in any other non-linguistic structure, e.g. in music. Yet, self-referentiality is seen in mathematics on its borders with a natural language, e.g. in *Gödel's incompleteness theorem*.

($\circ\circ$) The internal combinatorics of $\mathcal{TONGUE}_{ergo}$ may be insufficient for the *full* reconstruction of the structure of the corresponding language.

For example, linguistic signals a child receives are normally accompanied, not necessarily synchronously, by what come via all his/her sensory systems, mainly *visual* and/or *somatosensory* signals – feeling of touch, heat, pain, sense of the position of the body parts, as well as *olfactory* and *gustatory* perceived signals.

The full structure of $\mathcal{TONGUE}_{ergo}$ and/or the meaning of an individual word may depend on (ergo)combinatorics of $\mathcal{VISION}_{ergo}$ coupled with $\mathcal{TONGUE}_{ergo}$ not on $\mathcal{TONGUE}_{ergo}$ alone.

$\mathcal{VISION}_{ergo}$ is vast and voluminous – more than half of the primate (including human) cortex is dedicated to vision, but the depth of structure of "visual" within $\mathcal{TONGUE}_{ergo}$ seems limited, as it is witnessed by the ability of deafblind people to learn to speak by essentially relying on their *tactile* sensory system that is feeling of touch.

The role of *proprioception* (your body/muscle sense) and the *motor control system* in learning (and understanding?) language is more substantial than that of vision, since production of speech is set in motion by firing motor neurons that activate muscles involved in speech production – *laryngeal muscles, tongue muscles* and hordes of other muscles (hand/arms muscles of mute people); thus, an essential part of human linguistic memory is the memory of sequential organization of these firings.

(Proprioception, unlike vision, hearing and olfaction, has no independent structural existence outside your body; also it is almost 100% interactive – you do not feel much your muscles unless you start using them. The internal structure of proprioception is quite sophisticated, but, probably, it is by no means "discretized/digitalized" being far remote from what we see in language. It is hard to evaluate how much of language may exist independently of $\mathcal{PROPRIOCEPTION}_{ergo}$(+$\mathcal{TACTILE}_{ergo}$[?]) coupled with the motor control system, since a significant disfunction of these systems at early age makes one unable to communicate.)

[42]Does the "meaning" of the following sentence reside in the game being played or in the conjunction of syntactic self-referentiality loops in there?

*I thought I understood why X's white knight was placed on a1 square but his next move caught me by surprise.*

The above notwithstanding, (ergo)programs (as we see them) for learning chess and a language, and accordingly, the corresponding ideas of *meaning* and *understanding* have much in common.

To imagine what kinds of programs these may be, think of an *ergo-entity*, call it $\mathcal{EE}$, from another Universe to whom you want to communicate the idea/meaning of chess and with whom you want to play the game.

A preliminary step may be deciding whether $\mathcal{EE}$ is a *thinking* entity; this may be easy if $\mathcal{EE}$ possesses an ergobrain similar to ours, which is likely if ergo is universal.

For example, let $\mathcal{EE}$ have a mentality of a six-year-old Cro-Magnon child, where this "child" is separated from you by a wall and where the only means of communication between the two of you is by tapping on this wall.

Could you decide if the taps that come to you ears are produced by a possessor of an ergobrain – more versatile than yours if you are significantly older than six, or from a woodpecker?

If you happens to be also six year old, the two of you will develop a common tap language-game and enjoy *meaningfully* communicating by it, but possessors of two mature human minds separated by a wall will do no better than two adult woodpeckers.

To be a good teacher of chess (or of anything else for this matter), you put yourself into $\mathcal{EE}$'s shoes and think of what and how yourself could learn from (static) records of games and how much a benevolent and dynamic chess teacher would help. You soon realize that this learning/teaching is hard to limit to chess as it is already seen at the initial stage of learning.

Even the first (*ergo-trivial*) step – learning the *rules* of moves of pieces on the board will be virtually insurmountable in isolation, since these rules can not be guessed on the basis of a non-exhaustive list of examples, say, thousand samples, unless, besides ergo, you have a simple and adequate representation of the geometry of the chess board in your head.

If your are blind to the symmetries of the chessboard, the number of possible moves by the white rook ♖ in the presence of the white king ♔, that you must learn (in $64 \cdot 63$ positions), is $> 64 \cdot 63 \cdot 13 > 50\ 000$. "Understanding" space with its symmetries, be this "understanding" preprogramed or acquired by *a learning process of spacial structre(s)*, is a necessary prerequisite not only for learning chess but also for communication/absorbtion of the rough idea of chess.[43]

But if you have no ergo counterparts to such concepts as "*some piece on a certain line*"[44] in your head, you'll need to be shown the admissible moves of the rook in *all*($> 10^{45}$) possible chess positions.

And the more you think about it the clearer it becomes that the only realistic way to design a chess learning/understanding program goes via some general/universal mathematical theory equally applicable to learning chess and learning languages.

[43]The geometry of the board can be reconstructed from a moderate list of sample chess games with *Poincaré's-Sturtivant space learning algorithms* (see §4 in [4]), but these algorithms are slow.

[44]Such "abstractions" are probably acquired by the visual ergo-system of a child well before to something as "concrete" as *white knight in a particular position on a chessboard*, for example.

## 1.8 Seven Flows.

*... think of some step that flows into the next one, and the whole dance must have an integrated pattern.*
FRED ASTAIRE

Incoming flows of signals can be divided according to the sensory receptors and pathways by which they enter the brain: *visual, auditory* and *somatosensory* where the two relevant aspect of the latter are *proprioception* – the body sense, and *tactile*, i.e. *touch perception.*

(Perception of temperature, pain as well *gustatory* and *olfactory* signals are are not ergo-relevant as being comparatively structurally shallow, at least in humans.)

But from an ergo-learner perspective, signals differ by how one learns their "meanings", how one interacts with them, how one arrives at understanding of their structures.

1. *Spoken language* depends on the auditory and sensory-motor systems; ears to listen and sensory-motor systems to generate speech. However, deaf-mute people speak in sign language and deafblind people communicate in tactile sign language.[45]

2. *Written language* (whenever it naturally exists) is likely to have a huge overlap with the spoken one in the human brain (of a habitual reader) but it also makes a world of its own. It is not inherently interactive, at least not so superficially[46], and it is not bound to the flow of time. Persistence of written literature is hard to reconcile with a naive selectionist's view on co-evolution of language and the brain.

3. *Mathematics.* Learning mathematics is an interactive process but it is hard to say exactly in what sense.

The images a mathematician generates in his/her mind are neither of Language nor do they belong with any particular "sensory department". Thinking mathematics is like driving an imaginary bicycle or performing/designing a dance with elaborate movements entirely in your head. (This may differ from person to person.)

4. *Languages of games.* We are able to enjoy and to learn a variety of mental and physical games. Probably, these are divided into several (about dozen?) classes depending on how they are incorporated into our erghobrains. Written language and mathematics may be particular classes of games.

5. *Music.* People gifted in music replay melodies in their minds and they can reproduce melodies vocally and/or with musical instruments; the rare few may generate new melodies. But melodies, unlike sentences in a Language, can not talk about themselves and there is no general context where one can formulate what human (unlike that of birds) music is and/or what should be regarded as "understanding of music".

(An avalanche of superlatives that a music lover pours on you when he/she speaks about music tells you something about endorphins release into his/her

[45]Most amazingly, some deafblind people can understand spoken language by picking up the vibrations of the speaker's throat.

[46]Writing and reading is kind of talking to oneself.

blood triggered by music but nothing about music related (ergo)structures in his/her brain.)

6. *Proprioceptive/somatosensory system.* Running over a rough unpredictable terrain is kind of talking to the road with the muscles in your body. This is much simpler than the ordinary language but is still beyond the ability of computers that control robots. Neither a present day robot is able to hand sew a button on your shirt.

7. *Vision.* At least half of the neocortex in humans is dedicated to vision, but this may be mainly due to the sheer volume of the information that is being processed and stored, rather than to the structural depth of visual images. And amazingly, vision impairment, even vision+hearing impairment, do not significantly affect human ergo. The ergo is robust and independent of particular sensory inputs.

Three flows among these: *Language, Mathematics, Music* have an essential feature in common: the receiver of such a flow $F$ develops an ability, with no external reinforcement, to creatively generate a new flow $F'$ in the class of $F$. (In the case of Mathematics and Music this happens rarely, but the miracles of this having happened in the brains of Mozart and Ramanujan outweighs any statistics.)

Modeling the transformation $F \mapsto F'$ is one of the key aspects in our picture of the universal learning problem. (Possibly, there are counterparts of $F'$ for other incoming flows $F$, but they may be kind of *internal*.)

The most interesting object for the study among these is the learning mechanism of native languages by children that is, probably, similar to how mathematics is learned by mathematicians.

Of course, the structure of a most sophisticated mathematics we build in our minds is by far simpler than that of natural languages (not speak of the vision), but it is still quite interesting, while the corresponding learning process may be more accessible, due, besides its relative simplicity, to a great variance in people's abilities in learning mathematics[47] and a presence of criteria for assessing its understanding.

### Learning to Read by Learning to Speak.

The original form of signals carried by the above seven flows is different from what arrives at your sensory systems. For instance, visual images result from 2D *projections* of *three dimensional* patterns to the retina in your eyes; moreover, brain's analysis of these projections is coupled with the activity of the brain's motor system that controls movements of the eyes that continuously modify these projections.

Similarly, the flow of speech as it is being generated in one's mind is, according to tenets of *generative grammar* has a tree-like structure that is then "packed" into single time line.

Reconstruction of $F_{orig}$ from the flow $F$ you receive is an essential and most difficult aspect of understanding the message carried by this $F$. For example, understanding a flow $F$ of speech is coupled with one's ability to speak, i.e. to reconstruct/generate $F_{orig}$, or something close to it, in one's ergobrain.

[47] Every sane person understands his/her mother tongue and has an adequate visual picture of the world. This uniformity makes understanding of these "understandings" as difficult as would be understanding motion in the world where all objects moved in the same way.

This only aspect of this reconstruction we shall discuss is what can be expressed as an *annotation* to $F$.

For instance, upon receiving a flat image $F$ on its screen (retina), an ergo learner $\mathcal{L}$ must correctly resolve depth in *interpositions/occlusions* and/or "guess" relative values of the third coordinates at essential points of $F$.

And the background tree structure in a (record of a) flow $F$ of speech can be indicated with *parentheses* properly inserted into $F$. (An annotation may also include additional syntactic and/or semantic comments concerning particular words and sentences.)

Such annotations performed by a human ergobrain depend on an elaborate guesswork that is by no means simple or automatic and it is still poorly understood. And besides annotating flows of signals, the ergobrain augments them by something else.

For instance, formation of a visual image in one's mind depends on the activity of motor neurons involved in eye movements and "understanding" of these images depends on structural matching this activity with similar actions of these neurons in the past.

This active process of perception can be seen as a conversation or a kind of a game of the ergobrain with the environment. But such games, unlike anything like chess, are not easy to mathematically formalize.

## 1.9 Brain, Mind and Computations.

It had been realized millennia ago[48] that,

*the power of the brain to synthesize sensations makes it the seat of thought*[49]

but our understanding of the logic of the arrow

$$\textsc{Brain} \rightsquigarrow \textsc{Mind}$$

has advanced little, if at all, from the time of antiquity.

This is amazing. Ancients had no idea of *cell* and could not fathom how the brain works — the beauty of the cellular structure of the brain and the neuronal-synaptic principle of its function have been established only at the turn of the twenties century.[50]

But neither this principle, nor the enormity of the experimental data on the (electrochemical) neurophysiology of the brain accumulated to-day, helps us to make sense of this arrow "$\rightsquigarrow$" or just to find a suitable "name" for it.

Is the mind

*caused/produced/generated/constructed* or *determined/controlled/run*

by the brain?

Probably, none of the above and any attempt of compressing it to a single word strikes one as silly.[51] If you want to keep close to the truth you had better to resort to words of poetry.

[48] *A Hole in the Head, More Tales in the History of Neuroscience,* Charles G. Gross The MIT Press, 2009

[49] Attributed to Alcmaeon of Croton (≈450 BCE).

[50] This was expounded by Charles S.Sherrington in his 1906 publication *The Integrative Action of the Nervous System.*

[51] This is akin to the "impliction" *genotype* $\rightsquigarrow$ *phenotype* with the enormous machinery of embryonic development behind this arrow.

*The eye sends ... into the cell-and-fibre forest of the brain throughout the waking day continual rhythmic streams of tiny, individually evanescent, electrical potentials.*

*This throbbing streaming crowd of electrified shifting points in the spongework of the brain bears no obvious semblance in space pattern, and even in temporal relation resembles but a little remotely the tiny two-dimensional upside-down picture of the outside world which the eyeball paints on the beginnings of its nerve fibers to the brain.*

*But that little picture sets up an electrical storm. And that electrical storm so set up is one which effects a whole population of brain cells. Electrical charges having in themselves not the faintest elements of the visual - having, for instance, nothing of "distance," "right-side-upness," nor "vertical," nor "horizontal," nor "color," nor "brightness," nor "shadow," nor "roundness," nor "squareness," nor "contour," nor "transparency," nor "opacity," nor "near," nor "far," nor visual anything - yet conjure up all these.*

*A shower of little electrical leaks conjures up for me, when I look, the landscape; the castle on the height, or, when I look at him approaching, my friend's face, and how distant he is from me they tell me.*

*Taking their word for it, I go forward and my other senses confirm that he is there.*

SHERRINGTON, MAN ON HIS NATURE, 1942

But if you are on a quest for the scientific rather than a poetic truth you have to turn to mathematics. Below is an example.

Different types of brain injury produce different psychological/behaviorial impairments,[52] and advanced experimental/observational neurophysiology (ideally) delivers a correspondence between the "set" $M$ of "states $\mu$ of mind" and the set $\mathcal{N} = \mathcal{N}(\mu)$ of subsets of those neurons in the brain that are active in the presence of $\mu$.

Since the anatomy of the brain is, roughly, the same for all people, this allows a comparison of similar pattern $\mu$ and $\mu'$ in different individuals.

For instance, if experiencing particular colors $\mu$ were universally recognizable in terms of $\mathcal{N}(\mu)$ for all people (animals), one would be tempted to attribute the "predicate of existence" to *"qualia* of these colors".

More interestingly, the set $M$ inherits the *natural metric/distance*[53] from $\mathcal{N}$ via the map $M \to \mathcal{N}$ for $\mu \mapsto \mathcal{N}(\mu) \in \mathcal{N}$, and the resulting metric $dist_{\mathcal{N}}$ on $M$ makes an essential structural ingredient of the mind.

(If such a metric on $M$ were a reality, psychology would be equated with "*geometry of the mind(s)*" but deciding which particular metric is relevant for scientific understanding of human psychology, one needs to go deeper into its inner structure; this is not possible with the current state of knowledge. But some mathematics must be in the structure of the mind – it could not be so beautiful otherwise. And [without] *mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature* as Richard Fynmann says.)

[52] This has been known since about 3000 BCE and recorded in Edwin Smith Surgical Papyrus that was written about 1000 BCE and discovered in the middle 1800s.

[53] The distance between two subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ in a set of cardinality $N$ is defined as $\frac{1}{N}\left(card(\mathcal{N}_1 \cup \mathcal{N}_2) - card(\mathcal{N}_1 \cap \mathcal{N}_2)\right)$.

It has been attempted, since the advent of electronic computers, to model the Brain/Mind system in the language of *computations*. This might seem not a bad idea, since the language of computations universally apples to all time processes, such as the dynamics of the brain, for instance.

However, this very universality precludes a fine tuning of this language to specific problems and limits structural richness of mathematical theories based on computations. Nothing achieved in the "theory of algorithms and computations" comes close in its depth and structural beauty to, say, *Galois theory* or *Algebraic topology*.

If we want to understand this mysterious *nameless arrow* $\rightsquigarrow$, we have to borrow from *all kind* of ideas that mathematics has to offer.

The expressive power of the 21st century mathematical language exceeds anything dreamed of fifty years ago. But, probably, this language still falls short of accounting for this $\rightsquigarrow$, and, to be understood, the ergobrain structure that stands behind this arrow needs an input of yet unknown to us mathematics.

## 2 Mathematics and its Limits.

*Geometry is one and eternal shining in the mind of God.*
JOHANNES KEPLER.

We are no gods and our minds are not pure ergo. To build a mathematical frame for "ergo" we need to recognize what of our mathematics is ready to serve as "parts" of ergosystems, what should be rejected and what needs to be be made anew.[54] And "ergo-criteria" for these "ergo-parts" are exactly those we use everywhere in mathematics:

NATURALITY, UNIVERSALITY, LOGICAL PURITY and CHILDISH SIMPLICITY.

*Universality* of (many) learning programs in our ergobrains $\mathcal{EB}$ can be seen in the fact that we, humans, at least some of us, *enjoy and learn many* logically complicated games (and not only games). This suggests, for example, that a chess learning program in somebody's $\mathcal{EB}$ must be a specialization of a universal learning program for a rather generous concept of "universality".

On the other hand, why such programs should be simple? After all

*The human brain is the most complicated object in the Universe. Isn't it?*

But being mathematicians, we know that most general/universal theories are logically the simplest ones.[55] What is not simple is formulating/discovering such theories.

Also, as mathematicians we are ready to accept that we are hundred times stupider than the evolution is but we do not take it for the reason that evolution is able to make miracles, such as a logically complicated brain at birth. Believers into simplicity, we are compelled to seek our own solution to the *universal learning problem.*

As we aim at the very source of mathematics – ergobrain itself and try to develop a theory of ergosystems, purity and simplicity of the building blocks of

[54]Mathematics is the last born child of ergobrain and a *mathematician is an ergobrain's way of talking to itself* – as Niels Bohr would say.

[55]The simplicity of a universal idea, e.g. of *Gödel's incomleteness theorem*, may be obscured by plethora of technical details.

such theory becomes essential. It is not logical rigor and technical details that are at stake – without clarity you miss diamonds – they do not shine in the fog of an ego-pervaded environment.

But our thinking is permeated by ego that makes hard for us to tell "true and interesting" from "important" and that makes the (ergo)right choices difficult. For instance, in the eyes of the egomind, *simple and concrete* is what you see in front of you; much of mathematics appears *abstract and difficult*. But this simplicity is deceptive and unsuitable for "ergo-purposes": what your eyes "see" is *not* simple – it is an outcome of an elaborate image building by your visual ergosystem that is, probably, more abstract and difficult than most of our mathematics.

Evolution of mathematical concepts in their convergence to clear shapes suggests how one may design ergosystems. Our mathematical diamonds have been polished and their edges sharpened – century after century, by scratching away layers of ego from their facets, especially for the last fifty years. Some of what came out of it may appear as "abstract nonsense" but, as Alexander Grothendieck points out,

*The introduction of the cipher 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps.*

Yet, not all routes we explored had lead us to the promised land; understanding what and why did not work may be more instructive than celebrating our successes.

## 2.1 Logic and Rigor.

*Contrariwise, if it was so, it might be;*
*and if it were so, it would be;*
*but as it isn't, it ain't.*
*That's logic.* LEWIS CARROLL

According to tenets of *logicism* of Frege, Dedekind, Russell and Whitehead mathematics is composed of atomic *laws of thought* dictated by formal logic and the rigor of formal logic is indispensable for making valid mathematical constructions and correct definitions.

Admittedly, logicians participated in dusting dark corners in the foundations of mathematics but... most mathematicians have no ear for formal logic and for logical rigor.[56] We are suspicious of "intuitive mathematical truth" and we do not trust *meta*mathematical rigor of formal logic.

(Logicians themselves are distrustful one of another. For example, Bertrand Russell, pointed out that Frege's *Basic Law V* was self-contradictory, while in Gödel's words,

[Russel's] *presentation ... so greatly lacking in formal precision in the foundations ... presents in this respect a considerable step backwards as compared with Frege.*

[56]We happily embrace *model theory, set theory, theory of algorithms* and other logical theories that became parts of mathematics.

Russel's words *"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true"* apply to formal logic rather than to mathematics.)

Cleanness of things does not make them beautiful in the eyes of a mathematician. We care for logic no more than a poet for the rules of grammar.

Soundness of mathematics is certified by an ***unbelievably equilibrated*** *harmony* of its edifices rather than by the pedantry of the construction safety rules. Criticism of insufficient rigor in mathematics by George Berkeley (1734) as well the idea of "redemption" of Leibniz' calculus by Abraham Robinson (1966) strike us as nothing but puny in the presence of the miraculous formula

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...=\frac{\pi}{4}$$

for $\pi = 3.14159265...$ being one half of the length of the unit circle. (Leibniz, 1682).[57]

Historically, the system of calculus rolling on fuzzy wheels of *infinity* and *infinitesimals*, has been the main intellectual force driving the development of mathematics and science for more than three hundred years. But just a step away from mathematics, volumes of philosophical speculations on the "true nature" of infinity remain on libraries shelves covered in dust year after year.

(Yet, almost unbelievably, as recently as at the beginning of the 20th century, Florian Cajori, then a leading historian of mathematics, hailed *The Analyst* – the treatise by George Berkeley where he attacks "non-rigorous infinitesimals" as *the most spectacular event of the* [18th] *century in the history of British mathematics.*

The landscape of the 18th century British mathematics was, indeed, so bleak that even *The Analyst* was noticeable. But there were, however, two English mathematicians who left non-trivial imprints on the 18th century science –Thomas Bayes who suggested what is now called a *Bayesian* approach to empirical probability[58] and Edmond Halley, who is most famous for computing the orbit of *Halley's Comet*.[59])

We can not take seriously anything like $(a,b) := \{\{a\},\{a,b\}\}$[60] but for some inexplicable reason this century old foundational dust finds its way to our textbooks under pretext of rigor as, e.g., in the following definition of a *graph* $G$ as

*an ordered* [by whom?] *pair* $G = (V,E)$ *comprising a set* $V$ *of vertices...*

We better keep clear of this "rigor".

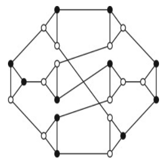
Not everything in logic is collecting, cleaning and classifying morsels of common sense. In 1931, the *logician* Kurt Gödel defied everybody's intuition, including that of the *mathematician* David Hilbert who formulated the question a few yeas earlier, by *mathematically* proving that

[57]The achievement of Robinson from a working mathematician perspective was not so much in justification of Leibniz' idea of infinitesimals but rather in a vast and powerful extension of this idea.

[58]Bayesian approach relies on continuous updating of conditional probabilities of events rather then on integrated frequencies; it is systematically used now-a-days in *machine learning*.

[59]Halley is the only short-period comet that is clearly visible from Earth when it returns to the inner solar system, approximately with 75 year intervals.

[60]This is the 1921 definition of an *ordered pair* by Kuratowski. To get "convinced" that this definition is worth making, you must accept logicians' appeal to metamathematical intuition.



*Every* ***formalization of mathematics*** *contains*
***unprovable*** *propositions that can not but be regarded as being "**true**".*

Geometrically speaking,

*the "body of mathematical truth" is* DISCONNECTED.

(In fact, this "body" consists of *infinitely many* islands with *no bridges of deductive logic* joining them.[61])

Here "formalization of mathematics", denoted $\mathcal{MATH}$, means a "formal mathematical system or theory" – a language with a prescribed vocabulary and grammar rules. An essential property of such a $\mathcal{MATH}$ needed for the validity of Gödel's theorem is that $\mathcal{MATH}$ contains a sufficient vocabulary for speaking about languages $\mathcal{Y}$ regarded as mathematical objects. Basically, what one needs is the concept of a certain *mathematical property* to be *satisfied* by a given word (a sentence if you wish) $y$ in $\mathcal{Y}$ and/or to have a *proof* in $\mathcal{MATH}$. Then what $\mathcal{MATH}$ says about itself translates to Gödel's proof of the theorem.

Nothing special about $\mathcal{MATH}$ is needed for Gödel's theorem – it is valid for all "reasonable formal systems". One does not even have to know what a formal language is; all one needs is to spell out "reasonable" in general terms and apply *Cantor's Diagonal Argument* to some function $F$ in two variables $p$ and $s$, where this $F$ says in effect that a certain "property" depending on $p$ is satisfied or not by an $s$.

And to be "rigorous" one has to suffer through half a page of (inevitably?) boring definitions.

The vocabulary of a $\mathcal{MATH}$ must include the following.

• A set $S$, the members $s \in S$ of which are called *sentences in the the language of* $\mathcal{MATH}$.

• A set $T$ called the *set of truth values for* $\mathcal{MATH}$. (In the "every day $\mathcal{MATH}$" this $T$ consists of two elements **true** and **untrue** where meaningless sentences $s$ are regarded as **untrue**.)

• A class $\mathcal{F}$ of $T$-valued functions $f(s)$ on $S$ called *functions defined in* $\mathcal{MATH}$. (In "real math", such an $f$ tells you whether a sentence $s$ is true or untrue/meaningless.)

• A subset $P \subset S$ where sentences $p \in P$ are called *proofs*.

• A *reduction* map $R : p \mapsto f \in \mathcal{F}$ from $P$ to $\mathcal{F}$ where the functions $f(s)$ in the image of $R$ are called *provably defined in* $\mathcal{MATH}$. (This means that every proof $p$ includes a "statement" of what it proves; this "statement" is called $R(p) \in \mathcal{F}$.)

Then GÖDEL'S INCOMPLETENESS THEOREM says that under the following assumptions (A) and (B)

[61]Udi Hrushovski was explaining to me several times (not that I understood it) that this metaphor applies only to "bridges on a given level". Also he pointed out to me that what we call and prove here under the heading of *Gödel's theorem* is what logicians call *Tarski's "undefinability of truth" theorem.*

*the map $R$ can not be onto*:

*there exist functions* **defined** *in* $\mathcal{MATH}$ *that* **can not be provably** *defined.*

(A) The "$P$-diagonal" $F(p,p)$ of the $T$-valued function in two variables $F(p,s) = R(p)(s)$ admits an extension to a function on $S \supset P$, say $f_R(s)$, that is *defined in* $\mathcal{MATH}$.

(B) There exists a transformation $\tau : T \to T$, such that

($B_1$) the composed functions $\tau \circ f : S \to T$ are *defined in* $\mathcal{MATH}$ *for all* $f \in \mathcal{F}$,

($B_2$) $\tau$ has *no fixed point*: $\tau(t) \neq t$ for all $t \in T$.

(Properties (A) and ($B_1$) are satisfied by the "real world math" almost by definition, while ($B_2$) says that no sentence can be simultaneously **true** and **untrue**.)

*Proof of Gödel's Theorem.* By (A), the function $f_\circ(s) = \tau \circ f_R(s)$ is defined in $\mathcal{MATH}$; this $f_\circ(s)$ is different from the functions $f_p(s) = R(p)(s)$ for all $p$, since $f_p(s) \neq \tau \circ f_R(s) = \tau \circ R(p)(s)$ at $s = p$ because of ($B_2$).

*Discussion.* (a) Cantor's diagonal argument was designed for showoing that the set (space) of *all* functions $f : S \to T$ is *greater* than $P$ for all $P \subset S$ and all $T$ of cardinality at least two. This *greater* is strengthened and "quantified" in many geometric categories as follows.

No family $f_p = f_p(s)$ of functions on $S$ contains *generic* $f = f(s)$.

This, applies, for instance, with several geometrically defined notions of genericity[62] for maps between Euclidean spaces where functions $f$ may be continuous, smooth analytic or algebraic (and where genericity is accompanied by *transversality*).

On the other hand, explicitly described functions that one finds in "real life" (e.g. on Google) are, as we mentioned earlier, more scarce than, say, natural numbers $n$, partly, because descriptive (less so graphical) presentation of "interesting" functions occupies more space than that for numbers. We shall see similar patterns in the hierarchical organization of our ergosystems.

(b) The childish simplicity of the proof of Gödel's theorem[63] does not undermine its significance. *Metamathematics* is similar in this respect to other non-mathematical sciences where a mathematical argument is judged not by its difficulty but by its applicability to "real life". Nontriviality of Gödel's theorem resides in a possibility of a meaningful metamathematical interpretation of the above "provably defined".

In logical practice, the truth value set $T$ usually (but not always) consists of two elements, say, *yes* and *no* with $\tau$ interchanging the two and, in Gödel's

[62]Geometry is non-essential here: concept of "genericity" belongs with mathematical logic. The universal logical power of "genericity" was *forcefully* demonstrated by Paul Cohen in his proof that the cardinality of "a generic subset" in continuum is *strictly* pinched between "countable" and "continuum".

[63]Originally, Gödel's theorem was stated for a certain formalization $\mathcal{ARITH}$ of arithmetic that was designed for talking about numbers rather than about languages; that necessitated a lengthy translation from the language of $\mathcal{ARITH}$ to the language in which one could formulate the theorem.

A transparant categorical rendition of Gödel's theorem is presented in "Conceptual Mathematics" by Lawvere and Schanuel [8]. This was pointed out to me by Misha Gavrilovich who also explained how the above proof may be seen as an adaptation of their argument.

case, one takes $P = S$. Our functions $f(s)$ are associated with "properties" $\Pi$ describable in the language of $\mathcal{MATH}$", with $f_\Pi(s)$, equal *yes* or *no*, depending upon whether $\Pi$ is satisfied or not by $s$, where, in general, the truth value comes without being accompanied by a proof.

For example, a sentence $s$ may describe an equation with $\Pi$ saying "solvable", where an equation, is either solvable or not regardless of an availability of a proof of this in a given $\mathcal{MATH}$. (The certainty of this "either *yes* or *no*" is debatable even for *Diophantine* equations $f(x_1, ..., x_k) = 0$, i.e. where $f$ is a *polynomial* with *integer* coefficients and where one speaks of *integer* solutions $(x_1, ..., x_k)$.)

By definition of $P$, a proof $p \in P$ that certifies correctness of the truth values $f_\Pi(s)$ at all $s$, "says" in particular, what is the property $\Pi$ that this $p$ proves; this information is extracted from $p$ by the reduction map $R$.

But anything that can be called "rigor" is lost exactly where the things become interesting and nontrivial – at the interface between mathematics and "logical reality". For instance, a variation of Gödel's theorem may tells you that there exists a mathematical proposition that can be written, say, on 10 pages but the proof of which will need between $10^{10^{10}}$ and $1000^{1000^{1000}}$ pages. This is perfectly acceptable *within* mathematics but becomes non-sensical if you try to apply it to mathematics "embedded into the real world".

To see what makes us preoccupied with these "logical trifles", look closely at what stands behind the following *kindergarten Ramsey theorem*:

Given a group of six chess players, then necessarily,

*either* there are *three of them* where *every two* of these players once met over a chessboard,

*or* there are *three* such that *no game* was ever played between any two of them.

A child may *instantaneously* visualize a *graph* with green (played a game) and a red (never played a game) strings/sticks/*edges* between the pairs of these six people for *vertices*. (The child does not have to know graph theory.) Then the proof of the existence of a *monochromatic triangle* will come after a few minutes (hours?) thought.

Moreover, a mathematically inclined child will soon generalize this to the full Ramsey theorem:

*if the subsets of an infinite set $X$ are colored either in green or in yellow, then, for every $k = 1, 2, 3, ...,$, this $X$ contains an infinite* ***$k$-monochromatic*** *subset $Y = Y(k) \subset X$, i.e. where all $k$-element subsets are of the same color.*[64]

No existing computer program is anywhere close to doing this. The main difficulty is not finding proofs of mathematically stated Ramsey level theorems - these may be within the range of "symbol crunching" programs. It is automatic translation of the "real world" problems to mathematical language what remains beyond our reach. Probably, only a *universal* ergoprogram that would teach itself by reading lots of all kinds of texts will be able to achieve such translation.

### Logic in Science.

Mathematical rigor and logical certainty are absent not only from logical foundations of mathjematics but also from all natural sciences even from theoretical physiscs. Einstein puts it in words:

[64]Graphs correspond to $k = 2$, while 6 and 3 are equated with $\infty$ in kindergartens.

*As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

But "the physical level of rigor" is higher on certainty than the logical one, since reproducible experiments are more reliable than anybody's, be it Hilbert's, Einstein's or Gödel's, intuition.

## 2.2 Polynomialas, Equations, Computations.

Formal languages do not walk on the streets occupying themselves with proving Gödel's style theorems one about another. But we humans *are* walking computers that are programmed, among other things, to guess and to imitate each other's mental computations.

*"Computation"* as it is used in the science of the brain and in science in general is a metaphor for elaborated, yet, structurally organized process. But there is no clarity with this notion.

Does, for instance, a planetary system perform a computation of, say, its total potential energy? You would hardly say so on the microsecond time scale but it may look as a "computation" if the time measured in million years.

In mathematics, there are several specific models of computation but there is no readymade language for describing all conceivable models.

Mind you, there is an accepted class $\mathcal{COMP}_{\mathbb{N}\to\mathbb{N}}$ (that parallels the class of provably defined functions) of what is called *computable* or *recursive functions*, $R(n)$ that send $\mathbb{N} \to \mathbb{N}$ for $\mathbb{N}$ being the set of *natural numbers* i.e. of positive integers $n = 1, 2, 3, 4, 5, \ldots$. Yet, there is no single distinguished natural description of this class as it is witnessed by the presence of *many* suggestions for its "best" description with the following five being most prominent.

*Recursion + Inversion* (Skolem, Gödel, Herbrand, Rózsa Péter),
*$\lambda$-calculus* (Church),
*Turing machines and programs* (Babbage, Ada Lovelace, Turing),
*cellular automata* (Ulam, von Neumann, Conway),
*string rewriting systems* (Markov).

These definitions of "computable" reflect their author's ideas on what is *"simple, useful, natural"* with the corresponding schemes of computation being quite different. None of them can be taken for the "normal" or "canonical" form of computation.[65] Besides, all these definitions of $\mathcal{COMP}$ are decades old and they have not undergone the post-Grothendieck *category theoretic* renovation.[66] But the traces of the following ideas, that underly the concept of computation, will be seen in our ergo-models.

• Compositions and Categories. *Composabilty* says that if a computation with the input from some (constructive set? class?) $X_1$ and output in $X_2$, denote it $C_1 : X_1 \rightsquigarrow X_2$, is followed by $C_2 : X_2 \rightsquigarrow X_3$, then the tautological composition $C_3 = C_2 \circ C_1 : X_1 \rightsquigarrow X_3$, where $C_2$ is performed right after $C_1$,

[65]It is hard to argue for or against something being "natural". *Feets*, *meters* and *miles* may seem natural physical units of distance for some people. But, probably, there is *neither a truly canonical* normal form *nor a convincing mathematical* concept of equivalence applicable to different models of computation.

[66]Computations are not bound to $\mathbb{N}$ but I am not certain if there is a proper definition of "computable objects", e.g. sets with some structures representing suitable functors, in (yet, hypothetical) "computable categories".

is a computation again.[67] Thus, computations make what one calls *categories* provided *the identity* [non]-computations are in there as we shall always assume.

(Composability is a fundamental but non-specific feature of computations – almost everything you do in mathematics can be "composed" if you think about it.)

Moreover, many computation schemes operate with functions in *several variables* with arguments and/or values in a certain set $X$, that may be the set $X = \mathbb{N}$ of *natural numbers*, the set $X = \mathbb{Z}$ of *integers* and the set $X = \mathbb{R}$ of *real numbers.*, where "complicated" computable functions are successively built from "simple modules" by composing these "modules".For instance, the following four functions:

two one variable functions: *the constant* $\mathbf{x} \mapsto \mathbf{1}$ and *the identity* $\mathbf{x} \to \mathbf{x}$
and two functions in two variables:
*subtraction* $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_1 - \mathbf{x}_2$ and *multiplication* $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_1 \cdot \mathbf{x}_2$

generate, in an obvious sense, all *polynomials* that are sums of products of constants (coefficients) and powers of variables:

$$P(x_1, ..., x_k) = \sum_{d_1,...d_k \le D} a_{d_1,,...d_k} x_1^{d_1} \cdot ... \cdot x_k^{d_k}.$$

with integer coefficients $a_{d_1,...,d_k}$ for all $k = 1, 2, ...$ and all degrees $D = 0, 1, 2, ....$

SUPERPOSITIONS, CLONES, MULTICATEGORIES, OPERADS.

The algebraic skeleta of sets of functions in several $x$-variables closed under compositions, also called *superpositions* in this context, go under the names of *"abstract clones"* in mathematical logic and universal algebra and and/or "*operads*" in algebraic topology. More generally, if the domains and ranges of maps are *not* assumed to coincide, than one speak of *multicategories*. These, similarly to ordinary categories, are described in terms of of classes of *diagrams of (multi)arrows* that mimic the obvious associativity-like properties of superpositions of functions.

The operad structures underly neural networks models of the brain. They will be also present in our ergosystems, where we shall insist on assigning specific structures to what goes under the heading "several" and/or "multi". (A newly born ergobrain does not know what the set $\{1, 2, ..., k\}$ is and it can not operate with functions presented as $f(x_1, x_2, ..., x_k)$).

• INVERSIONS. Inverting a function $y = P(x)$, that is finding $x$ that satisfy the equation $P(x) = y$ for all $y$ in the range of $P$, may be frustratingly difficult even for simple function $P : X \to Y$. An instance of this is computing (the integer part of) $\sqrt{y}$ for integer $y$ that is much harder than taking $y = x^2$.

In general, the inverse map $P^{-1}$ sends a point $x \in X$ *not to a single point* but to a (possibly empty) *subset* called $P^{-1}(x) \subset Y$, namely, to the set of those $y \in Y$ where $P(y) = x$. But a composition of such $P^{-1}$ with some map $Q : Y \to Z$ may be a bona fide point map denoted $R = Q \circ P^{-1} : X \to Z$. This happens if $P^{-1}(x)$ is *non-empty* for all $x \in X$, i.e. $P$ is *onto*, and if $Q$ is constant on the subsets $P^{-1}(x)$ for all $x \in X$. In this case

$R$ equals the *unique solution* of the equation $R \circ P = Q$.

[67]There is no consensus for writing $C_2 \circ C_1$ or $C_1 \circ C_2$. Although the *Zermelo Buridan's ass axiom* allows a choice of one of the two, remembering which one is impossible – how can one tell "←" from "→" in a symmetric Universe?

Thus, extensions of classes of maps by adding such inverses may be described in category theoretic terms as follows. Let $\mathcal{P}$ be a subcategory of a category $\mathcal{S}$, e.g a class of maps $p$ between sets that is closed under composition.

*The invertive extension* $\mathcal{R}$ of $\mathcal{P}$ in $\mathcal{S}$ is, by definition, obtained by adding to $\mathcal{P}$ the solutions $R \in \mathcal{S}$ of the equations $R \circ P = Q$ for all $P$ and $Q$ in $\mathcal{P}$ whenever such a solution exists and is unique.

(This $\mathcal{R}$ may be *non-closed* under composition of morphisms and it can be *enlarged further* by generating a subcategory in $\mathcal{S}$ out of it.)

Such an extension may be incomparably greater than $\mathcal{P}$ itself, where the BASIC EXAMPLE of this is as follows.

Let $\mathcal{S}$ be the category (*semigroup* in this case) of functions $\mathbb{N} \to \mathbb{N}$ for $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ and let $\mathcal{P}$ consist of *primitively recursive* functions.

*Then the invertive extension $\mathcal{R} \subset \mathcal{S}$ of $\mathcal{P}$ equals the set of all recursive (i.e. computable) functions $\mathbb{N} \to \mathbb{N}$.*

"Primitively recursive" is a currently accepted formalization of "given by an explicit formula". Such formalizations and the resulting $\mathcal{P}$ may be somewhat different but the corresponding $\mathcal{R}$ are all the same.

A convincing instance of this is

DPRM THEOREM.[68] *The invertive extension $\mathcal{R}$ of the subcategory $\mathcal{P}$ of polynomial maps $\mathbb{N}^k \to \mathbb{N}^l$, $k, l = 1, 2, 3, 4, 5, ...$, in the category $\mathcal{S}$ of all maps equals the category of recursive (i.e. computable) maps $\mathbb{N}^k \to \mathbb{N}^l$.*

In other words,

*every computable function $R : \mathbb{N} \to \mathbb{N}$, can be decomposed as $R = Q \circ P^{-1}$, where*

- *$P, Q : \mathbb{N}^k \to \mathbb{N}$ are integer polynomials,*
- *the map $P : \mathbb{N}^k \to \mathbb{N}$ is onto,*
- *$Q$ is constant on the subsets $P^{-1}(n) \subset \mathbb{N}^k$ for all $n \in \mathbb{N}$.*

*(Moreover, there is a universal bound on $k$, e.g. $k = 20$ suffices.)*

The way the theorem is proven allows an explicit construction of polynomials $P$ and $Q$, e.g. in terms of a Turing machine (defined later on) that presents an $R$. For instance, these $P$ and $Q$ can be actually written down for the *nth prime number function* $n \mapsto p_n$.

However, this theorem does not and can not shed any light on the structure of prime numbers[69]. All it shows is that Diophantine equations, that make a tiny fragment of the world of mathematics, have, however, a capability of "reflecting" all of $\mathcal{MATH}$ within itself: any given (properly formalized) mathematical problem $\Pi$ can be translated to the solvability problem for such an equation. This, in conjunction with Gödel's theorem, tells you that

*solvability of general equations $P(x_1, x_2, ..., x_k) = n$ is an intractable problem.*

There is a host of similar theorems for all kinds of (not at all Diophantine) "simple equations" that make mathematics, seen from a certain angle, look like a fractal composed of infinitely many "Gödel's fragments" where each "fragment" multiply reflects $\mathcal{MATH}$ as a curved fractal mirror with every reflected image of

[68] Conjectured by Martin Davis (Emil Post?) in 1940's and finalized by Matiyasevich in 1970 following Davis, Putnam and Robinson.

[69] Probably, *nothing what-so-ever* about prime numbers can be seen by looking at such $P$ and $Q$, not even that there are infinitely many of primes.

$\mathcal{MATH}$ being transfigured by a chosen translation of $\mathcal{MATH}$ to the language of this "fragment".

A translation of a "general difficult problem" $\Pi$ to a "concrete and simple" equation whenever such a translation is available by a DPRM kind of theorem, does not help solving $\Pi$ but rather shows that an apparent simplicity of the corresponding class of "equations" is illusory with Gödel's theorem guarding you from entering blind alleys of naive solvability problems.[70]

For instance, the solvability problem for a Diophantine equation $P(x_1,...,x_k) = 0$ transforms by a particular translation algorithm $\mathrm{ALG}_{part}$ built into a given proof of DPRM to the solvability problem for an equation $P_{new}(x_1,...,x_l) = 0$ with the (integer) polynomial $P_{new}$ being by far more complicated (i.e. with larger coefficients) than the original $P$, where it is virtually impossible to reconstruct $P$ back from $P_{new}$ even if you know $\mathrm{ALG}_{part}$.

The DPRM theorem itself was a response to David Hilbert who suggested in his 10th problem:

> *to devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.*

This idea of a possible effective resolution of all Diophantine problems was in line with Hilbert's (pre-Gödel) optimistic:

*Wir müssen wissen – wir werden wissen!*

[We must know – we will know!)]

(Hilbert also articulated this position is his 2nd problem: *a direct method is needed for the proof of the compatibility of the arithmetical axioms.*)

But the DPRM theorem showed that Hilbert's suggestion taken literally was unsound and, if followed, must be coupled with a search for *particular classes* of equations where integer solutions are well structurally organized.

The "true Diophantine beauty", as we see it to-day, resides not in integer solutions of $P(x_1,...,x_k) = 0$, but in *non-Abelian higher dimensional "reciprocity laws"* associated to integer polynomials $P$. Roughly, such laws can be seen as *analytic relations* between infinitely many numbers $N_p(P)$ for *all prime* $p = 2,3,5,7,11,13,17,...,$ where $N_p(P)$ equals the number of solutions of the congruence $P = 0 \mod p$.

[70] Even without Gödel, anything as easy to formulate as the solvability problem makes one wary, be these Diophantine or other kinds of equations.

Such relations are expected to generalize *Riemann's functional equation*

$$\frac{\zeta(1-s)}{\zeta(s)} = \frac{\alpha(s)}{\alpha(1-s)},$$

where

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \quad \text{and } \alpha(s) = \frac{1}{2}\pi^{-s/2}\int_0^\infty e^{-t}t^{\frac{s}{2}-1}dt \quad \text{for } s>1,$$

where both functions, $\zeta$ (that harbors the deepest mysteries of prime numbers) and $\alpha$ (an apparently insignificant child of simple minded analysis) admit meromorphic extensions to all complex $s$-plane and where the thus defined functional equation, applied at different $s$, encompasses infinitely many relations between the prime numbers $p = N_p(P)$ for $P = P(x_1, x_2) = x_1 - x_2$.

The above is just a hint at what is known as the *Langlands program*[71] that predicts a presence of unexpectedly strong and simple structural constrains (laws) that are satisfied by quite general and complex objects like the above $P$ and that is opposite to the spirit of the DPRM style theorems where one is keen at exhibiting special and apparently simple objects that display an arbitrarily complex behaviour not constrained by any "law".

The lesson we draw from the "Diophantine story", where properties $P$ of *unknown* objects $x$ are expressed by algebrac equations, is that

> *identifying essential properties $P$ of an $x$ and formulating structurally significant questions about these $P$ is more instructive than straightforward attempts to construct $x$.*

We believe, this as much applies to yet unknown objects $x$ that mathematically represent *thinking and learning processes* as to $k$-tuples of integers $x = (x_1, ..., x_k)$.

## Networks behind Formulas.

Arithmetic operations such as $x_1 + x_2$ and $x_1 \cdot x_2$ becomes progressively more and more elaborate as the numbers $x_1$ and $x_2$ grow, but they are decomposable into sequences of a few elementary operations *over the decimals* of these numbers as every schoolgirl knows.

And general "complicated computations" can be realized by networks of "elementary computational steps" with no explicit use of anything mathematically elaborate (e.g. addition and multiplication of integers) as it is done in our computers and, probably, in our brains (where "elementary steps" may go far from computers).

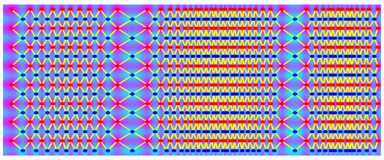
In fact, (almost?) all multivariable functions "in real life" come as sums of simple superpositions of functions in one and two variable such as (generalized) *tensorization* – the most common decomposition of functions into sums of products :

$$F(x_S) = \sum_{i\in I}\prod_{s\in S} f_{s,i}(x_s),$$

where

- $S$ is a finite set that enumerates the $x$-variables:

[71] This program goes along the lines of Hilbert's 9th and 12th problems on *the most general law of reciprocity in any number field* and on *Abelian extensions* of such fields.



• each $x$-variable $x_s$, $s \in S$, runs over a given *domain set* denoted $X \ni x_s$;
• $x_S$ is denotes the totality of the variables $x_s$, that is an element of the Cartesian $S$-power of $X$:

$$x_S \in X^S = \underbrace{X \times X \times ... \times X}_{S};$$

• $I \ni i$ is a finite set that enumerates functions $f_{s,i}$ on $X$;
• the functions $f_{s,i}$ on $X$ and $F$ on $X^S$ take values in a set called $Y$.

In order to be able to make "sums of products" $\sum_i \prod_s$, we endow the set $Y$ with two *binary operations* that are functions in two variables $Y \times Y \to Y$ denoted $y_1 \boxplus y_2$ and $y_1 \boxtimes y_2$ and we suppose that

*$Y$ contains two elements, called "zero", denoted $0 \in Y$ and "one" $1 \in Y$ with their usual arithmetic properties:*

$y \boxplus 0 = 0 \boxplus y$ for all $y \in Y$,
$y \boxtimes 0 = 0 \boxtimes y = 0$ for all $y \in Y$,
$y \boxtimes 1 = 1 \boxtimes y = y$ for all $y \in Y$, $y \neq 0$.

If the functions $f(x) = f_{s,i}(x_s)$ are *atomic* i.e. different from "zero" (at most) at a *single* element in $X$ (depending on $s$ and $i$) and if, for each $i \in I$, there is at most *single* $s \in S$ such that the non-zero value of $f_{s,i}$ is different from "one" (this is a kind of "multiplicative atomicity"), then the above $\sum_i \prod_s$ makes sense. Moreover, (this is obvious)

*every function $F(x_S)$ that equals "zero" away from a finite subset in $X^S$ admits a "tensorial" decomposition into atomic functions in the variables $x_s$.*

In fact, one does not truly need $\boxplus$ as it can be obtained by composing $\boxtimes$ with the one variable function $y \mapsto y^\perp$ that interchanges $0 \leftrightarrow 1$ and keeps all other $y \in Y$ unchanged. Indeed, the operation

$$y_1 "\boxplus" y_2 =_{def} (y_1^\perp \boxtimes y_2^\perp)^\perp$$

does satisfy the above properties required of $\boxplus$.

Polynomials in real variables $x_s \in \mathbb{R}$ are most common examples of such "tensors", but computationally/logically the simplest case is where $X = Y = \mathbb{F}_2 = \{0,1\}$, that is the *field of integers* mod 2 (where $1 + 1 =_{def} 0$), rather than $\mathbb{R}$. Here, observe, (this equally applies to all finite fields $\mathbb{F}_{p^n}$)

*every atomic $\mathbb{F}_2$-valued function on $\mathbb{F}_2$; hence, every $\mathbb{F}_2$-valued function on $\mathbb{F}_2^S$, is representable by a polynomial.*[72]

[72]One can *discard multiplication* in writing down polynomials over the fields $\mathbb{F}_{p^n}$ for the primes $p \neq 2$, because $x_1 x_2 = \frac{1}{4}\left((x_1+x_2)^2 - (x_1 - x_2)^2\right)$. This does not work for $p = 2$, e.g. in $\mathbb{F}_2$ where, instead, one can discard addition by expressing it as a superposition of the functions $x \mapsto x + 1$ and $(x_1, x_2) \mapsto x_1 \cdot x_2$.

The architecture of tensorial representations of functions is clearly visible. For instance, there are $2^{2^{card(S)}}$ different polynomials over $\mathbb{Z}_2$ in $x_s, s \in S$, that correspond to subsets in the power set $2^S$.

But general representations of functions by a superposition of two binary operations, say of "+" and "×" are combinatorially more elaborate: these are *formulas*, such as

$$((*o(*o*))o*)o(((*o*)o(*o*)o*)),$$

where

$$* \text{ substitutes for } x_s\text{-variables;}$$

and

$$o \text{ stands either for } + \text{ or for } \times.$$

These formulas express computations as *strings* in four types of symbols:

$$* \quad o \quad ( \quad )$$

but depicting computation by *one dimensional strings* is an artifact of the way we speak and write[73] that does not faithfully reflect the geo-combinatorial structure behind such formulas.

The essential reason for his is that the *rule of brackets* that restricts admissible configurations of right and left brackets symbols ")" and "(" is *non-local in the string geometry*[74] unlike, for instance, the prohibition of $oo$ or $)*$ for consecutive pairs of symbols in the strings.[75]

"Spaces" between pairs of brackets, (regardless of symbols written in there), such as ((...)... (...(..).. ...)), make a *nested* family of subsets (intervals). The natural *partial order* (by inclusion) between these subsets can be depicted by a tree – a binary tree[76] in the present case, where some vertices are labeled by two "colors" that are + and ×.

In general, let $(V, \vec{E})$ be a directed graph where there are three kinds of vertices:

- *input vertices* $v = v_{in}$: these have no incoming edge-arrows at them;
- *output vertices* $v = v_{out}$: these have no outgoing edge-arrows;
- *operational vertices* $v = v_o$ also called *o-vertices*: these have two incoming and several (a single one for the trees depicting the above string-formulas) outgoing edges.

An additional structure in this graph is an *o-labeling*:

> *o-vertices are labeled* either by + or by × and, accordingly, denoted by $v_+$ and $v_\times$.

Now, given a set $X$ with symmetric binary operations + and ×, this graph defines

A STATIONARY DESCRIPTION OF COMPUTATION
*over $X$-values of functions on the input vertices in $V$.*

This computation, that applies to arrays of values $x_{in} = x(v) \in X$, $v \in V_{in}$, is represented by

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[73]Like it or not, there is a non-trivial reason why languages are condemned to this one-dimensionality. Nature herself could not write her messages on 2D DNA.

[74]An essential (not the only) aspect of this rule is the equality:

*the number of the left brackets "(" $=$ the number the right ones ")" .*

[75]Only eight (out of 16) consecutive pairs are allowed: $*o \quad o* \quad )) \quad (( \quad (* \quad *) \quad )o \quad o($ .

[76]The (artificial) linear order structure in strings disappears when we pass to this tree.

*an $X$-valued function* on $V$, say $x(v)$ that *extends* $x_{in}$ from $V_{in}$ to all of $V$, such that

$$x(v_+) = x(v_1) + x(v_2) \text{ and } x(v_\times) = x(v_1) \times x(v_2)$$

for all +-vertices and ×-vertices in $V$ and the of vertices pairs of vertices $v_1$ and $v_2$ adjacent to them by $v_1 \longrightarrow v_o \longleftarrow v_2$, where $o$ substitutes for the corresponding + or × label at this $v_o$.

If such an extension exists, as it is the case for trees, the computation is truly defined and one may speak of

*the output* of the computation that is the restriction of $x(v)$ to the subset $V_{out} \subset V$ of the output vertices in the graph.

Notice that this output consists of a *single* $x \in X$ in the case of string-formulas where these graphs are *rooted trees* with $V_{out}$ being a single vertex – the root of this tree.

In order to arrive at the *polynomial* (rather than a computation of its particular value) defined by such a graph one needs three extra ingredients.

(1) Division of the input subset of vertices, $V_{in} \subset V$ into two disjoint subsets corresponding to constants and to variables $V = V_{in,const} \cup V_{in,var}$;

(2) Giving specific values, say $c_v$ to all $x(v)$, $v \in V_{in,const}$;

(3) Joining some vertices in $V_{in,var}$ by edges, representing equalities between the values of $x$ at the corresponding vertices, where the set of these edges is denoted $E^{=}_{in}$.

This latter serves to remove the set $S$ indexing the variables $x_s$ from the definition of a polynomial by (implicitly) replacing it with the set of the *connected components* of the graph $(V_{in,var}, E^{=}_{in})$:

The graph $(V, E)$ with extra (1), (2), (3) ingredients, defines a polynomial in variables indexed by the connected component of this $(V_{in,var}, E^{=}_{in})$, at least in the case where this $(V, E)$ itself is a rooted tree.

(Supression of "amorphous" sets like $S$ from the description of mathematical objects is very much in the spirit of ergo logic, where an alternative is giving "interesting structures" to such sets $S$.)

The following problem raised about half a century ago remans wide open.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two sets of polynomials over $\mathbb{F}_2 = \{0, 1\}$ that are defined by two "simply combinatorially describable" classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of $o$-labeled graphs $(V, E \cup E^{=}_{in})$, where, to simplify, we assume that $X_{in} = X_{in,var}$ and so the corresponding polynomials contain no constant terms.

*When can one tell that $\mathcal{P}_1$ equals $\mathcal{P}_2$ or that $\mathcal{P}_1$ is included into $\mathcal{P}_2$?*

In particular,

*can one bound from below the "complexity" of graphs from $\mathcal{C}_1$ (e.g. the minimal possible number of vertices in such a graph[77]) that represent the same polynomials as given graphs in $\mathcal{C}_2$?*

[77]The famous $P \neq_? NP$ dilemma is an instance of a precise formulation of this complexity problem.

## 2.3 Games of Life.

The idea of decomposing logical reasoning into elementary computational steps was suggested by Leibniz three centuries ago, who introduced what is nowadays called *the Boolean algebra* that is generated by three operations over *binary* variables $x$, i.e. with two possible values $\circ$ and $\bullet$. The one of the three operations, called $NO$, is *unary*, i.e. it maps $\{\circ,\bullet\}\to\{\circ,\bullet\}$ and the other two are binary, $\{\circ,\bullet\}^2\to\{\circ,\bullet\}$, called $AND$ and $OR$:

*NO:* $\bullet \overset{\perp}{\leftrightarrow} \circ$,

*AND:* $(x_1,x_2)\mapsto x_1\wedge x_2$; *this, by definition, equals* $\bullet$ *if* $x_1=x_2=\bullet$ *and* $x_1\wedge x_2=\circ$ *if either* $x_1$, *or* $x_2$, *or both equal* $\circ$,

*OR:* $(x_1,x_2)\mapsto x_1\vee x_2 =_{def} (x_1^\perp\wedge x_2^\perp)^\perp$; *this equals* $\circ$ *if and only if* $x_1=x_2=\circ$.

Since $x_1\wedge x_2$ equals the product $x_1\cdot x_2$ in the field $\mathbb{F}_2=\{0,1\}$ for $\{\circ\leftrightarrow 0,\bullet\leftrightarrow 1\}$ and since addition as a function $\mathbb{F}_2\times\mathbb{F}_2\to\mathbb{F}_2$ can be expressed via multiplication,[78] *all* binary functions $\{\circ,\bullet\}^S\to\{\circ,\bullet\}$ are superpositions of the above three. (This, apparently, was one of Leibniz' motivations for introducing these $\wedge$ and $\vee$.)

General purpose programmable computers based on similar decompositions were designed by Charles Babbage in the mid-1800's and the modern perspective on machine/network computations was opened by a 1936 paper by Alan Turing.

The idea of Turing was to show that *an arbitrary* computational process $X$ can be effectuated by a *"simple minded bug"* who crawls back and forth along a given string of letters (or digits) and performs a computation over such a string by modifying the letters along its path according to an instruction given to it. (A definition is given later on in this section.)

However, contrary to what Leibniz dreamed, decomposing "complicated processes, call them $\Psi^\star$ into "elementary (bug's) steps" *does not* render $\Psi^\star$ amenable to any kind of "simple algebraic analysis", but rather shows that the apparent simplicity of decomposed $\Psi^\star$ is illusory. In fact, even a simplest operation, say squaring of integers $n\mapsto n^2$, may become nearly incomprehensible when translated to the language of instructions executed by a "Turing bug".

The network "core" of the bug computational model is so simple that it is almost invisible: it is the ideal *memory tape* $[0,\infty)$ decomposed into unit segments $s=[i,i+1]$, $i=0,1,2,...$,

$$0 \;\text{-|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|— |— |—}\underset{i}{|}\text{—}\underset{i+1}{|}\text{—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—|—}...,$$

that is represented by the graph where the edges correspond to the pairs of adjacent segments.

A geometer as well as a "true bug" would feel rather constrained on the 1D tape and be more comfortable on the 2-plane divided into squares.

LANGTON ANT (1986).

This "ant" crawls from square to square on an infinite sheet $S$ of squared paper, with a letter **L** or **R** written in each square $s$.

The ant is depicted by an arrow that may be oriented in four different ways:

$$\{\uparrow,\downarrow,\to,\leftarrow\}.$$

[78] $OR$ is similar to the sum in in $\mathbb{F}_2$, except that $1\vee 1=1$ while $1+1=0$.

At a given moment, the ant makes one step ahead from its current position at $s$ to the *adjacent* square $s'$.

Upon stepping to $s'$, the ant turns either left (counterclockwise) or right (clockwise) by 90° depending on **L** or **R** written at $s'$.

Simultaneously, the letter written at $s'$ switches to the other one:

$$\mathbf{L} \mapsto \mathbf{R} \text{ or } \mathbf{R} \mapsto \mathbf{L}.$$

The path taken of by the ant in $S$ may be amazingly complicated even for an *initially constant* {**L**,**R**} valued function on $S$, where the bug starts by creating *rather symmetric*, {**L**,**R**} patterns that become progressively more and more *irregular*.

Eventually the ant runs to infinity along a "highway" made of 104 steps that repeat indefinitely. (This is seen in the picture below with white □ for **L** and ■ for **R** .)

Conjecturally, the bug eventually takes such a "highway" to infinity for *all* initial **L-R** distributions (functions) on $S$ that are *constant at infinity*, i.e. either with **L** written at all but finitely many squares or, similarly, with **R** written everywhere at infinity.

(It does not seem to be even known if the bug eventually goes to infinity, but it is clear that the route of the bug must be *unbounded*. In fact, if a square $s$ is consecutively visited *twice*, necessarily with *different* letters **l** and **r** written in $s$ at the two visits, then it must, obviously, have *three* (rather than only two) adjacent squares $s'$ also visited twice or more. This shows that the set $M$ of multiply visited squares can not haver corners, in particularly, must be unbounded. But, a priori, $M$ may be equal to all of $S$ or to something like $S$ minus a square.)

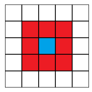
Langton ant, when positioned somewhere on $S$, starts moving and performs sequence of transformations, say $\Psi_1, \Psi_2, \Psi_3, ..., \Psi_i, ...$ of {**L**,**R**}-valued functions $\sigma = \sigma(s)$ on $S$. If the ant eventually goes to infinity, these transformation stabilize and define a transformation that can be denoted $\Psi_\infty$, such that $\Psi_\infty(\sigma)(s) = \Psi_i(\sigma)(s)$ for all sufficiently large $i$ depending on $s$. Notice that there are many such transformations depending on where the starting positions of the ant on $S$ are.

Then every correspondence $C$ between the set of function $\sigma$ on $S$ constant at infinity with the set $\mathbb{N}$ of integers $1, 2, 3, ...$ will make $\Psi_\infty$ act on $\mathbb{N}$, where such an action can be regarded as *numerical computation*.

There are lots of simple correspondences $C$ between our functions $\sigma$ on $S$ and numbers, but none of them is natural/canonical in any way. This makes it rather awkward to nicely formulate the question of which numerical computations can be performed by this ant, and this awkwardness persists with other models of computation.[79]

Langton ant is an instance of what is called a *cellular automaton*, but it is a very atypical one, since there is "true (unsolved) mathematical problem" associated to it. Usually (always?) what one proves or even conjectures about such automata is that they are "universal", i.e. may behave in an arbitrarily

[79]Apparently, there is no mathematical theory of "computable symmetries" of $\mathbb{N}$ responsible for this ambiguity that would be comparable in beauty and power to Galois theory in algebra or to the theory of fiber bundles in topology.



complicated manner.

Conway's Game.

Unlimited complexity arising from apparent simplicity does not make mathematicians happy: what we try to do is quite opposite – finding simple regularities in the sea of an apparently unlimited complexity.

But in 1970, a mathematician John Conway found an unexpected beauty among monsters of computational complexity, called

*Conway's Game of Life*

with an amazing balance between "chaotic" and "regular" behaviour resembling the real Game of Life on Earth.

Conway's Game is played on the same field where Lannton's ant roams, that is on the plane divided into unit squares $s$, but the network/graph structure on the set $S$ of these squares is different: all eight squares $s'$ that touch an $s$, either at a side or at a corner of $s$ are regarded as *adjacent* to (joint by an edge with) $s$.

The states of this Game are similar to what we saw before: these are functions $\sigma(s)$ on $S$ with values in a two element set, but the essential difference of this game from a wandering ant is that action may take place at many locations $s$ simultaneously: Conway's $S$ is inhabited not by a single live entity but by a dynamic ecology of interacting cells.

Formally, the game is defined as a transformation $\Psi$ acting on functions $\sigma = \sigma(s)$, where the value of $\Psi(\sigma)(s)$ depends *only* on the values $\sigma$ at $s$ *itself* and eight cells $\sigma'$ *adjacent to* $s$. The two possible values of $\sigma$ are seen as

$$\square = \textit{dead} \quad \text{and} \quad \blacksquare = \textit{live},$$

where $\Psi$ says what happens to the "life" $\sigma(s)$ at the moment $t+1$ depending on the immediate surrounding of $s$ at the time $t$.

This dependence is assumed being the same for all $s$; thus, $\Psi$ is defined by

*a single $\{\square,\blacksquare\}$-valued function $\psi$ on the set of binary (namely, $\{\square,\blacksquare\}$-valued) functions in nine variables.*

These variables correspond to the nine "modes" of adjacency between squares in $S$ and they may be depicted as $\{\bullet,\uparrow,\downarrow,\rightarrow,\leftarrow,\nearrow,\searrow,\swarrow,\nwarrow\}$, where $\bullet$ signifies the adjacency of a square to itself.

If you feel it is easy to find an "interesting cellular game" of this kind by a brute force computer search and/or by trial and error, just imagine how long it would take to single out any "interesting" binary function in nine variables,

$$\psi : \{\square,\blacksquare\}^9 \to \{\square,\blacksquare\},$$

out of $2^{2^9} = 2^{512} > 10^{150}$ (!) possibilities.

Apparently, human (ergo)brain is able to make such a choice by blinding its eyes to the enormity of the problem. Thus, closing his eyes, Conway takes his $\psi$ that *does not depend on the actual* □ *or* ■ *values* of the variables corresponding to $\{\uparrow, \downarrow, \rightarrow, \leftarrow, \nearrow, \searrow, \swarrow, \nwarrow\}$, but only *on the number of variables where these values equal* ■.

Since there are 9 possible values of this number: $0, 1, ..., 8$, the total number of possibilities for $\psi$ is reduced to $2 \cdot 2^9 \approx 1000$, where the extra "2" comes from two possible values for the • variable, that is the value $\sigma(s)$ itself.

From this moment on, conceivably, there remains a single "interesting" possibility – the one suggested by Conway:

$\psi(\square, n) = \blacksquare$ for $n = 3$, and
$\psi(\square, n) = \square$ for $n = 0, 1, 2, 4, 5, 6, 7, 8$;
(Dead come to life only in the presence of exactly three live neighbours.)

$\psi(\blacksquare, n) = \blacksquare$ for $n = 2, 3$, and
$\psi(\blacksquare, n) = \square$ for $n < 2$ as well as for $n > 3$.
(Both underpopulation and overpopulation is deadly for live cells.)

The dynamics of the *iterates* of resulting transformation $\Psi$, denoted

$$\Psi^{\circ N} = \underbrace{\Psi \circ \Psi \circ ... \circ \Psi}_{N}$$

is amazingly rich even if the action of $\Psi$; hence of $\Psi^{\circ N}$, is restricted to the space of *finite configurations/patterns* $\sigma$ that are functions $\sigma(s)$ that are "dead", i.e. equal □ at *all but finitely many* $s \in S$.

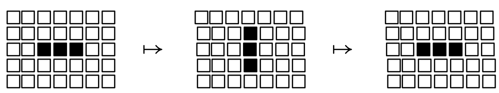
The following (taken from Wikipedia) gives a rough idea of possibilities of this dynamics.

• Some finite configurations, e.g. the following one that is localized on a single line,
... □□□□□□□■■■■■■■■□■■■■■□□□■■■□□□□□□■■■■■■■□■■■■■□□□□□□...
*grow indefinitely* in the Conway plane, (contrary to the original conjecture by Conway).

• There are many *N-periodic points* of $\Psi$, that are finite configurations $\sigma$, such that $\Psi^{\circ M}(\sigma) = \sigma$ exactly for $M$ being multiples of a given $N$.

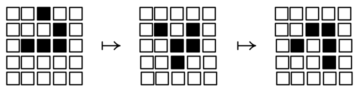
The simplest among these – *fixed points* of $\Psi$, that are $N$-periodic points with $N$=1, are represented by *stationary configurations* where
*no* ■ *has exactly three* □ adjacent to it (thus it remains "dead") and *each* □ *has either two or three* adjacent □.

The smallest 2-periodic pattern $\sigma$ is the triplet:



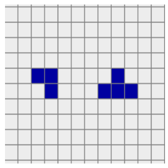
and examples of $N$-periodic $\sigma$ are known for *all but finitely many* $N$.

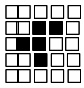
• Some configurations, called *gliders*, move regaining their shapes:



and they eventually drift away to infinity, necessarily along a straight line (that is a diagonal for the above "pentom").

• The following "pentom" discovered by Conway stabilizes in 1103 generations at population of 116.





• The 3+4 pattern sketched in blue on the above figure, call it $\sigma_{3+4}$, with seven ■ in it "dies" (that is a special kind of stabilization) after 130 generations: $\Psi^{130}(\sigma_{3+4})(s) = \square$ for all $s \in S$.

The above examples suggest that "almost everything" can be "seen" in Conway's game.... but it is hard to find a proper mathematical *language* for comprehensive expression/development of the following question:[80]

*What can and what can not happen in Conway's game?*

Should this question and expected answers be formulated in terms of the geometry/topology of the *space* of functions $\sigma(s)$ and the *dynamics* of $\Psi$ acting on this space?

What are interesting (sub)spaces of functions on $S$, that are invariant under the action of $\Psi$?

(Instances of these, besides "dead at infinity" are periodic functions $\sigma(s)$ and those depending in a "simple manner" only on the combinatorial distance from $s$ to some $s_0 \in S$.)

Evolution of what kind of observables (properties of configurations $\sigma$) do we want to understand?

What data, besides properties of the initial configurations, must be used for the answers?

Could formulating/answering this kind of questions be helpful for understanding the real world systems?

One knows (this took a while to prove) that any computational process, e.g. any other cellular automaton, such as Langton's ant, for instance, can be "simulated" by the Game of Life, but such a "simulation" the way this is usually understood, would not look at all as the "real" ant.

Yet, the REAL GAME OF LIFE does have this ability as it is witnessed by the *images* of computer simulations designed by human players of this GAME.

*Is there something mathematically discernible in the* REAL GAME OF LIFE *that is absent from Conway's game?*

This deficiency of Conway's game is shared by most models of universal computation.[81] The apparent (but not the only) reason for this is that "general modeling" suppresses the time (and space?) factor processes being modeled that is a most essential feature of a machine computation.

What would you make of an omniscient computer with $2^N$ minute time delay in answering your $N$th question?

[80]This is the fundamental problem we face everywhere in science.

[81]"Representation of reality" by the brain is no better in this respect.

Cellular Automata of Ulam and von Neumann.

A couple of decades prior to Conway's game, von Neumann described "automata" that can build other "automata" and later on modeled his "construction" by Conway's like game on the squared paper $S$. (The latter idea is attributed to Stanislaw Ulam.)

The "game" suggested by von Neumann, that mimicked his imaginary engineering of such automata, required a 29 letter alphabet $X$ representing possible states $x$ of the cells $s$ (instead of Conway's two) and the von-Neumann's $\Psi = \Psi_\psi$ acting on function $x(s)$ with values in this $X$, where this action depends, via $\psi$, only on $4+1$ (rather than $8+1$ as in Conway's game ) relevant neighborhood/adjacency relations between cells: $\uparrow, \downarrow, \rightarrow, \leftarrow$ and $\bullet$ for self-adjacency of $s$ to $s$.

There are $29^{29^{4+1}} > 10^{28\,000\,000}$ different $\psi : X^{4+1} \to X$ in this case – no surprise that all kind of beasts roam the super-duper universe of these $\Psi$-games.[82] Apparently, *everything* can be *faithfully* modeled by this kind of a "game" if there is no restriction on the number of states $x$ of cells, but... it is hard (impossible?) to formulate mathematically what these "everything" and "faithfully" signify. Even the original "self-replication construction" by von Neumann has not been *formulated* as a *true mathematical* theorem with such a formulation *not being tied up* to a specific class of models *beforehand*.[83]

Von Neumann-Conway "games", often called *cellular automata*,[84] are associated with the values of the following (Ulam-Neumann) *UN bifunctor* the two entries of which are:

($\mathbf{1}_{UN}$) *A $Y$-valued function $y = \psi(x_d)$ in $x$-variables that are indexed by $d \in D$,*

$$\psi : X^D \to Y,$$

*where $X$ and $Y$ and $D$ are given sets.*[85]

For instance, $X = Y = \{\square, \blacksquare\}$ in the Conway game and, in the von Neumann case, $X = Y$ is an alphabet with 29 letters.[86]

($\mathbf{2}_{UN}$) *A $D$-labeled bipartite $(S,T)$ graph, denoted $S \leftleftarrows_D T$.*

The vertex set of this graph is the disjoint union of the sets $S$ and $T$ where all edges go from $t \in T$ to $s \in S$ and where the edges issuing from each vertex $t$ are *$D$-labeled* by $d \in D$ for a given set $D$.

In other words, such a graph is defined by a map

$$G : T \times D \to S,$$

that we interpret as a $D$-family of maps

[82] What is more difficult, if possible at all, is finding *non-trivial* "laws" that would constrain the behaviors of all these beasts.

[83] Yet, on the technical side, it would be interesting to have a von Neumann-Conway style mathematical model of an ecology of *interacting replicators* with the resulting cut-off of the exponential growth of population.

[84] Compare http://mathworld.wolfram.com/CellularAutomaton.html.

[85] Our definitions apply to $X$ and $Y$ from (a class of) categories $\mathcal{X}$ different from the category $\mathcal{S}$ of sets, and, this is less straightforward, the exponents $D$, as sets with certain structures, also may be taken from categories $\mathcal{D}$ different from $\mathcal{S}$.

[86] Anything as large as this alphabet – 29 is a lot in ergo-terms – does not come as "just a set" but always endowed with some structure(s).

$$G_d : T \to S, \text{ for } G_d(t) = G(t,d), d \in D,$$

where the pairs $(t, G_d(t))$, $d \in D$, are seen as edges of the graph $S \leftleftarrows_D T$ that issue from $t \in T$. Thus, all vertices $t \in T$ have the same number, call it $k$, of edges issuing from them for $k = card(D)$ that is the number of elements in $D$.

Conversely, every graph with $k = card(D)$ edges at all $t \in T$ admits a $D$-labeling, with the Cartesian $T$-power $perm_k^T$ of the (permutation) group $aut(D) = perm_k$ of automorphisms of $D$ acting on the set of these labeling.

(One could equally interpret maps $G : T \times D \to S$, as *$T$-labeled $(D,S)$ graphs* that are represented by $T$-families $G_t : D \to S$, but in the "real life" the set $S$ may be large but $D$ is small. Moreover, in most examples, $S$ and $T$ are not just larger than $D$, but, as a objects with structures, they lie in a category that is *different* from the one to which $D$ belongs to.)

For instance, $T = S$ of Conway's game equals the (infinite) set of the unit squares in the plane, and $D$ can seen as the set of *distinguished directions* or *adjacency rules* that are depicted by the arrows $\uparrow, \downarrow, \rightarrow, \leftarrow, \nearrow, \searrow, \swarrow, \nwarrow$ and by $\bullet$ signifying the adjacency of a cell to itself.

### Definition of the Ulam-Neumann Bifunctor.

The value of the $UN$ bifunctor on the pair $(G, \psi)$ for the $D$-family of maps $G = G_d : T \to S$, $d \in D$, that represent the graph $S \leftleftarrows_D T$ and the function $y = \psi(x_d)$ is

*the map $\Psi$ from the set $X^S$ of $X$-valued functions $x(s)$ on $S$*
*to the set $Y^T$ of functions $y(t)$*

that is defined by composing $x(s)$ with $G_d(t)$ and $\psi(x_d)$ as

$$t \underset{G_d}{\mapsto} s \underset{x(s)}{\mapsto} x \underset{\psi}{\mapsto} y = y(t);$$

namely

$$y(t) = \Psi(x)(t) = \psi(x_d) \text{ for } x_d = x(s_d) \text{ and } s_d = G_d(t).$$

*Cellular automata* are defined as such maps $\Psi = \Psi_{G,\psi}$ under the assumptions $S = T$ and $X = Y$.

Usually one also assumes that

[1] *The set $X$ is finite.* In this case set $X^S$ with its product topology is compact, moreover, it is homeomorphic to the *Cantor set*.

[2] *The set $D$ is finite.* Then then the map $\Psi : X^S \to X^S$ is continuous in the Cantor set topology.

[3] *The automorphism group $\Gamma$ of the graph $S \leftleftarrows_D S$ is transitive on $S$.* Then the map $\Psi$ is $\Gamma$-*equivariant*, i.e. it *commutes with the natural action of* $\Gamma$ *on* $X^S$.

The games of von-Neumann and Conway, clearly, satisfy [1], [2], [3] with $\Gamma$ being the group $\mathbb{Z}^2$ of pairs of integers.[87]

[87]Since Conway's $\psi : \{\square, \blacksquare\}^D \to \{\square, \blacksquare\}$ is invariant under the group $perm_8$ acting on the eight arrows in $D = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \nearrow, \searrow, \swarrow, \nwarrow, \bullet\}$, the corresponding $UN$ bifunctor is defined for *all* graphs $G$ with *no labeling* of the edges where one does not even have to assume that there are exactly eight edges at all $s \in S$. This points toward a functorially finer version of our general definitions but a satisfactory concept of "computational network" still remains beyond our grasp.

### Being a bug

Tutng's bugs are described in the language of cellular automata as follows[88].

$*$ Turing's $S = T$ equals the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$, and the set $X$ comes with a $\{\circ, *\}$ valued function $\sigma(x) \in \{\circ, *\}$ such that $\sigma(x(s)) = *$ signifies that the cell $s$ is occupied by the bug.

$*$ The set $D$ of direction in the Turing case is taken to be $\{\bullet, \leftarrow, \rightarrow\}$ with the edge $\leftarrow 1 \in \mathbb{N}$ interpreted as the loop $1 \longleftarrow 1$ rather than as $0 \longleftarrow 1$ for 0 being not in $\mathbb{N}$.

$\star$ (*Localizaton.*) Functions $\psi : X^3 \to X$ admitted in this model must be such that the bug can move only one step at a time either to the left or to the right; it may stand still only at $s = 1$; the only way a cell $s$ may become occupied at the moment $t+1$, it is by the bug moving to $s$ from one of the adjacent locations $s+1$ or $s-1$ (with convention $1 - 1 = 1$) at this moment.

(This localization is exactly what turns a general cellular automaton into a "bug".)

### Eventualization.

Turing imposes these mathematically artificial conditions (in slightly different terms) in order to achieve *maximal specificity* of his computation model that he, shows, is powerful enough to perform all conceivable computations where "unlimited complexity" achieved by an repetition of a a simple operation *unspecified number* of times.

In general terms, let $\Psi$ be a map of a space $\mathcal{X}$ into itself, where Turing's main example is the space of $X$-valued functions $x$ on $S = \mathbb{N}$ that are *sequences* $x(s)$, $s = 1, 2, 3, ...$, and the map

$$\Psi = \Psi_\psi : \mathcal{X} \to \mathcal{X}$$

is defined by some function $\psi : X^3 \to X$, via the above (UN bifunctor) construction, where the exponent "3" is a shorthand for the three-element set $\{\bullet, \leftarrow, \rightarrow\}$.

Turing's transformations $\Psi = \Psi_\psi$ themselves are no more complicated than the underlying functions $\psi$ and if $X$ is a finite set one sees a finite level of complexity in them. But the iterates of these maps, that are

$$\Psi^{\circ N} = \underbrace{\Psi \circ \Psi \circ ... \circ \Psi}_{N} : \mathcal{X} \to \mathcal{X}$$

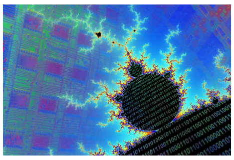
may become unexpectedly complicated, similarly to what we see in the familiar image of *the Mandelbrot set.*

Say that $\Psi$ *stabilizes on the orbit of* $x \in \mathcal{X}$ if

$$\Psi^{\circ N}(x) = \Psi^{\circ N_\star}(x) \text{ for all } N \geq N_\star \text{ and some } N_\star \text{ that depends on } x.$$

(A weaker concept of stabilization is possible for $x$ that are *functions* $x = x(s)$, where the equality $\Psi^{\circ N}(x)(s) = \Psi^{\circ N_\star}(x)(s)$ holds with $N_\star$ that may *depend on* $s$ as well as on $x$.

[88]This description may be unaccceptably abstract for a practically minded computer scientist, but being mathematicians we want to find a proper place for the Turing bug in a *maximally general*, hence *simplest possible* environment. An engineer may laugh at a mathematician who start designing a square box with making up an abstract theory of symmetries but, with a luck, we may be the ones who laugh in the end.



More generally, if $\mathcal{X}$ is a *topological space*, one can define stabilization as *convergence* of $\Psi^{\circ N}(x)$ to a *fixed pont* of $\Psi$ for $N \to \infty$.)

If $\Psi$ stabilizes on the orbits of all $x$ in (necessarily $\Psi$-invariant) subset $\mathcal{U}$ in $\mathcal{X}$, we say that $\Psi$ *stabilizes on* $\mathcal{U}$ and define the *stabilized map*

$$\Psi^\star : \mathcal{U} \to \mathcal{U}$$

by

$$\Psi^\star(u) = \Psi^{\circ N_\star}(u) \text{ for the above } N_\star = N_\star(u),$$

where an essential property of such $u_\star = \Psi^\star(u) \in X$ is being *fixed points* of $\Psi$.

(The stabilized map $\Psi^\star$, whenever it exists, can be described in purely algebraic – semigroup theoretic, terms as being $\Psi$*-bivariant*, i.e. such that

$$\Psi \circ \Psi^\star = \Psi^\star \circ \Psi = \Psi^\star$$

and such that all other $\Psi$-bivariant $\Phi$ are, necessarily, $\Psi^\star$-bivariant as well.)

Passing from $\psi$ to $\Psi^\star = \Psi^\star_\psi$ via $\Psi_\psi$, say in the Turing case, may lead to computations of mind-boggling complexity. The reason for this is *non-specificity* of what can be called *eventualization* $\Psi \rightsquigarrow \Psi^\star$ that makes it *impossible* to decide a priori

[A] *for which $x$ the sequence $\Psi^{\circ N}(x)$ stabilizes*;

and/or to give

[B] an "*effective*" upper bound on *the first* $N_\star$ *where the stabilization begins*.

This impossibility, is "equivalent" to Gödel's Incompleteness Theorem and when "impossibility" is understood as *non-computability* it comes as

TURING'S HALTING THEOREM.[89]

Another manifestation of the unlimited complexity inherent in $\Psi^\star$ is TURING MODELING THEOREM that says, in effect, that

*every computable (i.e. recursive) function can be "written" as $\Psi^\star_\psi$ for some finite set $X$ and a function $\psi : X^3 \to X$.*

In fact, there is a computable transformation $\mathcal{T}$ from the set of functions $\Phi$ that define recursive functions $x$ as solutions of equations "$\Phi(x) = 0$" to the space of Turing's $\psi$, such that the *functional equation* "$\Phi(x) = 0$" transforms to the *fixed point equation* $\Psi(x) = x$ where $\Psi = \Psi_\psi$ for $\psi = \mathcal{T}(\Phi)$, where, moreover, this fixed point, call it $x_\star \in \mathcal{X}$, is a *unique attractive* one.

The construction of Turing's $\mathcal{T}$ is straightforward except that one needs to specify how Turing's sequences $x(s)$, $s = 1, 2, 3, \ldots$ are related with numbers –

[89]The equivalence Gödel ~ Turing is rather obvious. On the other hand, there is no simple general framework where this "equivalence" would admit a mathematically acceptable formulation. The same applies to the even more obvious equivalence [A]~[B].

arguments of recursive functions. There are many simple ways of doing this and the conclusion of the theorem is valid (and pretty obvious) for any one of them.

Yet, there is no *natural*, *distinguished* or *canonical* $\mathcal{T}$ nor there is a natural correspondence between numbers and finite sequences;[90] also there is no *natural* extension of finite sequences that represent numbers to infinite ones where Turing's $\Psi^\star$ resides.

### Serial and parallel computations.

In general, a cellular automaton performs many (similar) computations in parallel, but the above *localization* property ($\star$) enforces such a computation to be sequential:

*Turing's "bug" executes a single procedure at a time*,

where this procedure can be one of the three Boolean operations $AND$, $OR$, $NO$ over binary variables: $x_1 \wedge x_2$, $x_1 \vee x_2$ and $x \mapsto x^\perp$.

On the other hand, the brain's "computations" are manifestly *parallel* with millions of neurons firing simultaneously, that, apparently allows your ergo-brain to absorb and understand flows of signals such as visual images and words/phrases with their internal structures being by far more elaborate than what is nodded for distinguishing binary digits, say for telling $x = \circ$ from $x = \bullet$.

This undeniable supremacy of the brain over machines may, however, be illusory, since, according to Turing's modeling theorem, *every* conceivable computation *can be* reprogrammed into a sequential form. Besides, allowing "*parallel*" *does not* ameliorate the computation time in many cases.

However,

there is *no natural* reprogramming *parallel* $\rightsquigarrow$ *sequential*.

Such reprogrammings destroy relevant structures in interesting non digital flows of signals while introducing some irrelevant artificial structures. Consequently, the *stability* under small random perturbations – an essential feature of algorithms that direct natural ergo-processes, will be lost when you go sequential.

Besides,

there is no *automatic* process for introducing *an ordering* of branches[91] of a parallel computation always preceding such a reprogramming.

Who, on Earth, can *order* millions of active neurons in the brain, where even their number is unknown to us?

A mathematician would try to bypass this issue by introducing the set $PO$ of *all* possible orders, but the enormous size of this $PO$ makes this idea computationally unusable. In fact, $card(PO) = N! = 1\times 2\times 3\times \ldots \times N$ for a computation running in $N$ parallel branches.[92]

[90] Digital representation of integers is practically convenient but there is nothing intrinsically nice and natural about it. This, possibly, is why it was not accepted by ergo-oriented Greek mathematicians, even though Archimedes came close to it in his *Sand Reckoner*.

[91] Rephrasing Hermann Weyl one may say that indiscriminatory ordering mathematical objects is *an act of violence whose only practical vindication is the special calculatory manageability.*

[92] If $N = 2$, the ordering problem, often being attributed to *Buridan's ass* (1340), goes sixteen centuries back to Aristotle. Today, we all know for sure that this problem admits *no algorithmic solution* as it follows by contradiction from the existence of *Möbvous' strip*. And it is amazon to see how "a geometric unfolding" of this "asinine idea" has turned into magnificent theory of *fibered spaces* accompanied by *gauge theories* while in algebra it has

Maybe it is the *equivalence parallel ~ sequential* that is illusory.

*Discouraging Conclusion.* Most (all?) classical statements concerning general classes of computations are easy to prove, yet, they are rarely (ever?) set into a *mathematically* (as opposed to "logically") satisfactory general framework. On the other hand there are many open problems, such as $P \neq_? NP$, with no progress being achieved toward their solutions.

This difficulty, probably, is *inseparable* from our non-understanding of the logic of Life and of the Mind: we do not know what are the right questions to ask.

As far as computations are concerned we do not know, for instance, what are the true objects to which computation should be applied: are they numbers, strings of digits, or we must allow general finite combinatorial structures such as finite graphs, or some kind of (controlled?) infinite recursive (non-recursive?) objects, or what?

Even though all such computation theories are mutually equivalent in some philosophical sense, such "equivalence" is useless when it comes to modeling ergo-systems by such objects.

## 2.4 Miracle of Numbers.

*All the mathematical sciences are founded on relations between physical laws and laws of numbers.*
JAMES CLERK MAXWELL.

The existence of Mathematics as we know it strikes one as improbable as emergence of Life on Earth. Nothing in the foundation of mathematics suggests such thing is possible, like nothing in the Earth chemistry suggests it can beget Life.

One may say that mathematics starts with *numbers.* We are so used to the idea that we forget how *incredible* properties of *real numbers* are. The seamless agreement of several different structures – *continuity, order, addition, multiplication, division* – embodied into this *single* concept is amazing.

Unbelievably perfect *symmetries* in geometry and physics – *Lie groups, Hilbert spaces, gauge theories...*–emerge in the world of numbers from the seed of the Pythagorean theorem. Mathematics and theoretical physics are the two facets of these symmetries that are both expressed in the essentially same mathematical language.

As Poincare says,

*... without this language most of the intimate analogies of things would forever have remained unknown to us; and we would never have had knowledge of the internal harmony of the world, which is, as we shall see, the only true objective reality.*

In the "harsh real world", away from pure mathematics and theoretical physics, the harmony of the full "symmetry spectrum" of numbers comes into play only rarely. It may even seem that there are several different kinds of numbers: some may be good for *ordering* objects according to their size and some

developed into the *Galois theory.*

may be used for *addition* of measured quantities. Using the all-powerful real numbers for limited purposed may strike you as wasteful and unnatural.

For example, *positive* numbers appear in classical physics as *masses* of bulks of matter while electric charges represent positive and negative numbers. The relevant *operation* with these numbers is *addition*, since mass and electric charge are naturally (nearly perfectly) additive: $(a,b) \mapsto a+b$ corresponds to bringing two physical objects together and making a single $(a+b)$-object out of the two corresponding to $a$ and to $b$.

But there is no comparably simple implementation of, say, $a \mapsto 2a$ – one can not just copy or double a physical object. And writing $2a = a+b$ for $a = b$ does not help, since mutually equal macroscopic physical objects do not come by themselves in physics.

In contrast, doubling is seen everywhere in Life. All of us, most likely, descend from a polynucleotide molecule which had successfully doubled about four billion years ago. Organisms grow and propagate by doubling of cells. Evolution is driven by doublings of genomes and of significant segments of the whole genomes (not by the so called "small random variations").

A true numerical addition may be rarely (ever?) seen in *biology proper* but, for example, additivity of electric charges in neurons is essential in the function of the brain. This underlies most mathematical models of the neurobrain, even the crudest ones such as neural networks. But the ergobrain has little to do with additivity and linearity.[93]

The apparent simplicity of *real numbers* represented by points on an infinite straight line is as illusory as that of visual images of the "real world" in front of us. An accepted detailed exposition (due to Edmund Landau) of real numbers by *Dedekind cuts* (that relies on the order structure) takes about hundred pages. In his book *On Numbers and Games*, John Conway observes (and we trust him) that such an exposition needs another couple hundred pages to become complete.

To appreciate this "problem with numbers", try to "explain" real numbers to a computer, without ever saying "obviously" and not resorting to anything as artificial as decimal/binary expansions. Such an "explanation computer program" will go for pages and pages with a little bug on every second page.

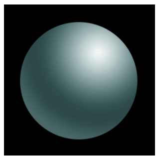
We shall not attempt to incorporate the full theory of real numbers in all its glory into our ergosystems, but some "facets of numbers" will be of use. For example we shall allow an ergo-learner the ability of distinguishing frequent and rare events, such as it is seen in behaviour of a baby animal who learns not to fear *frequently* observed shapes.

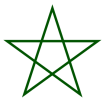
On the other hand, while describing and analyzing such systems we shall use real numbers as much as we want.

*The shape of the heaven is of necessity spherical.*
ARISTOTLE.

Numbers are not in your ergobrain but the idea of symmetry is in there. Much of it concerns the symmetries of our (Euclidean) 3-space, the essential ingredient of which – the group of the (*3-dimensional Lie) group* $O(3)$ of rotations

[93]"*Non-linear*" customary applies to systems that are set into the framework of numbers with their *addition structure* being arbitrarily and unnaturally contorted.





of the Euclidean *round 2-sphere* within itself – has been facinating mathematicians and philosophers for millennia. And not only "the haven" but also your eyes and some of your skeletal joints that "talk" to the brain are by necessity spherical; hence, *rotationally symmetric*.

(The rotation group $O(2,1)$ of the non-Euclidean *hyperbolic plane*, that is logically more transparent than $O(3)$ as it can be represented by symmetries of a calender [SLE, §2.1], was discovered less than two centuries ago. This group along with $O(3)$ serves as as a building block for other *simple Lie groups* that are representatives of essential geometric symmetries.)

A plausible (ergo)brain's strategy for *learning space*, in particular, for reconstruction of spacial symmetries from the retinal images of moving objects, was suggested by Poincaré in §IV of *La science et l'hypothèse*, where Poincaré indicates what kind of mathematics may be involved in *learning space by our visual system*. An aspect of our "ergo-approach" is an attempt to spell out what Poincaré might have in mind.[94]

Our ergobrain is also sensitive to *arithmetic symmetries* that issue from prime numbers as is seen in the recurrence of the *magical pentagram figure* depicting the *finite (Galois) field* $\mathbb{Z}_5$ with the miraculous symmetry of $20(=5\cdot(5-1))$ (affine) transformations acting on it.

A fantastic vision, unimaginable to ancient mystics and to mediaeval occultists, emerges in the *Langlands correspondence* between arithmetic symmetries and the *Galois symmetries* of algebraic equations, where much of it is still in the clouds of conjectures. It is tantalizing to trace the route by which the ergobrain has arrived at comprehension of this kind of symmetries.

## 2.5 Big and Small.

Mathematicians treat all numbers on equal footing, be these

$$2,\ 3,\ 4,$$

or $$20,\ 30,\ 40,$$

or $$1\,000,\ 10\,000,\ 100\,000,\ 1\,000\,000,$$

or $$10^{10},\ 10^{20},\ 10^{30},\ 10^{40},$$

or $$10^{10^{2}},\ 10^{10^{30}},\ 10^{10^{400}}$$

But "democracy of numbers" breaks down in the "real world", be it the physical Universe or the human ergo-world.

[94] A similar idea can be seen in Sturtevant's 1913 construction of the first *genetic map* as we explain in $4 of [4].

For example, the grammars of some languages, e.g. of Russian, distinguish the numbers 2, 3 and 4, while 5, 6, ..., 20, 30, 40,..., 100, 200,...(but not, say 23, 101 and 202) are put, syntactically speaking to "the same basket as infinity.[95]

One instantaneously evaluates the cardinality of • • • •, one needs a fraction of a second to identify "unstructured five" • • • • •, it takes a couple of seconds for • • • • • • • (it is much faster if the symmetry is broken, for instance, as in • • •  • • • •) and it is impossible with

• • • • • • • • • • • • • • •.

But a little structure helps:

• •   • •   • •
 •     •     •
• •   • •   • •

And slightly larger numbers, such as

 •     •     •
• •   • •   • •
 •     •     •
• •   • •   • •
 •     •     •
• •   • •   • •
 •     •     •

if perceived, then through a lens of mathematics.

Our intuition does not work anymore when it comes to thousands, millions, billions. Answer fast:

Do you have more hairs on your head (assuming you are not bald) than the number of people an Olympic stadium may contain?[96]

What is greater the number of bacteria living in your guts or the number of atoms in a bacterium?[97]

Below are ergo-relevant numbers.

• TIME. Hundred years contain < 3.2 billion seconds. With the rate three words per second you vocalize *less than ten billion* ($10^{10}$) words in the course of you life.

Ten billion garrulous individuals all together[98] will utter *at most*

$$10^{10} \times (3 \times 3.2 \cdot 10^{7}) \times 5 \cdot 10^{9} < 5 \cdot 10^{27}$$

words until Sun turns into a *red giant* in about five billion years.

Speaking more realistically, humanity *can not* come up with more than $10^{12}$ -$10^{18}$ *different ideas* — poems, theorems, computer programs, descriptions of particular numbers, etc.[99]

[95] Amusingly, there is also a chasm in essential properties between geometric spaces of dimensions 1,2,3,4, and those of dimension 5 and more.

[96] Both numbers are about 100 000.

[97] There about $10^{11}$-$10^{14}$ atoms in bacteria and more than $10^{12}$-$10^{13}$ bacteria living in your body, mainly in your guts.

[98] The human population on Earth today is slightly above seven billion.

[99] LIFE on Earth, in the course of its $\approx 3.9 \cdot 10^{9}$ year history, has generated a comparable number of "ideas" and recorded them in DNA sequences of organisms inhabiting the planet.

$10^{15}$ years of possible duration of the Universe is made of less than $10^{46} = 10^{15} \times 3 \cdot 10^7 \times \frac{1}{3} 10^{24}$ *jiffie-moments*.[100]

• Brain. The number of neurons in the human brain is estimated between ten and hundred billion neurons with hundreds synaptic connections per neuron, somewhere $10^{12}$-$10^{14}$ synapses all together.

This gives an idea on the volume of the memory stored in the brain, that is comparable to that on a computer hard disk of about $10^{12}$-$10^{13}$ bits.

The (short time) brain performance is limited by the *firing rates* of neurons – something about 100 times per second. Thus, say hundred million active neurons can perform $10^{10}$ "elementary operations" per second[101] that is what an average computer does.[102]

• Space. A *glass of water* contains about $10^{25}$ molecules, the *planet Earth* is composed of about $10^{50}$ atoms and the *astronomically observable universe* contains, one estimates today, $10^{80}$ particles.[103]

Thus, there are (significantly) less than $10^{130}$ *classical* (as opposed to quantum) "events" within our space-time and this grossly overestimated number makes

*an unquestionable bound of what will be ever achieved by any conceivable (non quantum) computational/thinking device of the size of the Universe.*

But... there are at least $2^{10^{10}} > 10^{3\,000\,000\,000} >> 10^{130}$ possible "texts" that you, a humble 21st century human being, *can* (?) write in sequences $s$ of $10^{10}$ bits on the hard disc of your tiny computer. Can't you?

How comes that only a negligible percentage, less than $\frac{1}{10^{10^9}}$ of possibilities, can be actualized?

Worse than that, it is *impossible* to pinpoint a *single instance* of non-realizable sequence $s$: indicating an $s$ will make this very $s$ actual.

It is far from clear whether such inconsistency between "can" and "will" admits a clean mathematical reformulation or this belongs with the *paradox of the heap*. Yet, there are a few purely mathematical theorems and open problems that address this issue, albeit not satisfactorily.

• The oldest is the so called *Scolem's paradox*, that is a theorem in the mathematical logic saying that *uncountably* many mathematical objects (sets) can be "adequately represented" by *countably* many "verbal descriptions".

• It is often (almost always?) quite difficult to *explicitly construct* a single mathematical object $O$ that satisfies a certain (non-trivial!) property $P$, despite (because of?) a presence of a counting or similar argument showing that a "predominate majority" of objects $O$ do satisfy $P$.

The most annoying open class of such problems concerns explicit construction of "simple" functions $f(n)$, $n = 1, 2, 3, ....$, evaluation of which needs *long*

[100] *Jiffy* $\approx 3 \cdot 10^{-24} s$ is the time needed for light to travels the proton-sized distance.

[101] But the rate of learning is measured not in seconds but in hours, days, months, years. This is so, partly, because modification of the strength of synaptic connections is slow.

[102] The speed of modern *supercomputers* is measured in *petaflops* corresponding to $10^{15}$ (*floating point*) operations per second. This is achieved with particularly designed network architectures of processors that allow thousands (not millions as in the brain) operations performed *in parallel*.

[103] Archimedes evaluated the number of sand grains that would fill the Universe by $\approx 10^{60}$ where *exponential representation of numbers* was invented by him for this purpose.

*computations.*

*Ad hoc Example.* Let $d_n(\pi)$ be defined as the $10^n$-th digit of

$$\pi = 3.1415\ 92653\ 58979\ 32384\ 62643\ 38327\ 95... \ .$$

Here, one sees that $d_1(\pi) = 3$, $d_2(\pi) = 4$, $d_3(\pi) = 7$. As for today, *ten trillion* ($10^{13}$) digits of $\pi$ were computed[104] with a use of versions of the

RAMANUJAN MYSTERIOUS FORMULA

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}} = \frac{2\sqrt{2}}{9801}\left(1103 + \frac{24\cdot 27493}{396^4} + ...\right).$$

Thus, one knows the values $d_n(\pi)$ up to $n = 13$.

One can envisage a similar brut force computation of $d_n(\pi)$ up to $n = 16$ or even $n = 19$ but, probably, the sequence $d_1(\pi)$, $d_2(\pi)$, $d_1(\pi)$, ..., $d_{1000}(\pi)$ is not "written" anywhere in the space-time continuum of *our* Universe and the number $d_{1000}(\pi)$ will be never determined by any superhuman civilization.[105]

Beside such artifacts as $d_1(\pi), d_2(\pi), d_3(\pi), ...$ there are mathematically meaningful objects in the "real world" that are also beyond our present day computational prowess. For instance, it is unlikely that a genome sequence of a viable organism, e.g. of a photosynthesizing plant, can be (re)constructed with the only input/knowledge being "*fundamental laws of physics*" (if these exist) by a computation composed of $10^{25}$-$10^{30}$ "elementary steps".[106]

There is a "natural" class of "simply describable" functions that is called $NP$. The famous $P \neq NP$ *problem*/conjecture says that there are functions $f = f(n)$ from $NP$ that *can not* be computed in ***polynomial** time*, say, in at most $const_f \cdot n^3$ steps for some constant depending on $f$. The stark failure in solving this problem shows how limited our vision of the basic structure of *computations in mathematics* is.[107]

On the positive side, despite the overall lack of progress in understanding *"generic-versus-effective"* in the mathematical world, there were a few successes, e.g. specific constructions of graphs with "random, properties" e.g. of *expander graphs.*

## 2.6 Probabilities Old and New.

*The true logic of this world is the calculus of probabilities.*
JAMES CLERK MAXWELL.

*The notion of a probability of a sentence is an entirely useless one, under any interpretation of this term.* NAUM CHOMSKY.

[104]The computation took more than a year.

[105]A miraculous advance in quantum computing or an equally miraculous discovery in mathematics of $\pi$ can make it possible but this would hardly help, for instance, in deciding wether the number digitally represented by $0.d_1(\pi)d_2(\pi)d_3(\pi), ...,$ is transcendental.

[106]Nature managed doing this in something like $10^{50}$ steps.

[107]The concept of *"computational polynomiality"* makes little sense outside pure math: a mathematician may be happy with anything like $10^9 \cdot (10^6)^3$ but this is as bad (good?) as infinity when it comes to the "real life". Learning algorithms in the (ergo)brain must be effectively *log×linear,* rather than polynomial.

Human languages carry imprints of the mathematical structure(s) of the ergobrain and, at the same time, learning a natural (and also a mathematical[108]) language is a basic instance of the universal learning process by the human ergobrain. We hardly can understand how this process works unless we have a fair idea of what LANGUAGE is. But it is hard to make a definition that would catch the *mathematical essence* of the idea of LANGUAGE.

But isn't a language, from a mathematical point of view, *just*

*a set of strings of symbols from a given alphabet,*

or, more generally,

*a probability distribution on the set of such strings?*

A linguist would dismiss such definitions with disgust, but if you are a mathematician these *effortlessly* come to your mind. Paradoxically, this is why we would rather *reject* than accept them:

*Mathematics is shaped by definitions of its fundamental concepts, but there is no recipe for making "true definitions". These do not come to one's mind easily, nor are they accepted by everybody readily.*

For example, the idea of an *algebraic curve* that is a *geometric* representation of

*solutions of a polynomial equation* $P(x_1, x_2) = 0$ *in the* $(x_1, x_2)$*-plane*

by something like , originated in the work by Fermat and Descartes in 1630's and these curves have been studied in depth by generation after generation of mathematicians ever since.

But what is now seen as the simplest and the most natural definition of such a curve – the one suggested by Alexander Grothendieck in 1950s in the language of *schemes*, would appear absurd, if understood at all, to anybody a few decades earlier.

Defining "language" and/or "learning" is, non-surprisingly, more difficult than "algebraic curve", since the former have non-mathematical as well as purely mathematical sides to them. They are similar in this respect to the concept of *probability* that by now is a well established mathematical notion.

It is instructive to see how "*random*" crystallized to *"probability"*, what was gained and what was lost in the course of this "crystallization".

Also, we want to understand how much of "random" in languages in (ergo)learning process (including learning languages) is amenable to what Maxwell calls "the calculus of probabilities".

The concept of *chance* is centuries old as is witnessed by some passages in Aristotle (384– 322 BCE) and also in Talmud.[109] And Titus Lucretius (99 –55 BCE), a follower of Democritus, describes in his poem *De Rerum Natura* what is now called *Einstein-Smoluchowski stocahstic model* of Brownian motion[110].

But mathematics of "random" was originally linked to gambling rather than to science.

*I of dice possess the science and in numbers thus am skilled*

[108] *Mathematical language* for us is the language used for communication between mathematicians but not a mathematical language of formal logic.

[109] Our sketchy outline of the history of probability relies on [10] [2], [14], [7], [6], [15] with additional *References for Chronology of Probabilists and Statisticians* on Ming-Ying Leung's page, http://www.math.utep.edu/Faculty/mleung/mylprisem.htm

[110] This is the collective random movements of particles suspended in a liquid or a gas that should be rightly called *Ingenhousz' motion*.

said Rituparna, a king of Ayodhya, after estimating the number of leaves on a tree upon examining a single twig. (This is from *Mahabharata*, about 5 000 years ago; also 5 000 years old dice were excavated at an archeological site in Iran.)

What attracts a mathematician to random dice tossing and what attracts a gambler are the two complementary facets of the *stochastic symmetry*.

Randomness *unravels and enhances* the *cubical symmetry* of dice (there are $3! \times 2^3 = 48$ symmetries/rotations of a cube) – this is what fascinates a mathematician.

But randomness also *breaks* symmetries: the only way for a donkey' ergo-brain (and ours as well) to solve Bouridan's ass problem is to go random.[111] Emanation of the "miraculous decision power of random" intoxicates a gambler's ergo.[112]

The first(?) documented instance of the *calculus* of probabilities – "*measuring chance*" by a European[113] appears in a poem by Richard de Fournival (1200-1250) who lists the *numbers* of ways three dice can fall. (The symmetry group in the case of $n$ dice has cardinality $n! \times (48)^n$ that is 664 552 for $n = 3$.)

Next, in a manuscript dated around 1400, an unknown author correctly solves an instance of *the problem of points*, i.e. of division of the stakes.

In 1494, the first(?) treatment of the problem of points appears in *print*[114] in Luca Paccioli's *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*.[115]

Paccoli's solution was criticized/analized by Cardano in *Practica arithmetice et mensurandi singularis* of 1539 and later on by Tartaglia in *Trattato generale di numerie misure*, 1556.

### About Cardano.

Gerolamo Cardano was the second after Vesalius most famous doctor in Europe. He suggested methods for teaching deaf-mutes and blind people, a treatment of syphilis and typhus fever. Besides, he contributed to mathematics, mechanics, hydrodynamics and geology. He wrote two encyclopedias of natural science, invented *Cardan shaft* used in the to-days cars and published a foundational book on algebra. He also wrote on gambling, philosophy, religion and music.

The first(?) systematic mathematical treatment of statistic in gambling appears in Cardano's *Liber de Ludo Aleae*, where he also discusses the psychology of gambling, that written in the mid 1500s, and published in 1663.

In a short treatise written between 1613 and 1623, Galileo, on somebody's

[111] No deterministic algorithm can select one of the two points in the (empty) 3-space as it follows from the existence of the *Möbius strip*. And a general purpose robot that you can ask, for instance, *bring me **a** chair* (regardless of several available chairs being identical or not) needs a "seed of randomness" in its software.

[112] In the same spirit, the ABSOLUTE ASYMMETRY of an individual random ± sequence of outcomes of coin tosses complements the ENORMOUS SYMMETRY of the whole space $S$ of dyadic sequences that is acted upon by the compact Abelian group $\{-1,1\}^{\mathbb{N}}$ for $\mathbb{N} = \{1,2,3,4,5,...\}$ and by automorphisms of this group.

[113] Some "calculus of probabilities", can be, apparently, found in the *I Ching* written about 31 centuries ago.

[114] The first book printed with movable metal type was Gutenberg Bible of 1455.

[115] Paccioli became famous for the system of *double entry bookkeeping* described in this book.

request, effortlessly explains why upon tossing three dice the numbers (slightly) more often add up to 10 than to 9. Indeed, both

$$9 \underset{1}{=} 1+2+6 \underset{2}{=} 1+3+5 \underset{3}{=} 1+4+4 \underset{4}{=} 2+2+5 \underset{5}{=} 2+3+4 \underset{6}{=} 3+3+3$$

and

$$10 \overset{1}{=} 1+3+6 \overset{2}{=} 1+4+5 \overset{3}{=} 2+2+6 \overset{4}{=} 2+3+5 \overset{5}{=} 2+4+4 \overset{6}{=} 3+3+4$$

have six decompositions, but 10=3+ 3 +4=3+4+3=4+3+3 is thrice as likely as 9=3+3+3.

(If you smile at the naivety of people who had difficulties in solving such an elementary problem, answer, instantaneously,

*What is the probability of having two girls in a family with two children where one of the them is known to be a girl?*[116])

Formulation of basic probabilistic concepts is usually attributed to Pascal and Fermat who discussed gambling problems in a few letters (1653-1654) and to Huygens who in his 1657 book *De Ratiociniis in Ludo Aleae* introduced the idea of *mathematical expectation*.

But the key result – *the Law of Large Numbers* (hinted at by Cardano) was proved by Jacob Bernoulli only in 1713.

This, along with the *Pythagorian theorem* and the *quadratic reciprocity law*[117] stands among the ten (±2) greatest mathematical theorems of all time. To appreciate its power look at the following example relevant to some (ergo)-learning algorithms.

Let $X$ be a finite set, e.g. the set of numbers $1, 2, 3, ..., N$ and let $\Theta$ be a collection of (test) subsets $T \subset X$. Say that a subset $Y \subset X$ is $\Theta$-*median* if the cardinalities of the intersections of $Y$ with the members $T$ of $\Theta$ satisfy

$$\frac{1}{3} card(T) \le card(T \cap Y) \le \frac{2}{3} card(T) \text{ for all } T \in \Theta.$$

A slightly refined version of the Law of Large Numbers implies that if $\Theta$ contains *at most* $2^{M/10}$ (test) subsets $T \subset X$, for $M = \min_{T \in \Theta} card(T)$, i.e. if

$$card(\Theta) \le 2^{card(T)/10} \text{ for all } T \in \Theta,$$

then,

*for "large" $M$, "most" subsets $Y \subset X$ with $card(Y) = \frac{1}{2} card(X)$ are $\Theta$-median.*

(If $card(X)$ happened to be odd, let $card(Y) = \frac{1}{2} card(X) + \frac{1}{2}$.)

In particular,

*if $M \ge 10$ and $card(\Theta) \le 2^{M/10}$ then $X$ contains a $\Theta$-median subset $Y \subset X$.*

What is interesting is that even if a collection $\Theta$ is defined by "simple explicit rules", say in the case $X = \{1, 2, 3, ..., N\}$, there may be *no* "simple description" of any $\Theta$-median subset $Y$, albeit we do know that such a $Y$ does exist. (This is

[116]This would take half a second for Galileo – the answer is 1/3 ($\pm\varepsilon$).

[117]Let $p, q$ be odd primes and $q^* = (-1)^{(q-1)/2} q$. Then $n^2 - p$ is divisible by q for *some* integer $n$ if and only if $m^2 - q^*$ is divisible by $p$ for *some* $m$.

a characteristic instance of the poorly understood *generic-versus-effective* phenomenon mentioned in the previous section.)

*Example.* Let $X = X_N$ equal the set of integers $1, 2, ..., N$ and $\Theta = \Theta_M$ be the set of all arithmetic progressions $T$ of length $M$ in this $X_N$.

If $M \geq 1\,000$ and $N \leq 10^{20}$, then $\Theta$-median subsets $Y \subset \{1, 2, ..., 10^N\}$ exist.

But exhibiting any single one of them, say for $M = 1000$ and $N = 10^{12}$ seems difficult.[118] And effective description of $M$-median subsets $Y \subset X = \{1, 2, ..., N\}$ becomes progressively harder for tricker, yet, explicitly described $\Theta$.

*"Continuous probability"* was invented in 1733 by Buffon who thought of

> *a needle of unit length (instead of dice) randomly thrown on the plane, where this plane was divided into parallel strips of unit width.*

He proved that

> *the probability of crossing a line between two strips by the needle equals $2/\pi$ for $\pi = 3.14...$ being one half the length of the unit circle*

About Georges-Louis Leclerc Buffon.

Besides opening the fields of *geometric probability* and *integral geometry*, Buffon also contributed to optics: lenses for lighthouses and concave mirrors of his design have been in use for two centuries afterwards.

But his major contribution was to what he called "natural history" – a development of a synthetic picture of Life on Earth, where he outlined many essential interactions between organisms and their environment, much of which is now goes under the heading of "biogeography".

Buffon emphasized the preeminence of biological reproduction barriers between different groups of organisms over the obvious geographical ones that suggested a definition of *species* that has withstood the attempts to "improve" it by later natural philosophers including some 20th century post Darwinian evolutionary thinkers.

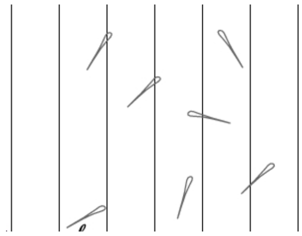
Buffon was the first(?) who articulated the main premise of the evolutionary biology – the concept of the *common ancestor of all animals*, including humans.

Buffon's view on Nature and Life, expounded in his *Histoire naturelle, génèrale et particuliere* published between 1749 and 1789 in 36 volumes, became a common way of thinking among educated people in Europe for two centuries afterwards.

With the Buffon's needle, "random" merged with "analysis of continuum" and were empowered by "calculus of infinitesimals". This is what was hailed by Maxwell and exploited by generations of mathematicians and physists after Buffon.[119]

[118] Conjecturally, if $N \geq 10^M$, then *no $\Theta$-median* subset $Y \subset \{1, 2, ..., N\}$ exists for this $\Theta = \Theta_M$ (made of arithmetic progressions of length $M$), but this is known only for much larger $N$, e.g. for $N \geq 2^{2^{2^{2^{2^N}}}}$ by Gowers' refinement of *the Baudet-Schur-Van der Waerden-Szemeredi theorem.*

[119] The brightest supernova in the 19th century sky of science, as it is seen from the position of the 21st century, was the 1866 article *Versuche über Pflanzen- Hybriden* by Gregory Mendel who derived the *existence of genes* – atoms of heredity by a statistical analysis of the results of his experiments with pea plants. The world remained blind to the light of this star for more than 30 years.



This calculus comes at a price: probability is a "full fledged number" with the addition/multiplication table behind it. But assigning a *precise specific* numerical value of probability to a "random event" in "real life", e.g. to a sentence in a language, is not always possible.

Apparently, the elegance and success of probabilistic models in mathematics and science (always?) depends on (often tacitly assumed and/or hidden) symmetry.

(A bacterium size speck of matter may contain, say, $N_{AT} = 10^{12}$ atoms and/or small molecules in it and the number $N_{BA}$ of bacteria residing in your colon is also of order $10^{12}$. If there are two possible states for everyone – be they atoms or bacteria – then the number of the *conceivable* states of the entire system, call it $S$, is the monstrous

$$M = M(S) \geq 2^{10^{12}} > 10^{3\,000\,000\,000}$$

where its reciprocal

$$\frac{1}{M} < 0.\underbrace{000...000}_{3\,000\,000\,000}1$$

taken for the probability of $S$ being in a particular state is too small for making any experimental/physical/biological sense.

However, the assignment of the $\frac{1}{M}$-probabilities to the states is justified and will lead to meaningful results IF, there is a symmetry that makes these tiny meaningless states "probabilistically equivalent", where the nature of such a symmetry, if it is present at all, will be vastly different in physics and in biology.[120]

### On Symmetry in Randomness.

Essentiality of "equiprobable" was emphasized by Cardano and *parametrization* of random systems by "independent variables" has always been the main tenet of the probability theory. Most (all?) of the classical mathematical probability theory was grounded on *(quasi)invariant Haar(-like) measures* and the year 2000 was landmarked by the most recent triumph of "symmetric probability" – the discovery of (essentially) *conformally invariant* probability measures in spaces of planar curves (and curves in Riemann surfaces) parametrized by increments of *Brownian's processes* via the *Schram-Loewner evolution equation.*

[120]It is not fully accidental that the numbers $N_{AT}$ and $N_{BA}$ are of the same order of magnitude. If atoms were much smaller or cells much bigger, e.g. if no functional cell with less than $10^{20}$ atoms (something slightly smaller than a Drosophila fly) were possible, then, most probably, LIFE, as we know it, could not have evolved in our short lived Universe with hardly $10^{80}$ atoms in it.

But if there is not enough symmetry and one can not *postulate* equiprobability (and/or something of this kind such as *independence*) of certain "events", then the advance of the classical calculus stalls, be it mathematics, physics, biology, linguistic or gambling.

### On Randomness in Languges.

Neither unrealistic smallness of probabilities, nor failure of "calculus with numbers" preclude a use of probability in the study of languages and of learning processes. And if you are too timid to contradict Chomsky, just read his "*under any interpretation of this term*" as *"under any interpretation of the term probability you can find in a 20th century textbook"*.

Absence of numbers for probabilities in languages is unsurprising – numbers are not the primary objects in the ergoworld. Numbers are not there, but there is a visibly present *partial order* on "plausibilities" of different sentences in the language. This may look not much, but a *hierarchical use* of this order allows recovery of many linguistic structures as we shall see later on.

An essential problem with probability is a mathematical definition of "events" the probabilities of which are being measured.

The now-a-days canonized solution, suggested in 1933 by Kolmogorov in his *Grundbegriffe der Wahrscheinlichkeitsrechnung*, is essentially as follows.

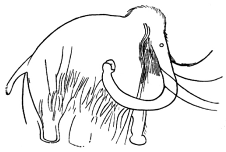
*Any kind of randomness in the world can be represented (modeled) geometrically by a subdomain $Y$ in the unit square $\blacksquare$ in the plane. You drop a points to $\blacksquare$, you count hitting $Y$ for an* ***event*** *and define the probability of this event as $area(Y)$.*

However elegant this *set theoretic* frame is, (with $\blacksquare$ standing for a *universal probability measure space*) it must share the faith of André Weil's *universal domains* from his 1946 book *Foundations of Algebraic Geometry*. The set theoretic language introduced in mathematics by Georg Cantor that has wonderfully served us for almost 150 years is now being supplanted by a more versatile language of *categories* and *functors*. André Weil's varieties were superseded by Grothendieck's schemes, and Kolmogorov's definition will eventually go through a similar metamorphosis.

A particular path to follow is suggested by Boltzmann's way of thinking about statistical mechanics – his ideas invite a use of *non-standard analysis* as well as of a Grothendieck's style *category theoretic* language. (This streamlines Kolmogorov"s $\blacksquare$ in certain applications as we explain in [5].) But a mathematical interpretation of the idea of probability in *languages* and in *learning* needs a more radical deviation from (modification? generalization of?) this $\blacksquare$.

Cardano, Galileo, Buffon. The very existence of these people challenges our vision on the range and spread of the human spirit. There is no apparent wall between the ergos and egos in the minds of these men.

Where are such people to-day? Why don't we see them anymore? Nobody in the last 200 years had a fraction of Cardano's intellectual intensity combined with his superlative survival instinct. Nobody since Buffon has made long lasting contributions to domains as far-distant one from another as pure mathematics and life sciences. What needs to be done to bring Galileos back to us?



# 3 Language in the Brain.

*The limits of my language means the limits of my world.*
LUDWIG WITTGENSTEIN.

• Da-Da, Ma-Ma, Pa-Pa, Ba-Ba.
• Neanderthals are mobbing a mammoth; their shouts fly throw the air and reappear as cuneiform writings on clay.
• The "run and yell" program in your brain switches to the "sit and read" mode.

What is LANGUAGE? Is it *conversing, writing, reading?*

A mammoth hunter scratches his head and pronounces after a minute of concentration:

"Semiosis that relates *signs* with *things* I can eat."[121]

This translates in our terms to

a *bipartite graph*[122] $\Sigma$ on two vertex sets[123], call them $Th$ and $Si$ – the sets of "things" $th$ and of "signs" $si$.

(The edges of $\Sigma$ correspond to the pairs $(th, si)$, where $th$ and $si$ are related by semiosis).

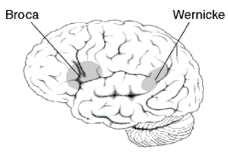
The hunter interrupts and raises his ax:

"This mathematics, it is not REAL LANGUAGE."

Since the hunter is safely far away, I may state openly what I think:

*If we want to say something general and structurally interesting about*

[121]However cruel, employment of semiosis by these people in hunting was indispensable for filling their bellies with meat and their life with meaning.
[122]This is not the same as an *existential graph* of Peirce.
[123]On may argue whether "signs" constitute a *set* in the proper meaning of the word and this is even more dubious for "things" and "meanings".



LANGUAGE, *we ought to speak* MATHEMATICS – *the only language available to Homo sapience for this purpose.*

(Accidentally, there is a semiotic turn to it:

*Mathematics is the art of giving the same name to different things.*
HENRI POINCARÉ.

Of course, this a metaphor, it is not meant to be taken literally, nor is it intended as a definition. This sentence brings to one's mind the image of an exquisitely elaborate graph depicting the artful arrangements of patterns of the body of mathematical ideas and where one sees bushes of edges sprouting from buds of *universal* "abstract" concepts toward many seemingly different specific "concrete things".)

But does it make any difference if you say "graph" rather than "semiosis"?

Different names you give to things channel your stream of thought in different directions.[124]

"Semiosis" brings to you mind the images of "meaningful things" of "real world" and "meaningless" scribbles on paper – the signs associated to these objects .

When you say "graph", you focus on the *structure of association* and close your eyes to all what you naively[125] perceived as being real and meaningful.[126]

However, whatever you say, the bare semiosis model is more applicable to vervet monkey alarm calls rather than to human languages.

Languages have intricate internal structures, and formation of a language in a developing human brain plays its role in *compression and structuralization* of flows of information carried by the senses under the stimuli coming from "real things".[127]

*Signs, objects, meanings*, etc. are all about appearances, not about structures and/or structuralizing processes.

And *semiosis* per se for LANGUAGE is like *applications* for MATHEMATICS: it carries only a secondary and rather shallow structure. Yet, thinking in terms of graphs is instructive.

For instance, one may notice that "signs" and "things/meanings" appear on an equal footing in the semiosis model and ask:

Can one tell *who is who*, i.e. which vertices represent *signs* and which *things*, by the combinatorics of the graph $\Sigma$?

Also one observes by looking at graphs that a semiosis can support an interesting structure only if a typical sign $si \in Si$ has *multiple meanings* – the same name for different things, and, moreover, if most signs come in significantly numerous groups of *synonyms.*

(The multiplicity should not be excessive: even a monkey would figure our that there is something wrong if *all* things $th$ are connected with *all* $si$ by semiotic edges of $\Sigma$. A paradigm of a logically perfect language in the eyes of

[124]And they appeal to different groups of people: you hardly find (non-existential) "graph" in a semiotics text or "semiosis" in an article on graph theory.

[125]A mammoth hunter will have another idea of who is being naive.

[126]This is how a Homo sapience child approaches the kindergarten Ramsey, see 2.1.

[127]Superficially, this is like mathematics: there have been thousands of *different* stones that have crossed your field of vision (have been stored in you visual memory?) but the *same* word *stone* stands for all of them in your brain with probably only a few neurons occupied for this purpose.

a vervet monkey would be a *one-to-one* correspondence between "signs" and "things", that is if each signs has a unique well defined meaning, and also every meaning is encoded by a single sign. But such a language will be no good for hunting animals as intelligent as mammoths.)

Also, our graph suggests how one may measure distances between different "things/meanings" in terms of signs associated to them.

Namely, given a thing $th \in Th$, one assigns to it the subset $S_{th} \subset Si$ of all signs $si$ associated to $th$ by an edge in $\Sigma$ and define the *Hamming distance* between $th_1$ and $th_2$ in terms of the cardinalities of the corresponding sets and their intersections as follows.

$$dist_{Ham}(th_1, th_2) = card(S_{th_1}) + card(S_{th_2}) - card(S_{th_1} \cap S_{th_2}),$$

and similarly, one defines a distance on $Si$ by representing all $si$ by subsets $T_{si} \subset Th$.

Now one can approach the above "who is who" question by comparing the geometries of $Th$ and $Si$ with respect to these distances.

Are, for instance, the shapes of spaces $(Si, dist_{Ham})$ of signs tend to be round or oblong?

Do the spaces $(Th, dist_{Ham})$ of things look smooth or hairy?

Taking the geometry of a space like $(Si, dist_{Ham})$ seriously may strike you as silly. But we shall see next that this acquires significance for a class of graphs that are *intrinsically* associated to LANGUAGE with *no reference* to "things".

## 3.1 Words, Graphs, Categorization and Co-clustering.

*There can be no isolated sign. Moreover, signs require at least two Quasi-minds.*

CHARLES SANDERS PEIRCE.

Let us assume that a language we study admits a simple general definition of *word-unit* and where we possess a universal rule for identification of *word boundaries*. (In real life defining what is a word and devising an algorithm for identifying them in a flow of signals is by no means easy.)

Let us try to classify words according to their *functions* where two words $w_1$ and $w_2$ are regarded *functionally similar* if the other words with which they systematically "cooperate" are themselves tend to be similar.

The condition

$w_1$ *is similar to* $w_2$ *if coworkers of* $w_1$ *are often similar to coworkers of* $w_2$

may strike you as being circular, but this is easily taken care of by the formal definition below with the apparent circularity making co-clustering mathematically so nice.

What is more difficult is to define and/or identify *togetherness* of "doing something" for pairs (or larger groups) of words. But it is relatively easy to decide, without any reference to "meaning" or "function" whether two given words, say $w_1$ and $w_2$, *often* come *close together* or, on the country they come close relatively rarely.[128]

[128]This preassumes that we know what it means to be *"same"* for words positioned at *different* locations in flows of speech or in written texts.

This gives you what is called the *co-occurrence graph* on the set $W$ of words,[129] where $w_1$ is joined with $w_2$ by an edge if the two often come close together, where, moreover, one may vary "often" (measured by a *frequency* an evaluation of which may need some care) and "close" (in some *positional distance*) and thus obtain a family of graphs depending on two parameters.

The remarkable fact is that such graphs, if they come from "real life", have huge redundancy in them – they are very far from anything that can be regarded as "random".

More specifically such a $G$, typically admits *approximate reductions* to certain much smaller graphs $\underline{G}$.

On terminology.

Division of "objects" into classes is called *categorization* in linguistic and in psychology, while doing this by means of a $G$ is called *co-clustering* in linguistics and *bi-clustering* as well as *two mode clustering* in data mining and in bioinformatics where one says *clusters* rather than of "classes".

This kind of analysis, probably, has been used in other branches of science/statistics under different names that makes it hard to find out when and by whom this idea was originally introduced. (Not impossibly, this was understood and implicitly used by Aristotle.)

*Humble Example of Bi-Clustering.* Let $W$ consist of letters (kind of) representing phonemes of the English language and let the edges in $G$ represent those pairs of letters that *often* appear *next to each other*, where "often" for $(w_1, w_2)$ signifies that the frequency of this pair is significantly higher than what one would expect from a random sequence of letters. That is

$$prob(w_1, w_2) \geq (1 + S) \cdot prob(w_1) \cdot prob(w_2)$$

in terms of probabilities, where $S > 0$ is a positive constant the specific value of which depends on what "significant" signifies.[130]

Since this typically happens when one of the letters is *vowel* and another one is *consonant*, this $G$ (approximately) "reduces" to the two vertex graph $\bullet\!-\!\bullet$ by dividing the vertex set $W$ into two classes/clusters

$W$=**vo***wels* & **co***nsonants*

$$\overset{\textbf{vo}}{\bullet}\!-\!\overset{\textbf{co}}{\bullet}$$

Let us emphasize that this partition of $W$ does not depend on any a priori knowledge of the "nature" of letters, but only on the relative frequencies of letters and pairs of letters in texts; the idea of *meaning* we attribute to these classes comes along with the *names* we assign to them.

To define graph reduction in general, it is convenient to think of $G$ as a $\{0,1\}$-function on the vertex set (of words) $W$ of $G$, written as $G(w_1, w_2)$, where "reduction" is a representation of $G$ as a composition, sometimes called *superposition*, of a surjective (i.e. onto) *reduction map* $R : W \to V$ for some set $V$, usually significantly smaller than $W$, and a $\{0,1\}$-function on $V$, say $\underline{G}(v_1, v_2)$,

$$G(w_1, w_2) = \underline{G}(R(w_1), R(w_2)).$$

[129] We assume here that words constitute *sets*.

[130] Being vague here poses no danger of lulling ourselves into a false sense of understanding, as it frequently happens to people carrying out speculative discussions with their intuition unaided by mathematics.

It is unrealistic to expect the existence of such "perfect reduction" if $V$ is much smaller than $W$. All we may hope for is a *good approximation* of $G(w_1, w_2)$ by $\underline{G}(R(w_1), R(w_2))$ for some $R$ and $\underline{G}$, where "approximation" (usually in this context) means that the above equality holds for a "significant majority" of the pairs $(w_1, w_2) \in W \times W$ that correspond to closeness in the $l_1$*-metric* in the space of real functions on the set $W \times W$.[131]

To get a rough idea of a possible magnitude and efficiency of such a reduction let the set $W$ contain 200 000 of (generously understood) "words", including common *di-grams* (pairs of words ). Then, in general, the description of $G$ needs about

$(2 \cdot 10^5)^2 = 40\ 000\ 000\ 000$ – *forty billion* – bits of information.

This is a pretty big number, you can not learn that much during your lifetime of three billion seconds.

On the other hand, if you reduce $W$ to a set $V$ with, say 300 elements – classes of words – in it, then the pair $(R, \underline{G})$ can be encoded with only

$(\log_2 300) \cdot 200\ 000 + 300^2 < 2\ 000\ 000$ – *two million* bits.

This is more than *twenty thousand-fold* reduction of information!

It is the (ergo)-grammar, of course, not texts themselves written in our language, that may admit such incredible compression. (The maximal compression of texts is believed to be well below *ten*.) But this is exactly what ergo-learning is about: it is not remembering whatever enters you brain, but classifying, forgetting, compressing and further structuralizing the incoming "information".

Now there are three questions that beg for answers.

1. What do you gain by this compression of information if you need to know *all of* $G$ to start with in order to construct $R$ and $\underline{G}$?

2. Even if you are given $G$, say written down in your computer memory, how can you find a reduction in practice?

3. If there are several different "reasonably looking" reductions, how can you trust any one of them?

*Ansver to* 1. In order to find $R$ and $\underline{G}$ you do not need to know all of $G$ but only a part of it that carries somewhat more than 2 000 000 bits fn information. For example, a child who learns a language may recognize, in the course of first fifteen years of his/her life, with 20% of this time being exposed to the language (20% of 15 years make about hundred million seconds), ten million pairs of words $(w_1, w_2)$ frequently (decided according to some criterion) coming together.

You extend $G(w_1, w_2) = 1$ at these pairs by zero everywhere else on $W \times W$ and search for a reduction for this incomplete version of $G$ to a $300 \times 300$ function $\underline{G}$.

*Answer to* 2. In general, there is no simple and fast *co-clustering algorithm* for finding a reduction, but certain (some are not fully understood) features of human languages make such algorithms possible.[132]

For example, a presence of small body of *core words* that have exceptionally high frequencies makes naive iteration algorithms quite efficient.

[131] There is much to be done in order to make this more specific or, on the contrary, more general, e.g. by using other metrics on spaces of functions in *two* variables.

[132] Co-clustering programs must be present in the image processing systems of animals (all vertebrates?) and these kind of ready made algorithms in our brain had directed the path taken by the evolution of the "language instinct" in Homo sapience.

*Answer to* 3. The existence of a sufficiently sharp approximate reduction, i.e. with $\underline{G}(R(w_1), R(w_2))$ sufficiently close to $G(w_1, w_2)$, is rather exceptional and miracles do not happen twice: if there are, say, two such reductions $R_1 : W \to V_1$ and $R_2 : W \to V_1$, where the set $V_2$ has cardinality $card(V_2) \leq card(V_1)$, then, most likely $R_2$ equals a reduction of $R_1$.

This means, there exists a reduction $R_{12} : V_1 \to V_2$ of the graph $\underline{G}_1$ to $\underline{G}_2$, such that $R_2$ equals the composition of the two maps, $R_2 = R_{12} \circ R_1$, i.e. $R_2(w) = R_{12}(R_1(w))$, or at least $R_2$ is "quite close" to $R_{12} \circ R_1$.

All of the above being said, a doubt may linger in you mind.

Isn't this $\underline{G}(R(w_1), R(w_2))$ too simple to teach you anything substantial about learning and understanding?

Do you need mathematics to express the idea of two words being similar if they have similar surroundings in texts?

Prior to responding to this let us make it clear that the above kind of co-clustering is neither the final product of building a structure from "flows of words'" nor is it an "atomic unit" of such a structure.

One rather should picture it as a large molecule with simple, yet, non-trivial, internal architecture where this molecule, in turn, serves as a building block for more elaborate syntactic structures.

The simplicity of this "mathematical molecule" makes it quite versatile: one can modify it in many ways and adjust it to building a variety of different global structures.

For instance:

• One may apply co-clustering to functions in more than two variables (this is why we prefer "co-clustering" to "bi-clustering") where these functions may take values in more interesting sets than $\{0, 1\}$.

• Instead of a single reduction one may bring forth *diagrams* of several of them, such as the above $W \to V_1 \to V_2$, or something combinatorially more elaborate and interesting than that.

• On may extract more subtle and/or more substantial information about the structure of a language by looking closer at the geometry of the set of words $W$ with respect to the metric (distance) induced from the space $\mathcal{F}$ of functions $f = f(w)$ on $W$, for the (tautological) imbedding of $W$ to $\mathcal{F}$ defined as in the previous section for the semiosis graph:

a word $w_0 \in W$ is assigned $f_{w_0} \in \mathcal{F}$, such that $f_{w_0}(w) = G(w_0, w)$.

Here, unlike how it was with the semiosis graph, the geometry of $(W, dist_{\mathcal{F}})$ for the metric induced from the space $\mathcal{F}$ (with a suitable Hammnig-like metric on it) is significant.

For example, co-clustering can be seen in the light of this geometry as an ordinary (mono)clustering of $(W, dist_{\mathcal{F}})$ into "loosely connected pieces" with respect to $dist_{\mathcal{F}}$.

On the other hand, the geometry of $(W, dist_{\mathcal{F}})$ also suggests another classification of words $w \in W$, namely, according to the geometry of small balls around them in $(W, dist_{\mathcal{F}})$.[133]

[133]Different prepositions in English, e.g. under and over, may be accompanied by *different* kinds of words, say, nouns and/or verbs; yet the geometries/combinatorics of the balls around them in $(W, dist_{\mathcal{F}})$ look, nevertheless, quite *similar* that may serve as in indicator of the two belonging to the same class (cluster).

And this classification is not the end of the story – many structural features of "flows of signals" are encoded by (not-quite) *pseudogroups* of approximate isometries of spaces like $(W, dist_{\mathcal{F}})$.

One may continue indefinitely along these lines but one has to stop somewhere. Wings of imagination supplied by the power of mathematics can bring you beyond of whatever can be reached by a more pedestrian kind of thinking. But if you fly too high in the sky of math you may miss your destination down on Earth.

## 3.2 Similarity, Co-functionality, Reduction.

Let us make a short (and incomplete) list of four "logically (quasi)atomic constituents" of (ergo)operations applied to flows of signals, in particular of those we used for co-clustering. But we do not attempt at the present point to give precise definitions of these "atoms", to justify their reality, and/or to explain how one finds them in flows of signals.

1. Segmentation and Parsing. The first step in structuralizing flows of signals is identifying/isolating *units* in these flows, where the simplest (but not at all simple) process serving this purpose is *segmentation*: dividing a flow into non-overlapping "geometrically simple" parts.

These may be small and frequently appearing signals, such as phonemes, words and short phrases in the flow of speech or basic visual patterns such as *edges* and *T-junctions*. But these may be as long as sentences, internet pages, chapters in books or intrinsically coordinated visual images of such objects as animals, trees, forests, buildings, mountains.

Our formalism must apply to general "flows of discretely discernible units" where the learning consists in building structures out of "internal units" that may be similar or dissimilar to the units of incoming flows.

Naively, *unit* is anything that can be given a name and/or characterized in a few simple words, but... these words may be of very different kinds depending not only on the intrinsic properties of such a unit, but also on how it is being processed by a particular ergosystem, e.g. a human ergobrain.

*Sentences, words, morphemes, graphemes (letters)* are all units but they belong to different categories, where, "macro-units" such as sentences come as "parametric families of a kind" or as "formulas with free micro-unit variables in them".[134]

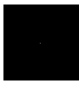
Similarly, elaborate paintings and simple figures both may regarded as units but they are unlikely to be filed by your visual ergosystem in the same "units-directory".

(Processing of linguistic and visual inputs by your (ergo)brain, probably, relies on natural parsing of incoming flows of signals followed by a combinatorial organization of the resulting "units".

But *proprioception sensory system*[135] and motor control of skeletal muscles may also depend on *continuity*, since the incoming signals may be not(?) naturally decomposable into "discrete units".)

[134]Straightforward attempts to make this precise may confine you to the Procrustean bed of the traditional mathematical/logical language.

[135]This is the perception of motion, of stresses and of position of parts of the body.



2. Similarity, Equivalence, Equality, Sameness. There are several *similarity relations* between units of languages/images where these relations may differ in kind and in strength.

For example, images may be similar in shape, size. color, subjects they depict, etc. while two sentences may be similar in the kind and style of words they employ, the idea they convey or in their syntax. The strongest similarities in texts *are letter-wise equalities* of *different* strings.

There is a discrepancy between how the concept of *equality* is treated in mathematics/logic and in in natural languages: we happily say:

*2+3 equals 5*

but:

*5 equals 5*

appears non very informative even to a most logically minded mathematician – these two *"equal"* are not mutually equal and the common language has no means to express this inequality. For example, if you try

*5 is the the same as 5*

this does not make it to look better. But this can be settled if we introduce an ergosystem in the picture, where equalities as well as weaker similarities result from certain *processes*, that are qualitatively different from how one arrives at *sameness*.[136] (We discuss some of this in our [SLE]-paper.)

On Composability of Similarities. Customary, one defines an *equivalence* as a *symmetric binary relations* on a set[137] $S$, denoted, say by $s_1 \sim s_2$, that satisfies the transitivity property:

$$s_1 \sim s_2 \ \& \ s_2 \sim s_3 \Rightarrow s_2 \sim s_3.$$

It is more convenient to depict equivalences (and similarities) of signals $s$ in a category theoretic style by arrows with "names" attached to them, such as $s_1 \overset{f}{\leftrightsquigarrow} s_2$, where one think of such an arrow as an "*implementation of* $\sim$" by some "logical/computational process", e.g. by some co-clustering algorithm.

Then one may compose arrows

$$s_1 \overset{f}{\leftrightsquigarrow} s_2 \overset{g}{\leftrightsquigarrow} s_3 \text{ with the composition denoted } s_1 \overset{f \boxdot g}{\leftrightsquigarrow} s_3.$$

This allows one, for instance, to say that

[136] The spirit of this is close to how different levels of "equivalence" are treated in *the n-category theory*.

[137] This definition does not cover equivalencies between theories and/or between categories since these are *are not* relations on *sets*.

*the composition $f \circ g$ of two "strong similarities" $f$ and $g$ is itself a "weak similarity".*

Also one can now speak of certain equivalencies $f$ and $g$, e.g. one in color and another one in size, being *incomposable*.

3. Classification, Clustering, Reduction. Equivalence relations $E$ on a set $S$ go hand in hand with partitions of this $S$ into the corresponding *equivalence classes* $c \subset S$ *of* $E$, where, in turn, such partitions are "*essentially the same*" as *reduction maps* $R = R_E$ from $S$ *onto* sets $C$, and where, conversely, such a map $R$ defines an equivalence relation $E = E_R$ by

$$s_1 \underset{E}{\sim} s_2 \text{ if and only if } R(s_1) = R(s_2).$$

However, implementations of binary relations $s_1 \underset{E}{\sim} s_2$ and of unary operations $R(s)$ are quite different from a working ergosystem point of view.[138] For instance, it is much harder to record $\approx N^2$ bits encoding an equivalence relation on a set $S$ of cardinality $N$, than $\approx N \log N$ bits needed for defining $R(s)$, where $N$ may be somewhere between $10^4$ and $10^7$. Because of this, *similarities* and *reductions* must be treated separately.

An essential feature of *reductions* from our perspective is *compression of information* and

*"creation" of new units $c$ from the original unites $s$, that are $c = R(s)$.*

A more general and less cleanly defined class of operations is called *clustering* that is based on similarities that are not sharply defined and are not perfectly *transitive* unlike what is usually required of "equivalence".

The tautological map $R : s \mapsto c$ associated to a given clustering that assigns to each member $s$ of $S$ the cluster $c$ in $S$ that contains $s$ (this $R$ may be defined not for all $s$) is still called the *quotient map* or *reduction* from the original set $S$ to the set $C$ of clusters. The reduction that defines co-clustering is an instance of this.

Compression, Morphisms, Functors. Besides the above, there are reductions of quite different type that correspond to "non-local" *compression of texts with limited loss of information*, where one forgets non-essential in a text (or in a visual image) while preserving the significant structure/content of it; this is a hallmark of understanding.

It may happen, of course, that a text has little redundancy in it, such as a telephone directory, for instance. Then no significant reduction and no understanding of such text is possible.

In fact, "perfect texts" with no redundancy in them are indistinguishable from random sequences of symbols, while every meaningful text $T$ admits many reductions, depicted by arrows, say $T \overset{r'}{\to} T'$, $T \overset{r''}{\to} T''$, where the bulk of the the process of understanding a text consists of a multi-branched cascade of such reductions.

An example of a significant commonly used reduction is making a *resume* or *summary* of a text. Also giving a *title* is an instance of a reduction – a *terminal reduction*: you can not reduce it any further.

If we agree/assume/observe that consecutive performance of reductions, say $T_1 \overset{r_{12}}{\to} T_2$ and $T_2 \overset{r_{23}}{\to} T_3$, make a reduction again, denoted $T_1 \overset{r_{13}}{\to} T_3$, also written

[138] This is discussed at length in the context of cognitive linguistics by George Lakoff in *"Fire Women and Dangerous Things"* where *classification* is called *categorization*.

as *composition*

$$r_{13} = r_{12} \circ r_{23},$$

then reductions between texts can be regarded as *morphisms*, of the **category** (in the mathematical sense) of *texts and reductions* where, strictly speaking the word *"reduction"* suggests these arrow $r$ being *epimorphisms*, i.e. they add *no new information* to texts they apply.

It may be amusing to encode much (all?) information about a language $\mathcal{L}$ – *syntax, semantics, pragmatics,* in terms of such a category $\mathcal{R} = \mathcal{R}(\mathcal{L})$ of reductions in $\mathcal{L}$, with translations from one language to another, $\mathcal{L}_1 \rightsquigarrow \mathcal{L}_2$, being seen as *functors* between these categories; but we shall not try to force categories into languages at this point.

Reduction and Agglomeration of Similarities. There are circular relationships between similarities of different types and/or of different strengths. For instance two signals $s_1$ and $s_2$ that have equivalent or just strongly similar reductions may be regarded as weakly similar.

Conversely, if there are "many independent" weak similarity relations between $s_1$ and $s_2$ then $s_1$ and $s_2$ are strongly similar and possibly, equal. For instance if the numbers $N_i$ and $N_i'$ of the letters on the $i$th pages of two books $B$ and $B'$, say with 200 pages each, satisfy $N_i \neq 0$ and $|N_i - N_i'| \leq 2$ for somehow chosen hundred numbers $i$, then one can bet that $B$ and $B'$ are copies of the same book.

4. Co-functionality. Some units in a text $T$ or in another kind of flow of signals form relatively tightly knit groups where we say that these units *perform a common function.*

A priori, co-functionality is not a binary relation (albeit we assumed so defining co-clustering in the previous section); it can be, however, made binary by give "names" to these *"functions"* and by regarding functions as new kind of units.

Then we say that *unit $s$ performs function $f$* and depict this by a directed edge $s \longleftarrow f$. Alternatively, we depict $f$-co-functional units as being joined by $f$-colored edges $s_1 \underset{f}{\longleftrightarrow} s_2$.

## 3.3 Structure of Annotations.

If we identify and indicate all of the above in a text $T$ we shall endow it with a graph-like combinatorial structure $\mathcal{T}$ with the following ingredients.

A. *Colored nods – vertices of the graph.* Nods in the annotated text $\mathcal{T}$ correspond to units and colors represent different types of units, where most units correspond to words and other distinguished strings and groups of strings in $T$ and where their total number is only slightly greater than the number of words in the text $T$.

B. *Colored edges.* Some of edges in $\mathcal{T}$ correspond to similarities between units, where the edges indicating equalities between strings, e.g. words, in different positions in the text are most essential. But also there are other kinds of edges, e.g. corresponding to reductions that are depicted by arrows, say as $s \mapsto c$.

C. *Structure on Colors.* There are significantly fewer colors in our examples than of sets of nods and edges, and they carry much simpler structures. For instance, colors may be hierarchically organized: a color of an edge may signify, for example,

[1.] *This is an equivalence*;

[2.] *This equivalence is of a particular kind, say $\mathcal{F}$, within which composition is allowed.*

One can (should?) treat such colors as units in $\mathcal{T}$, thus erasing the distinction between nods, edges and *colors* and enforcing *circular self-referentiality* structure into $T$.[139] Then, for instance, one may replace [2.] by introducing equivalence relation between "[1.]-colors" where equivalent colors are, by definition, composable.[140]

The above does not account for the full structures in texts and/or in visual images, where an essential omission is the category theoretic interpretation of textual compression as morphisms. Also, besides, structural annotations one may need(?) contextual ones that add information to a text, with something like illustrations by images.

But combinatorics of the above ABC annotations gives you a fair idea of how rich the structures carried by what we call *flows of signals* can be. This combinatorics may harbor many other interesting structures with the above co-clustering being a simple instance of these.

$EX$. Perception of language (also of vision) depends, besides ABC that is derived from the structure of texts themselves, on a class of *external annotations* that represent (reductions of) links between $\mathcal{TONGUE}_{ergo}$ and other (ergo)brain systems, most significantly with proprioception coupled with the motor system and with vision.

Essential questions are:

How much of $EX$ can be reconstructed from $\mathcal{TONGUE}$ itself?

How much of external, call them $EX$, annotations is needed
for understanding $\mathcal{TONGUE}$?

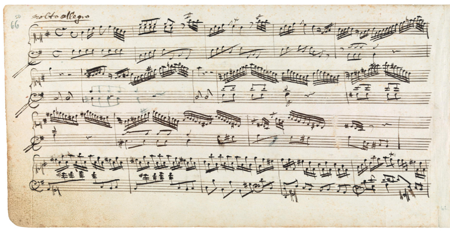
How elaborate is the structure of the "$EX$-language".

Does understanding of $\mathcal{TONGUE}$ stripped of $EX$
*qualitatively* differ from the "full understanding"?

One can not answer these questions at the present stage of knowledge but we conjecture that

*despite many links to other structurally elaborate systems,*
$\mathcal{TONGUE}$ *is essentially autonomous*
and that the
*structure of $EX$ representing these links" is rather simple.*
Thus,
*awareness of a presence of $EX$ in $TONGUE$ enriches one's **knowledge** but adds little to structural **understanding** of $\mathcal{TONGUE}$.*

[139]Self-referentiality is indispensable for languages but there is no(?) reason to bring it into non-linguistic flows of signals.

[140]This may look as silly pedantry, but an appropriate language is essential for correctly designing working models of ergo-learning.

For example, upon reading about an orchestra playing *Mozart's Molto Allegro, Concerto in G* you may get a tune in your head and/or an image of the score. The corresponding $EX$-link transports you from $\mathcal{TONGUE}$ to another domain in your ergobrain but it does not significantly contribute to your understanding of what your red as far as $\mathcal{TONGUE}$ per se is concerned.[141]

Similarly, a gymnast who watches another gymnast's performance mentally undergoes the movements he/she sees – apparently, firing *mirror neurons* make one feel this way. But an untrained person understands essentially the same in what he/she *sees*, since activity of the mirror/motor neurons does not add anything to the structure of the *visual image* in his/her mind.

The ABC structure that is supported on totality of (very) many texts/images, even more so when it is augmented by $EX$, is vast – it *can not*, as it stands, *be incorporated* into the human ergobrain and/or into any realistic ergosystems in general. One has to decide/guess how such an annotated text could be "condensed" to a more compact internal structure befitting an ergo-learner.

## 3.4 Frozen Flows.

We shall not attempt to approach any realistic learning problem and/or structuralization of general "flows of information" but we shall look at a model problem of learning an *imaginary non-human language*. We assume that one has a vast library of texts and may use a computer to analyses these but where one has not even a whiff of knowledge of the semantic and the function of this language. It may be not even known beforehand whether this is a language or a digitalized record of music or of a visual image.

To see things in perspective, let us look at *freezing and putting on record* non-linguistic "flows of signals."

Visual and auditory signals in the outside world are "written" on the rigid space-time $(x, t)$-*background*. The stationary images "carved " on the $x$-space are most essential in human vision[142] while hearing is mostly associated with the time $t$-coordinate[143].

Since stationary landscapes are commonly seen in life, visual images are,

[141]Feeling of touch plays a most essential role in the formative years of human life and can not be dismissed so easily.

[142]Vision of many animals is more $dx/dt$ dependent than ours. For instance, frogs seem to respond only to moving objects.

[143]People with fine hearing (often with impaired vision) distinguish some spacial features around them similarly to animals with acute hearing, such as owls, for instance. But the best are *echolocating animals* – bats, dolphins, porpoises, some whales and certain birds, e.g *South American oilbirds*.

psychologically speaking, the easiest to "freeze". And technology for doing this have been existing for, probably, more than 50 000 years.[144]

The idea of "frozen sound" is not so obvious: we often see snapshots of beautiful sceneries frozen in time but we hear no interesting stationary sounds. (But light is harder to understand than sound: the idea of sound waves goes back to Aristotle while the idea of light being of waves had to wait another two thousand years.)

The first sound recorder was, probably, *phonautograph* (1857) of Scott de Martinville, while the idea of inversion of this recording was suggested in 1877 by Charles Cros, and implemented in *phonograph* by Thomas Edison in 1878; this allowed historically the first recording of human voice.[145] But freezing and recording speech in writing goes more than 5000 years back.

### No Ergo in the Nose.

Humans can distinguish about 10 000 different scents. We have about thousand of different kinds of *olfactory receptors* in our nose that are proteins[146] where each kind of proteins is coded by a particular gene.

But the internal library of smells has, apparently, no ergo in its architecture being organized simpler (this is seen not only by introspection) than how we remember visual images, sounds, words and ideas.

Scents, unlike images and sounds that can be written and rewritten on a variety of backgrounds, are supported by *specific to each of them* physical/chemical substances. No, rewriting, no simple digitalization, no universally organized library of smells comparable to that of images or sounds seem possible.

Since olfaction, unlike vision, does not depend on your muscles, it is disconnected from proprioception system and we have no means of (re)producing scents at will, albeit we we think we can recall them.

There aren't many clearly identifiable *universal smells* common to large groups of object. Non-surprisingly, languages (of urban populations?) have few specific names for smells — about ten in English:
*musky, putrid, rotten, floral, fruity, citrus, vegetative, woody, herbaceous, spicy.*
(There are slightly more smell-words in certain languages, for example there are about fifteen of them in the *Kapsiki of Cameroon*.)

Natural languages do not waste words for naming *individual* objects/properties but rather exercise the art of giving the same name to *many* different things,

[144]No known terrestrial or aquatic animal and no *randomly taken* (untrained) contemporary human is able adequately/artistically record visual images in paintings or otherwise. But this could have been different with Neanderthals and/or Cro-Magnon people.

[145]Some people and birds are good as remembering and imitating strings of sounds; also there are records of 9th century mechanical music playing devices invented by Banu Musa brothers.

[146]A single class of receptors may bind a range of odor molecules, and same kind of molecules may bind to several different receptors.

very much as mathematical theories do. There is no grammar of scents, no books in the language of odors, no "frozen" flows of olfactory perception.[147]

## Frozen Assymetries.

Digital arrays on a magnetic tape have, physically speaking, little in common with flows of real sounds and/or images but recording preserves how spacial/temporal symmetries are broken by these flows.

"Ideal tapes" where signals are recorded make by themselves, without recordings on them, a rather "symmetric community":

a *directed tape* segment $T_1$ of length $l_1$ can be placed into a segment $T_2$ of length $l_2$ whenever $l_2 \geq l_1$, where every such direction preserving placement $\tau : T_1 \to T_2$ where $\tau$ is identified with a number $\tau \leq l_2 - l_1$ that is the distance between the left ends, $t_2$ of $T_2$ and $t_1$ of $T_1$ inside $T_2$.

|——$\tau$—— |-----------------|————————————|
$t_1$ $t_2$

Such placements, say $T_1 \to T_2$ and $T_2 \to T_3$, can be obviously composed with the result being a placement $T_1 \to T_3$; thus, placements make a (rather trivial) *category* in mathematical sense, denoted $\mathcal{PT}$ that is is closely (and obviously) related to the additive group of real numbers.

Now, if $T_1$ and $T_2$ record something encoded by strings of symbols, then there are distinguished placements, called *matchings* $T_1 \to T_2$ that *agree* with what is recorded in there. For instance if $T_1$ with $length(T_1) = 5$ caries the "word" ⊡⊟⊞⊠⊟ then it admits exactly three (rather than 50) placements[148] into $T_2$ of $length(T_2) = 55$ with the following string in the four symbol-letters: ⊡, ⊟, ⊞, ⊠:

⊡⊟⊠⊡⊟⊞⊠⊟⊞⊠⊟⊡⊟⊡⊡⊟⊞⊡⊠⊠⊟⊡⊟⊞⊠⊟⊡⊞⊟⊞⊟⊡⊠⊡⊡⊟⊞⊠⊟⊟⊞⊞⊡⊡⊟⊡⊟⊠⊟⊡⊠⊟⊡⊡⊟
written on it.[149]

Matching records constitute a *subcategory* of $\mathcal{PT}$, denoted $\mathcal{MR}$. This $\mathcal{MR}$ carries the same information as the records themselves, and it has the advantages of being insensitive to the nature and to the size of your "alphabet".

However, "alphabets" often carry significant structures of their own; such a structure may be lost[150] unless it is incorporated into the category theoretic setting.

Besides, perfect matching is too restrictive when it comes to real life signals where one should deal with "approximate matchings" and other (sharp and approximate) category theoretic arrow-morphisms, such as *similarities* and/or *reductions* in languages.

Such "approximate morphisms" are *not always composable* and the category theoretic language must be augmented with "certainty weights" assigned to compositions of arrows.

---

[147] Perfumes do not count.

[148] In the fully randomly equidistributed case the probability of this "three" would be $(50/4^5)^3 \approx 0.008$ that is less than 1%, but *meaningful* flows of signals are NEVER stochastically equidistributed.

[149] Searching for ⊡⊟⊞⊠⊟ in ⊡⊟⊠⊡⊟⊞⊠⊟⊞⊠... requires a concentrated effort on your part but the "equivalent" problem of locating the familiar pattern *pasta* (stored in your long-term memory) in

patpastastapappaspttapastapsasaptppastaassppapataptappa,

is solved by your visual system semiautomatically with rapid, 20-50ms, saccadic movements of the eyes.

[150] Often structures in alphabets, e.g. the options of upper/lower cases in many languages, can be eventually reconstructed from the global structures of texts.

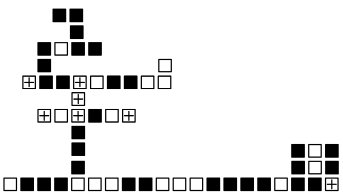
*On Alignment of Sequences.* Approximate matching is mathematically similar to how bioinformaticians compare sequences of DNA, RNA and proteins where these sequences are structurally different from what one encounters, say, in "flows" of spoken and/or written human languages. Also, one practices *multiple alignments* of k-tuples of sequences, for $k > 2$ that are kind of "integral curves" of alignment (similarity) "equations" in Cartesian products $T_1 \times T_2 \times T_3 \times ... \times T_k$.[151]

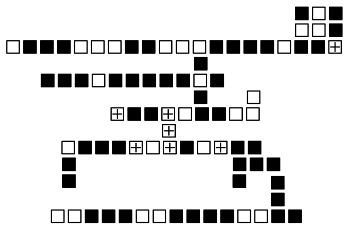
### Maximal matching in vision.

A category theoretic style "matching records" description also applies to visual images "engraved" on spacial+temporal domains $S$ that may be taken of dimensions $d = 2, 3, 4$.

If $d > 1$ it is impractical to "match records" for all kind of domains $S$, since there are two many different shapes of them but one can keep track of *partial matchings* between $S_1$ and $S_2$ e.g. of *maximal connected* subsets, say $S_1' \subset S_1$ and $S_2' \subset S_2$, where the records match by a map from some class of transformations with a given precision threshold and where one concentrates only on sufficiently informative/representative maximal matching pairs $S_1'$ and $S_2'$ rather independently of the ambient $S_1 \supset S_1$ and $S_2 \supset S_2$.

Pattern matching, is, probably, the most essential class of operations performed by our visual ergo-system (look at the two □-■ figures below) but the mathematical (probably, neuronal as well) mechanisms implementing these matchings remain unknown.





*Ergo-irrelevant*(?) *Definitions of Geometric Transformations.* If $d > 1$, the counterpart of the above $\mathcal{PT}$ may be taken by a category $\mathcal{PS}$ for domains $S$ in the Euclidean space $\mathbb{R}^d$ and some class of geometric "placements" that are transformations (maps) $P : S_1 \to S_2$, where relevant classes of placements are associated with the following transformations of Euclidean spaces.

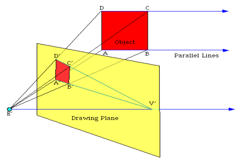
$\bullet_{tra}$ *translations* of $\mathbb{R}^d$, where $x \mapsto x + p$ for all $x \in \mathbb{R}^d$ and some $p \in \mathbb{R}^d$ (which brings us back to $\mathcal{PT}$ for $d = 1$);

$\bullet_{iso}$ *isometries* that are *translations* & *rotations*;

$\bullet_{sim}$ *similarity transformations* that are *translations & rotations & homotheties*, where the latter are maps $x = (x_1, x_2, ..., x_d) \mapsto \lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_d)$ for some number $\lambda$ (sometimes assumed $> 0$);

$\bullet_{aff}$ *affine transformations* where translations & rotations may be additionally composed with *anisotropic scaling maps* $(x_1, x_2, ... x_d) \mapsto (\lambda_1 x_1, \lambda_2 x_2 ... \lambda_d x_d)$;

[151]It is questionable if human ergo unaided by mathematical+biological knowledge could master such alignments.




$\bullet_{pro}$ *projective transformations* between domains $S_1, S_2 \subset \mathbb{R}^d$ that, by definition, send *straight linear segments* from $S_1$ to such segments in $S_2$.

Affine transformations of $\mathbb{R}^d$ are instances of these and, although it is counterintuitive, there are lots of *non-affine projective* transformations, that are, however, *never defined on all* of $\mathbb{R}^d$.

Non-affine projective equivalence is not (?) detectable by the human visual system, but projective transformations are used to create perspective in drawings. In fact, such a transformation for $d = 2$ can be seen as a *radial projection map* between planar regions $S_1$ and $S_2$ in the 3-space.

Namely, if $p$ is a point in $\mathbb{R}^3$ away from the planes containing these regions then the radial projection, say $Pr_p : S_1 \to S_2$, is defined, provided the straight lines between $p$ and all $s_1 \in S_1$ *meet* $S_2$, say at

$$s_2 = s_2(s_1, p) =_{def} Pr_p(s_1).$$

Notice that such a projection is an affine (in fact, similarity) transformation, if $S_1$ and $S_2$ lie in *parallel* planes.

All these transformations, even parallel translations of $\mathbb{R}^d$ for $d > 2$ unlike translation of $\mathbb{R} = \mathbb{R}^1$, do not admit natural discretizations and the groups of these transformations are non-commutative.

It remains unclear how the human visual system developes internal models of these groups, where the coupling with the somatosensory and motor systems is, probably, essential, at least for the isometry group, as it was suggested by Poincaré (see section 2.4).

### Libraries of Moves.

There are no *external* libraries storing sequences of positions, velocities and accelerations/stresses of parts of your body along with memories of impulses sent by motor neurons to your skeletal muscles in these positions, but each of us has such a library in the brain.[152]

Structural organization of this information kept by your proprioception system, e.g. associated with locomotion is different from how strings of words are stored inside and outside our brains, where each "proprioceptive word-move"[153] comes from a multidimensional space. (For instance, fifteen finger joints on your hand contribute twenty active degrees of freedom to the positional part of this space.)

Only members of a tiny *sample subset* $\mathcal{S}$ of the full "space of motions" $\mathcal{M}$ of your body can be experienced during your lifetime and stored in your brain, with an essential part of the "neuronal motor memory" being accounted for by the temporal/sequential organization of $S \in \mathcal{S}$.

[152]The nature and mechanisms of human and animal memory are still shrouded in cloaks of mystery, but we know that even the long-term memory is fluid rather than frozen.

[153]These "words" are kind of moves in the game your body plays with the surrounding space.

Apparently, this *discrete* sample set $\mathcal{S}$ of motion mathematically naturally extrapolates to a *continuous hierarchically organized* "manifold" $\Sigma \subset \mathcal{M}$ decomposable into simple(?) low dimensional blocks, where the geometric structure of this block structure of $\Sigma$ admits experimental/observational study. However, a mathematical (ergo) model of $\Sigma$ is indispensable for understanding this structure and/or for design of agile robots.

The tactile systems involved in handling objects and perproprioception, both coupled with the motor control system, are the most "active" of your perception systems[154] but all perceptions are far from being entirely passive.

For instance, formation of visual images in you brain depends on elaborate muscular movements of the eye that explores its field of view. (E.g. generation of visual images in dreams is associated with these movements.) If this exploration is interfered with, you visual perception suffers.

(If a lecturer blocks your field of vision on the screen and displays one word after another rather than showing longish strings of words, these words do not integrate into meaningful sentences in your mind. Many of us have suffered through such lectures.

But, amazingly, one can comprehend spoken language by apparently passive process of listening, although it is hard to listen to somebody else speaking without opening one's own mouth every so often.)

## 3.5 Understanding Structures and the Structure of Understanding.

*If there was a parrot which could answer every question,*
*I should say at once that it was a thinking being.*

DIDEROT, PENSEES PHILOSOPHIQUES, 1746.

But...

*It never happens that it* [an automaton] *arranges*
*its speech in various ways, in order to reply appropriately*
*to everything that may be said in its presence,*
*as even the lowest type of man can do.*

DESCARTES, DISCOURSE ON METHOD, 1637.

Is Descarte justified in his belief that no machine can pass what is now-a-days called *Turing Test*, i.e. *to reply appropriately to everything that may be said in its presence*?

Does passing such a test certify one as a THINKING BEING who UNDERSTANDS what is being said, as Didereot maintains?

What does it mean to UNDERSTAND, say a language or any other flow $\mathcal{FS}$ of signals?

Diderot indicates a possible answer:

*the continuity of ideas, the connection between propositions,*
*and the links of the argument that one must judge if a creature thinks.*

[154] Auditory systems of echolocating animals are (at least) as "active" and as elaborate as perproprioception. Designing an (ergo)system with echolocating + auditory abilities of a bat, not to speak of a bat's flying agility, remains a robotist's dream.

In general terms, UNDERSTANDING includes:

$[\bullet]_U$ *a certain mathematical (logical?)* ***structure*** *$U$ in the understander's mind/brain/program;*

$[\bullet]_{[IU]}$ *a* ***process*** *$IU$ of implementation of $U$ by an ergosystem representing "an understander";*

$[\bullet]_{[RU]}$ *the* ***result*** *$RU$ of such implementation, $RU = [IU](\mathcal{FS})$, where $[IU]$ is seen as a transformation applied to flows of signals.*

The catch is that nobody has a clear idea of what kind of structure it could be; this precludes any speculation on how and where such a structure can be implemented. Besides such a structure is by no means unique but rather different $U$ are organized as a structural community that can be partly described in category theoretic terms.

An essential feature, one may say the *signature* of a $U$, is its space/time characteristics: $U$ is much smaller in the volume content than the totality of the flow $\mathcal{FS}$ it "understands" and application of $[IU]$ to $\mathcal{FS}$ is much faster than achieving $U$.

It takes, probably, $\approx l \log l$ elementary steps for learning $\mathcal{FS}$ of length $l$ that translates to months or years when it comes to learning a language or a mathematical theory.[155] But when learning is completed, it takes a few seconds to realize, for instance, that a certain string of symbols in the language of your $\mathcal{FS}$ is completely meaningless.

On the other hand, the space/volume occupied by an understanding program $U$ is a few orders of magnitude greater than a learner's program $\mathcal{L}$, where such a program is *universally* (independently of the total number of signals from $\mathcal{FS}$ received/inspected by a learner) bounded by something like $10^6$ bits. Picturesquely,

$$\overset{\star}{\mathcal{L}} \qquad \underset{U}{\circledast \equiv \blacksquare - \blacksquare - \blacksquare - \blacksquare - \blacksquare - \blacksquare - \blacksquare -}$$

where $\circledast$ represents the "core understanding" – a few thousand page "dictionary+grammar" of $\mathcal{FS}$ that is augmented by several (tens, hundreds or thousands depending on $\mathcal{FS}$) "volumes" $\blacksquare$ of loosely (imagine RAM on your commuter) organized "knowledge", while the available $\mathcal{FS}$ itself may number in tens or even hundreds of millions of nearly unrelated units – volumes, internet pages, images memorized by your visual system, etc:

... ●●●●●●●●●●●●●●●●●●●●●●●●
●●●●●●●●●●●●●●●●●●●●●●●...

(We do not know for sure if *understanding* is a formalizable concept, since the only convincing argument in favor of this would be designing a functional *thinking machine/program*, while the only conceivable NO might come from an incredible discovery of a hitherto unknown fundamental property of the live matter of the brain.

But impossibility of resolving the UNDERSTANDING and the *thinking machine* problems by speculative reasoning does not abate our urge to make the world

[155] The true measure of time, call it *ergo-time*, should be multi-(two?)-dimensional, since it must reflect parallelism in programs modeling learning and other mental processes.

know what our *gut feeling* tells us about these issues.[156]

Amusingly, the *gut feeling* itself, at least the one residing in dog's guts, unlike the ideas propagated from human guts to human minds, was experimentally substantiated by A. N. Drury, H. Florey[157] and M. E. Florey in their study of *The Vascular Reactions of the Colonic Mucosa of the Dog to Fright*, 1929.)

When we say "UNDERSTANDING" we mean *understanding structural entities* where such an *understanding* is seen as a structural entity in its own right that admits a non-trivial mathematical model/description.

We conjecture that most (all?) structures we encounter in life, such as natural languages, mathematical theories, etc. are *understandable*[158] and we search for mathematics that can describe this *understanding*.

Answers to the following questions, let these be only approximate ones, may serve to narrow the range of this search.

QUESTION 1. What are essential (expected? desired?) features/architectures of mathematical models of structural understanding?

QUESTION 2. If such a model exists should it be essentially unique? In particular, are the hypothetical structures of understanding, say of a language and of chess must necessarily be closely resembling one another?

QUESTION 3. How elaborate such a model need to be and, accordingly, how long should be a computer program implementing such a model?

QUESTION 4. What is an expected time required for finding such a model and writing down the corresponding program?

QUESTION 5. What percentage of this time may be delegated to machine (ergo)learning with a given level of supervision?

QUESTION 6. How much the supervision of such learning can be automated?

QUESTION 7. What are criteria/tests for performance of "I UNDERSTAND" programs?[159]

QUESTION 8. Can Turing-like tests be performed with *algorithmically* designed questions that would trick a computer program to give senseless answers?

QUESTION 9. Are there simple rules for detecting senseless answers?

QUESTION 10. Can the human learning (teaching?) experience be of use for designing clever learning algorithms?

QUESTION 11. Does ergo logic help answering the above questions?

*Discussion.* When we say "mathematical structure" we do not have in mind any particular branch of the continuously growing and mostly hidden from us enormous tree that is called MATHEMATICS. Whatever a relevant branch can be its structure, probably, is quite elaborated; very likely this branch has not grown yet on this tree.

But sometimes things are simple, e.g. for the vervet monkey *"Alarm Call Language"* that matches a few (four?) word-signs – their alarm calls, with

[156] An attempt to explain the reason for the incessant flow of publicized expressions of YES and of NO opinions on this subject matter is made in section 6.5 of our SLE-paper.

[157] If there is a single person in the human history responsible for saving nearly hundred million lives – this is *Howard Florey* whose titanic efforts had brought penicillin to the therapeutic use by mid-40s.

[158] Overoptimistic? Yet, in line with the remark *"... mystery of the world is its comprehensibility"* by Einstein.

[159] Designer's own ability to pass a test is a poor criterion for designing such a test.

object-events, that are particular predators – leopards, eagles, pythons, baboons.

However, no monkey would think that a mathematical *one-to-one correspondence*, call it $ACL$, between two 4-element sets *understands* the meaning of the alarm calls even if this $ACL$ is implemented by a monkey shaped robot that properly reacts to predators by correctly emitting the corresponding calls and, thus, passes the vervet monkey Turing test.

Why then do Decartes and Diderot, not to speak of Turing himself, attach such significance to the Turing test?

Is there an *essential* difference between the correspondence *"questions"* $\longrightarrow$ *"correct answers"* and the $ACL$ correspondence?

The answer is:

*Yes, there is an essential difference, an* ENORMOUS DIFFERENCE.

Operating with tiny sets, e.g. composed of four uterings – alarm calls of vervet monkeys and with correspondences between such sets needs no structure in these sets. But one CAN NOT manipulate human uterings and even less so longish strings of utterings and/or written texts in a structureless way.

It is tacitly assumed by scientifically minded people — Decartes, Diderot, Turing... , that the above correspondence "$\longrightarrow$" must be compatible with the essential structure(s) of the human language, call it $\mathcal{HL}$, used in a particular Turing test, where the basic (but not the only) structure in $\mathcal{HL}$ is that of an *exponential/power set*:

an uttering/sentence, say in thirty words, in a language with dictionary $D$ is seen as a member of the HUGE *power set*

$$D^{30} = \underbrace{D \times D \times D \times ... \times D}_{30}.$$

Such structurality is indispensable for an implementation of a "thinking automaton" and/or the program running it in a *realistic* space time model [160] that *necessarily excludes*, for instance, "imaginary programs" containing in their memories lists of more than, say of $N^{15}$, sentences with number $N$ being comparable with the cardinality $card(D)$ of the dictionary.[161]

("Large sets", be they finite or infinite, have no independent existence of their own, but only as carriers of structures in them, similarly how the space-time in physics makes no sense without energy-matter in it. This is not reflected, however, in the set theoretic notation that may mislead a novice. For instance, it is rarely stated in elementary textbooks that the "correspondence" $x \mapsto y$ in the "definition" of a *real variable function* $y = f(x)$ is only a metaphor and that a function $f(x)$, if it claims the right to exist in mathematics, must *"respect" some structure* in the set of real numbers.[162])

[160] The property of being physically realistic is often missing in philosophical discourses on artificial intellegence.

[161] This very sentence: "Such structurality is... of the dictionary" contains forty words with roughly half of then being nouns, verbs and adjectives. By varying these, one "can" generate more than $1000^{20} = 10^{60}$ grammatical sentences. Can one evaluate the number of meaningful ones among them? Would you expect *thousand* of them or, rather, something closer to *ten thousand*? Is it conceivable that "weakly meaningful" sentences number in $10^6$, or there are more than $10^{10}$, or even greater than $10^{18}$ of them?

[162] There is no accepted definition of *"function"* that would separate the *wheat*: $\sin x$,

A program that would imitate a human conversing in a natural language and that is seen as "realistic" from the ergo-perspective must be within $10^9$-$10^{12}$ bits in length. If such a program would fool somebody like Diderot, then its level of structurality must necessarily be comparable to that of the human ergobrain and one would be justified in saying that this program *understands* what is being said.[163]

### From Libraries to Dictionaries.

Let us limit the concept of *language* to that of *library* – a collection of written or spoken texts – recorded strings of words pictured above as ... ●●● ... The length of this may be as small as $10^6$-$10^7$ "words in strings", something (implicitly) kept in the memory of a youngster or as large as $10^{12}$-$10^{13}$ words comprising what was ever recorded in the English language.

Even if such a library, call it $\mathcal{LIB}$, is disconnected from non-linguistic flows of signals and, being "frozen", it is not apparently interactive – this is very much unlike how it is with "true language", an ergo learner (e.g. a human child) that is run by a *universal* program would build a certain understanding $U$ of such a library by formally manipulating strings of symbols comprising $\mathcal{LIB}$.

We think of the implementation $IU$ of $U$ as augmentation of texts-strings from $\mathcal{LIB}$ by their structural annotations that would include identification of *linguistic units* in texts, their *functional associations* as well as corresponding *reductions* and *clusters* of these units.

It is unrealistic at this point to develop a clear idea of the structure of $U$ in an ergo-learner's mind/program but it is easier to think of what we call *learner's dictionary* $\mathcal{LD}(\mathcal{LIB})$, that is a kind of a "concentrated extract" of $\mathcal{LIB}$ that contains basically the same "information" as $U$ and where $\mathcal{LD}$ by itself stands for such "extraction/concentration" process/opearation(s). But unlike $U$ such an $\mathcal{LD}(\mathcal{LIB})$, may be written in the language of $\mathcal{LIB}$. Presence of an $\mathcal{LD}(\mathcal{LIB})$ would significantly facilitate building understanding $U$ of $\mathcal{LIB}$ by an ergo-learner, since most of this understanding is encrypted in $\mathcal{LD}(\mathcal{LIB})$.

To get an idea, imagine yourself in a position of such a learner with $\mathcal{LIB}$ written by thinking entities that are far culturally removed from us but who have their ergobrains organized similarly to ours.[164]

Alternatively, think of $\mathcal{LIB}$ as the totality of internet pages in English and try to comprise $\mathcal{LD}(\mathcal{LIB})$ for a use by an ergo-learner, with a six year old Crog-Magnon child in mind, where $\mathcal{LD}$ must be constructed of some *universal operations* that should apply not only to $\mathcal{LIB}$ but to many other "flows of signals", such as libraries and /or records of speech of all kinds of human languages, collections of images, series of mathematical theorems and theories, lists of chess games and chess problems, etc.

Doing this, say for your native language, is mach harder than it seems: we have as little insight into the *ergo*-structure[165] of our mother tongues as fish have

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$\arctan x$, $\sqrt{x}$, $\log x$, Rieman's $\zeta(x)$, Dirac's $\delta(x)$,..., from the *chaff* , such as the characteristic function of the subset of rational numbers.

[163]Beware of ELIZA type programs that respond to everything you say by: "You are right, it is very profound what you say. You must be very intelligent".

[164]One does not need another physical Universe for this – just think of a possible language of blind aquatic echolocating creatures.

[165]It is hard to give definition of this "ergo" but non-ergo examples are plentiful, e.g. *cats* defined as "carnivorous feline mammals" and *nouns* as "members of a class of words that typically can be combined with determiners...". Ask a Cro-Magnon child what he/she thinks

in the singularity structure of solutions of *Navier-Stokes equations* for motion of liquid.[166] Yet we do see, albeit rarely, such insights in textbooks written by some people, such as Albert Sidney Hornby[167] and Gilbert Taggart.[168]

But unlike to how it is usually done, the a grammar of a language encoded in our $\mathcal{LD}$-dictionary must be fully expressible in structure terms of $\mathcal{LIB}$. For instance, an "explanation" of distinction between *"she reads"* and *"she is reading"* should depend entirely on combinatorial positions of such strings/phrases in $\mathcal{LIB}$, rather than on the concept of *aspect* that expresses how an action relates to the flow of time. The sole perspective on the time structure within $\mathcal{LIB}$ that we admit, is an interpretation of combinatorics of certain "grammatical forms" employed in $\mathcal{LIB}$.

Designing algorithms for making such "dictionaries" that would convey the essential grammar and semantic rules of the corresponding languages will by no means solve the fundamental ergo-learning problem, but thinking about such $\mathcal{LD}$ brings us a step closer to this goal.[169]

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of these definitions.

[166] Don't be arrogant: human mathematicians and physicists do not understand it either.

[167] The most famous is *The Advanced Learner's Dictionary of Current English by A.S. Hornby, E.V. Gatenby, H. Wakefield.* Its second edition, Oxford 1963, contains $\approx 2 \cdot 10^4$ word-entries and $\approx 1.5 \cdot 10^5$ sample phrases selected with unprecedented perspicacity.

[168] Taggart wrote several textbooks on French language published in Quebec.

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