Preprint typeset in JHEP style - HYPER VERSION

DAMTP-2010-12, IPHT-T-10/012, IHES/P/10/05, UB-ECM-PF-10/7

String theory dualities and supergravity divergences

Michael B. Green

Department of Applied Mathematics and Theoretical Physics
Wilberforce Road, Cambridge CB3 0WA, UK
M.B.Green@damtp.cam.ac.uk

Jorge G. Russo

Institució Catalana de Recerca i Estudis Avançats (ICREA)
Departament ECM and Institut de Ciencies del Cosmos,
University de Barcelona, Facultat de Fisica
Av. Diagonal, 647, Barcelona 08028 SPAIN
jrusso@ub.edu

Pierre Vanhove

Institut des Hautes Etudes Scientifiques
Le Bois-Marie
F-91440 Bures-sur-Yvette, France
and
Institut de Physique Théorique,
CEA, IPhT, F-91191 Gif-sur-Yvette, France
CNRS, URA 2306, F-91191 Gif-sur-Yvette, France
pierre.vanhove@cea.fr

Abstract: We demonstrate how duality invariance of the low energy expansion of the four-graviton amplitude in type II string theory determines the precise coefficients of multiloop logarithmic ultraviolet divergences of maximal supergravity in various dimensions. This is illustrated by the explicit moduli-dependence of terms of the form $\partial^{2k}\,\mathcal{R}^4$, with $k \leq 3$, in the effective action. Furthermore, we show that in the supergravity limit the perturbative contributions are swamped by an accumulation of non-perturbative effects of zero-action instantons.

Keywords: string theory; supergravity; ultraviolet divergences.

Contents

1. Introduction

It is well known that string theory provides an ultraviolet completion of supergravity – there are no ultraviolet divergences in perturbative string theory. Since perturbative quantum supergravity arises as the low energy limit of superstring theory it is of interest to see how the field theory ultraviolet divergences appear in the appropriate limit.

This paper will describe how these field theory divergences are encoded in logarithmic terms that arise in coefficients of the low energy expansion of the type II superstring four-graviton amplitude compactified to D dimensions on a $d = (10 - D)$-torus, $\mathcal{T}^d$. These scattering amplitudes have a dependence on the moduli that is highly constrained by dualities [1], which relate their perturbative and non-perturbative properties. For example, the low energy expansion of the four-graviton amplitude generates terms in the effective action of the form $\partial^{2k}\,\mathcal{R}^4$, where $\mathcal{R}^4$ is a specific contraction of four curvature tensors. The coefficients of such terms are functions of the moduli that are invariant under discrete symmetries associated with the groups $E_{d+1(d+1)}(\mathbb{Z})$ (which are discrete versions of real split

forms of the corresponding Lie groups of rank $d+1$)[1], and contain the complete perturbative and non-perturbative information about the amplitude. In contrast to string theory, classical maximal supergravity is invariant under the continuous version of these groups, which implies that the Feynman rules are independent of the moduli. As a consequence, perturbative supergravity amplitudes do not depend on the moduli. However, this ignores the presence of infinite towers of non-perturbative charged BPS black hole states, which probably invalidates the use of the perturbative approximation, whether or not there are ultraviolet divergences, as we will argue later.

In a recent paper [2], which will be summarised in section 2, we determined the non-perturbative expressions for the coefficients of a number of terms in the low energy expansion of the four-graviton amplitude of maximally supersymmetric string theory compactified from 10 dimensions to $D = (10 - d)$ dimensions on a d-torus. The most detailed analysis was for the analytic part of the amplitude with $d \leq 3$, although certain features of the nonanalytic terms and the $3 < d \leq 7$ cases were also determined. The simplest interactions considered in some detail in [2] (extending earlier work in [3–15], see also recent discussions in [16, 17]) were $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$, for which the coefficients are special combinations of Eisenstein series of the kind considered in [18]. The coefficient of the $\partial^6\mathcal{R}^4$ interaction coefficient is a more general automorphic function [8,9]. A thorough analysis of these coefficients demonstrated that they reduce to the correct expressions in three different limits: (i) String perturbation theory; (ii) Decompactification from D to $D+1$ dimensions when a radius of $\mathcal{T}^d$ becomes large; (iii) The semi-classical eleven-dimensional supergravity limit, in which the M-theory torus, $\mathcal{T}^{d+1}$, becomes large and loop calculations in eleven-dimensional supergravity are valid. It was also argued that in certain 'critical' dimensions, D_L, the leading logarithmic ultra-violet divergences of L-loop maximal supergravity are reproduced[2]. As remarked in [2], particular examples of such behaviour arise for the $\mathcal{R}^4$ interaction with $(D_1 = 8,\ L = 1)$, the $\partial^4\mathcal{R}^4$ interaction with $(D_2 = 7,\ L = 2)$ and the $\partial^6\mathcal{R}^4$ interaction with $(D_3 = 6,\ L = 3)$. The structure of the coefficients determined in [2] will be reviewed in section 2.

In the following section we will present a detailed argument that the logarithmic factors that arise in the automorphic coefficients of the string theory higher derivative interactions indeed determine the values of logarithmic ultraviolet divergences in loop amplitudes of maximal supergravity. To be precise, we will see in section 3 that the logarithmic terms in the coefficients of $\partial^{2k}\mathcal{R}^4$ interactions with $k = 0$ in $D = 8$, $k = 2$ in $D = 7$, and $k = 3$ in $D = 6$ are equal to the logarithmic terms that arise in maximal supergravity after subtracting the ultraviolet divergences. The $\partial^6\mathcal{R}^4$ coefficient function was not determined in [2] and so, for completeness, it will be obtained in appendix A.

In addition, there are 'non-leading' logarithmic terms that arise in dimensions $D > D_L$, which are identified with further logarithmic ultraviolet divergences in maximal supergrav-

[1] For $d \leq 5$ $E_{1(1)}(\mathbb{R}) = SL(2,\mathbb{R})$, $E_{2(2)}(\mathbb{R}) = SL(2,\mathbb{R}) \times GL(1)$, $E_{3(3)}(\mathbb{R}) = SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$, $E_{4(4)}(\mathbb{R}) = SL(5,\mathbb{R})$, $E_{5(5)}(\mathbb{R}) = SO(5,5,\mathbb{R})$ and for $5 \leq d \leq 7$ $E_{d+1(d+1)}$ is the real split form of the exceptional Lie group E_{d+1}.

[2] The critical dimension at L loops is the lowest (possibly non-integer) dimension in which the theory has ultraviolet divergences.

ity. For example, there is a single pole, $1/\epsilon$, and a double-pole, $1/\epsilon^2$, in dimensionally regularised two-loop maximal supergravity in $D = 8$ dimensions that contributes to $\partial^6\mathcal{R}^4$ (whereas the $D_3 = 6$ single pole contributes to $\partial^6\mathcal{R}^4$). Another new feature arises in the field theory since the one-loop $\mathcal{R}^4$ divergence requires a counterterm. This contributes to a one-loop 'triangle' diagram in which one vertex is the counterterm, which results in another $1/\epsilon^2$ contribution [6], which we will also evaluate in section 3. The sum of these contributions gives rise to log and $\log^2$ terms that are reproduced by the string theory coefficient of this interaction. In order to compare the field theory and string theory expressions it is important to use consistent normalisation conventions, which are briefly outlined in appendix B.

In section 4 a connection will be made with the issue of whether quantum supergravity might be a consistent theory that can be obtained as a decoupling limit of closed-string string theory, much as $\mathcal{N} = 4$ super Yang–Mills in four dimensions can be obtained as a decoupling limit of open string theory. It was pointed out in [19] that this is probably far from the case even if the individual terms of the perturbative expansion are finite. The problem is due to the presence of infinite towers of non-perturbative states, which correspond in toroidally compactified string theory to massive Kaluza–Klein modes, winding modes, Kaluza–Klein monopoles and wrapped p-branes of various kinds. It was shown in [19] that the supergravity limit is one in which towers of states becomes massless and the restriction of the spectrum to the massless perturbative states – the basic assumption in supergravity – is not a sensible approximation to the theory. In an analogous fashion the simple examples in this paper involve a condensation of zero-action instantons, as will be demonstrated in section 4, based on the explicit expressions for the coefficients of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions.

Although the complete structure of the automorphic coefficient functions has not been determined beyond order $\partial^6\mathcal{R}^4$, a certain amount is known about higher order terms based on analysis of one and two loop amplitudes in eleven-dimensional supergravity compactified to $D = 9$ nine dimensions on $\mathcal{T}^2$ in [9]. This will be used as the basis of a speculative discussion in section 5 suggesting that the $\partial^8\mathcal{R}^4$ interaction is not protected by supersymmetry against perturbative corrections at genus five and higher, which would have significant implications for the onset of ultraviolet divergences in perturbative maximal supergravity.

The paper will end with a short discussion of these results in section 6.

2. Summary of duality invariant coefficients in the low energy expansion

In [2] we were concerned with properties of the low-momentum expansion of the four-graviton amplitude. It is useful to separate the D-dimensional amplitude into the sum of analytic and non-analytic terms,

$$A_D(s,t,u) = A_D^{analytic}(s,t,u) + A_D^{nonan}(s,t,u)\,, \tag{2.1}$$

where the analytic part has a low energy expansion in powers of the Mandelstam variables

$(s = -(k_1 + k_2)^2,\, t = -(k_1 + k_4)^2,\, u = -(k_1 + k_3)^2)$ of the form

$$A_D^{analytic} = \sum_{p=0}^{\infty} \sum_{q=-1}^{\infty} \mathcal{E}^{(D)}_{(p,q)}(\phi_{K\backslash G})\, \sigma_2^p\, \sigma_3^q\, \mathcal{R}^4\,. \tag{2.2}$$

This is the general symmetric polynomial in the Mandelstam invariants, which enter in the dimensionless combinations

$$\sigma_n = (s^n + t^n + u^n)\, \frac{\ell_D^{2n}}{4^n}\,, \tag{2.3}$$

where ℓ_D is the Planck length in D dimensions. The coefficient functions, $\mathcal{E}^{(D)}_{(p,q)}(\phi_{K\backslash G})$, are functions of the symmetric space moduli, $\phi_{K\backslash G}$, which are the scalar fields, of the coset space $K\backslash G$ appropriate to compactification on a $d = (10 - D)$-torus (where G is $E_{d+1(d+1)}(\mathbb{R})$ and K is its maximal compact subgroup). They are required to be automorphic functions that are invariant under the D-dimensional duality group, $E_{d+1(d+1)}(\mathbb{Z})$, The expansion is one in which $k_i \cdot k_j\, r^2 \ll 1$ and $k_i \cdot k_j\, \ell_D^2 \ll 1$, where r is any radius of the toroidal dimensions, ℓ_D is the D-dimensional Planck length, and k_i and k_j are any of the external momenta. The nonanalytic term, A_D^{nonan}, contains singularities due to thresholds in which internal lines of the perturbative contributions to the amplitude are on-shell. The separation of the amplitude into the two pieces in (2.1) is well defined at low orders in the low-energy expansion, where there are few perturbative contributions to the amplitude.

It is convenient to express the analytic part of the amplitude in terms of a local one-particle irreducible effective action,

$$S_D^{local} = \sum_{p\geq 0, q\geq -1} \ell_D^{8+2k-D} \int d^D x\, \sqrt{-G^{(D)}}\, \mathcal{E}^{(D)}_{(p,q)}\, \partial^{2k}\mathcal{R}^4 \tag{2.4}$$

where $k = 2p + 3q$ and $G^{(D)}$ is the determinant of the space-time metric in the Einstein frame.

2.1 Constraints on the coefficients

It is clear that maximal supersymmetry imposes strong constraints on the structure of the coefficient functions. In particular, it was shown in [7] that type IIB supersymmetry requires the coefficient of the $\mathcal{R}^4$ interaction in ten dimensions to satisfy a Laplace eigenvalue equation (with a particular eigenvalue), for which the unique solution compatible with string perturbation theory is a nonholomorphic Eisenstein series, $\mathcal{E}^{(10)}_{(0,0)}(\Omega) = \mathbf{E}_{\frac{3}{2}}(\Omega)$, where Ω is the complex modulus of the IIB theory. So far there has been no progress in generalising this supersymmetry argument to higher order interactions (see, however, [20, 21]) or higher-rank groups, but the following indirect arguments (given in [2]) lead to appropriate generalised Laplace eigenvalue equations satisfied by the coefficient functions in the compactified theory. It was argued in [2] that in the decompactification limit $r_{10-D}/\ell_{D+1} \to \infty$ the Laplace operator, $\Delta^{(D)}$, on $K\backslash G$ becomes

$$\Delta^{(D)} \to \Delta^{(D+1)} + \frac{D-2}{2(D-1)}\, (r_{10-D}\partial_{r_{10-D}})^2 + \frac{D^2 - 3D - 58}{2(D-1)}\, r_{10-D}\partial_{r_{10-D}}\,, \tag{2.5}$$

and the eigenvalues $\lambda^{(D)}_{(p,q)}$ of the interaction coefficients $\mathcal{E}^{(D)}_{(p,q)}$ satisfy the equation

$$\lambda^{(D)}_{(p,q)} - \lambda^{(D+1)}_{(p,q)} = \frac{2p+3(q+1)}{(D-1)(D-2)}\,(D^2 - 3D - 52 + 4p + 6q)\,. \tag{2.6}$$

Using the ten dimensional values $\lambda^{(10)}_{(0,0)} = 3/4$, $\lambda^{(10)}_{(1,0)} = 15/4$ and $\lambda^{(10)}_{(0,1)} = 12$ determined in [3, 6–8, 20], we deduce that the coefficients of the terms discussed in [2] satisfy the following set of Laplace eigenvalue equations with source terms,

$$\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right)\,\mathcal{E}^{(D)}_{(0,0)} = 6\pi\,\delta_{D-8,0}\,, \tag{2.7}$$

$$\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right)\,\mathcal{E}^{(D)}_{(1,0)} = 40\zeta(2)\,\delta_{D-7,0}\,, \tag{2.8}$$

$$\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2}\right)\,\mathcal{E}^{(D)}_{(0,1)} = -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 120\zeta(3)\,\delta_{D-6,0}\,, \tag{2.9}$$

where the coefficient of the $\delta_{D-6,0}$ in equation (2.9), which was not determined in [2], is derived in appendix A. Although most of the discussion in [2] focused on explicit solutions of these equations with $7 \le D \le 10$, the iterative argument linking dimensions D and $D+1$ shows that they hold more generally for all dimensions $D \ge 3$.

The structure of equations (2.7) and (2.8) generalizes the Laplace equation satisfied by the $\mathcal{R}^4$ interaction in $D = 10$ dimensions [3]. A notable feature of these eigenvalue equations is the presence of the Kronecker delta sources which are non-zero in the dimensions in which the eigenvalues vanish. These are the critical dimensions, D_L, which are the lowest dimensions in which L-loop maximal supergravity has ultraviolet divergences. Equation (2.9), satisfied by the coefficient of the $\partial^6\,\mathcal{R}^4$ interaction, has a source term that is quadratic in the coefficient of the $\mathcal{R}^4$ interaction, which can also be interpreted to be a consequence of maximal supersymmetry [8]. In addition the Kronecker delta contributes in $D_3 = 6$ dimensions, which is again the dimension in which the eigenvalue vanishes and is also the lowest dimension in which $L = 3$ supergravity has an ultraviolet divergence. Interactions of higher order will not be discussed here in any detail. However, some of their properties in $D = 9$ dimensions were determined in [9], which indicated that the coefficients are *sums* of automorphic functions that satisfy equations that are generalisations of (2.9).

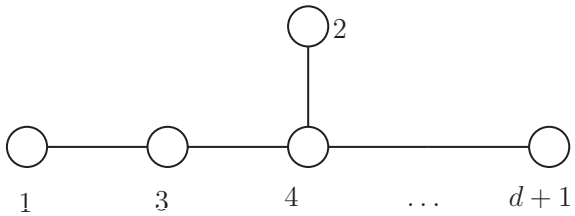

Figure 1: The Dynkin diagrams of the U-duality groups $E_{d+1(d+1)}$ $(0 \le d \le 7)$

The solutions of (2.7)–(2.9) are highly constrained by imposing boundary conditions that require them to reproduce known features of string/M theory in various limits. These limits are:

(i) The limit in which one radius, r_d, of the string theory torus, $\mathcal{T}^d$, becomes large, $r_d \gg \ell_{D+1}$ so that the amplitude effectively decompactifies from $D = 10 - d$ to $D + 1$ dimensions.[3] Since the external momenta, k_i $(i = 1, 2, 3, 4)$, are fixed, this is a limit in which $k_i \cdot k_j \, r_d^2 \gg 1$, which lies outside the range of validity of the original expansion. In order for the low energy expansion to be valid in $D+1$ dimensions it is necessary that $k_i \cdot k_j \, \ell_{D+1}^2 \ll 1$. Although this interchange of limits might generally be expected to pose problems, it does not at low orders in the derivative expansion that are considered here because only a finite number of powers of r_d occur. To be precise, the $\mathcal{R}^4$ coefficient, $\mathcal{E}^{(D)}_{(0,0)}$ has two distinct powers of r_d in its expansion, so (ignoring coefficients) the expansion has the form

$$\frac{r_d}{\ell_{D+1}} \, \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{8-D} . \tag{2.10}$$

The term that grows linearly with r_d gives the finite contribution to the $\mathcal{R}^4$ interaction in the large r_d/ℓ_{D+1} limit. The second term is the $n = 1$ term of an infinite series of the schematic form $r_d^{8-D} \, (s \, r_d^2)^n \, \mathcal{R}^4$, which resums in a manner that converts the first nonanalytic threshold of the D-dimensional amplitude to that of the $(D + 1)$-dimensional amplitude. For simplicity, we have suppressed a $\log r_d/\ell_{D+1}$ factor that multiplies the second term when $D = 7$ and $D = 8$.

The $\partial^4 \mathcal{R}^4$ coefficient, $\mathcal{E}^{(D)}_{(1,0)}$, has three power-behaved terms in its expansion,

$$\frac{r_d}{\ell_{D+1}} \, \mathcal{E}^{(D+1)}_{(1,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{12-D} + \left(\frac{r_d}{\ell_{D+1}}\right)^{6-D} \mathcal{E}^{(D)}_{(0,0)} . \tag{2.11}$$

Again the term linear in r_d gives the finite contribution to the interaction in the large-r_d limit, while the second term contributes the $n = 2$ term of the series $r_d^{8-D} \, (s \, r_d^2)^n \, \mathcal{R}^4$ that resums to give the first nonanalytic threshold. The last term contributes the first term of a second infinite series that resums to give the second $(D + 1)$-dimensional nonanalytic threshold. We have suppressed a $\log(r_5/\ell_6)$ factor multiplying the second term when $D = 5$.

The $\partial^6 \mathcal{R}^4$ coefficient, $\mathcal{E}^{(D)}_{(0,1)}$, has four terms in its expansion

$$\frac{r_d}{\ell_{D+1}} \, \mathcal{E}^{(D+1)}_{(0,1)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{14-D} + \left(\frac{r_d}{\ell_{D+1}}\right)^{8-D} \mathcal{E}^{(D)}_{(0,0)} + \left(\frac{r_d}{\ell_{D+1}}\right)^{4-D} \mathcal{E}^{(D)}_{(1,0)} . \tag{2.12}$$

The term linear in r_d again gives the finite contribution to the interaction in the large-r_d limit, the second term contributes the $n = 3$ term of the series that resums to give the first nonanalytic threshold and the third term contributes a second term to the series that sums to the second threshold. The last term contributes the first term to a new infinite series that resums to give the third $(D+1)$-dimensional nonanalytic threshold. Again, we have ignored logarithmic factors that arise for $D = 4$ and $D = 8$.

[3] This limit is equivalent to $r_d \gg \ell_s$ with the $D + 1$-dimensional string coupling y_{D+1} held fixed.

(ii) The limit of string perturbation theory. This is the limit in which the D-dimensional string coupling becomes small, so that $y_D = g_s^2\, \ell_s^d / V^{(d)} \ll 1$, where $V^{(d)} = r_1 r_2 \dots r_d$ is the volume of $\mathcal{T}^d$ and g_s is the string coupling. In this limit each coefficient possesses a finite set of terms that are power behaved in y_D. In string frame a term of order y_D^{-1+h} corresponds to a term of genus-h in closed string perturbation theory. In addition there is an infinite set of exponentially suppressed instanton contributions. A great deal is known about the low-energy expansion of the four-graviton amplitude directly from string perturbation theory at genus-one and genus-two, and a certain amount at genus-three.

(iii) The limit in which the M-theory torus becomes large, $\mathcal{V}_{(d+1)} \gg \ell_{11}^{d+1}$. In this limit, $r_d \gg \ell_{11}$, with $k_i \cdot k_j \ell_{11}^2 \ll 1$ the semi-classical (Feynman diagram) approximation to eleven-dimensional supergravity is expected to be a good approximation. A variety of calculations in compactified eleven-dimensional supergravity at one loop and two loops provide much information about this limit [5, 6, 9, 10, 22].

In each of these three cases a specific parameter becomes large and there is a finite number of terms that are power-behaved in this parameter, together with an infinite series of exponentially suppressed terms. The sum of power behaved terms contributes the zero Fourier mode, or 'constant' term with respect to the angular parameters that enter in the off-diagonal entries of the matrix N (the unipotent radical) of the standard Levi decomposition of a maximal parabolic subgroup of G, $P = MN$, where M is the Levi factor for the corresponding subgroup of G. Such constant terms are are obtained by deleting specific nodes of the $E_{d+1(d+1)}$ groups. Numbering the $E_{d+1(d+1)}$ nodes as indicated in figure 1, in limit (i) node $d+1$ is deleted, in limit (ii) node 1 is deleted, and in limit (iii) node 2 is deleted. The exponentially suppressed terms in each case have the interpretation of BPS instanton contributions due to D-instantons and a variety of wrapped euclidean p-branes. Although these contributions have not been analysed in detail they should correspond to 1/2-BPS states in the $\mathcal{R}^4$ case, 1/4-BPS states in the $\partial^4\mathcal{R}^4$ case, and 1/8-BPS states in the $\partial^6\mathcal{R}^4$ case (see for example [16] for a recent viewpoint of such contributions in the 1/2- and 1/4-BPS cases). A novel feature appears in the $\partial^6\mathcal{R}^4$ case, where D-instanton/anti D-instanton pairs with zero net instanton number arise, giving exponentially suppressed contributions to the constant terms.

The coefficient functions discussed in [2] are in precise agreement with all the boundary data in these three limits and also satisfy the Laplace equations in (2.7)–(2.9). In the case of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions the solutions are combinations of Eisenstein series for the rank-$(d+1)$ duality groups. In the case of the $\partial^6\mathcal{R}^4$ interaction the solution is a less familiar automorphic function. Although we have not proved that these solutions are unique, given the number of conditions that need to be satisfied it seems unlikely that there are ambiguities (although we cannot rule out the possibility of cusp forms). We will briefly review the kinds of series that enter into the solutions (more details are given in appendix B of [2]).

2.2 Definition and properties of Eisenstein series.

The 'minimal parabolic' Eisenstein series for a group G is defined with respect to a complex vector λ in the weight space of the Lie algebra $\mathfrak{g}$ as [18]

$$\mathbf{E}^G_\lambda(g) = \sum_{\gamma\in G(\mathbb{Q})\backslash B(\mathbb{Q})} e^{\langle\lambda+\rho,H(g\gamma)\rangle}\,, \tag{2.13}$$

where ρ is half the sum of the positive roots, $\langle\cdot,\cdot\rangle$ is the inner product on the root system of G, $H(g)$ is a vector in the Cartan subalgebra, and B is a Borel subgroup of G. These Eisenstein series are eigenfunctions of the invariant differential operators of $K\backslash G$. In particular, they are eigenfunctions of the Laplacian,[4]

$$\Delta_{K\backslash G}\,\mathbf{E}^G_\lambda(g) = \langle\lambda,\rho+\lambda\rangle\,\mathbf{E}^G_\lambda(g)\,. \tag{2.14}$$

Whereas the $SL(2)$ Eisenstein series depends on a single complex parameter s, for higher-rank groups there are $r = \mathrm{rank}(G)$ such parameters, s_k $(k = 1,\ldots,r)$, that are related to the entries in λ. The minimal parabolic Eisenstein series has a poles for various values of λ [18], but the special cases of interest to us are ones that are obtained by taking the multiple residue on the poles at $s_k = 0$ for all $k \neq \alpha$, so only $s \equiv s_\alpha$ is non-zero, where α is a particular node of the Dynkin diagram of G. In other words we set

$$\begin{aligned}&\lambda_{d-\alpha+1} - \lambda_{d-\alpha} - 1 = 2s\,,\\ &\lambda_{d-k+1} - \lambda_{d-k} - 1 = 0\,, \qquad \text{all } 1\leq k\neq\alpha\leq d-1\,.\end{aligned} \tag{2.15}$$

This defines the *maximal* parabolic Eisenstein series for a particular parabolic subgroup of G associated with the Dynkin label $[0^{\alpha-1}\,1\,0^{r-\alpha}]$, which will be denoted by[5] $\mathbf{E}^{(G)}_{[0^{\alpha-1}\,1\,0^{r-\alpha}];s}$.

These Eisenstein series can be expressed as sums over integer lattices. In the simplest cases these sums can be analysed directly. For example, the $SL(n)$ series $\mathbf{E}^{SL(n)}_{[0^{\alpha-1}\,1\,0^{n-\alpha-1}];s}$ is given by

$$\mathbf{E}^{SL(n)}_{[0^{\alpha-1}\,1\,0^{n-\alpha-1}];s} = {\sum_{\{m_i\}\in\mathbb{Z}^d}}' \frac{1}{(d^{[i_1\ldots i_\alpha]}\,g_{i_1j_1}\ldots g_{i_\alpha j_\alpha}\,d^{[j_1\ldots j_\alpha]})^s}\,, \tag{2.16}$$

where g_{ij} $(i,j=1,\ldots,n)$ is the $SL(n)$ metric, $d^{[i_1\ldots i_\alpha]}$ is the antisymmetrized product of α integer vectors, $d^{[i_1\ldots i_\alpha]} = m_1^{[i_1}m_2^{i_2}\ldots m_\alpha^{i_\alpha]}$ and the sum excludes the values at which the denominator vanishes.

However, for other duality groups these lattice sums are more subtle. This is illustrated by the case of the $SO(d,d)$ series[6] $\mathbf{E}^{SO(d,d)}_{[1,0^{d-1}];s}$, which has the representation (motivated by the expression for one-loop perturbative amplitude for string theory compactified on $\mathcal{T}^d$ [2])

$$\mathbf{E}^{SO(d,d)}_{[1\,0^{d-1}];s} = \frac{\pi^s}{2\zeta(2s+2-d)\Gamma(s)}\int_{\mathcal{F}_{SL(2,\mathbb{Z})}}\frac{d^2\tau}{\tau_2^2}\mathbf{E}_{s+1-\frac{d}{2}}(\tau)\,(\Gamma_{(d,d)}(\tau)-V_{(d)})\,, \tag{2.17}$$

[4]Invariance under K implies that the eigenvalue of the Laplacian is the same as the value of the second-order Casimir of G.

[5]The conventional $SL(2)$ Eisenstein series will be denoted by $\mathbf{E}_s \equiv \mathbf{E}^{SL(2)}_{[1];s}$.

[6]The $d=5$ case is of relevance as the $D=6$ U-duality group $SO(5,5)$, which also arises as the duality symmetry of perturbative string theory in $D=5$ with a different interpretation of the moduli.

where $\Gamma_{(d,d)}(\tau)$ is the standard lattice factor for compactification of the one-loop string amplitude on $\mathcal{T}^d$, $V_{(d)}$ is the volume of $\mathcal{T}^d$ and the integral is over the fundamental domain of $SL(2,\mathbb{Z})$. The corresponding representations of the other $SO(d,d)$ series, as well as the $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ series have not been determined (as far as we know). However, it is possible to analyse all the series from their definition (2.13). This procedure has been carried out and will be reported in detail elsewhere.

The arguments of [2] (and earlier work reviewed therein) lead to the $\mathcal{R}^4$ coefficients that enter the Einstein-frame action (2.4)[7],

$$\begin{array}{lll}
D=10 & \mathcal{E}^{(10B)}_{(0,0)} = \mathbf{E}_{\frac{3}{2}}(\Omega) & d=0\ , \\
D=9 & \mathcal{E}^{(9)}_{(0,0)} = \mathbf{E}_{\frac{3}{2}}(\Omega)\,\nu_1^{-\frac{3}{7}} + 4\zeta(2)\,\nu_1^{\frac{4}{7}} & d=1\,, \\
D=8 & \mathcal{E}^{(8)}_{(0,0)} = \lim_{\epsilon\to 0}\left(\mathbf{E}^{SL(3)}_{[10];\frac{3}{2}+\epsilon} + 2\mathbf{E}_{1-2\epsilon}(U)\right) = \hat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\hat{\mathbf{E}}_1(U) & d=2\,, \\
3\le D<8 & \mathcal{E}^{(D)}_{(0,0)} = \mathbf{E}^{E_{d+1(d+1)}}_{[10\cdots 0];\frac{3}{2}} & 2<d\le 7\,.
\end{array} \tag{2.18}$$

Each Eisenstein series in these equations is a function of the moduli that parametrize the coset space $K\backslash G$ of the U-duality group $G = E_{d+1(d+1)}$ by its maximal compact subgroup K. In the following we will omit the arguments of the Eisenstein series unless this is likely to lead to confusion. The quantity ν_1 is defined in terms of the radius of the circular dimension in the type IIB theory, r_B, by $\nu_1 = (r_B/\ell_{10})^2$. The individual series in the third line have poles at $\epsilon = 0$ but these poles cancel in their sum. The symbol $\hat{\mathbf{E}}$ indicates a series that is regularised by subtracting a pole in ϵ. In [2] it has been explicitly verified that these coefficients satisfy all the required boundary conditions, as well as the Laplace eigenvalue equations (2.7) for $D \ge 6$ (and is extended to $D \le 5$ in a forthcoming paper in collaboration with Stephen Miller [23]).

The coefficients of the $\partial^4\mathcal{R}^4$ interactions in dimensions $7 \le D \le 10$ are given by

$$\begin{array}{lll}
D=10 & \mathcal{E}^{(10B)}_{(1,0)}(\Omega) = \frac{1}{2}\,\mathbf{E}_{\frac{5}{2}}(\Omega) & d=0 \\
D=9 & \mathcal{E}^{(9)}_{(1,0)} = \frac{1}{2}\,\nu_1^{-\frac{5}{7}}\,\mathbf{E}_{\frac{5}{2}}(\Omega) + \frac{2\zeta(2)}{15}\,\nu_1^{\frac{9}{7}}\,\mathbf{E}_{\frac{3}{2}}(\Omega) + \frac{4\zeta(2)\zeta(3)}{15}\,\nu_1^{-\frac{12}{7}} & d=1 \\
D=8 & \mathcal{E}^{(8)}_{(1,0)} = \frac{1}{2}\,\mathbf{E}^{SL(3)}_{[10];\frac{5}{2}} - 4\,\mathbf{E}^{SL(3)}_{[10];-\frac{1}{2}}\,\mathbf{E}_2(U) & d=2 \\
D=7 & \mathcal{E}^{(7)}_{(1,0)} = \lim_{\epsilon\to 0}\left(\frac{1}{2}\,\mathbf{E}^{SL(5)}_{[1000];\frac{5}{2}+\epsilon} + \frac{3}{\pi^3}\mathbf{E}^{SL(5)}_{[0010];\frac{5}{2}-\epsilon}\right) = \frac{1}{2}\,\hat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{3}{\pi^3}\hat{\mathbf{E}}^{SL(5)}_{[0010];\frac{5}{2}} & d=3
\end{array} \tag{2.19}$$

The poles in the last line again cancel, yielding a finite expression. These expressions satisfy all the boundary conditions in the three degeneration limits described earlier, as well as the Laplace eigenvalue equations (2.8). The extension of these expression for $D \le 6$ will be presented in reference [23].

The solutions of the inhomogeneous equations for the coefficients, $\mathcal{E}^{(D)}_{(0,1)}$, of the $\partial^6\mathcal{R}^4$ interaction are more complicated and given in [2] for $D \ge 7$. Some details of the $D = 6$ case are presented in appendix A since it is of particular interest to this paper.

[7] 10B indicates the ten-dimensional type IIB theory.

3. Logarithmic terms and ultraviolet divergences in supergravity

One of the intriguing features of the expressions for the coefficients in [2] is the manner in which potential divergences cancel. The Eisenstein series that enter into the coefficients, $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$, have singularities at specific values of the parameter s. This reflects the pole at $s = 1$ in the Riemann zeta function, $\zeta(s)$. However, the precise combinations of Eisenstein series that enter are ones for which the pole residues cancel. This is a manifestation of the consistency of string perturbation theory. Although the poles cancel, there are residual terms that are logarithms of a modulus, which are important elements in the structure of the amplitude. We will here focus on logarithms of the coupling constant, $\log y_D$. These enter in cases where the low energy supergravity limit has a logarithmic ultraviolet divergence, manifested as a pole in dimensional regularisation.

These logarithmic terms are the origin of the Kronecker delta terms on the right-hand side of (2.7)–(2.9). Roughly speaking this follows from the fact that part of the Laplace operator acting on $\mathcal{E}^{(D)}_{(p,q)}$ contains $y^2\partial_y^2 \log y = -1$. The simplest example of this phenomenon is seen in the $\mathcal{R}^4$ coefficient, $\mathcal{E}^{(8)}_{(0,0)}$ in $D = 8$ dimensions in (2.18), the next being in the $\partial^4\mathcal{R}^4$ coefficient, $\mathcal{E}^{(7)}_{(1,0)}$ in $D = 7$ dimensions (2.19). The third example is the $\partial^6\,\mathcal{R}^4$ coefficient in $D = 6$ dimensions, which is presented in appendix A.

3.1 Logarithmic thresholds in the Einstein frame

Closed string perturbation theory is an expansion in the D-dimensional coupling constant, in which the genus-h term is proportional to y_D^{h-1} when evaluated in the string frame. The four-graviton amplitude contains terms that are non-analytic in the Mandelstam invariants due to massless thresholds that are determined by unitarity. Up to the order in the low energy expansion that we are considering in this paper these are the same thresholds as those of maximal supergravity where they arise at L loops in dimensions $D_L = 4+6/L$ [22]. In the string amplitude these are schematically of the form,

$$\ell_s^{8-D_L}\, y_{D_L}^{L-1} (\ell_s^2\, s)^{n_L}\, h_L(x)\, \mathcal{R}^4\, \log(-\ell_s^2\, s\, f_L(x))\,, \qquad n_1 = 0\,,\ n_2 = 2\,,\ n_3 = 3\,, \tag{3.1}$$

where f_L and h_L are complicated functions of the dimensionless variable $x = -t/s = 1 + u/s$, the details of which do not concern us (see [22] for a discussion of these contributions). The power of ℓ_s in the overall factor is fixed by the power of the Mandelstam invariants and the dimension D_L. Importantly, apart from the explicit power of the string coupling, y_{D_L}, there is no dependence on the moduli in the overall factor multiplying these nonanalytic terms, although $f_L(x)$ does depend on the moduli other than y_{D_L}. Transforming from the string frame to the Einstein frame is equivalent to replacing ℓ_s by ℓ_D using $\ell_D^{D-2} = \ell_s^{D-2}\, y_D$. This implies that the Mandelstam invariants are rescaled so that

$$\ell_s^2\, s = y_D^{-\frac{2}{D-2}}\, \ell_D^2\, s, \qquad \text{or} \qquad \log(-\ell_s^2\, s) = \log(-\ell_D^2\, s) - \frac{2}{D-2}\, \log y_D\,. \tag{3.2}$$

The contribution to the amplitude in (3.1) is therefore equal to the Einstein frame expression

$$\ell_{D_L}^{8-D_L}\, (\ell_{D_L}^2\, s)^{n_L}\, h_L(x)\, \mathcal{R}^4\, \left(\log(-\ell_{D_L}^2 s f_L(x)) - \frac{2}{D_L-2}\, \log y_{D_L}\right) \tag{3.3}$$

So we see that when the Mandelstam invariants are expressed in Einstein frame units the non-analytic $\log s$ term in the amplitude leads to a term proportional to $\log y_D$ in the analytic part. In this discussion there is an ambiguity in the scale of the logarithms, but this does not affect the overall coefficient and is independent of the moduli, so for our purposes it can be ignored. In other words, the coefficient of the $\log y_D$ term in Einstein frame is $-2/(D_L-2)$ times the coefficient of the $\log(-\ell_D^2 s)$ term.

On the other hand, in supergravity the factor of $\log s$ arises as an infrared threshold singularity accompanied by a logarithmic ultraviolet divergence. If this is regulated by an ultraviolet momentum cutoff Λ, it results in a term of the form $\log(-s/\Lambda^2)$, where the $\log\Lambda$ can be subtracted by addition of a local counterterm. In dimensional regularisation the ultraviolet divergence appears an ϵ pole in the amplitude evaluated in $D = D_L + 2\epsilon$ dimensions. The logarithm appears after subtracting the pole and using $\lim_{\epsilon\to\infty}((-s/\mu)^{2\epsilon}-1)/\epsilon \sim \log(-s/\mu)$, where μ is an arbitrary scale, Needless to say, since the coefficient of the log is determined by unitarity it is not sensitive to the regularisation scheme adopted.

The conclusion is that the logarithmic terms in the automorphic functions, determine the coefficients of the $\log s$ factors in A_D^{nonan}, and hence the logarithmic terms that represent the ultraviolet divergences in supergravity. The following examples illustrate this feature of the amplitudes in the three cases, $D_1 = 8$, $D_2 = 7$ and $D_3 = 6$. The conventions used to compare the amplitudes in string theory and supergravity are exhibited in appendix B.

• **The $\mathcal{R}^4$ interaction in $D = 8$ dimensions**

It was shown in [2,11] that the coefficient $\mathcal{E}^{(8)}_{(0,0)}$ in (2.18) has the perturbative expansion

$$\mathcal{E}^{(8)}_{(0,0)} = \frac{2\zeta(3)}{y_8} + 2(\hat{\mathbf{E}}_1(T) + \hat{\mathbf{E}}_1(U)) + \frac{2\pi}{3}\log y_8 + O(e^{-(y_8T_2)^{-\frac{1}{2}}}, e^{-(y_8/T_2)^{-\frac{1}{2}}})\,. \tag{3.4}$$

In this case there is no overall power of ℓ_8 in (2.4) so this expression is also the coefficient in the string frame and the power-behaved terms are identified with tree-level ($h = 0$) and genus-one ($h = 1$) contributions, together with the $\log y_8$ term. The latter is a signal of a genus-one $\log(-s\,\ell_s^2)$ term in the string frame, where there can be no $\log y_8$, as argued above.

This expression can be compared with the expression that arises in dimensionally regularised one-loop maximal supergravity in $D = 8 + 2\epsilon$, where the ϵ pole is associated with an ultraviolet divergence. The field theory amplitude given in [24] is

$$A_R^{tree} + A_{R^4}^{1-loop} \propto \mathcal{R}^4 \left(\frac{64}{stu\,\ell_8^6} + \hat{I}_1(\ell_8^2 k_i\cdot k_j)\right)\,, \tag{3.5}$$

where we have included the tree-level term proportional to $\mathcal{R}^4/stu$ in order to display the relative normalisations (we refer to appendix B for details) and

$$\hat{I}_1(\ell_8^2 k_i\cdot k_j) = I_1(s,t) + I_1(t,s) + I_1(s,u) + I_1(u,s) + I_1(t,u) + I_1(u,t)\,, \tag{3.6}$$

with

$$I_1(s,t) = \lim_{\epsilon\to 0}(I_1^\epsilon(s,t) + \frac{\pi}{\epsilon})\,, \tag{3.7}$$

and

$$I_1^\epsilon(s,t) = \frac{2\pi}{3}\left(\frac{1}{2\epsilon} + \log\left(-\frac{\ell_8^2 t}{\mu}\right)\right)\int_0^1 dx \frac{t(1-x)}{sx - t(1-x)} + \frac{2\pi}{3}\int_0^1 dx \frac{t(1-x)\,\log(1-x)}{sx - t(1-x)} + O(\epsilon) \tag{3.8}$$

(μ is an arbitrary constant). It is easy to see that this expression contains a logarithmic term. Summing over the terms in (3.6) and rescaling the metric to the string frame using the identity $\ell_8^2 = y_8^{1/3}\,\ell_s^2$ gives

$$\hat{I}_1(\ell_s^2 k_i \cdot k_j) = \hat{I}_1(\ell_8^2 k_i \cdot k_j) + \frac{2\pi}{3}\log y_8\,. \tag{3.9}$$

Therefore, the $\frac{2}{3}\pi \log y_8$ contribution in the coefficient $\mathcal{E}^{(8)}_{(0,0)}$ in (3.4) implies the presence of the threshold logarithm, which is given in supergravity by the dimensionally regularised expression $\hat{I}_1(\ell_8^2 k_i \cdot k_j)$. So the coefficient of the logarithmic ultraviolet divergence associated with the field theory pole in (3.8) is precisely the coefficient of the $\log y_8$ required by U-duality.

• **The $\partial^4\mathcal{R}^4$ interaction in $D = 7$ dimensions**

The coefficient of this interaction is $\ell_7^5\,\mathcal{E}^{(7)}_{(1,0)}$ which is defined by (2.19) and was shown in [2] to have the small-y_7 the expansion

$$\begin{aligned}\mathcal{E}^{(7)}_{(1,0)} = {} & \frac{\zeta(5)}{y_7^2} + \frac{3}{\pi^3 y_7}\,\mathbf{E}^{SL(4)}_{[010];\frac{5}{2}} + \frac{2}{3}(\hat{\mathbf{E}}^{SL(4)}_{[100];2} + \hat{\mathbf{E}}^{SL(4)}_{[001];2}) + \frac{8\pi^2}{15}\log y_7 \\ & + O(e^{-(y_7 v_3)^{-\frac{1}{2}}}, e^{-(y_7 \ell_s/r_i)^{-\frac{1}{2}}})\,,\end{aligned} \tag{3.10}$$

where $v_3 = (r_1 r_2 r_3)/\ell_s^3$. The various powers of y_7 in this expression correspond to tree-level ($h = 0$), genus-one ($h = 1$) and genus-two ($h = 2$) terms. This is seen by transforming to the string frame where the terms are of order y_7^{-1+h} using the fact that $\ell_7^5 = \ell_s^5\,y_7$. The logarithmic term here implies the existence of a genus-two threshold term of the form $4\pi^2/3\,y_7 \log(-s\ell_s^2)$ in string frame using (3.2) again.

We can compare the coefficient of the $\log y_7$ term in (3.10) with the ultraviolet divergence of two-loop maximal supergravity in $D = 7$ dimensions, which was evaluated using dimensional regularisation in [25] and gave (once again including the tree-level term in order to compare normalisations),

$$A_R^{tree} + A_{\partial^4 R^4}^{2-loop} \propto \mathcal{R}^4\, y_7^{\frac{1}{5}}\left(\frac{64}{stu\,\ell_7^6} + \hat{I}_2(\ell_7^2 k_i \cdot k_j)\right)\,, \tag{3.11}$$

where $\hat{I}_2(\ell_7^2\,k_i \cdot k_j)$ is the two-loop contribution, which is given by a sum of terms

$$\hat{I}_2(\ell_s^2\,k_i \cdot k_j) = (\ell_7^2\,s)^2\,(I_2^P(s,t) + I_2^P(s,u) + I_2^{NP}(s,t) + I_2^{NP}(s,u)) + \text{perms}(s,t,u)\,. \tag{3.12}$$

Here P and NP denote contributions from planar and nonplanar double-box Feynman integrals, which are defined via dimensional regularisation in $D = 7 + 2\epsilon$ dimensions using equation (4.3) of [25]

$$I_2(s,t) = \lim_{\epsilon\to 0}(I_2^\epsilon + \frac{\ell_7^4}{2\epsilon}\,\frac{\pi^2}{12}\,(s^2 + t^2 + u^2))\,, \tag{3.13}$$

where

$$I_2^{X\,\epsilon}(s,t) = \frac{1}{4^2}\,(4\pi)^{D-6}\,\Gamma(7-D)\,(-\ell_7^2 s)^{D-7}\,\int_0^1 d^7\nu\delta(1-\sum_{i=1}^{7}\nu_i)\,\Delta_X^{\frac{14-3D}{2}} + \cdots\,, \quad (3.14)$$

where $X = P$ or NP and $\ldots$ indicates terms that do not contribute to the logarithm and Δ_X is given by [25]

$$\begin{aligned}\Delta_P &= (\nu_1+\nu_2+\nu_3)(\nu_4+\nu_5+\nu_6)+\nu_7(1-\nu_7)\,,\\ \Delta_{NP} &= (\nu_1+\nu_2)(\nu_3+\nu_4)+(\nu_1+\nu_2+\nu_3+\nu_4)(\nu_5+\nu_6+\nu_7)\,.\end{aligned} \quad (3.15)$$

Expanding (3.14) one gets (see appendix C of [25])

$$\hat{I}_2 = \ell_7^4\,\frac{\pi^2}{12}\,(s^2\log(-\frac{\ell_7^2 s}{\mu})+t^2\log(-\frac{\ell_7^2 t}{\mu})+u^2\log(-\frac{\ell_7^2 u}{\mu}))+\cdots \quad (3.16)$$

Using the relation $\ell_7^2 = \ell_s^2\,y_7^{2/5}$

$$\hat{I}_2(\ell_7^2 k_i\cdot k_j) = \hat{I}_2(\ell_s^2 k_i\cdot k_j)+\frac{8\pi^2}{15}\,\log y_7\,\sigma_2\,. \quad (3.17)$$

So we see that the $\log y_7$ term in the coefficient $\mathcal{E}^{(7)}_{(1,0)}$ in (3.10) represents a two-loop threshold term in the Einstein frame which is absent in the string frame provided the string amplitude possesses a logarithmic threshold with precisely the same coefficient as in $L = 2$ supergravity.

In other words, as with the $\mathcal{R}^4$ interaction, we can identify the precise coefficient of the logarithm associated with an ϵ pole in dimensional regularisation of two-loop maximal supergravity in $D = 7+2\epsilon$ dimensions with the coefficient of the logarithm in the duality-invariant coefficient, $\mathcal{E}^{(7)}_{(1,0)}$.

• **The $\partial^6\mathcal{R}^4$ interaction in $D = 6$ dimensions**

In this case the coefficient, $\mathcal{E}^{(6)}_{(0,1)}$ is an automorphic function for the U-duality group $SO(5,5)$ that satisfies the inhomogeneous equation (2.9), which has vanishing eigenvalue but non-zero Kronecker delta term when $D = 6$. The solution of this equation is less straightforward than the earlier cases. Since this case was hardly discussed in [2] (whereas the $\partial^6\mathcal{R}^4$ coefficients for $D > 6$ were obtained in [2, 8, 15]), a discussion is included in the appendix, from which we see that the coefficient $\mathcal{E}^{(6)}_{(0,1)}$ has the perturbative expansion

$$\begin{aligned}\mathcal{E}^{(6)}_{(0,1)} = {}& \frac{2\zeta(3)^2}{3\,y_6^3}+\frac{1}{y_6^2}\,(\frac{2\zeta(3)}{3}\,\mathbf{E}^{SO(4,4)}_{[1000];1}+\frac{8\zeta(4)}{69\pi}\mathbf{E}^{SO(4,4)}_{[1000];4})+\frac{1}{y_6}\,F_2^{SO(4,4)}\\ &+\frac{4\zeta(2)}{105}\,(\hat{\mathbf{E}}^{SO(4,4)}_{[0001];3}+\hat{\mathbf{E}}^{SO(4,4)}_{[0010];3})+15\zeta(3)\,\log y_6+n.p.\,.\end{aligned} \quad (3.18)$$

where $n.p.$ stands for various non perturbative contributions evaluated in appendix A where the function $F_2^{SO(4,4)}$ is also discussed. In this case the powers of the string coupling, y_6, correspond to tree-level, genus-one, genus-two and genus-three. The three-loop contribution involves the regularized $SO(4,4)$ series $\hat{\mathbf{E}}^{SO(4,4)}_{[0001];3}$ and $\hat{\mathbf{E}}^{SO(4,4)}_{[0010];3}$. In particular, the

logarithmic term is a sign of a genus-three logarithm associated with a term in the string frame of the schematic form $y_6^2\zeta(3)\log(-\ell_s^2 s)$.

Once again this can be compared with dimensionally regularised supergravity, which has a three-loop contribution to the $\partial^6\mathcal{R}^4$ amplitude in $D = 6$ dimensions of the form (again adding in the tree-level amplitude to compare normalisations)

$$A_R^{tree} + A_{\partial^6 R^4}^{3-loop} \propto \mathcal{R}^4\, y_6^{\frac{1}{2}} \left(\frac{64}{stu\,\ell_6^6} + \hat{I}_3(\ell_6^2\, k_i\cdot k_j)\right) . \tag{3.19}$$

The function $\hat{I}_3$ is a sum of many contributions [26,27] that is given by using equation (5.12) of [26], which gives

$$\hat{I}_3 = \lim_{\epsilon\to 0}(I_3^\epsilon + \frac{15\zeta(3)}{\epsilon}\sigma_3)\,. \tag{3.20}$$

The expression for $\hat{I}_3$ can be deduced from equation (5.14) of [27] (using the $D = 7$ two-loop result in equation (5.19) to establish normalisations). Transforming from Einstein frame to the string frame (and using the relation $\ell_6^2 = \ell_s^2\, y_6^{1/2}$) gives

$$\hat{I}_3(\ell_6^2\, k_i\cdot k_j) = \hat{I}_3(\ell_s^2\, k_i\cdot k_j) + 15\zeta(3)\,\log y_6\,\sigma_3\,. \tag{3.21}$$

Therefore, the coefficient of the $\log y_6$ term in $\mathcal{E}^{(6)}_{(0,1)}$ in (3.18) determines the coefficient of the logarithmic terms associated with the ϵ pole.

• **The $\partial^6\mathcal{R}^4$ interaction in $D = 8$ dimensions**

The examples discussed so far are ones in the critical dimensions, $D_L = 4 + 6/L$, for $L = 1, 2, 3$. There are, however, other ultraviolet logarithms that arise in dimensions $D > D_L$ for any value of L. The simplest of these appears to arise in the one-loop in ten dimensions, where there is a threshold that is schematically of the form $s\,\ell_{10}^2\,\log(-s\,\ell_{10}^2)\,\mathcal{R}^4 + \mathrm{perm}(s,t,u)$. However, under the rescaling $\ell_{10}^2 = \ell_s^2\, y_{10}^{1/4}$ the shift is $(s+t+u)\,\log y_{10} = 0$, so the logarithmic term vanishes.

The simplest nontrivial example is the two-loop amplitude in $D = 8$ dimensions, which has both log and $\log^2$ divergences associated with a singe and double pole multiplying $\partial^6\,\mathcal{R}^4$ in dimensional regularisation in $D = 8+2\epsilon$ dimensions. The presence of these supergravity divergences is again encoded in the duality invariant $\partial^6\,\mathcal{R}^4$ coefficient function, $\mathcal{E}^{(8)}_{(0,1)}$, which satisfies (2.9) with $D = 8$. In this case the source term on the right-hand side of (2.9) is the square of the $\mathcal{R}^4$ coefficient, $\mathcal{E}^{(8)}_{(0,0)}$, which itself has a one-loop $\log y_8$, as exhibited in (3.4). The solution of this equation has the perturbative expansion given in equation (5.20) in [2], which has the logarithmic terms,

$$\mathcal{E}^{(8)}_{(0,1)} = \cdots + \frac{\pi}{9}(\frac{\pi}{6} + \mathbf{E}^{pert}_{(0,0)})\log y_8 - \frac{\pi^2}{27}\log^2 y_8\Big) + n.p.\,, \tag{3.22}$$

where $\mathbf{E}^{pert}_{(0,0)}$ is the perturbative part of the $\mathcal{R}^4$ interaction which has the expansion given in (3.4). The term in (3.22) involving the tree-level part of $\mathbf{E}^{pert}_{(0,0)}$ is a stringy threshold effect that was discussed in [28]. It contains the factorisation of the string loop into the

product of a tree-level $\mathcal{R}^4$ factor and a massless pole factor. The one-loop contribution to $\mathbf{E}^{pert}_{(0,0)}$ gives both $\log y_8$ and $\log^2 y_8$ contributions that add to the explicit $\log y_8^2$ term in (3.22).

In this case comparison with two-loop field theory in $D = 8$ dimensions requires us to take into account the presence of two kinds of diagrams: (i) The two-loop diagrams evaluated in [25]. (ii) A contribution involving the $\mathcal{R}^4$ counterterm that cancels the one-loop divergence - it is necessary to include the diagram in which this counterterm, is inserted as a vertex in a one-loop diagram.

In the first of these contributions, (i), the double ϵ pole of the dimensionally regulated two-loop amplitude of maximal supergravity in (3.14) leads to a $\log^2 s$ term given in equation (4.4) of [25]

$$\begin{aligned}\hat{I}_2^{(i)} &= \lim_{\epsilon\to 0}(I_2^\epsilon + \ell_8^6\,(\frac{\pi^2}{48\epsilon^2} + \frac{\pi^2}{288\epsilon})(s^3+t^3+u^3) \\ &\quad + \ell_8^6\,\frac{\pi^2}{24\epsilon}(s^3\log(-\frac{s\ell_8^2}{\mu}) + t^3\log(-\frac{t\ell_8^2}{\mu}) + u^3\log(-\frac{u\ell_8^2}{\mu}))\,.\end{aligned} \tag{3.23}$$

The $\log^2 y_8$ term should correspond to the double-pole in ϵ in the two-loop supergravity amplitude in $D = 8$ [25].

However, in eight dimensions the complete amplitude also includes contribution (ii) due to the one-loop $\mathcal{R}^4$ counterterm, which has an ϵ pole, inserted into a one-loop diagram. This results in a triangle diagram, which makes an additional contribution to I_2 that was not considered in [25], but makes an essential contribution to the complete $\partial^6\mathcal{R}^4$ ultraviolet divergence in eight dimensions. Its overall normalisation is difficult to determine, so we will fix it by unitarity, which, as explained above, guarantees that it matches the string theory result. Although this is not a completely independent check of the normalisation (unlike the previous cases), it shows the precise origin of the different structures that contribute to give the string theory result. With this proviso, the counterterm contribution is

$$I_2^{(ii)\,\epsilon} = \frac{240}{(4\pi)^3}\,\frac{\pi}{\epsilon}\,\mathcal{R}^4\,\ell_8^4\,(s^2\,I_\triangleright^\epsilon(s) + t^2\,I_\triangleright^\epsilon(t) + u^2\,I_\triangleright^\epsilon(u))\,, \tag{3.24}$$

where

$$I_\triangleright^\epsilon(s) = \int \frac{d^D\ell}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2} = \frac{2^{5-D}\pi^{\frac{D+3}{2}}}{(D-4)}\,\frac{(-\ell_D^2 s)^{D/2-3}}{\Gamma\left(\frac{D-3}{2}\right)\sin(\pi(3-\frac{D}{2}))} \tag{3.25}$$

in which one vertex is the $\mathcal{R}^4$ counterterm and in which the loop integral generates a second power of $1/\epsilon$. As a result this contribution has the form

$$\hat{I}_2^{(ii)} = \lim_{\epsilon\to 0}\Big(I_2^{(ii)\,\epsilon} - \ell_8^6\,\frac{5\pi^2}{32}\,\frac{s^3+t^3+u^3}{\epsilon^2} - \ell_8^6\,\frac{5\pi^2}{32}\,\frac{s^3\log(-\frac{\ell_8^2 s}{\mu}) + t^3\log(-\frac{\ell_8^2 t}{\mu}) + u^3\log(-\frac{\ell_8^2 u}{\mu})}{\epsilon}\Big) \tag{3.26}$$

which gives another contribution to the double pole.

The total contribution obtained by adding (i) and (ii) is given by

$$\begin{aligned}\hat{I}_2^{(i)} + \hat{I}_2^{(ii)} = &-\ell_8^6 \frac{\pi^2}{192}(s^3 \log^2(-\frac{s\ell_8^2}{\mu}) + t^3 \log^2(-\frac{t\ell_8^2}{\mu}) + u^3 \log^2(-\frac{u\ell_8^2}{\mu})) \\ &- \ell_8^6 \frac{\pi^2}{72}\left(s^3 \log(-\frac{s\ell_8^2}{\mu}) + t^3 \log(-\frac{t\ell_8^2}{\mu}) + u^3 \log(-\frac{u\ell_8^2}{\mu})\right).\end{aligned} \tag{3.27}$$

which transforms as (using $\ell_8^2 = \ell_s^2\, y_8^{1/3}$)

$$\hat{I}_2(\ell_8^2 k_i \cdot k_j) = \hat{I}_2(\ell_s^2 k_i \cdot k_j) - \frac{\pi^2}{27} \log^2 y_8\, \sigma_3 + \cdots\,, \tag{3.28}$$

where $\cdots$ denotes terms with a single power of $\log y_8$. So we see that there is agreement between the coefficient of the $\log^2 y_8$ term in the automorphic coefficient $\mathcal{E}^{(8)}_{(0,1)}$ and the supergravity calculation. As is evident from (3.22), the string theory coefficient automatically includes the term with the single logarithm, $\mathbf{E}^{pert}_{(0,0)} \log y_8$, which corresponds to the one-loop term that has to be subtracted in the field theory calculation in [25].

4. The supergravity limit and instanton corrections

We turn now to consider the particular low energy limit of string theory that should relate to perturbative quantum supergravity in D dimensions, which is an expansion in powers of $k_i \cdot k_j\, \ell_D^2 \ll 1$, where the D-dimensional Planck length is fixed while $\ell_s \to 0$, so the string excitation masses become large. Since

$$\ell_D^{D-2} = y_D\, \ell_s^{D-2}\,, \tag{4.1}$$

it follows that the limit of interest is one in which the D-dimensional string coupling becomes large,

$$\lim_{\ell_s \to 0} y_D = \frac{g_s^2\, \ell_s^d}{r_1 \cdots r_d} \to \infty\,. \tag{4.2}$$

In addition, in order to arrive at the the field theory limit in which there is a single massless supermultiplet, the masses of all other massive states must become large and decouple. This requires, in particular, $r_i \to 0$ so that non-zero Kaluza–Klein masses are large, and $\ell_s^2/r_i \to 0$ for the winding masses to become large.

4.1 The perturbative terms

In the $y_D \to \infty$ limit the perturbative term with the highest power of y_D dominates the others. For $D > D_L = 4 + 6/L$ this is a positive power of y_D so the leading term diverges, signifying a power-behaved divergence in supergravity. The simplest example of this is in $D = 10$ string theory, where the genus-one term corresponds, in this limit, to a term of the form $\ell_s^{-2}\mathcal{R}^4 = y_{10}^{1/4}\ell_{10}^{-2}\,\mathcal{R}^4$. This diverges in the large-y_{10} limit, signifying the quadratic divergence of the one-loop term in supergravity.

When $D = D_L$ the dominant *perturbative* term in the $y_D \to \infty$ limit is the $\log y_{D_L}$ term, which gives the supergravity logarithm for each of the three interactions described in equations (3.4), (3.10), (3.18).

For $D < D_L$ the perturbative terms vanish in the field theory limit since they involve inverse powers of y_D that arise in the translation from string frame to Einstein frame. This is clearly seen from the specific examples of the $\mathcal{R}^4$ interaction in $D = 7$ and $D = 6$ dimensions, as follows.

• The $\mathcal{R}^4$ interaction in $D = 7$ dimensions has perturbative terms that are given by [2],

$$\mathcal{E}^{(7)}_{(0,0)}\Big|_{pert} = y_7^{-\frac{1}{5}} \left(\frac{2\zeta(3)}{y_7} + 2\pi\, \mathbf{E}^{SO(3,3)}_{[100];\frac{1}{2}} \right) , \tag{4.3}$$

where the factor of $y_7^{-1/5}$ comes from the relation $\ell_7 = \ell_s\, y_7^{1/5}$ in converting from string units to Planck units in seven dimensions.

• The $\mathcal{R}^4$ interaction in $D = 6$ dimensions has the perturbative terms [2],

$$\mathcal{E}^{(6)}_{(0,0)}\Big|_{pert} = y_6^{-\frac{1}{2}} \left(\frac{2\zeta(3)}{y_6} + 2\mathbf{E}^{SO(4,4)}_{[1000];1} \right) , \tag{4.4}$$

where the factor $y_6^{-1/2}$ again arises from the conversion from string frame to Einstein frame (using $\ell_6 = \ell_s\, y_6^{1/4}$).

In both these examples the perturbative terms vanish in the $y_D \to \infty$ limit, which is a statement of the well-known fact that there is no local $\mathcal{R}^4$ interaction in maximal supergravity for $D < 8$. In these dimensions the leading contribution beyond the tree-level term is a non-local interaction roughly of the form $s^{\frac{D-8}{2}}\, \mathcal{R}^4$ (although its precise details are more complicated [24]). A similar argument shows that the perturbative parts of the $\partial^4 \mathcal{R}^4$ coefficients, $\mathcal{E}^{(D)}_{(1,0)}$, vanish in the $y_D \to \infty$ limit for $D < 7$. The same is true for $\mathcal{E}^{(D)}_{(0,1)}$ when $D < 6$. Whether analogous statements apply to higher orders in the derivative expansion has not been demonstrated.

However, there are important non-perturbative effects in the string amplitude that swamp the perturbative contribution [19] as will be demonstrated next.

4.2 Supergravity limit including the instanton terms

Nonperturbative effects are, of course, suppressed in string perturbation theory, in which y_D is small and other moduli are fixed. However, the $y_D \to \infty$ limit produces an infinite series of instanton terms with actions that become small in the limit under consideration. For example, consider the exponential terms in the expansion of $\mathcal{E}^{(8)}_{(0,0)}$ in (3.4), which correspond to a series of D-instanton terms (with action $(y_8\, T_2)^{-1/2}$) and of wrapped D-string instanton terms (with action $(y_8/T_2)^{-1/2}$). Although these are both suppressed when y_8 is small, at least one of these series is unsuppressed for large y_8. This is an instanton manifestation of the effect described in [19], where it was shown that in dimensions $D > 3$ there are necessarily towers of non-perturbative particle states that become massless in the supergravity limit. This will now be demonstrated in our explicit examples.

• **The $\mathcal{R}^4$ interaction in $D = 8$**

In this case we will reexamine the exact expression for $\mathcal{E}^{(8)}_{(0,0)} = \hat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\hat{\mathbf{E}}^{SL(2)}_{[1];1}(U)$ in (2.18) in the limit $y_8 \to \infty$. Consider first the expansion of $\mathbf{E}^{SL(3)}_{[10];s}$ in the limit $y_8 \to \infty$,

which is defined by [2, 11]

$$\mathbf{E}^{SL(3)}_{[10];s} = \sum_{(m_1,m_2,m_3)\neq(0,0,0)} \frac{y_8^{\frac{s}{3}}}{\left(y_8\Big(m_1+m_2\Omega_1+m_3(B_{RR}+\Omega_1 T_1)\Big)^2+\frac{|m_2+m_3T|^2}{T_2}\right)^s}\,. \quad (4.5)$$

The limit $y_8 \to \infty$ can be studied by separating the leading piece, which is the term with $m_1 = 0$ in (4.5), and then perform Poisson resummations. This expansion is analogous to the one in (B.52) in [2], but with the substitution $(\nu_2, \Omega) \to (y_8, T)$. For $s \neq 3/2$ this gives

$$\begin{aligned}\mathbf{E}^{SL(3)}_{[10];s} &= y_8^{\frac{s}{3}}\,\mathbf{E}_s(T) + 2\pi\,\frac{\Gamma(s-1)}{\Gamma(s)}\,\zeta(2s-2)y_8^{\frac{3-2s}{3}} \\ &+ \frac{2\pi^s}{\Gamma(s)}\,y_8^{\frac{3-s}{6}}\,T_2^{\frac{1-s}{2}}\sum_{\substack{m_1,m_2\\ m_3\neq 0}}\left|\frac{m_2-m_1T}{m_3}\right|^{s-1} \\ &\times\, K_{s-1}\left(2\pi|m_3|\sqrt{\frac{m_2^2}{T_2}+m_1^2T_2}\,y_8^{1/2}\right)\,e^{2i\pi m_3(m_1B_{\rm RR}+m_2B_{\rm NS})}\,.\end{aligned} \quad (4.6)$$

Regularising the pole at $s = 3/2$ gives

$$\hat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} = y_8^{\frac{1}{2}}\,\mathbf{E}_{\frac{3}{2}}(T) - \frac{4\pi}{3}\,\log y_8 + O(e^{-\sqrt{y_8T_2}}, e^{-\sqrt{y_8/T_2}})\,. \quad (4.7)$$

The exponential terms in this expression are suppressed for fixed T_2 – the Poisson resummation has resummed the effect of light wrapped branes and non-perturbative objects. The net result is that the effect of including these non-perturbative effects has swamped the perturbative term and the leading piece is the term proportional to $y_8^{1/2}$ (and the coefficient of the subleading logarithmic term appears with a different coefficient from the one in the perturbative expansion discussed earlier).

- **The $\partial^4\mathcal{R}^4$ interaction in $D = 7$**

The perturbative part of the $\mathcal{E}^{(7)}_{(1,0)}$ given in last line of (2.19) was derived in [2]. We are now interested in the limit, $y_7 \to \infty$. This gives (see (B.78) and (B.93) in [2] with the replacement $r^4 = y_7$)

$$\begin{aligned}\mathbf{E}^{SL(5)}_{[1000];s} &= y_7^{\frac{s}{5}}\mathbf{E}^{SL(4)}_{[100];s} + \frac{2\pi^2\Gamma(s-2)}{\Gamma(s)}\,\zeta(2s-4)\,y_7^{2-\frac{4s}{5}} + O(e^{-(y_7v_3)^{\frac{1}{2}}}, e^{-(y_7\ell_s/r_i)^{\frac{1}{2}}})\,, \\ \mathbf{E}^{SL(5)}_{[0010];s} &= y_7^{\frac{3s}{5}}\zeta(2s-1)\mathbf{E}^{SL(4)}_{[001];s} + \frac{\pi\Gamma(s-1)}{\Gamma(s)}\,y_7^{1-\frac{2s}{5}}\mathbf{E}^{SL(4)}_{[010];s-\frac{1}{2}} + O(e^{-(y_7v_3)^{\frac{1}{2}}}, e^{-(y_7\ell_s/r_i)^{\frac{1}{2}}})\,.\end{aligned} \quad (4.8)$$

As before we have resummed all the instanton effects so that the exponential terms in this expression are suppressed in the large-y_7 limit. In particular, the series of relevance to the $\partial^4\mathcal{R}^4$ interaction arise in the combination $\mathbf{E}^{SL(5)}_{[1000];\frac{5}{2}+\epsilon}$ and $\mathbf{E}^{SL(5)}_{[0010];\frac{5}{2}-\epsilon}$ in the limit $\epsilon \to 0$.

The poles in the individual series cancel and the combination has the expansion,

$$\begin{aligned}\mathcal{E}^{(7)}_{(1,0)} &= \frac{1}{2}\hat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{3}{\pi^3}\hat{\mathbf{E}}^{SL(5)}_{[0010];\frac{5}{2}} \\ &= \frac{\pi}{30}y_7^{\frac{3}{2}}\mathbf{E}^{SL(4)}_{[001];\frac{5}{2}} + \frac{1}{2}y_7^{\frac{1}{2}}\mathbf{E}^{SL(4)}_{[100];\frac{5}{2}} + \frac{2}{\pi^2}\hat{\mathbf{E}}^{SL(4)}_{[010];2} - \frac{4\pi^2}{5}\log y_7 + O(e^{-(y_7v_3)^{\frac{1}{2}}}, e^{-(y_7\ell_s/r_i)^{\frac{1}{2}}})\,.\end{aligned} \tag{4.9}$$

The leading behaviour is dominated by the term that behaves as $y_7^{3/2}$ (and, once more, the coefficient of the logarithmic term is different from the one in the perturbative expansion).

These expressions illustrate that the perturbative supergravity logarithms are dominated by the "non-perturbative" instanton contributions. Furthermore, the result of summing these contributions leads to expressions that diverge badly in the $y_D \to \infty$ limit. This is a sign that the low energy expansion in powers of $k_i \cdot k_j\, \ell_D^2$ is invalid. As pointed out in [19], string dualities relate this limit to a limit which may be described by trans-Planckian scattering in a decompactified dual of the original string theory.

5. Comments on higher-order interactions and higher-loop supergravity.

The structure of the terms in the low energy expansion of string theory that we have discussed is presumably highly constrained by a combination of duality and maximal supersymmetry, even though this has not been explicitly used in determining the coefficients. It would obviously be of interest to discover the detailed structure of such terms and to what extent they are protected by maximal supersymmetry. Although this has not been understood in detail, there is some information about higher-order terms (i.e., terms of order $\partial^8\,\mathcal{R}^4$ and higher) in $D = 9$ dimensions.

This comes from evaluating the amplitude in the limit of large volume, $\mathcal{V}_{(d+1)}$, of the M-theory torus (limit (iii) described in the section 2) where the Feynman diagram approximation to eleven-dimensional supergravity compactified on $\mathcal{T}^{d+1}$, should be a valid approximation. The contributions to the $\partial^8\mathcal{R}^4$ coefficient, $\mathcal{E}^{(9)}_{(2,0)}$, from one-loop and two-loop Feynman diagrams compactified on $\mathcal{T}^2$ is given in equation (4.25) of [9]. This is a *sum* of automorphic functions satisfying inhomogeneous Laplace equations with source terms that are quadratic in the lower order coefficients, generalising the equation that determines the $\partial^6\mathcal{R}^4$ coefficient (2.9). Although the expression is incomplete since it undoubtedly gets contributions from higher-loop Feynman diagrams, it is striking that its perturbative expansion terminates at genus five, rather than genus four.

The occurrence of a five-loop contribution to $\partial^8\mathcal{R}^4$ is novel since it breaks the pattern set by $\partial^{2k}\mathcal{R}^4$ interactions with $k = 2, 3$, for which there are no contributions with genus larger than k for any value of D. Similar statements also apply to the other higher order terms considered in [9], namely, the $\partial^{10}\mathcal{R}^4$ coefficient (equation (4.31) of [9]) which contains terms up to genus seven, and the $\partial^{12}\mathcal{R}^4$ coefficient (equations (4.32) and (4.33) of [9]), which includes terms up to genus nine. This pattern shows that the claim [29], that supersymmetry protects $\partial^{2k}\mathcal{R}^4$ interactions with $1 < k \leq 5$ from renormalisation in $D = 10$ dimensions, must be modified in lower dimensions. Furthermore, there are indications based on technical issues in the pure spinor formalism [30] that even in ten dimensions

the non-renormalisation property only holds up to $k = 3$. If that were the case the $\partial^8\mathcal{R}^4$ interaction would be unprotected and would be expected to have contributions to all orders in perturbation theory.

Following the earlier considerations of this paper, a genus-five term in the complete $\mathcal{E}^{(D)}_{(2,0)}$ coefficient would imply a five-loop logarithmic ultraviolet divergence in maximal supergravity in critical dimension 24/5. This contrasts with the value that follows if the five-loop amplitude first contributes at order $\partial^{10}\mathcal{R}^4$, in which case the critical dimension would satisfy $D_L = 4 + 6/L$ with $L = 5$, or $D_5 = 26/5$. Furthermore, if $\partial^8\mathcal{R}^4$ is indeed not protected by supersymmetry, so its complete coefficient contains terms to all orders in perturbation theory, the critical dimension at L loops would be $D_L = 2+14/L$. This would lead to a seven-loop logarithmic ultraviolet divergence in maximal supergravity in $D = 4$ This is in line with the suggested presence of a seven-loop counterterm [31]. This conflicts with an earlier argument by the present authors, based on [29], that the first divergence would not occur until at least nine loops [32].

6. Summary and discussion of higher-order contributions

This paper has demonstrated several main features of the structure of the duality invariant coefficients, $\mathcal{E}^{(D)}_{(p,q)}$, of terms up to the order $\partial^6\,\mathcal{R}^4$ (or $2p + 3q \leq 3$) in the low-energy expansion of the four-graviton amplitude in type II string theory compactified to $D = 10-d$ dimensions on a d-torus, $\mathcal{T}^d$. The explicit expressions for these coefficients were derived and their properties analysed in [2] (where earlier work is reviewed).

• The perturbation expansions of these coefficients in certain critical dimensions – $D_1 = 8$ for $\mathcal{R}^4$, $D_2 = 7$ for $\partial^4\,\mathcal{R}^4$, $D_3 = 6$ for $\partial^6\,\mathcal{R}^4$ – contains logarithms of the string coupling $\log y_{D_L}$. Their presence is required by the duality invariance of the analytic part of the amplitude and arises from the presence of poles in Eisenstein series, although the poles themselves cancel, leaving a finite amplitude. Such non-analytic behaviour in the coupling constant cannot be present in perturbative string theory so it must disappear when the amplitude is transformed from the Einstein frame to the string frame using the relation of the D-dimensional Planck scale to the string scale, $\ell_D^{D-2} = \ell_s^{D-2}\,y_D$. In order for this to happen there must be specific terms that are logarithmic in the Mandelstam invariants $\sim \log(-s\,\ell_s^2/\mu)$ (where μ is an arbitrary constant), which correspond to threshold terms in the amplitude. These are precisely the threshold $\log(-s\,\ell_d^2)$'s that arise in supergravity field theory as ultraviolet divergences, or poles in dimensional regularisation. In other words, we have obtained the coefficients of the ultraviolet divergences of maximal supergravity at $L = 1$ loop in $D = 8$, $L = 2$ loops in $D = 7$ and $L = 3$ loops in $D = 6$ as a consequence of U-duality rather than calculating the supergravity loop diagrams explicitly.

• The coefficient functions also contain more subtle effects associated with logarithmic divergences in supergravity amplitudes in dimensions $D > D_L$. For example, we saw that the normalisation of the double-pole, $1/\epsilon^2$, in three-loop supergravity in $D = 8 + 2\epsilon$ dimensions is in correspondence with the coefficient of $\log^2 y_8$ in the perturbative expansion of the automorphic coefficient of the $\partial^6\mathcal{R}^4$ interaction, $\mathcal{E}^{(8)}_{(0,1)}$, which satisfies (2.9) with $D = 8$. In this case the source term on the right-hand side of (2.9) is the square of

the $\mathcal{R}^4$ coefficient, $\mathcal{E}^{(8)}_{(0,0)}$, which itself has a one-loop $\log y_8$, as exhibited in (3.4). There are plenty of further examples of logarithmic divergences in field theory in dimensions $D > D_L = 4 + 6/L$, but they are all associated with interactions $\partial^{2L}\mathcal{R}^4$ with $L > 3$.

• The supergravity limit of string theory, $\ell_s \to 0$ with ℓ_D fixed requires $y_D \to \infty$. In this limit the highest-genus perturbative term (the highest power of y_D) dominates the lower-genus contributions. However, an accumulation of an infinite number of unsuppressed instanton contributions dominates the amplitude. These are terms that are exponentially small in the string perturbation theory limit. The precise consequences of summing over such zero-action instanton contributions were deduced by explicitly expanding the coefficient functions in the $y_D \to \infty$ limit. In the cases considered here, where the torus dimension $d \leq 4$, the instantons correspond (in type IIB language) to wrapped (p, q)-string world-sheets and D-instantons in $D = 8$, as well as wrapped $D3$-brane world-sheets in the $D = 6$ case. One lesson to draw from this is that, as discussed in [19], supergravity cannot be decoupled from string theory[8].

As was emphasised in section 5, understanding the systematics of higher derivative terms is intimately related to understanding the order at which ultraviolet divergences of four-dimensional $N = 8$ supergravity first arise and the stringy origin of such divergences.

Acknowledgments

We are grateful to Stephen Miller for many insights concerning Eisenstein series and to Jonas Bjornsson, Nick Dorey, Sergio Ferrara, Francisco Morales and Augusto Sagnotti for useful comments. PV would like to thank the INFN laboratory at Frascati and the LPTA of Montpellier for hospitality when this work was being finalized. MBG is grateful for the support of a European Research Council Advanced Grant No. 247252. J.R. acknowledges support by MCYT Research Grant No. FPA 2007-66665 and Generalitat de Catalunya under project 2009SGR502.

A. The $\partial^6\mathcal{R}^4$ interaction in $D = 6$ dimensions.

Since the coefficient $\mathcal{E}^{(6)}_{(0,1)}$ was not discussed in [2] its properties will be discussed in this appendix. As explained in section 3, this coefficient satisfies the Poisson equation (2.9)

$$\Delta^{(6)}\mathcal{E}^{(6)}_{(0,1)} = -\left(\mathbf{E}^{SO(5,5)}_{[10000];\frac{3}{2}}\right)^2 + c\,, \tag{A.1}$$

where c is a numerical constant to be determined. We have used the fact that the coefficient of $\mathcal{R}^4$ is $\mathcal{E}^{(6)}_{(0,0)} = \mathbf{E}^{SO(5,5)}_{[10000];3/2}$, which was discussed in detail in [2].

We begin by discussing the perturbative expansion, which is associated with the parabolic subgroup P_{α_1}, with Levi component $GL(1) \times SO(4, 4)$. In expanding the source term in (A.1) in powers of y_6 we need the expansion (see (3.54) of [2]),

$$\int_{P_{\alpha_1}} \mathbf{E}^{SO(5,5)}_{[10000];\frac{3}{2}} = 2\zeta(3)y_6^{-\frac{3}{2}} + 2y_6^{-\frac{1}{2}}\,\mathbf{E}^{SO(4,4)}_{[1000];1}\,, \tag{A.2}$$

[8] For alternative ideas on this subject see [33].

where the notation indicates an integration over the instanton phases associated with the unipotent radical, N, associated with the maximal parabolic subgroup P_{α_1}, as defined in [2], thereby projecting onto the zero Fourier mode. The solution of (A.1) can be found in perturbation theory, by expanding the automorphic function $\mathcal{E}^{(6)}_{(0,1)}$ as a power series in y_6,

$$\mathcal{E}^{(6)}_{(0,1)}\Big|_{\rm pert.} = y_6^{-3}\sum_{k=0}^{2} y_6^k F_k^{SO(4,4)} + F_3^{SO(5,5)} \ , \tag{A.3}$$

where $F_k^{SO(4,4)}$ are perturbative genus $k=0,1,2$ contributions and

$$\Delta^{SO(5,5)}\, F_3^{SO(5,5)} = c \ . \tag{A.4}$$

We now use the decomposition of the Laplace operator (also discussed in [2]),

$$\Delta^{SO(5,5)} \to \Delta^{SO(4,4)} + 2(y_6\partial_{y_6})^2 + 8(y_6\partial_{y_6}) \ . \tag{A.5}$$

Substituting (A.2), (A.3) and (A.5) into (A.1), we find the following equations

$$\begin{aligned} &6F_0^{SO(4,4)} = 4\zeta(3)^2 \ , \\ &\big(\Delta^{SO(4,4)} - 8\big)F_1^{SO(4,4)} = -8\zeta(3)\mathbf{E}^{SO(4,4)}_{[1000];1} \ , \\ &\big(\Delta^{SO(4,4)} - 6\big)F_2^{SO(4,4)} = -4(\mathbf{E}^{SO(4,4)}_{[1000];1})^2 \ . \end{aligned} \tag{A.6}$$

which determine the coefficients of $F_k^{SO(4,4)}$. In particular, it follows immediately that the tree-level and one-loop coefficients are

$$\begin{aligned} F_0^{SO(4,4)} &= \frac{2\zeta(3)^2}{3} \ , \\ F_1^{SO(4,4)} &= \frac{2\zeta(3)}{3}\,\mathbf{E}^{SO(4,4)}_{[1000];1} \ . \end{aligned} \tag{A.7}$$

The genus-two function $F_2^{(SO(4,4)}$, satisfying the last equation in (A.7), is more complicated but its properties can be analysed following the same procedure as in [8], although we will not need its properties here.

We now turn to $F^{SO(5,5)}$, which, as we will see later, generates a logarithm that is related to the $1/\epsilon$ pole in $D=6$ three-loop supergravity. The most general solution of (A.4) is a particular solution plus a solution of the homogeneous equation, where the homogeneous solution is a linear combination of $SO(5,5)$ Eisenstein series. These satisfy Laplace equations with eigenvalues given by (2.14). The two series of relevance are $\mathbf{E}^{SO(5,5)}_{[00010];s}$, $\mathbf{E}^{SO(5,5)}_{[00001];s}$, which satisfy

$$\begin{aligned} \Delta^{SO(5,5)}\mathbf{E}^{SO(5,5)}_{[00001];s} &= \frac{5}{2}\,s(s-4)\ \mathbf{E}^{SO(5,5)}_{[00001];s} \ , \\ \Delta^{SO(5,5)}\mathbf{E}^{SO(5,5)}_{[00010];s} &= \frac{5}{2}\,s(s-4)\ \mathbf{E}^{SO(5,5)}_{[00010];s} \ . \end{aligned} \tag{A.8}$$

The other possible series, $\mathbf{E}^{SO(5,5)}_{[00100];s}$, $\mathbf{E}^{SO(5,5)}_{[01000];s}$ and $\mathbf{E}^{SO(5,5)}_{[10000];s}$, need not be considered because they do not have perturbative expansions that contain powers of y_6 that are consistent

with string perturbation theory. In order for (A.8) to have zero eigenvalues as required by (A.4), we set $s = 4$ (the choice $s = 0$ gives equivalent solutions). Each series has a pole in ϵ at $s = s + \epsilon$, which needs to be subtracted, leaving an automorphic function that satisfies the Poisson equation with a constant source. The Eisenstein series with the pole subtracted will be denoted by a hat in the conventional manner. We are thus led to the ansatz

$$\begin{aligned} F_3^{SO(5,5)} &= a_0 \lim_{\epsilon\to 0} \big(\mathbf{E}^{SO(5,5)}_{[00001];4+\epsilon} + \mathbf{E}^{SO(5,5)}_{[00010];4-\epsilon}\big) \\ &= a_0 \big(\hat{\mathbf{E}}^{SO(5,5)}_{[00001];4} + \hat{\mathbf{E}}^{SO(5,5)}_{[00010];4}\big) \ , \end{aligned} \tag{A.9}$$

where a_0 is a numerical constant discussed below.

We are now interested in the constant term of $\hat{F}^{SO(5,5)}$ on the parabolic subgroup P_{α_1}, corresponding to string perturbation theory. Expanding for small y_6 gives an expansion of the form

$$\int_{P_{\alpha_1}} \mathbf{E}^{SO(5,5)}_{[00010];s} = \pi^2 \, \frac{\zeta(2s-4)\Gamma(s-2)}{\zeta(2s)\Gamma(s)} \, y_6^{\frac{1}{2}(s-4)} \mathbf{E}^{SO(4,4)}_{[0010];s-1} + y_6^{-\frac{s}{2}} \mathbf{E}^{SO(4,4)}_{[0010];s} \ , \tag{A.10}$$

and the functional relation

$$\mathbf{E}^{SO(5,5)}_{[00010];s} = \pi^5 \, \frac{\Gamma(s-\frac{7}{2})\Gamma(s-\frac{5}{2})\zeta(2s-7)\zeta(2s-5)}{\Gamma(s-1)\Gamma(s)\zeta(2s)\zeta(2s-2)} \, \mathbf{E}^{SO(5,5)}_{[00001];4-s} \tag{A.11}$$

and we are interested in $s \to 4$. The first term is a genus three term which will contribute to the $\log y_6$ piece, whereas the second term is a genus one contribution that will not concern us in this discussion.

The triality symmetry of $SO(4,4)$ implies that the series $\mathbf{E}^{SO(4,4)}_{[1000]s}$, $\mathbf{E}^{SO(4,4)}_{[0010]s}$ and $\mathbf{E}^{SO(4,4)}_{[0001]s}$ all have eigenvalues equal to $2s(s-3)$. Therefore, for $s = 3$ these Eisenstein series solve a Laplace equation with zero eigenvalue. In this case, the Eisenstein series have poles, as can be seen, for example, from the expansion in (C.7) of [2] ,

$$\mathbf{E}^{SO(4,4)}_{[1000];3+\epsilon} = V_{(4)}^{\frac{3}{2}} \mathbf{E}^{SL(4)}_{[001];3} + \frac{15}{2\pi^2}\,\zeta(3)\big(\frac{\pi^2}{\epsilon} + \hat{\mathbf{E}}^{SL(4)}_{[100];2} - \frac{\pi^2}{4}\log V_{(4)}\big) + O(\epsilon) + n.p. \tag{A.12}$$

where we have also used the ϵ expansion of $\mathbf{E}^{SL(4)}_{[100];2+\epsilon}$ given in equation (B.12) of [2]. The series $\mathbf{E}^{SO(4,4)}_{[0010];3+\epsilon}$, $\mathbf{E}^{SO(4,4)}_{[0001];3+\epsilon}$ also have poles at $\epsilon \to 0$ with the same residue.

It is now straightforward to obtain the regularised series $\hat{F}^{SO(5,5)} = a_0(\hat{\mathbf{E}}^{SO(5,5)}_{[00001];4} + \hat{\mathbf{E}}^{SO(5,5)}_{[00010];4})$ from $\mathbf{E}^{SO(5,5)}_{[00001];4+\epsilon} + \mathbf{E}^{SO(5,5)}_{[00010];4-\epsilon}$, and hence, from $F^{SO(5,5)}$ defined by (A.9). Concentrating on the $\log y_6$ piece this gives

$$F_3^{SO(5,5)} \to \frac{525}{2\pi^2}\ a_0\,\zeta(3)\,\log y_6 + \cdots \tag{A.13}$$

Finally, the value of a_0 can be determined by the decompactification limit $r_3 \to \infty$, where we must recover the $D = 7$ genus-three automorphic functions. One must have (see (5.41) in [2])

$$F_3^{SO(5,5)} \to\ 2r_3^3\ \big(\mathbf{E}^{SL(4)}_{[100];3} + \mathbf{E}^{SL(4)}_{[001];3}\big) \ . \tag{A.14}$$

In this limit

$$\hat{\mathbf{E}}^{SO(4,4)}_{[0010];3} + \hat{\mathbf{E}}^{SO(4,4)}_{[0001];3} \to r_3^3 \left(\mathbf{E}^{SL(4)}_{[100];3} + \mathbf{E}^{SL(4)}_{[001];3}\right) + \cdots \tag{A.15}$$

which requires $a_0 = 6\pi^2/35$. Thus

$$F_3^{SO(5,5)} \to 15\zeta(3)\, \log y_6 + \cdots \tag{A.16}$$

This means, in particular, that

$$c = 8 \times 15\zeta(3) \tag{A.17}$$

B. Normalisations

This appendix gives a brief definition of the conventions used for the normalisations of the amplitudes.

The normalisations of the supergravity field theory amplitude calculations at from tree level to three loops are given by [25–27]

$$A_D^{sugra} = \mathcal{R}^4 \left(\frac{\kappa_{(D)}}{2}\right)^2 \left(\frac{64}{stu\,\ell_D^6} + \left(\frac{\kappa_{(D)}}{2}\right)^2 I_1 + \left(\frac{\kappa_{(D)}}{2}\right)^4 I_2 + \left(\frac{\kappa_{(D)}}{2}\right)^6 I_3 + \cdots\right) . \tag{B.1}$$

By convention, the Newton constant in dimension $D \leq 10$, , κ_D, is related to the Planck length, ℓ_D, by $2\kappa^2_{(D)} = (2\pi)^{D-3}\,\ell_D^{D-2}$.

For the purpose of comparing the field theory and string theory normalisations it is useful to recall the expansion of the tree-level amplitude string in ten dimensions,

$$\begin{aligned} A_{tree}^{string} &= -\frac{1}{y_{10}^2}\mathcal{R}^4\, \frac{\Gamma(-\frac{\ell_s^2 s}{4})\Gamma(-\frac{\ell_s^2 t}{4})\Gamma(-\frac{\ell_s^2 u}{4})}{\Gamma(1+\frac{\ell_s^2 s}{4})\Gamma(1+\frac{\ell_s^2 t}{4})\Gamma(1+\frac{\ell_s^2 u}{4})} \\ &= -\frac{1}{y_{10}^2}\mathcal{R}^4 \left(\frac{64}{stu\,\ell_s^6} + 2\zeta(3) + \zeta(5)\,\hat{\sigma}_2 + \frac{2\zeta(3)^2}{3}\,\hat{\sigma}_3 + \cdots\right) . \end{aligned} \tag{B.2}$$

where $\hat{\sigma}_n = (s^n + t^n + u^n)\,\ell_s^{2n}/4^n$.

References

[1] C. M. Hull and P. K. Townsend, "Unity of Superstring Dualities," Nucl. Phys. B **438** (1995) 109 [arXiv:hep-th/9410167].

[2] M. B. Green, J. G. Russo and P. Vanhove, "Automorphic properties of low energy string amplitudes in various dimensions", arXiv:1001.2535 [hep-th].

[3] M. B. Green and M. Gutperle, "Effects of D-instantons," Nucl. Phys. B **498**, 195 (1997) [arXiv:hep-th/9701093].

[4] M. B. Green and P. Vanhove, "D-Instantons, Strings and M-Theory," Phys. Lett. B **408** (1997) 122 [arXiv:hep-th/9704145].

[5] M. B. Green, M. Gutperle and P. Vanhove, "One loop in eleven dimensions," Phys. Lett. B **409**, 177 (1997) [arXiv:hep-th/9706175].

[6] M. B. Green, H. h. Kwon and P. Vanhove, "Two loops in eleven dimensions," Phys. Rev. D **61**, 104010 (2000) [arXiv:hep-th/9910055].

[7] M. B. Green and S. Sethi, "Supersymmetry constraints on type IIB supergravity," Phys. Rev. D **59** (1999) 046006 [arXiv:hep-th/9808061].

[8] M. B. Green and P. Vanhove, "Duality and higher derivative terms in M theory," JHEP **0601** (2006) 093 [arXiv:hep-th/0510027].

[9] M. B. Green, J. G. Russo and P. Vanhove, "Modular properties of two-loop maximal supergravity and connections with string theory," JHEP **0807** (2008) 126 [arXiv:0807.0389 [hep-th]].

[10] J. G. Russo and A. A. Tseytlin, "One-loop four-graviton amplitude in eleven-dimensional supergravity," Nucl. Phys. B **508** (1997) 245 [arXiv:hep-th/9707134].

[11] E. Kiritsis and B. Pioline, "On R^4 threshold corrections in type IIB string theory and (p,q) string instantons," Nucl. Phys. B **508**, 509 (1997) [arXiv:hep-th/9707018].

[12] B. Pioline and E. Kiritsis, "U-Duality and D-Brane Combinatorics," Phys. Lett. B **418** (1998) 61 [arXiv:hep-th/9710078].
N. A. Obers and B. Pioline, "Eisenstein series and string thresholds," Commun. Math. Phys. **209** (2000) 275 [arXiv:hep-th/9903113].

[13] N. D. Lambert and P. C. West, "Coset symmetries in dimensionally reduced bosonic string theory," Nucl. Phys. B **615** (2001) 117 [arXiv:hep-th/0107209].
N. Lambert and P. C. West, "Duality groups, automorphic forms and higher derivative corrections," Phys. Rev. D **75** (2007) 066002 [arXiv:hep-th/0611318].

[14] A. Basu, "The D^4 R^4 Term in Type IIB String Theory on T^2 and U-Duality," Phys. Rev. D **77** (2008) 106003 [arXiv:0708.2950 [hep-th]].

[15] A. Basu, "The D^6R^4 term in type IIB string theory on T^2 and U-duality," arXiv:0712.1252 [hep-th].

[16] B. Pioline, "R^4 Couplings and Automorphic Unipotent Representations," arXiv:1001.3647 [hep-th].

[17] N. Lambert and P. West, "Perturbation Theory from Automorphic Forms," arXiv:1001.3284 [hep-th].
F. Gubay, N. Lambert and P. West, "Constraints on Automorphic Forms of Higher Derivative Terms from Compactification," arXiv:1002.1068 [hep-th].

[18] Robert Langlands, "On the functional equations satisfied by Eisenstein series", L.N.M. 544, Springer, 1976

[19] M. B. Green, H. Ooguri and J. H. Schwarz, "Decoupling Supergravity from the Superstring," Phys. Rev. Lett. **99** (2007) 041601 [arXiv:0704.0777 [hep-th]].

[20] A. Sinha, "The G-hat**4 lambda**16 term in IIB supergravity," JHEP **0208**, 017 (2002) [arXiv:hep-th/0207070].

[21] A. Basu and S. Sethi, "Recursion Relations from Space-time Supersymmetry," JHEP **0809**, 081 (2008) [arXiv:0808.1250 [hep-th]].

[22] M. B. Green, J. G. Russo and P. Vanhove, "Non-renormalisation conditions in type II string theory and maximal supergravity," JHEP **0702** (2007) 099 [arXiv:hep-th/0610299].

[23] M. B. Green, S. D. Miller, J. G. Russo and P. Vanhove (in preparation).

[24] M. B. Green, J. H. Schwarz and L. Brink, "N=4 Yang-Mills And N=8 Supergravity As Limits Of String Theories," Nucl. Phys. B **198** (1982) 474.

[25] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, "On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences," Nucl. Phys. B **530** (1998) 401 [arXiv:hep-th/9802162].

[26] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of N=8 Supergravity," Phys. Rev. D **78** (2008) 105019 [arXiv:0808.4112 [hep-th]].

[27] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, "Three-Loop Superfiniteness of $\mathcal{N} = 8$ Supergravity," Phys. Rev. Lett. **98** (2007) 161303 [arXiv:hep-th/0702112].

[28] M. B. Green, J. G. Russo and P. Vanhove, "Low energy expansion of the four-particle genus-one amplitude in type II superstring theory," JHEP **0802** (2008) 020 [arXiv:0801.0322 [hep-th]].

[29] N. Berkovits, "New higher-derivative R**4 theorems," Phys. Rev. Lett. **98** (2007) 211601 [arXiv:hep-th/0609006].

[30] N. Berkovits, M. B. Green, J. G. Russo and P. Vanhove, "Non-renormalization conditions for four-gluon scattering in supersymmetric string and field theory," JHEP **0911** (2009) 063 [arXiv:0908.1923 [hep-th]].

[31] P. S. Howe and U. Lindstrom, "Higher Order Invariants In Extended Supergravity," Nucl. Phys. B **181** (1981) 487.

[32] M. B. Green, J. G. Russo and P. Vanhove, "Ultraviolet properties of maximal supergravity," Phys. Rev. Lett. **98** (2007) 131602 [arXiv:hep-th/0611273].

[33] M. Bianchi, S. Ferrara and R. Kallosh, "Perturbative and Non-Perturbative $\mathcal{N} = 8$ Supergravity," arXiv:0910.3674 [hep-th].
M. Bianchi, S. Ferrara and R. Kallosh, "Observations on Arithmetic Invariants and U-Duality Orbits in $\mathcal{N} = 8$ Supergravity," arXiv:0912.0057 [hep-th].