Étude for linear Dyson–Schwinger Equations

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March 28, 2006

Abstract

We discuss properties of linear Dyson–Schwinger equations.

1 Introduction

In this short note we want to focus on some elementary properties of linear Dyson–Schwinger equations. In particular we want to emphasize their solutions by Mellin transforms. This is a simple étude in non-perturbative quantum field theory which we present here as the precursor to a much more substantial discussion of nonperturbative quantum field theory which is upcoming.

Acknowledgments

It is my pleasure to thank Christoph Bergbauer, Spencer Bloch, David Broadhurst and Karen Yeats for many stimulating discussions in Bures and elsewhere during the last couple of months. They crucially helped to shape my view on Dyson–Schwinger equations.

I also want to express my regret for being unable to attend the workshop on *Traces in Geometry, Number Theory and Quantum Fields* and thank Matilde Marcolli and Sylvie Paycha for generously inviting me to contribute to these proceedings.

2 Solving a linear Dyson–Schwinger equation

The distinguishing feature of a linear Dyson–Schwinger equation is that it evaluates a group-like element in the corresponding cocommutative and commutative (sub-) Hopf algebra which describes its perturbation theory [7, 2].

2.1 A group-like element in H

Our starting point is the very simple Hopf algebra H with formal generators t_i, $i \in \mathbb{N}_{\geq 0}$, $t_0 = \mathbb{I}$, and coproduct

$$\Delta t_i = \sum_{j=0}^{i} t_j \otimes t_{i-j},\ j \geq 0. \tag{1}$$

We have the shift operator

$$B_+ : H \to H,\ B_+(t_i) = t_{i+1} = B_+^{i+1}(\mathbb{I}), \tag{2}$$

where we write B_+^j for the j-fold iteration of the shift and consider the formal series

$$X(\alpha) = \mathbb{I} + \alpha B_+ \left(X(\alpha) \right). \tag{3}$$

Hence,

$$X(\alpha) = \mathbb{I} + \sum_{j=1}^{\infty} \alpha^k B_+^j(\mathbb{I}). \tag{4}$$

The shift operator is a very simple Hochschild one-cocycle [2, 7]

$$\Delta B_+ = B_+ \otimes \mathbb{I} + \{\mathrm{id} \otimes B_+\}\Delta. \tag{5}$$

We have

Proposition 1 $\Delta(X(\alpha)) = X(\alpha) \otimes X(\alpha)$.

Feynman rules ϕ_L typically need some input parameter L say, (a logarithm say of an external momentum) and provide for a chosen such parameter a map

$$\phi_L : H \to \mathbb{C} \tag{6}$$

and we define them recursively: $\phi_L(\mathbb{I}) = 1$ and

$$\phi_L(B_+(h)) = \int d_+(k, q)\phi_x(h). \tag{7}$$

In this highly symbolic notation k refers to variables to be integrated out and q to parameters whose logarithm provides the parameter L above. The measure $d_+(k, q)$ is defined as

$$\phi_L(B_+(\mathbb{I})) = \int d_+(k, q). \tag{8}$$

Note that the map ϕ_L implies the existence of a map

$$\phi : \mathbb{G}_a \to \mathrm{Spec}\ H, \tag{9}$$

which assigns to an element L in the additive group an element ϕ_L in Spec H.

2.2 The linear Dyson–Schwinger equation

Next, the Green function $G(\alpha, q)$ is defined by its Dyson–Schwinger equation

$$\begin{aligned} G(\alpha, q) &:= \phi_L\left(X(\alpha)\right) = 1 + \alpha \int d_+(k,q) G(\alpha, x), \\ &= 1 + \alpha \int d_+(k,q) + \alpha^2 \iint d_+(k,q) d_+(k',k) + \cdots, \end{aligned} \tag{10}$$

which iterates the kernel d_+.

Typically, the measure d_+ is logarithmically divergent and so the above equation is just about ill-defined, and hence an object of interest.

Indeed, $\int d_+$ can often be regarded as an expression which can be naturally studied from the viewpoint of log-polyhomogenous symbols, an approach which we will not pursue any further at this moment.

A Dyson–Schwinger equation has a Birkhoff decomposition in the sense used in previous work with Alain Connes [4, 5] which is evident if we write the equation for the renormalized Green function

$$G_R(\alpha, q, \mu) = Z(\alpha, \mu) + \int d_+(k,q) G_R(\alpha, k, \mu) \tag{11}$$

where

$$Z(\alpha, \mu) = S_R^\phi(X(\alpha)), \tag{12}$$

and the μ dependence is implicit in the choice of a renormalization scheme R: here we subtract at $q^2 = \mu^2$, so R is an evaluation map and this choice fixes the boundary conditions in our Dyson–Schwinger equation: $G_R(\alpha, \mu, \mu) = 1$.

Note that $\int d_+ G_R(\alpha, x)$ gives the contribution of Bogoliubov's $\bar{R}$ operation on $X(\alpha)$.

2.3 An example

The reader not familiar with renormalization theory will be lost already, so let me try to illuminate these concepts in a very simple example. We hence turn to massless Yukawa theory where we start with the simple one-loop graph

$$\phi_{\ln q^2}(B_+(\mathbb{I})) = \int d^4k \frac{1}{k^2(k+q)^2}, \tag{13}$$

to be regarded as an elementary example for $\int d_+(k,q)$, generated by the convolution of two massless propagators, inverse of the Euclidean Klein-Gordon operator each.

We get the integral equation

$$G(\alpha, q^2) = 1 + \alpha \int d^4k \frac{G(\alpha, k^2)}{k^2(k+q)^2}. \tag{14}$$

This is just about ill-defined as it stands but reason is restored the moment we renormalize, say by a simple subtraction at $q^2 = \mu^2$:

$$\begin{aligned} G_R(\alpha, q^2, \mu^2) = 1 \quad &+ \quad \int d^4k \frac{G_R(\alpha, k^2, \mu^2)}{k^2(k+q)^2} \\ &- \quad \int d^4k \frac{G_R(\alpha, k^2, \mu^2)}{k^2(k+q)^2}|_{q^2=\mu^2}. \end{aligned} \tag{15}$$

We then have

$$Z(\alpha, \mu^2) = 1 - \int d^4k \frac{G_R(\alpha, k^2, \mu^2)}{k^2(k+q)^2}|_{q^2=\mu^2}. \tag{16}$$

2.4 The use of a Mellin transform

It is time to introduce the Mellin transform of the integral $\int d_+(k, q)$:

$$F(\rho) = \int [k^2]^{-\rho} d_+(k, 1), \tag{17}$$

where the notation indicated that we evaluate the external scale at unity (zero in the additive group). In our example, we have

$$F(\rho) = \int d^4k \frac{1}{[k^2]^{1+\rho}(k+q)^2}|_{q^2=1} = \frac{1}{\rho(1-\rho)}. \tag{18}$$

Before we use the Mellin transform to solve our equation to all orders in α, let us make contact with perturbation theory. For if

$$G_R(\alpha, \ln q^2, \ln \mu^2) = 1 + \sum_{j=1}^{\infty} \alpha^j c_j(\ln q^2, \ln \mu^2), \tag{19}$$

then

Theorem 2 $S_R^\phi \star \phi(t_j) = c_j(\ln q^2, \ln \mu^2)$.

The reader should work that out himself, using the above iterations of kernels for the renormalized kernel $d_+^R(k, q, \mu) = d_+(k, q) - d_+(k, \mu)$.

A similar result holds in total generality for a renormalizable quantum field theory and its Hopf algebra of graphs.

Actually, the sub Hopf algebra provided by the sum of all graphs at a given loop order is the simplest one for which such a theorem holds for the full Green function, a crucial fact in circumstances where the sum of all graphs of a given degree has features which are lacking in single graphs, as is the case in gauge theories [8].

More interesting here is that we can solve for $G(\alpha, q^2, \mu^2) = G_R(\alpha, L)$ in one stroke.

Theorem 3 $G_R(\alpha, q^2, \mu^2) = \left(\frac{q^2}{\mu^2}\right)^{-\gamma_1(\alpha)} = e^{-L\gamma_1(\alpha)}$, *where the series* $\gamma_1(\alpha)$ *is determined by the Mellin transform*

$$1 = \alpha F(\gamma_1(\alpha)). \tag{20}$$

This is readily confirmed by plugging the above scaling Ansatz into the linear Dyson–Schwinger equation, and indeed holds for any such equation which is linear and fulfills some very mild assumptions on the measure d_+, routinely available in a renormalizable quantum field theory.

Also, we remark that a non-linear Dyson–Schwinger equation, reduced to depend on one kinematical parameter L, has a solution which can be written as

$$G_R(\alpha, L) = e^{-\sum_{k=1}^{\infty} \gamma_k(\alpha) L^k}, \tag{21}$$

where

$$\gamma_k(\alpha) = \sum_{j \geq k} \gamma_{k,j} \alpha^j. \tag{22}$$

In our linear example we find

$$\gamma_1(\alpha) = \frac{1}{2}\left(1 - \sqrt{1 - 4\alpha}\right), \tag{23}$$

where the constraint $\gamma_1(0) = 0$ fixes the root in the quadratic equation for γ_1.

There is a simple relation between the Taylor coefficients of the Mellin transform

$$F(\rho) = \frac{r}{\rho} + \sum_{j=0}^{\infty} f_k \rho^k \tag{24}$$

(r, the residue of the graph $B_+(\mathbb{I})$, is an object of remarkable mathematical interest [1]) and the Taylor coefficients of the anomalous dimension

$$\gamma_1(\alpha) = \sum_{j=1}^{\infty} \alpha^j \gamma_{1,j}. \tag{25}$$

We have, by Thm. 3, $\gamma_{1,1} = r$ and for $n > 1$

Proposition 4

$$\gamma_{1,n} = \sum_{j=0}^{n-1} f_j \prod_{i=1}^{j+1} \sum_{n_1 + \cdots + n_i = n-1} \gamma_{1,n_i}.$$

Our example is very special indeed: for $F(\rho) = 1/[\rho(1-\rho)]$, so that $1 = r = f_0 = f_1 = \cdots$, we get Catalan numbers: $\gamma_1 = \gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 5, \ldots$ and so on. The invariance $\rho \to 1-\rho$ of the Mellin transform is universal and reflects the conformal invariance of the measure d_+ at unit scale $q^2 = 1$.

2.5 The log of a linear DSE is linear

Let us reflect for a moment on the structure of the above result. The existence of a scaling solution can be most easily interfered from the fact that our fix-point equation is group-like. Indeed, there is a hierarchy in the expansion of the Green function in terms of $\ln q^2/\mu^2$ [3] described by operators $\sigma_n := \phi \circ S \star Y^n$ for which a solution of a linear Dyson–Schwinger equation only sees the first term (the residue) as follows.

If we set $\ln G(\alpha, L) = \sum \gamma_k L^k$ as before, then $\gamma_k = 0$ for $k \geq 2$ in the linear case. Indeed, as $X(\alpha)$ is group-like and starts with $\mathbb{I}$, the formal logarithm $\ln X(\alpha)$ is primitive and annihilated when acted upon by $\sigma_1 \otimes \sigma_1$, and hence there are no higher powers of L in $\ln G_R$ by the scattering type formula [5].

2.6 A remark on dimensional regularization

What happens if we were to use a different renormalization scheme? In the above, we never introduced a regulator and found that this infinite sum of graphs regulates itself, by the anomalous dimension which the Green function develops.

If we were to use dimensional regularization instead, together with minimal subtraction, we actually loose the simple scaling Ansatz and our renormalized Dyson–Schwinger equation looks like

$$\begin{aligned} G_R(\alpha, q^2, \mu^2, z) = 1 \quad &+ \quad \int d^D k \frac{G_R(\alpha, k^2, \mu^2, z)}{k^2 (k+q)^2} \\ &- \quad \left\langle \int d^D k \frac{G_R(\alpha, k^2, \mu^2, z)}{k^2 (k+q)^2} |_{q^2=\mu^2} \right\rangle, \quad (26) \end{aligned}$$

where the limit $z \to 0$ exists but on the rhs can not be taken inside the integrands. This can not be solved in terms of a scaling solution directly, but needs further information which is provided by the analysis at the critical fiber $z = 0$.

Let us finish this short paper with a more upbeat observation concerning dimensional regularization.

Let us look in dimensional regularization at the measure d_+^z. For our example we have

$$\int d_+^z(k,q) = \int d^D k \frac{1}{k^2(k+q)^2} = \left(\frac{q^2}{\mu_{\mathrm{DR}}^2}\right)^{\frac{D-4}{2}} \frac{\Gamma(z)\Gamma^2(1-z)}{\Gamma(2-2z)}, \quad (27)$$

where $D = 4 - 2z$.

How can we obtain the result for the renormalized Green function $G_R(\alpha, L)$ if we use dimensional regularization instead of a Mellin transform, but maintain the same renormalization condition? Consider

$$G_R(\alpha, L) = \left(\frac{q^2}{\mu^2}\right)^{-\gamma_1(\alpha)} = 1 + \alpha \int \left\{ d_+^z(k,q) - d_+^z(k,\mu) \right\}. \quad (28)$$

This has a solution

$$z = \gamma_1(\alpha) \quad (29)$$

and

$$\mu_{\mathrm{DR}}^2 = \mu^2 A(\gamma_1), \quad (30)$$

where

$$A(\gamma_1) = -\frac{1}{\gamma_1} \ln\left(\frac{(1-\gamma_1)}{(1-2\gamma_1)} \frac{\Gamma(1-\gamma_1)^2 \Gamma(1+\gamma_1)}{\Gamma(1-2\gamma_1)} \right), \quad (31)$$

where $A(0)$ exists and this is universally true for any linear Dyson–Schwinger equation, as the Mellin transform and the integral $\int d_+^z$ have the same residue.

A rather interesting phenomenon appears: we note that upon setting

$$z = \gamma_1(\alpha) \quad (32)$$

an integration in the α-dependent dimension

$$D = D(\alpha) = 4 - 2\gamma_1(\alpha) \quad (33)$$

of d_+ reproduces the infinite sum of amplitudes given by the Dyson–Schwinger equation in four dimensions. This possibility to obtain the solution of a recursive problem in four dimension as a solution to a non-recursive integral in non-integer dimensions clearly deserves further thought, in particular in comparison with [6].

3 Some Remarks

Things one should remember is that linear Dyson–Schwinger equations can be solved by scaling Ansatz, that the shift operator which defines their underlying fix-point equation is a closed Hochschild one-cocycle, that the fix-point equation has a group-like solution and that the corresponding integral equation can be solved by a Mellin transform. Amazingly, the non-linear case has very similar properties to be exhibited in due course.

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