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Algebra for quantum fields

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Abstract. We give an account of the current state of the approch to quantum field theory via Hopf algebras and Hochschild cohomology. We emphasize the versatility and mathematical foundation of this algebraic structure, and collect algebraic structures here in one place which are either scattered over the literature, or only implicit in previous writings. In particular we point out mathematical structures which can be helpful to farther develop our mathematical understanding of quantum fields.

1. Free QFT, interacting QFT

Classical geometry does not rule the day when it comes to quantum fields. Much to the contrary, the often beautiful classical geometry of fields, gauge fields in particular, must emerge as a classical limit of quantum field theory. Hence we speak about QFT without taking recourse to classical fields. We ignore the geometry of the classical spacetime manifold over which we want to construct QFT, and just memorize that it has a four-dimensional tangential and cotangential space locally, isomorphic to flat Minkowski space. It is over such local fibres that one formulates QFT.

Our first concern is to understand the elementary amplitudes which we use to describe the observable physics which results from quantum field theory.

They come in two garden varieties: amplitudes for propagation, and amplitudes for scattering. The former are provided by free quantum field theory: free propagators, in momentum space, are obtained as the inverse of the free covariant wave equations. Hence, for Minkowski space, its Wigner's representation theory of the Lorentz and Poincaré groups which rules the day, providing us with free propagators for massless and massive bosons and fermions.

The latter, amplitudes for scattering, are again provided by representation theory of the Poincaré group, augmented by the requirement for locality.

Let us look at a simple example to see how this comes about. Assume we take from free quantum fields the covariances for a free propagating electron, positron and photon. Assume we want to couple those in a local interaction. Representation theory tells us that this interaction will have to transform as a Lorentz vector v_μ, coupling the spin-one photon to a spin-1/2 electron and positron. Also, knowing the

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scaling weights of free photons, electrons and positrons as determined from the accompanying free field monomials, such a vertex must have zero scaling weight itself, as the scaling weights of those monomials add up to the dimension of spacetime. Indeed,

$$[\bar{\psi}\partial\!\!\!/\psi] = 4 \quad \Rightarrow \quad [\bar{\psi}] = [\psi] = 3/2, \tag{1}$$

$$\frac{1}{4}[F^2] = 4 \quad \Rightarrow \quad [A] = 1, \tag{2}$$

$$[v_\mu \bar{\psi} A_\mu \psi] = 4 \Rightarrow [v_\mu] = 0. \tag{3}$$

So what would be the Feynman rule, in momentum space say, for such an amplitude? If the electron has momentum p_1, the positron momentum p_2, and the photon momentum $q = -p_1 - p_2$, the vertex can be a linear combination of twelve invariants

$$v_\mu = c_1 \gamma_\mu + c_2 \frac{q_\mu q\!\!\!/}{q^2} + \cdots . \tag{4}$$

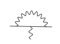
But if we have to have a local theory, any graph for a quantum correction for the unknown vertex built from that unknown vertex and the known propagators will be, by a simple powercounting exercise, -we know the scaling weight of our unknown vertex at least-, logaritmic divergent.

If we are to absorb this logarithmic divergence by a local counterterm, this gives us information on the desired Feynman rule. Let us work it out. To keep the example simple, let us assume we suspect that the vertex is of the form

$$v_\mu = v_\mu(q) = c_1 \gamma_\mu + c_2 \frac{q_\mu q\!\!\!/}{q^2}. \tag{5}$$

Let us consider the one-loop 1PI graph -the lowest order quantum correction- to find the sought after Feynman rule.

With three vertices in the graph we have $2^3 = 8$ integrals to do which appear in the limit

$\lim_{\Lambda\to+\infty} \frac{1}{\ln\Lambda}\Phi\big($  $\big)$

$$\sim \lim_{\Lambda\to+\infty} \frac{1}{\ln\Lambda} \int_{-\Lambda}^{\Lambda} v_\alpha(k) \frac{1}{k\!\!\!/} v_\mu(k) \frac{1}{k\!\!\!/} v_\beta(k) D_{\alpha\beta}((k+q)^2) d^4k = f(c_1, c_2)\gamma_\mu.$$

In a local theory, the coefficient of $\ln\Lambda$ from the integral must be proportional to the desired vertex. Hence, dividing and taking the limit, we confirm that the term $\sim q_\mu q\!\!\!/$ vanishes like $1/\ln\Lambda$ in all eight terms. We hence conclude $c_2 = 0$, and this gives a good idea how locality is needed for quantum field theory to stabilize at dintinguished Feynman rules in a self-similar manner. Similarly, if we had done the example with the full 12^3 terms of the full vertex, as it must be for a renormalizable theory.

We also conclude that the price for Feynman rules determined by locality is that we indeed pick up local short-distance singularities. That leaves us the freedom to set a scale, which is no big surprise: looking only at quantum fields for a typical fibre -the cotangential space-, we hence miss the only parameter around to set a scale: the curvature of the underlying manifold. The extension of such local notions to the whole manifold awaits understanding of quantum gravity. This might well start from understanding how gravity with its peculiar powercounting behaves as a Hopf algebra [**15**].

Let us now proceed to see what comes with those edges and vertices as prescribed above - graphs, obviously.

2. 1PI graphs, Hopf algebras

Having hence elementary scattering and propagation amplitudes available, we can set up a quantum theory: we define incoming and outgoing asymptotic states, and sum over all unobserved intermediate states. This is standard material for a physicist, and we leave it to the reader to acquaint himself with the necessary details on the LSZ formalism and other such aspects [**34, 7, 6, 14**].

While many textbooks on contemporary physics proceed using the path integral to define Green functions for amplitudes, for connected amplitudes and for 1PI amplitudes, we emphasize that these Green functions can be given mathematical precise meaning through the study of the Hopf algebra structure underlying the graphs constructed from the representation theory mentioned above.[1]

So having Feynman rules for edges and vertices the above gives us Feynman rules for n-PI graphs, graphs which do not disconnect upon removal of any n internal edges. Amplitudes for connected graphs are obtained from 1-PI Green functions by connecting them via free covariances, and disconnected graphs finally by exponentiation. Its for 1-PI graphs that the underlying algebraic structure of field theory becomes fully visible.

The basic such algebraic structure then at our disposal are:
i) the Hopf- and Lie algebras coming with such graphs,[**29, 9, 10, 11**]
ii) the correponding Hochschild cohomology and the sub-Hopf algebras generated by the grading,[**17, 13**]
iii) the co-ideals corresponding to symmetries in the Lagrangian,[**22, 32**]
iv) the coradical filtration and Dynkin operators governing the renormalization group and leading log expansion,[**8, 20, 31**]
v) the semi-direct product structure between superficially convergent and divergent graphs, [**10**]
vi) and finally the core Hopf algebra [**5, 16**], suggesting co-ideals leading to recursions à la BCFW, showing that loops and legs speak to each other in many ways: it is indeed the hope of the author that the rather disparate structures we observe in experience with multi-loop vs multi-leg expansions combine finally in a common mathematical framework [**18**].

We omitted in this list Rota–Baxter algebras [**12**], which are useful for MS schemes but less so in renormalization schemes based on on-shell or momentum space subtractions. The reader can find detailed study of Rota–Baxter algebras in the above-cited work of Ebrahimi-Fard and Manchon, while the use of momentum space subtractions was exhibited recently beyond perturbation theory in [**21**]. We also omit the algebraic structure of field theory in coordinate space, see [**2, 4, 3**] for a clarification how to connect it with the approach described here.

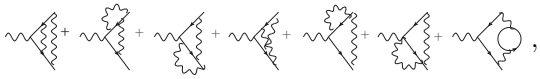
In this contribution, we will mainly review combinatorial and algebraic aspects developed in recent years. We include a few results only implicit in published work

[1]This might implicitly define the path integral, which has to be seen in future work. Too often in the authors opinion, the path integral is in the context of quantum field theory only a reparametrization of our lack of understanding, giving undue prominence to the classical Lagrangian.

so far. A summary of analytic and algebro-goemetric achievements has to be given elsewhere.

Let us now illustrate these algebraic structures. For that we strengthen our muscle on quantum electrodynamics (QED) graphs for the vertex, fermion- and photon-selfenergy, up to two loops each. Here they are:

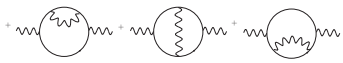
(6) $c_1^{\bar\psi \not{A} \psi} =$  ,

(7) $c_2^{\bar\psi \not{A} \psi} =$  ,

(8) $c_1^{\bar\psi \psi} =$  ,

(9) $c_2^{\bar\psi \psi} =$  ,

(10) $c_1^{\frac{1}{4}F^2} =$  ,

(11) $c_2^{\frac{1}{4}F^2} =$  .

2.1. The Hopf algebra. We define a family of Hopf algebras $\mathcal{H}$. Each Hopf algebra $H \in \mathcal{H}$ is generated by generators given by 1-PI graphs and its algebra structure is given as the free commutative $\mathbb{Q}$-algebra over those generators, with the empty graph furnishing the unit $\mathbb{I}$.

For a graph Γ, we let $\Gamma^{[0]}$ be the set of its vertices, $\Gamma^{[1]}_{\rm int}$ the set of its internal edges, and $\Gamma^{[1]}_{\rm ext}$ be the set of its external edges. Each edge is assigned an arbitrary orientation (all physics is independent of that choice), so that we can speak of a source $s(e)$ and target $t(e)$ for an edge e. For each internal edge $e \in \Gamma^{[1]}_{\rm int}$, $s(e) \in \Gamma^{[0]}$ and $t(e) \in \Gamma^{[0]}$. We do not require that $s(e) \neq t(e)$. For each $e \in \Gamma^{[1]}_{\rm ext}$, $t(e) \in \Gamma^{[0]}$ but $s(e) \notin \Gamma^{[0]}$.

To each internal edge e we assign a weight $w(e)$, to each vertex v we assign similarly a weight $w(v)$. We wite $\sum_{w\in\Gamma} w$ for the sum over all these edge and vertex weights. Then, we define

$$\omega_{2n}(\Gamma) := -2n|\Gamma| + \sum_{w\in\Gamma} w. \tag{12}$$

Also, a grading $|\Gamma|$ is provided by the number of independent cycles in a graph Γ, -its lowest Betti number-, and we hence write

$$H = \underbrace{H^0}_{\mathbb{Q}\mathbb{I}} \oplus \underbrace{\left(\oplus_{j=1}^{\infty} H^j\right)}_{{\rm Aug}(H)}. \tag{13}$$

So H is reduced to scalars off the augmentation ideal ${\rm Aug}(H)$. We let $\langle\Gamma\rangle$ be the linear span of generators.

We distinguish these Hopf algebras $H = H_{2n}$ by an even integer $0 \leq 2n$, $n \in \mathbb{N}$. They are all based on the same set of generators, hence have an identical algebra structure. There are slight differences in their coalgebra structure though, as we give them a coproduct depending on $2n$:

$$\Delta_{2n}(\Gamma) := \Gamma\otimes\mathbb{I} + \mathbb{I}\otimes\Gamma + \sum_{\gamma=\prod_i \gamma_i \subset \Gamma, \omega_{2n}(\gamma_i)\leq 0} \gamma\otimes\Gamma/\gamma. \tag{14}$$

The sum is over all disjoint unions of 1-PI subgraphs γ_i such that for each γ_i, $\omega_{2n}(\gamma_i) \leq 0$. In the limit $n \to \infty$, we hence obtain the core Hopf algebra H_{core} with coproduct

$$\Delta_{\text{core}}(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma = \prod_i \gamma_i \subset \Gamma} \gamma \otimes \Gamma/\gamma. \tag{15}$$

We also use the reduced coproducts

$$\Delta'_{2n}(\Gamma) := \sum_{\gamma = \prod_i \gamma_i \subset \Gamma, \omega_{2n}(\gamma_i) \leq 0} \gamma \otimes \Gamma/\gamma. \tag{16}$$

This gives us a tower of quotient Hopf algebras [**5**]

$$H_0 \subset H_2 \subset H_4 \subset \cdots \subset H_{2n} \cdots \subset H_{\text{core}}. \tag{17}$$

In the following, we often omit the subscript $_{2n}$ as it is either clear which integer we speak about, or the statement holds for arbitrary $2n$ in an obvious manner.

Note that H_0 is the trivial Hopf algebra in which every graph is primitive. It is the free commutative and cocommutative bialgebra of polynomials in all its generators $\in \langle\Gamma\rangle$. Fittingly, its use in zero-dimensional field theory is an excellent tool to count graphs [**1**].

For any other such $H_{2n} \in \mathcal{H}$, $n < \infty$, we find that the Hopf algebra decomposes into a semi-direct product

$$H_{2n} = H_{2n}^{\text{ren}} \times H_{2n}^{\text{ab}}, \tag{18}$$

where H_{2n}^{ren} is generated by graphs Γ such that $\omega_{2n}(\Gamma) \leq 0$ and H_{2n}^{ab} is the abelian factor generated by graphs such that $\omega_{2n}(\Gamma) > 0$. See [**10**].

Let us explain the above tower a bit more. The core Hopf algebra allows to shrink any 1-PI subgraph γ_i to a point, and hence is built on graphs with internal vertices of arbitrary valence, coupling an arbitrary numbers of edges and all types of edges for which we had free covariances. Again, locality and representation theory provide for such vertices Feynman rules as before, which are in general a sum over all local operators which are in accordance with the quantum numbers of those covariances. We can distinguish those operators by labeled vertices, which does not hinder us to set up the Hopf algebra as before. For the core Hopf algebra, all primitives which we find in the linear span $\langle\Gamma\rangle$ have degree one,

$$\Delta'_{\text{core}}(\Gamma) \neq 0 \Rightarrow |\Gamma| > 1 \tag{19}$$

Note that for any chosen finite $2n$, the results can be very different. A renormalizable theory is distinguished by the fact that for some finite n_0,

$$\omega_{2n_0}(\Gamma) = \omega_{2n_0}(\Gamma'),\ \forall\, \Gamma, \Gamma' \text{ with } \text{res}(\Gamma) = \text{res}(\Gamma'). \tag{20}$$

Here, res(Γ) is the map which assigns to a graph Γ the vertex obtained by shrinking all internal edges to zero length. What remains is the external edges connected to the same point. If the number of external edghes was greater than two, this gives us a vertex. If it was two, we identify those two connected edges to a single edge.

If such a n_0 exists, we call $2n_0$ the critical dimension of the theory. Particle physics so far is concerned with theories critical at $n_0 = 2$, ie. in four dimensions of spacetime.

In such a case, all graphs with the same type of external edges evaluate to the same result under evaluation by ω_{2n_0}. $\omega_{2n_0}(\Gamma)$ then takes values $\in \{-r_0, \cdots, +\infty\}$,

where $-r_0$ is the value achieved for vacuum graphs, and we obtain arbitrary positive values on considering graphs with a sufficient number of external edges.

For $n > n_0$, for any configuration of external edges we find, at sufficiently high degree $|\Gamma|$, graphs such that $\omega_{2n}(\Gamma) \leq 0$. The theory becomes non-renormalizable.

If $n < n_0$, only a finite number of graphs fulfils $\omega_{2n}(\Gamma) \leq 0$ and the theory is super-renormalizable.

In any case, for a Hopf algebra H_{2n}, continuing our appeal to self-similarity, we consider graphs made out of vertices such that $\omega_{2n}(\Gamma) \leq 0$. This defines a Hopf algebra $H_{2n}^{\rm ren}$. Graphs made out of such vertices but with sufficiently many external edges such that $\omega_{2n}(\Gamma) > 0$ then provide a semi-direct product $H_{2n} = H_{2n}^{\rm ren} \times H_{2n}^{\rm ab}$. This Hopf algebra is a quotient of the core Hopf algebra, eliminating any graph with undesired vertices.

So already at this elementary level, there is a nice interplay between the beforementioned representation theory of the Lorentz and Poincaré groups and such towers of Hopf algebras, as it is this representation theory which determines the covariances and their possible local vertices, and hence the quotient algebras we get.

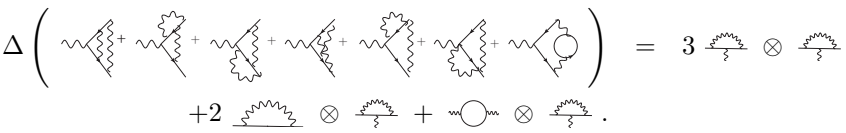
Let us now continue to list the other structural maps of those Hopf algebras. An antipode:

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subset \Gamma} S(\gamma) \otimes \Gamma/\gamma. \tag{21}$$

A counit $\bar{e} : H \to \mathbb{Q}$ and unit $E : \mathbb{Q} \to H^0 \subset H$:

$$\bar{e}(q\mathbb{I}) = q, \bar{e}(X) = 0, X \in {\rm Aug}(H),\ E(q) = q\mathbb{I}. \tag{22}$$

Finally, an example:



As a final remark, note that there are many more quotient Hopf algebras, by restricting generators to planar, or parquet, or whatever graphs. Also, we will find all the Hopf algebras needed for an operator product expansion as quotient Hopf algebras, using that for monomials (in operator-valued fields and their derivatives) $\Phi_1(x), \Phi_2(y)$, the expansion of vacuum expectation values (vev's) of products of two monomials at different spacetime points x, y into localized field monomials

$$\langle \Phi_1(x)\Phi_2(y)\rangle = \sum_{\mathcal{O}} f_{\mathcal{O}}(\frac{(x-y)}{2})\langle \mathcal{O}\rangle, \tag{23}$$

proceeds on a set of graphs having as local vertices the tree-level vev's of the operators $\mathcal{O}$, again in accordance with Wigner's representation theory. Note that all such vertices appear naturally in the core Hopf algebra (as we have quotients Γ/γ), and hence the core Hopf algebra is the endpoint in this tower of Hopf algebras which allows to formulate a full field algebra in the sense of operator product expansions.

2.2. The Lie algebra L such that $U(L)^* = H$. As a graded commutative Hopf algebra ([**24**]), any $H \in \mathcal{H}$ can be regarded as the dual $U^*(L)$ of the universal enveloping algebra $U(L)$ of a Lie algebra L. The tower $\mathcal{H}$ of quotient Hopf algebras

H_{2n} corresponds to a tower $\mathcal{L}$ of sub-Lie algebras L_{2n}. We write for each $L \in \mathcal{L}$,

$$U(L) = Q\mathbb{I} \oplus L \oplus_{k=2}^{\infty} L^{\hat{\otimes}^k}, \tag{24}$$

where $L^{\hat{\otimes}^k}$ indicates the symmetrized k-fold tensor product of L as usual for an universal enveloping algebra, obtained by dividing the tensor algebra $L^{\otimes^n}$ by the ideal $l_1 \otimes l_2 - l_2 \otimes l_1 - [l_1, l_2] = 0$.

We manifest the duality by a pairing between generators of L and generators of H,

$$(Z_\gamma, \Gamma) = \delta_{\gamma,\Gamma}, \tag{25}$$

the Kronecker pairing. It extends to $U(L)$ thanks to the coproduct.

There is an underlying pre-Lie algebra structure:

$$[Z_{\Gamma_1}, Z_{\Gamma_2}] = Z_{\Gamma_1} \otimes Z_{\Gamma_2} - Z_{\Gamma_2} \otimes Z_{\Gamma_1} \tag{26}$$

with

$$[Z_{\Gamma_1}, Z_{\Gamma_2}] = Z_{\Gamma_2 \star \Gamma_1 - \Gamma_1 \star \Gamma_2}. \tag{27}$$

Here, $\Gamma_i \star \Gamma_j$ sums over all ways of gluing Γ_j into Γ_i, which can be written as

$$\Gamma_i \star \Gamma_j = \sum_{\Gamma} n(\Gamma_i, \Gamma_j, \Gamma)\Gamma. \tag{28}$$

For any $\Gamma \in H$, we have

$$([Z_{\Gamma_1}, Z_{\Gamma_2}], \Gamma) = (Z_{\Gamma_1} \otimes Z_{\Gamma_2} - Z_{\Gamma_2} \otimes Z_{\Gamma_1}, \Delta(\Gamma)), \tag{29}$$

for consistency.

With such section coefficients $n(\Gamma_i, \Gamma_j, \Gamma)$ we then have

$$\Delta(\Gamma) = \sum_{h,g} n(h, g, \Gamma)\Gamma. \tag{30}$$

The (necessarily finite, as Δ respects the grading) sum is over all graphs h including the empty graph and all monomials in graphs g.

Note that we can regard a graph Γ obtained by inserting Γ_j into Γ_i as an extension

$$0 \to \Gamma_j \to \Gamma \to \Gamma_i \to 0. \tag{31}$$

A proper mathematics discussion of this idea has been given recently by Kobi Kremnizer and Matt Szczesny [**23**].

2.3. Hochschild cohomology. The Hochschild cohomology is encaptured by non-trivial one-cocycles $B_+^\gamma : H \to \mathrm{Aug}(H)$. The one cocycle condition (see [**17**]) means

$$bB_+^\gamma = 0 \Leftrightarrow \Delta B_+^\gamma(X) = B_+^\gamma(X) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^\gamma)\Delta(X). \tag{32}$$

We define $\forall \gamma \in \langle\Gamma\rangle$, such that $\Delta'(\gamma) = 0$, linear maps

$$B_+^\gamma(X) := \sum_{\Gamma \in \langle\Gamma\rangle} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{|X|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{(\gamma|X)}\Gamma. \tag{33}$$

Here, the sum is over the linear span $\langle\Gamma\rangle$ of generators of H. Furthermore,
i) $\mathrm{maxf}(\Gamma)$ is the number of maximal forests of Γ defined as the integer

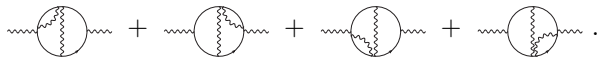
$$\mathrm{maxf}(\Gamma) = \sum_{p,\gamma \in \langle\Gamma\rangle, \Delta'(p)=0} (Z_\gamma, \Gamma')(Z_p, \Gamma''), \tag{34}$$

ii) $|X|_\vee$ is the number of distinct graphs obtained by permuting external edges of a graph,
iii) $\mathbf{bij}(\gamma, X, \Gamma)$ is the number of bijections between the external edges of X and half-edges of γ such that Γ results,
iv) and finally $(\gamma|X)$ is the number of insertion places for X in γ.
Finally, for any r which can appear as a residue $\mathbf{res}(\Gamma)$, we define

$$B_+^{r;k} = \sum_{\mathbf{res}(\gamma)=r, |\gamma|=k} \frac{1}{\mathrm{Aut}(\gamma)} B_+^\gamma, \tag{35}$$

which sums over all B_+^γ with a specified external leg structure and loop number, weighted by the rank $\mathrm{Aut}(\gamma)$ of their automorphism group.

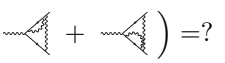
We want to understand these notions. We will do so by going through an example (see [**22**] for a more thorough exploration):

(36) $\Gamma =$  .

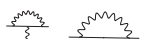
We will investigate

(37) B_+  $=?$

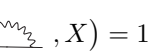
and

(38) B_+  $=?$

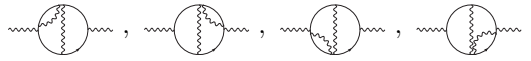
Let us start with (37). We have

(39)  $= 1.$

As fermion lines are oriented and hence all external edges distinguished, we can not permute external edges and obtain a different graph contributing to the same amplitude. Now let us count the bijections.

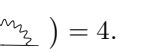
(40) $\mathbf{bij}\big($  $, X\big) = 1,$

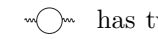
for all

$X \in \Big\{$  $\Big\}$.

Indeed, to glue the argument X of $B_+^\gamma(X)$ into γ, we identify the factors $X = \prod_i \gamma_i$. The multiset $\mathrm{res}(\gamma_i)$ identifies a number of edges and vertices. From the internal edges and vertices of γ we choose a corresponding set m which contains the same type and number of internal edges and vertices.

We then consider the external edges of elements γ_i of X and count bijections between this set and the similar set defined from m. Summing over all choices of m and counting all bijections at a given m such that Γ is obtained gives **bij** by definition. In the example, there is just a unique such bijection for each of the four different graphs X.

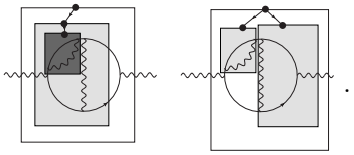
(41) $\big($  $\big) = 4.$

This counts the number of insertion places.  has two internal vertices and two internal edges, hence four possible choices of an insertion place.

Next, the maximal forests: we count the number of different subsets γ of 1PI subgraphs such that Γ/γ is a primitive element, $\Delta'(\Gamma/\gamma) = 0$.

(42) $$\mathrm{maxf}\,(X) = \mathrm{maxf}\left(\right) = 2,$$

for any of the four graphs X as above. For each of the four graphs there are two such possibilities. We indicate them in a way which makes the underlying tree structure ([**29, 9**]) obvious:

(43) 

This is one major asset of systematically building graphs from images of Hochschild closed one-cocycles: it resolves for us overlapping divergences into rooted trees.

Let us now collect:

(44) $$B_+^{\bigcirc}\ (\ \ ) =$$

(45) $$= \frac{1}{8}\left(\quad + \quad + \quad + \quad \right).$$

The reader will notice that this fails to satisfy the desired cocycle property. To understand the reason for this failure and the solution to this problem, we turn to (38). We have

(46) $$\Big| \Big|_{\vee} = \Big| \Big|_{\vee} = 1,$$

as before.

(47) $$\mathbf{bij}\left(\ ,\ , X\right) \ = \ 1,$$

(48) $$\mathbf{bij}\left(\ ,\ , X\right) \ = \ 1,$$

where X can still be any of the four graphs defined above.

Next,

(49) $$\left(\ | \ \right) = 2 = \left(\ | \ \right).$$

There are now two insertion places for the vertex graph to be inserted into the one-loop photon self-energy graph.

The maximal forests remain unchanged as we are generating the same graphs X in the two examples. Hence

(50) $$B_+^{\bigcirc}\left(\ + \ \right) =$$

(51) $$= \frac{1}{4}\left(\quad + \quad + \quad + \quad \right).$$

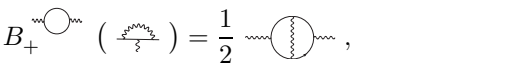
Again, this fails to satisfy the cocycle property. But let us now consider

(52) $$B_+^{\bigcirc}\left(4 \ \ + 2\left(\ + \ \right)\right)$$

(53) $$= \left(\quad + \quad + \quad + \quad \right).$$

We see that with these weights we do fulfill the cocycle condition. For this, it is actually sufficient that the ratio of the weights is two-to-one. Taking those weights to be four and two gives the result with the proper weights needed in the perturbative expansion of the photon propagator. It was a major result of [**22**] that these weights always work out in field theory such that we do have the desired perturbative expansion and cocycle properties. So while the maps B_+^γ are one - cocycles for Hopf algebras generated by dedicated subsets of graphs, one finds that the maps $B_+^{r;k}$ are proper cocycles for a Hopf algebra generated by sums of graphs with given external leg structure and loop number.

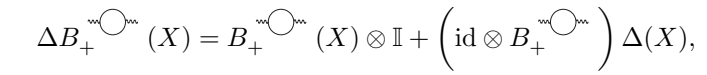
So working out

(54) 

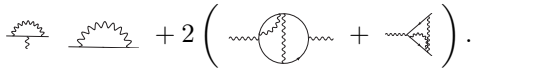
and

(55) 

we indeed confirm

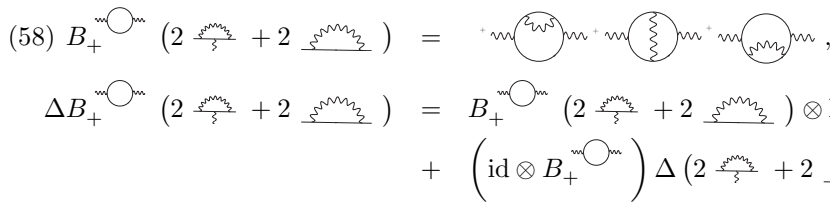
(56) 

for

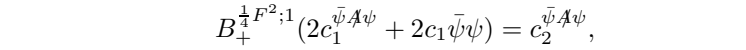
(57) 

We will understand soon how the weights 4 and 2 in (53) come about.

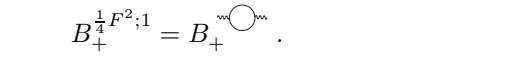
As a final exercise the reader might finally wish to confirm

(58) 

To put it shortly:

(59) 

where we indicated the residue **res**( ) of the one-loop primitive graph by its corresponding monomial $\frac{1}{4}F^2$ in the Lagrangian of QED, and there is indeed only one primitive at first loop order,

(60) 

2.4. Sub-Hopf algebras. In the example above, we looked at the sum of all 1-PI graphs contributing to a choosen amplitude r at a given loop order k . This gives us Hopf algebra elements $c_k^r \in H^k$ as particular linear combinations of degree-homogenous elements. Such Hopf algebra elements generate a sub-Hopf algebra.

For example in QED we have

$$\Delta'(c_k^{\bar\psi \not{A} \psi}) = \sum_{j=1}^{k-1} \Big[(2(k-j)+1)c_j^{\bar\psi \not{A} \psi} + 2(k-j)c_j^{\bar\psi\psi} + (k-j)c_j^{\frac{1}{4}F^2}\Big] \otimes c_{k-j}^{\bar\psi \not{A} \psi} + \text{terms non} - \text{linear on the lhs} \tag{61}$$

$$\Delta'(c_k^{\bar\psi\psi}) = \sum_{j=1}^{k-1} \Big[(2(k-j))c_j^{\bar\psi \not{A} \psi} + (2(k-j)-1)c_j^{\bar\psi\psi} + (k-j)c_j^{\frac{1}{4}F^2}\Big] \otimes c_{k-j}^{\bar\psi\psi} + \text{terms non} - \text{linear on the lhs} \tag{62}$$

$$\Delta'(c_k^{\frac{1}{4}F^2}) = \sum_{j=1}^{k-1} \Big[(2(k-j))c_j^{\bar\psi \not{A} \psi} + (2(k-j)-1)c_j^{\bar\psi\psi} + (k-j)c_j^{\frac{1}{4}F^2}\Big] \otimes c_{k-j}^{\frac{1}{4}F^2} + \text{terms non} - \text{linear on the lhs.} \tag{63}$$

We omit to give explicit expressions for the terms non-linear on the lhs of the co-product. They are not really needed, as we will soon see when we study the Dynkin operator. Similar to these sub-Hopf algebras, one can determine the corresponding quotient Lie algebras.

2.5. Co-ideals. Often, sub-Hopf algebras like above only emerge when divide by suitable co-ideals. An immediate application is a derivation of Ward–Takahashi and Slavnov–Taylor identities in this context [**22, 32**]. Lifting the idea of capturing relations between Green functions to the core Hopf algebra leads to the celebrated BCFW recursion relations [**18**]. All this needs much further work. The upshot is that dividing by a suitable co-ideal I, Feynman rules $\Phi : H \to \mathbb{C}$ can be well-formulated as maps

$$\Phi : H/I \to \mathbb{C}. \tag{64}$$

Let us consider as an example (following van Suijlekom [**33**]) the ideal and co-ideal I in QED given by

$$i_k := c_k^{\bar\psi \not{A} \psi} + c_k^{\bar\psi\psi} = 0, \forall k > 0. \tag{65}$$

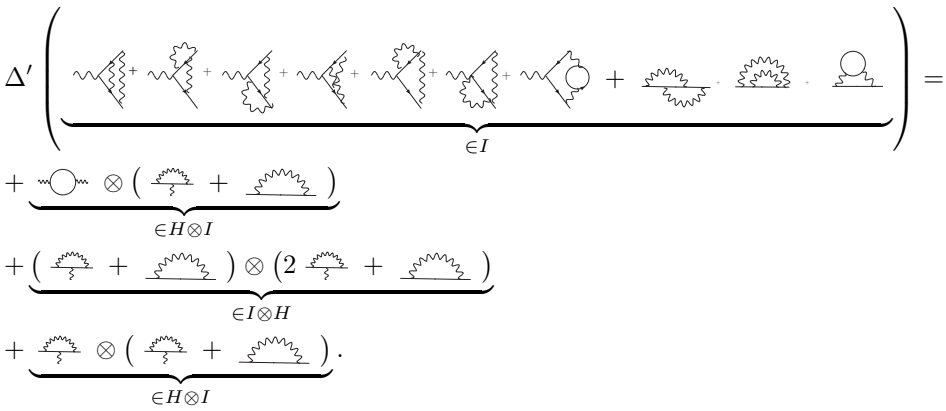
So, for example,

$$i_1 = \quad + \quad . \tag{66}$$

For I to be a co-ideal we need

$$\Delta(I) \subset (H \otimes I) \oplus (I \otimes H). \tag{67}$$

Let us look at $\Delta(i_2)$ for an example:



For a thorough discussion of the role of co-ideals and their interplay with Hochschild cohomology in renormalization and core Hopf algebras, see [**18**] and references there.

2.6. Co-radical filtration and the Dynkin operator. For our graded commutative Hopf algebras H there is a co-radical filtration. We consider iterations $[\Delta']^k : H \to \mathrm{Aug}(H)^{\otimes(k+1)}$ of the map $\Delta' : H \to \mathrm{Aug}(H) \otimes \mathrm{Aug}(H)$, and filter Hopf algebra elements by the smallest integer k such that they lie in the kernel of such a map. We can write the Hopf algebra as a direct sum over the corresponding graded spaces $H^{[j]}$,

$$H = \oplus_{j=0}^{\infty} H^{[j]}. \tag{68}$$

Elements $q\mathbb{I}$ are in $H^{[0]}$, primitive elements are in $H^{[1]}$, and so on.

A Hochschild one-cocycle is now a map

$$B_+^\gamma : H^{[j]} \to H^{[j+1]}. \tag{69}$$

Note that for example in $H^{[2]}$,

$$B_+^x(\mathbb{I})B_+^y(\mathbb{I}) = B_+^x \circ B_+^y(\mathbb{I}) + B_+^y \circ B_+^x(\mathbb{I}), \tag{70}$$

with the difference between the lhs and the rhs being an element in $H^{[1]}$.

In [**5**] this was used to reduce the study of renormalization theory to the study of flags of subdivergent sectors. This is closely connected to the Dynkin operator [**8, 20, 31**]

$$D : H \to \langle \Gamma \rangle,\ D := S \star Y = m(S \otimes Y)\Delta. \tag{71}$$

Here, $Y(\Gamma) = |\Gamma|\Gamma$ for all homogenous elements, extended by linearity.

Indeed, the above difference can be calculated as

$$D(B_+^x \circ B_+^y(\mathbb{I}) + B_+^y \circ B_+^x(\mathbb{I})) = B_+^x \circ B_+^y(\mathbb{I}) + B_+^y \circ B_+^x(\mathbb{I}) - B_+^x(\mathbb{I})B_+^y(\mathbb{I}). \tag{72}$$

In physics, this leads to the next-to-leading log expansion, see [**8**], upon recognising that the Feynman rules send elements in H to polynomials in suitable variables $L = \ln q^2/\mu^2$ say such that elements in $H^{[k]}$ are mapped to the terms $\sim L^k$.

There is an interesting remark to be made concerning the fact that the Dynkin operator vanishes on products. This allows for all things concerning renormalization

(including for example the derivation of the renormalization group [**11**]) to rely on a linearized coproduct

$$\Delta_{\rm lin} := (P_{\rm lin} \otimes {\rm id})\Delta : H \to H \otimes H, \tag{73}$$

with $P_{\rm lin} : H \to \langle \Gamma \rangle$ the projector into the linear span of generators.

Obviously, this is not a coassociative map.

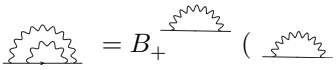
$$(\Delta_{\rm lin} \otimes {\rm id})\Delta_{\rm lin} \neq ({\rm id} \otimes \Delta_{\rm lin})\Delta_{\rm lin}. \tag{74}$$

To control this loss of associativity is a fascinating task on which we hope to report in the future.

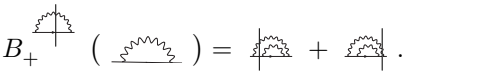
2.7. Unitarity of the S-matrix. A fact which will need much more attention in the future from the viewpoint of mixed Hodge structures is the fact that Feynman amplitudes are boundary values of analytic functions. We hence have dispersion relations available, and can relate, in the spirit of the Cutkosky rules, branchcut ambiguities to cuts on diagrams.

In particular, following guidance of the core Hopf algebra whose primitives are the one-loop cycles in the graph, the structure of the following matrix should reveal the desired relation between Feynman amplitudes and (variations of) mixed Hodge structures.

Actually, let us study a simple example where the renormalization Hopf algebra suffices (as the extra co-graphs in the core Hopf algebra would all be tadpoles [**16**]):

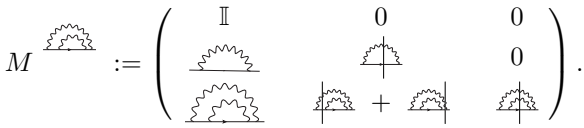
(75) 

Then, the two-particle cuts on $\Gamma :=$  are given by the two-particle cuts on the primtives appearing in the one-cocycles:

(76) 

The whole imaginary part can be obtained from this plus the three-particle cut  .

This can be combined into a nice matrix M^Γ which indeed suggests to study the connection to mixed Hodge structures more deeply.

(77) 

In each column we cut one loop at a time, such that suitable linear combinations of columns will express the branchcut ambiguities of the first column.

We hope that such matrices come in handy in an attempt to deepen the connection between Hodge theory and quantum fields, which started with the study of limiting mixed Hodge structures and renormalization in a recent collaboration between Spencer Bloch and the author [**5**]. While there it was the nilpotent orbit theorem which was at work in the back, we hope that the reader gets an idea from the above how we hope to farther the connection to Hodge structures. This hopefully succeeds in giving a precise mathematical backbone to renormalizability and unitarity simultanously, a feast notoriously missing in all attempts at quantum field theory (and gauge theories in particular) at present.

2.8. Fix-point equations. Let us finish this paper by listing the final fix-point equations (we give them for QED, and refer the reader to [**22, 32, 18**] for the general case) which generate the whole Feynman graph expansion of QED. We discriminate between the two formfactors of the massive fermion, $m\bar{\psi}\psi$ for its mass and $\bar{\psi}\partial\!\!\!/\psi$ for its wave function renormalization. Let

$$\mathcal{R}_{\rm QED} := \{\bar{\psi}\partial\!\!\!/\psi, m\bar{\psi}\psi, \bar{\psi}A\!\!\!/\psi, \frac{1}{4}F^2\}. \tag{78}$$

Then

$$X^r(\alpha) = \mathbb{I} \pm \sum_{k=1}^{\infty} \alpha^k B_+^{r;k}(X^r(\alpha)Q^{2k}(\alpha)), \tag{79}$$

where we take the plus sign for $r = \bar{\psi}A\!\!\!/\psi$ and the minus sign else, if r corresponds to an edge. We let

$$Q = \frac{X^{\bar{\psi}A\!\!\!/\psi}}{X^{\bar{\psi}\partial\!\!\!/\psi}\sqrt{X^{\frac{1}{4}F^2}}}. \tag{80}$$

Upon evaluation by renormalized Feynman rules it delivers the invariant charge of QED. The resulting maps $B_+^{r;k}$ are Hochschild closed

$$bB_+^{i,K} = 0. \tag{81}$$

Dividing by the (co-)ideal I simplifies Q;

$$Q = \frac{1}{\sqrt{X^{\frac{1}{4}F^2}}}. \tag{82}$$

See for example [**21**] for a far-reaching application of these techniques in QED.

Let us finally mention that upon adding suitable exact terms, $B_+^{r;k} \to B_+^{r;k} + L_0^{r;k}$ with $L_0^{r;k} = b\phi^{r,k}$, b being the Hochschild differential $b^2 = 0$, $\phi^{r,k} : H \to \mathbb{C}$, we can capture the change of parameters in the Feynman rules by suitable such coboundaries [**30**].

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