r_∞-Matrices, Triangular L_∞-Bialgebras, and Quantum$_\infty$ Groups

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Abstract. A homotopy analogue of the notion of a triangular Lie bialgebra is proposed with a goal of extending the basic notions of theory of quantum groups to the context of homotopy algebras and, in particular, introducing a homotopical generalization of the notion of a quantum group, or quantum$_\infty$-group.

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1. Introduction

1.1. Conventions and Notation

We will work over a ground field k of characteristic zero. A differential graded (dg) vector space V will mean a complex of k-vector spaces with a differential of degree one. The degree of a homogeneous element $v \in V$ will be denoted by $|v|$. In the context of graded algebra, we will be using the Koszul rule of signs when talking about the graded version of notions involving symmetry, including commutators, brackets, symmetric algebras, derivations, *etc.*, often omitting the modifier *graded*. For any integer n, we define a *translation* (or *n-fold desuspension*) $V[n]$ of V: $V[n]^p := V^{n+p}$ for each $p \in \mathbb{Z}$. For two graded vector spaces V and W, we define grading on the space $\mathrm{Hom}(V, W)$ of k-linear maps $V \to W$ by $|f| := n - m$ for $f \in \mathrm{Hom}(V^m, W^n)$.

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1.2. Quantum Groups

Recall that a *quantum group* in the sense of Drinfeld and Jimbo is a noncommutative, noncocommutative Hopf algebra A subject to the condition of being *quasitriangular* [5]. The latter implies, in particular, the existence of a solution $\mathcal{R}$ to the *quantum Yang-Baxter equation* $\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$ set up in A. More conceptually, the quasitriangularity condition provides data needed to put a braided structure on the monoidal category of left A-modules.

The most basic examples of quantum groups appear as *quantizations* or certain types of *deformations* (in the sense of Hopf algebras) of universal enveloping algebras and algebras of functions on groups. In the first case, starting with a Lie algebra $\mathfrak{g}$ and a Hopf-algebra deformation $U_h(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$, one passes to the "(*semi*)*classical limit*" $\delta(x) := \frac{\Delta_h(x)-\Delta_h^{op}(x)}{h}$ thus equipping $U(\mathfrak{g})$ with a *co-Poisson-Hopf structure* with $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ being the *co-Poisson cobracket*. In particular, the restriction $\delta|_{\mathfrak{g}}$ becomes a well-defined cobracket on $\mathfrak{g}$ turning it into a Lie bialgebra.

1.3. Quantization of Triangular Lie Bialgebras

The above process can be reversed: as it was shown in [13], any finite-dimensional Lie bialgebra $(\mathfrak{g}, [,], \delta)$ can be *quantized*, meaning that one can always come up with a Hopf-algebra deformation $U_h(\mathfrak{g})$ whose classical limit, in the sense of the above formula, agrees with δ. While a priori $U_h(\mathfrak{g})$ is just a Hopf algebra, one would really be interested in having a quasitriangular structure on it. As a special case, it was shown in [4] that such a structure exists, when $\mathfrak{g}$ is a *triangular* Lie bialgebra. This class of Lie bialgebras is defined as follows: let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$ ("*a classical r-matrix*") be a skew-symmetric element satisfying the *classical Yang-Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{23}, r_{13}] = 0,$$

which can be conveniently restated in the form of the *Maurer-Cartan equation*

$$[r, r] = 0$$

taking place in the graded Lie algebra $S(\mathfrak{g}[-1])[1]$ with respect to the *Schouten bracket* for elements r of degree one: $r \in (S(\mathfrak{g}[-1])[1])^1 = (S(\mathfrak{g}[-1]))^2 = S^2(\mathfrak{g}[-1])[2] = \mathfrak{g} \wedge \mathfrak{g}$. Such an element r, called a *Maurer-Cartan element*, gives rise to a Lie cobracket on $\mathfrak{g}$ in the form of the coboundary $\partial_{\mathrm{CE}}(r) : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ of r taken in the cochain *Chevalley-Eilenberg complex* of $\mathfrak{g}$ with coefficients in $\mathfrak{g} \wedge \mathfrak{g}$ (here, r is regarded as a 0-cocycle). The compatibility with the Lie-algebra structure on $\mathfrak{g}$ is packed into the relation $\partial^2_{\mathrm{CE}}(r) = 0$, thus guaranteeing that $\mathfrak{g}$ with such a cobracket is indeed a Lie bialgebra. The *co-Jacobi identity*, which could be rewritten as

$$[\partial_{\mathrm{CE}}(r), \partial_{\mathrm{CE}}(r)] = 0, \tag{1.1}$$

follows from the following statement, which is an odd version of the Hamiltonian correspondence in Poisson geometry, if one regards $S(\mathfrak{g}[-1])[1]$ as the shifted Gerstenhaber algebra of functions on the odd Poisson manifold

$(\mathfrak{g}[-1])^*$ and $\mathrm{Hom}(\mathfrak{g}[-1], S(\mathfrak{g}[-1]))$ as the graded Lie algebra of vector fields on $(\mathfrak{g}[-1])^*$.

Proposition 1.1. *The linear map*

$$\partial_{\mathrm{CE}}; S(\mathfrak{g}[-1])[1] \to \mathrm{Hom}(\mathfrak{g}[-1], S(\mathfrak{g}[-1]))$$

is a graded Lie-algebra morphism.

A *triangular Lie bialgebra* is a Lie algebra $\mathfrak{g}$ provided with a Lie cobracket $\partial_{\mathrm{CE}}(r)$ coming out of an r-matrix r. A basic statement concerning this class of Lie bialgebras is that $U(\mathfrak{g})$ can be quantized to a *triangular Hopf algebra* $U_h(\mathfrak{g})$. This condition is stronger than being quasitriangular, and in particular, the category of (left) modules over a triangular Hopf algebra turns out to be symmetric monoidal, as opposed to just being braided.

1.4. The Homotopy Quantization Program

The upshot of the above construction is that there is a source of quantum groups coming from the data of a Lie algebra $\mathfrak{g}$ and a solution of the Maurer-Cartan equation in $S(\mathfrak{g}[-1])[1]$. The goal of our project is to promote this construction to the realm of homotopy Lie algebras. In particular, this would generalize the work [2] done for the case of Lie 2-bialgebras. Here is an outline of our program:

1. Develop the notion of a *triangular L_∞-bialgebra* extending the classical one. In analogy with the classical case, the input data for this construction consists of an L_∞-algebra $\mathfrak{g}$ and a solution r of a generalized Maurer-Cartan equation set up in an appropriate algebraic context;
2. Show that the universal enveloping algebra $U(\mathfrak{g})$ of a triangular L_∞-bialgebra $\mathfrak{g}$ admits a natural *homotopy co-Poisson-Hopf* structure;
3. Extend the *Drinfeld twist* construction [5], which equips a cocommutative Hopf algebra with a new, triangular coproduct, to the homotopical context. Apply it to the case of universal enveloping algebra $U(\mathfrak{g})$ of the previous step to obtain a *quantum$_\infty$ group.*

The current paper is dedicated to describing the first step of the construction, which we believe might be interesting on its own.

The second step is work in progress. While the universal enveloping algebra $U(\mathfrak{g})$ of an L_∞-algebra $\mathfrak{g}$ is a strongly homotopy associative (or A_∞-) algebra that also turns out to be a cocommutative, coassociative coalgebra object in Lada-Markl's, see [10], symmetric monoidal category of A_∞-algebras, we would be interested in verifying that $U(\mathfrak{g})$ is actually a *Hopf$_\infty$ algebra*, that is, possesses an antipodal map satisfying certain compatibility conditions. We would also need to translate an L_∞-bialgebra structure on an L_∞-algebra $\mathfrak{g}$ into a co-Poisson$_\infty$ structure on the Hopf$_\infty$ algebra $U(\mathfrak{g})$. This would result in providing $U(\mathfrak{g})$ with the structure of a cocommutative co-Poisson$_\infty$ coalgebra object in Lada-Markl's symmetric monoidal category of A_∞-algebras.

Furthermore, we would like to develop deformation theory of homotopy Hopf algebras and use it to quantize homotopy Lie bialgebras. A different approach to quantization of homotopy Lie bialgebras (using the framework of PROPs) was taken in [12], in which a different notion of a homotopy Hopf algebra (or rather, homotopy bialgebra) was used. That notion depends on the choice of a minimal resolution of the bialgebra properad. The notion we outline above appears to be more canonical.

In the future we would also be interested in investigating what this program produces for L_∞-algebras arising in the geometric context, such as generalized Poisson geometry, L_∞-algebroids, or BV_∞-geometry.

2. The Big Bracket and L_∞-Bialgebras

Recall that the structure of a (strongly) homotopy Lie algebra (or *L_∞-algebra*) on a graded vector space $\mathfrak{g}$ may be given by a *codifferential*, *i.e.*, a degree-one, square-zero coderivation D such that $D(1) = 0$, on the graded cocommutative coalgebra $S(\mathfrak{g}[1])$ equipped with the shuffle comultiplication. The date given by D is equivalent to a collection of "higher Lie brackets" $l_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$, $k \geq 1$, of degree one obtained by restriction of D to the kth symmetric component of $S(\mathfrak{g}[1])$ followed by projection to the cogenerators. The condition $D^2 = 0$ is equivalent to the *higher Jacobi identities*, homotopy versions of the Jacobi identity. Outside of deformation theory, nontrivial examples of homotopy Lie algebras are known to arise in the context of multisymplectic geometry [14, 1], Courant algebroids [15], and closed string field theory [17, 7, 11].

In order to discuss the structure of an L_∞-bialgebra on a graded vector space $\mathfrak{g}$, we need to mix the graded Lie algebra $\mathrm{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1])$, used to define the structure of an L_∞-algebra on $\mathfrak{g}$, with the graded Lie algebra $\mathrm{Hom}(\mathfrak{g}[-1]), S(\mathfrak{g}[-1]))$, used to define the cobracket on $\mathfrak{g}$ in the classical, Lie algebra setting in Section 1.3. Consider the graded vector space

$$\mathcal{B} := \prod_{m,n \geq 0} \mathrm{Hom}(S^m(\mathfrak{g}[1]), S^n(\mathfrak{g}[-1]))[2]$$

and provide it with the structure of a graded Lie algebra given by the graded commutator

$$[f, g] := f \circ g - (-1)^{|f| \cdot |g|} g \circ f$$

under the *circle*, or $\cup_1$ *product*, cf. [6] and [16]:

$$(f \circ g)(x_1 \dots x_n) := \sum_{\sigma \in \mathrm{Sh}_{k,l}} (-1)^\varepsilon f(x_{\sigma(1)} \dots x_{\sigma(k)} g(x_{\sigma(k+1)} \dots x_{\sigma(n)})_{(1)})\, g(x_{\sigma(k+1)} \dots x_{\sigma(n)})_{(2)},$$

where $x_1, \dots, x_n \in \mathfrak{g}[1]$,

$$f \in \prod_{m \geq 0} \mathrm{Hom}(S^{k+1}(\mathfrak{g}[1]), S^m(\mathfrak{g}[-1]))[2],$$

$$g \in \prod_{m \geq 0} \mathrm{Hom}(S^l(\mathfrak{g}[1]), S^m(\mathfrak{g}[-1]))[2],$$

$n = k + l$ — otherwise we set $(f \circ g)(x_1 \dots x_n) = 0$, $\mathrm{Sh}_{k,l}$ is the set of (k,l) *shuffles*: permutations σ of $\{1, 2, \dots, n\}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+1) < \dots < \sigma(n)$, $\varepsilon = |x_\sigma| + |g|(|x_{\sigma(1)}| + \dots + |x_{\sigma(k)}|)$, $(-1)^{|x_\sigma|}$ is the *Koszul sign* of the permutation of $x_1 \dots x_n$ to $x_{\sigma(1)} \dots x_{\sigma(n)}$ in $S(\mathfrak{g}[1])$, and we use Sweedler's notation to denote the result $g_{(1)} \otimes g_{(2)}$ of applying to $g \in S(\mathfrak{g}[-1])[2]$ the (shifted) comultiplication $S(\mathfrak{g}[-1])[2] \to S(\mathfrak{g}[-1])[2] \otimes S(\mathfrak{g}[-1])$ followed by the projection $S(\mathfrak{g}[-1])[2] \to \mathfrak{g}[1]$ onto the cogenerators in the first tensor factor. This graded Lie algebra $\mathcal{B}$, under the assumption that $\dim \mathfrak{g} < \infty$ and in a slightly different incarnation, was introduced by Y. Kosmann-Schwarzbach [8] in relation to Lie bialgebras and later used by O. Kravchenko [9] in relation to L_∞-bialgebras. The Lie bracket $[,]$ on $\mathcal{B}$ is called the *big bracket*. The graded Lie algebra $\mathcal{B}$ has the property that its Maurer-Cartan elements represent L_∞ brackets and cobrackets on $\mathfrak{g}$, as well as mixed operations, comprising the structure of an L_∞-bialgebra on $\mathfrak{g}$. Here we adopt Kravchenko's approach and define an *L_∞-bialgebra structure* on $\mathfrak{g}$ as a Maurer-Cartan element in the subalgebra

$$\mathcal{B}^+ := \prod_{m,n \geq 1} \mathrm{Hom}(S^m(\mathfrak{g}[1]), S^n(\mathfrak{g}[-1]))[2]$$

of the graded Lie algebra $\mathcal{B}$.[1] This means

$$\mu = \sum_{m,n \geq 1} \mu_{mn},$$

$$\mu_{mn} : S^m(\mathfrak{g}[1]) \to S^n(\mathfrak{g}[-1])[2] \quad \text{of degree 1},$$

$$[\mu, \mu] = 0.$$

3. r_∞-Matrices and Triangular L_∞-Bialgebras

For an L_∞-algebra $\mathfrak{g}$, one can generalize the Schouten bracket to an L_∞-structure on $S(\mathfrak{g}[-1])[1]$ by extending the higher brackets l_k on $\mathfrak{g}$ as graded multiderivations of the graded commutative algebra $S(\mathfrak{g}[-1])$. This L_∞-structure may also be described via *higher derived brackets* (in the semiclassical limit) on the BV_∞-algebra $S(\mathfrak{g}[-1])$, see [3, Example 3.4]. This L_∞-structure can be naturally extended to the completion

$$\widehat{S}(\mathfrak{g}[-1])[1] := \prod_{n \geq 0} S^n(\mathfrak{g}[-1]))[1].$$

While investigating the deformation-theoretic meaning of solutions $r = r(\lambda) \in \lambda \widehat{S}(\mathfrak{g}[-1])[1][[\lambda]]$, where λ is the *deformation parameter*, that is to say, a (degree-zero) formal variable, of the *generalized Maurer-Cartan equation*

$$l_1(r) + \frac{1}{2!} l_2(r \odot r) + \frac{1}{3!} l_3(r \odot r \odot r) + \dots = 0, \tag{3.1}$$

where $\odot$ refers to multiplication in $S(V)$ for $V = \widehat{S}(\mathfrak{g}[-1])[2]$, certain analogies can be drawn with basic constructions of the theory of quantum groups.

[1] Maurer-Cartan elements in $\mathcal{B}$ would correspond to more general, *curved* L_∞-bialgebras.

An (*unpointed*) L_∞-*morphism* φ from $\widehat{S}(\mathfrak{g}[-1])[1]$ to $\mathcal{B}^+$, that is to say, a morphism $S(\widehat{S}(\mathfrak{g}[-1])[2]) \to S(\mathcal{B}^+[1])$ of dg coalgebras, amounts to defining a series of degree-zero linear maps $\varphi_n : S^n(\widehat{S}(\mathfrak{g}[-1])[2]) \to \mathcal{B}^+[1]$, $n \ge 0$, satisfying the following compatibility conditions for all $n \ge 0$:

$$\frac{1}{2}\sum_{k=0}^{n} \sum_{\sigma \in \mathrm{Sh}_{k,n-k}} (-1)^{\varepsilon}[\varphi_k(x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k)}), \varphi_{n-k}(x_{\sigma(k+1)} \odot \cdots \odot x_{\sigma(n)})] =$$

$$\sum_{m=1}^{n} \sum_{\tau \in \mathrm{Sh}_{m,n-m}} (-1)^{|x_\tau|} \varphi_{n-m+1}(l_m(x_{\tau(1)} \odot \cdots \odot x_{\tau(m)}) \odot x_{\tau(m+1)} \odot \cdots \odot x_{\tau(n)}),$$

where $x_1, \dots, x_n \in \widehat{S}(\mathfrak{g}[-1])[2]$ and $\varepsilon = |x_\sigma| + |x_{\sigma(1)}| + \cdots + |x_{\sigma(k)}|$. There is a *canonical* L_∞-*morphism* $\varphi : \widehat{S}(\mathfrak{g}[-1])[1] \to \mathcal{B}^+$, which may be defined by the maps

$$\varphi_n(x_1 \odot \cdots \odot x_n)(y) := l_{n+p}(x_1 \odot \cdots \odot x_n \odot N(y)),$$

where $x_1, \dots, x_n \in \widehat{S}(\mathfrak{g}[-1])[2]$, $y \in S^p(\mathfrak{g}[1])$, $N(y) \in S^p(\widehat{S}(\mathfrak{g}[-1])[2])$, $p \ge 1$, and $N : S(\mathfrak{g}[1]) \to S(\widehat{S}(\mathfrak{g}[-1])[2])$ is the graded algebra morphism induced by the obvious linear map $\mathfrak{g}[1] \hookrightarrow \widehat{S}(\mathfrak{g}[-1])[2] \hookrightarrow S(\widehat{S}(\mathfrak{g}[-1])[2])$. The following theorem generalizes Proposition 1.1 to the L_∞ setting.

Theorem 3.1. *The maps* φ_n*,* $n \ge 0$*, define an* L_∞*-morphism*

$$\varphi : \widehat{S}(\mathfrak{g}[-1])[1] \to \mathcal{B}^+$$

from the L_∞*-algebra* $\widehat{S}(\mathfrak{g}[-1])[1]$ *to the graded Lie algebra* $\mathcal{B}^+$*.*

The proof of the theorem is a straightforward checkup that reduces the statement to the higher Jacobi identities for the L_∞ brackets on $\widehat{S}(\mathfrak{g}[-1])[1]$. This theorem also generalizes Kravchenko's result [9, Theorem 19], which provides an L_∞-morphism from an L_∞-algebra $\mathfrak{g}$ to the graded Lie algebra $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$.

An r_∞-*matrix* r is a (*generalized*) *Maurer-Cartan element* $r = r(\lambda)$ in the L_∞-algebra $\lambda\widehat{S}(\mathfrak{g}[-1])[1][[\lambda]]$, *i.e.*, a degree-one solution of the generalized Maurer-Cartan equation (3.1). Sending an r_∞-matrix to the subalgebra $\mathcal{B}^+$ of the big-bracket Lie algebra $\mathcal{B}$ under an L_∞-morphism $\varphi : \widehat{S}(\mathfrak{g}[-1])[1] \to \mathcal{B}^+$ would yield a Maurer-Cartan element in $\lambda\mathcal{B}^+[[\lambda]]$, giving rise to an L_∞-bialgebra structure

$$\mu(\lambda) := \varphi(e^r) = \varphi_0 + \varphi_1(r) + \frac{1}{2!}\varphi_2(r \odot r) + \frac{1}{3!}\varphi_3(r \odot r \odot r) + \dots,$$

depending on the deformation parameter λ, on $\mathfrak{g}$ as per Section 2, in analogy with $\partial_{\mathrm{CE}}(r)$ giving rise to an ordinary Lie cobracket in the classical, nonhomotopical case, see also Example 3.2 below.

We call the L_∞-bialgebra $(\mathfrak{g}, \mu)$ produced out of an L_∞-algebra $\mathfrak{g}$ and an r_∞-matrix r by transferring it to a Maurer-Cartan element $\mu = \varphi(e^r)$ in $\mathcal{B}^+$ via the canonical L_∞-morphism φ a *triangular* L_∞*-bialgebra.*

Example 3.2. In the case of a classical Lie algebra $\mathfrak{g}$, the graded Lie algebra $\widehat{S}(\mathfrak{g}[-1])[1]$ is just the completed graded Lie algebra of (right-)invariant multivector fields on the corresponding local Lie group (or, equivalently, up to a degree shift, functions on the formal odd Poisson manifold $(\mathfrak{g}[-1])^*$) with the Schouten bracket, and the L_∞-morphism $\varphi = \varphi_0 + \varphi_1$ is a combination of a "constant part" $\varphi_0 = l_2$, equal to the Lie bracket on $\mathfrak{g}$ up to a degree shift and a sign, and a "linear part" $\varphi_1 = \partial_{\mathrm{CE}}$, equal to the Chevalley-Eilenberg differential of 0-cochains of the Lie algebra $\mathfrak{g}$ with coefficients in the graded $\mathfrak{g}$-module $\widehat{S}(\mathfrak{g}[-1])[2]$. The fact that $\varphi_0 + \varphi_1$ is an L_∞-morphism translates into three compatibility conditions

$$
\begin{aligned}
[\varphi_0, \varphi_0] &= 0, \\
[\varphi_0, \varphi_1(x)] &= 0, \\
[\varphi_1(x), \varphi_1(y)] &= \varphi_1([x, y]).
\end{aligned}
$$

The first condition is equivalent to the Jacobi identity for the Lie bracket on $\mathfrak{g}$, the second condition means that $\partial_{\mathrm{CE}}(\varphi_1(x)) = \partial^2_{\mathrm{CE}}(x) = 0$, and the third states that ∂_{CE} is a Lie-algebra morphism, which is the assertion of Proposition 1.1. Thus, the transfer $\varphi(e^r) = \varphi_0 + \varphi_1(r)$ of an r-matrix $r \in (\widehat{S}(\mathfrak{g}[-1])[1])^1 = (S^2(\mathfrak{g}[-1])[2])^0 = \mathfrak{g} \wedge \mathfrak{g}$ comprises a Lie bracket φ_0 and a Lie cobracket $\varphi_1(r)$ on $\mathfrak{g}$, satisfying the co-Jacobi identity (1.1) and the compatibility condition with the Lie bracket, thereby resulting in the structure of a triangular Lie bialgebra in $\mathfrak{g}$. Here we ignored the deformation parameter λ, because $\widehat{S}(\mathfrak{g}[-1])[1]$ is just a graded Lie algebra and the (generalized) Maurer-Cartan equation $[r, r] = 0$ in $\widehat{S}(\mathfrak{g}[-1])[1]$ is homogeneous.

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