

A REMARK ON QUANTUM GRAVITY*

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ABSTRACT. We discuss the structure of Dyson–Schwinger equations in quantum gravity and conclude in particular that all relevant skeletons are of first order in the loop number. There is an accompanying sub Hopf algebra on gravity amplitudes equivalent to identities between n -graviton scattering amplitudes which generalize the Slavnov–Taylor identities. These identities map the infinite number of charges and finite numbers of skeletons in gravity to an infinite number of skeletons and a finite number of charges needing renormalization. Our analysis suggests that gravity, regarded as a probability conserving but perturbatively non-renormalizable theory, is renormalizable after all, thanks to the structure of its Dyson–Schwinger equations.

1. INTRODUCTION

A renormalizable theory poses a computational problem for a theoretical physicist: even if only a finite number of amplitudes need renormalization, the quantum equations of motion –the Dyson–Schwinger equations (DSE)– ensure that these amplitudes must be calculated as iterated integrals based on a skeleton expansion for the Green functions. There is an infinite series of skeletons, of growing computational complexity, and thus a formidable challenge at hand. Order is brought to this situation by the fact that the skeletons can be organized in terms of the underlying Hochschild cohomology of the Hopf algebra of a renormalizable theory, the computational challenge remains though in the analytic determination of the skeletons and their Mellin transforms [1, 2, 3]. This approach, combining the analysis of the renormalization group provided in [4] with the analysis of the mathematical structure of DSE provided in [5, 6, 1], has led to new methods in solving DSE beyond perturbation theory [5, 2, 3].

A nice fact is that internal symmetries can be systematically understood in terms of this Hochschild cohomology: Slavnov–Taylor identities are equivalent to the demand that multiplicative renormalization is compatible with the cohomology structure [7], leading to the identification of Hopf ideals generated by these very Ward and Slavnov–Taylor identities [8].

For a non-renormalizable theory the situation is worse: the computational challenge for the theorist is repeated infinitely as there is now an infinite number of amplitudes demanding renormalization, each of them still based on an infinite number of possible skeleton iterations.

But the interplay with Hochschild cohomology leads to surprising new insights into this situation, which in this first paper we discuss at an elementary level for the situation of pure gravity.

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2. THE STRUCTURE OF DYSON–SCHWINGER EQUATIONS

To compare the situation for a renormalizable QFT with the situation for an unrenormalizable one, we consider QED in four and six dimensions of spacetime.

2.1. QED₄. Let us consider as a typical example quantum electrodynamics in four dimensions. The DSE involves a sum over superficially convergent skeleton kernels which are in one-to-one correspondence with the primitives of the Hopf algebra underlying perturbation theory. There is an infinite number of primitive

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skeleton graphs, a finite number of them at each loop order. Furthermore, there is a finite number of monomials, tree-level amplitudes represented graphically as elements of a finite set $\mathcal{R}$

$$(1) \quad \mathcal{R} = \left\{ \text{wavy}, \frac{\text{---}}{m}, \frac{\text{---}}{\partial}, \text{wavy} \text{---} \right\},$$

$|\mathcal{R}| = 4 < \infty$. The Lagrangian is

$$(2) \quad \mathcal{L} = \sum_{r \in \mathcal{R}} \hat{\phi}(r) = -\bar{\psi}[\not{\partial} + ie\not{A}]\psi - m\bar{\psi}\psi - \frac{1}{4}F^2,$$

with coordinate space Feynman rules $\hat{\phi}$,

$$(3) \quad \hat{\phi}(\text{wavy}) = \frac{1}{4}F^2, \quad \hat{\phi}\left(\frac{\text{---}}{m}\right) = m\bar{\psi}\psi, \quad \hat{\phi}\left(\frac{\text{---}}{\partial}\right) = \bar{\psi}\not{\partial}\psi, \quad \hat{\phi}(\text{wavy} \text{---}) = \bar{\psi}\not{A}\psi.$$

The Dyson–Schwinger equations take the corresponding form [1] $\forall r \in \mathcal{R}$,

$$(4) \quad G^r(\alpha, L) = 1 \pm \sum_{k=1}^{\infty} \alpha^k \phi_R \left(B_+^{r;k}(X^r Q^k) \right),$$

with renormalized Feynman rules ϕ_R in momentum space with subtractions at an Euclidean momentum $q^2 = \mu^2$, $L = \ln q^2/\mu^2$, normalized to unity on tree-level terms $\in \mathcal{R}$. We restrict to zero-momentum transfer in vertex functions such that Green functions $G^r(\alpha, L)$ are functions of the single kinematical variable L and projection onto form-factors is understood in accordance with the set $\mathcal{R}$.

The plus sign above is taken for the vertex functions, and the minus sign for inverse propagators. X^r is a solution to the corresponding combinatorial DSE in Hochschild cohomology,

$$(5) \quad X^r = \mathbb{I} \pm \sum_{k=1}^{\infty} \alpha^k B_+^{r;k}(X^r Q^k),$$

with $G_R(\alpha, L) = \phi_R(X^r)$ and

$$(6) \quad Q_R(\alpha, L) = \phi_R(Q) = \frac{1}{\sqrt{\phi_R(X \text{wavy})(\alpha, L)}}, \quad Q = \frac{1}{\sqrt{X \text{wavy}}},$$

defines the invariant running charge α/Q_R . We emphasize that the above are DSE for 1PI Green functions.

The primitives

$$(7) \quad B_+^{r;k}(\mathbb{I}), \quad bB_+^{r;k} = 0 \Leftrightarrow \Delta B_+^{r;k} = B_+^{r;k} \otimes \mathbb{I} + (\text{id} \otimes B_+^{r;k})\Delta,$$

are such that upon application of the renormalized Feynman rules, $\phi_R(B_+^{r;k})$, they give the Dyson kernels for the vertex $\text{wavy} \text{---}$, the quenched β function and light-by-light scattering amplitudes for the photon wavy , suitable mass derivatives for the fermion propagator $\frac{\text{---}}{m}$, while the kinetic part of the fermion propagator $\frac{\text{---}}{\partial}$ is taken care off by the Ward identity. Here are the primitives up to two loops:

$$(8) \quad \begin{array}{ll} r = \text{wavy} : & \begin{array}{c} \text{Diagram 1: Circle with wavy lines at top and bottom, dots at } \partial_q \text{ and } \partial_q. \\ \text{Diagram 2: Circle with wavy lines at top and bottom, dots at } \partial_q \text{ and } \partial_q. \end{array} \\ r = \frac{\text{---}}{m} : & \begin{array}{c} \text{Diagram 3: Wavy line with a dot at } \partial_m. \\ \text{Diagram 4: Wavy line with a dot at } \partial_m. \end{array} \\ r = \text{wavy} \text{---} : & \begin{array}{c} \text{Diagram 5: Triangle with wavy lines at top and bottom, dot at } \partial_q. \\ \text{Diagram 6: Triangle with wavy lines at top and bottom, dot at } \partial_q. \end{array} \end{array}$$

Labeled dots indicate derivatives with respect to external momenta q or fermion masses m , such as to render these integral kernels logarithmically divergent.

In general,

$$(9) \quad B_+^{r;k} = \sum_{|\gamma|=k, \text{res}(\gamma)=r} B_+^\gamma,$$

where B_+^γ is defined via pre-Lie insertion into the primitive γ such that $B_+^{r;k}$ is Hochschild closed [7, 1]. As a result we get a sub Hopf algebra which is generated by a single one-cocycle $B_+^{r;k}$ in each loop-degree k , [3].

The degree of divergence of a graph Γ with f external fermion lines and m external photon lines in D dimensions is

$$(10) \quad \omega_D(\Gamma) = \frac{3}{2}f + m - D - (D-4)(|\Gamma| - 1) \Rightarrow \omega_4(\Gamma) = \frac{3}{2}f + m - 4.$$

This is independent of the loop number for QED₄, $D = 4$, and is a sole function of the number and type of external legs. $\omega_D(\Gamma)$ determines the number of derivatives needed to render a graph logarithmically divergent.

This finishes our summary of QED₄ as a typical renormalizable theory.

2.2. QED₆. Here, $|\{\mathcal{R}\}| = \infty$, as each amplitude is superficially divergent in sufficiently high loop number.

Searching for primitives, there is a maximal loop order for each amplitude after which graphs contributing to any given amplitude have sub-divergences. We have $\omega_6(\Gamma) = \frac{3}{2}f + m - 4 - 2|\Gamma|$, and this is now a function of the number and type of external legs and the number of loops.

For example, the Dyson kernels (the $e^+e^- \rightarrow e^+e^-$ scattering graphs 2PI in the forward channel, [9]) have power-counting degree

$$(11) \quad \omega(\Gamma) = 2 - 2|\Gamma|,$$

and one immediately proves that no such graph is primitive beyond one loop.

But to set up our Dyson–Schwinger equations correctly, we need to investigate the integral kernels obtained by taking a sufficient number of derivatives, with respect to masses or external momenta, so as to obtain log-divergent integral kernels. This can be graphically indicated by (labeled) dots on the graph as in [10], and as we did in (8) above already. One immediately proves that these top-degree contributions provide an infinite number of skeleton kernels. Upon taking such derivatives we create actually new integral kernels free of subdivergences. There are log-divergent kernels for each amplitude, at each loop number.

Corollary 1. *For each $r \in \mathcal{R}$ and each positive integer k , there exists non-trivial one-cocycles $B_+^{r;\gamma}$ with $|\gamma| = k$.*

Proof: It suffices to give, for each amplitude, a single series of dotted graphs Γ with $\omega_6(\Gamma)$ dots each which has a primitive member at each loop order. We do so for the example $e^+e^- \rightarrow e^+e^-$ by the following figure.

$$(12) \quad \begin{array}{ccccccc} \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \dots \end{array}$$

where dots represent derivatives wrt an external momentum. For other amplitudes similar series are easily constructed. $\square$

Summarizing, the DSE take a form similar to QED₄,

$$(13) \quad G^r(\alpha, L) = 1 \pm \sum_{k=1}^{\infty} \alpha^k \sum_{|\gamma|=k, \text{res}(\gamma)=r} \phi_R(B_+^\gamma(X_\gamma)),$$

$$(14) \quad X_\gamma = \prod_{v \in \gamma^{[0]}, e \in \gamma_{\text{int}}^{[1]}} \frac{\Gamma^v}{\Gamma^e},$$

only that the number of amplitudes needing renormalization, $|\mathcal{R}|$, is infinite, and for each amplitude we have an infinite number of kernels. Note that we can not write this in terms of $B_+^{r;k}$, as we are lacking the crucial

identity

$$(15) \quad X_\gamma = X_{\gamma'},$$

for all primitive γ, γ' with the same loop number, and contributing to the same amplitude [7].

Indeed, note that we obtain new primitives γ as in the case of the initial photon vertex, which now has two one-loop kernels, one involving a new Green function $G^{e^+e^- \rightarrow e^+e^-}$ contributing to $\mathcal{R}$, given here with their dressings:

$$(16) \quad \begin{array}{c} \text{Diagram 1: A vertex } \Gamma \text{ (represented by a large black circle) with four external wavy lines. The top-left and bottom-right lines are labeled } \Gamma \text{ with a wavy line. The top-right and bottom-left lines are labeled } [\Gamma^-]^{-1} \text{ with a wavy line.} \\ \text{Diagram 2: A vertex } \Gamma \text{ (represented by a large black circle) with two external wavy lines. The top-left line is labeled } \Gamma \text{ with a wavy line. The bottom-right line is labeled } \Gamma^{e^+e^-e^+e^-} \text{ with a wavy line.} \\ \text{Diagram 3: A vertex } \Gamma \text{ (represented by a large black circle) with two external wavy lines. The top-left line is labeled } \Gamma \text{ with a wavy line. The bottom-right line is labeled } [\Gamma^-]^{-1} \text{ with a wavy line.} \end{array}$$

We would hence need a relation like

$$(17) \quad \frac{[\Gamma \text{ with wavy line}]^3}{[\Gamma^-]^{-2} \Gamma \text{ with wavy line}} = \frac{\Gamma \text{ with wavy line} \Gamma^{e^+e^-e^+e^-}}{[\Gamma^-]^{-2}},$$

and infinitely many more of them, to render the theory renormalizable by renormalization of a single charge.

It is this double infinity which renders such a theory non-predictive: there is an infinite amount of work for a theorist (work out the kernels) and an infinite amount of work for an experimentalist: measure an infinite number of charges to render the theory predictive.

3. GRAVITY

We consider pure gravity understood as a theory based on a graviton propagator and n -graviton couplings as vertices. A fuller discussion incorporating ghosts and matter fields is referred to future work.

An immediate observation concerns powercounting in such a theory. If we work with Feynman rules as given in [11], we see that each n -graviton vertex is a quadric in momenta attached to the vertex. This has an immediate consequence.

Corollary 2. *Let $|\Gamma| = k$. Then $\omega(\Gamma) = -2(|\Gamma| + 1)$.*

Proof: A 1PI one-loop graph has as many internal edges as vertices. Their contributions to the superficial degree of divergence hence cancel. We conclude that one-loop graphs are quadratically divergent, in accordance with the corollary. Increasing the loop order by one in a 1PI graph introduces one more propagator than vertices, the net gain in powercounting is hence -2 , $+2$ from counting propagators and vertices, and -4 from counting loops. $\square$

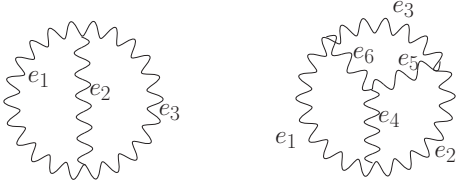
We have now a situation dual to a renormalizable theory: The superficial degree of divergence depends on the loop number, but not on the number and type of external legs. While in a renormalizable theory like massless QED₄ we have a single amplitude which needs renormalization -thanks to the Ward identity-, and primitives at each loop order, we have found a nice loop-to-leg duality: we have primitives only at first loop order, but for any number of external legs.

Investigating further, the crucial result of this paper is that quantum gravity is sufficiently non-renormalizable such that its skeletons are all one-loop.

3.1. All skeletons are one-loop. As powercounting is determined by the loop number instead of by the external leg structure, there is only one two-loop graph, made of three carrying edges. The notion of a carrying edge takes account of the fact that the powercounting of edges and vertices cancels. We hence ignore external edges and consider a string of edges, interrupted only by vertices with couplings to external legs, as a single carrying edge. We extend this notion to subgraphs. External edges are not explicitly given from now on, as they play no role for powercounting.

For a graph Γ , we let $d_{\omega(\Gamma)}$ be the set of graphs obtained by distributing $\omega(\Gamma)$ dots over Γ , indicating the action of derivatives wrt to external momenta or masses. Derivatives on vertices will likewise be indicated by dots on internal propagators in accordance with the leg which is involved.

Let us look at the the following two graphs.

(18) 

For the two loop graph on the left, we can distribute six markings over its three internal edges e_1, e_2, e_3 to render it log-divergent. We have to do so such that each of the one-loop subgraphs $\{(e_1, e_2)\}$, $\{(e_2, e_3)\}$, $\{(e_3, e_1)\}$ has five markings, as it has to be finite. But there is no partition of six into three integers such that the sum of any two of those integers is greater than four.

So there is no primitive two-loop graph. The first integer which has such a partition is $8 \rightarrow (3, 3, 2)$. We hence have to require that at higher loop orders, every two-loop graph, which necessarily is based on three carrying edges, obtains eight dots.

Let us now look at a three loop graph Γ . We have $\omega(\Gamma) = -8$. So we have eight markings at our disposal. We need to distribute them in a way such that each of its two-loop subgraphs contains all of them, as we would have to render all its two-loop subgraphs finite. This is impossible as two such two-loop subgraphs will not share all carrying edges. In the figure, we have six edges $e_1, \dots, e_6$. Any five of those six edges form a two-loop subgraph on three carrying edges. For example edges e_1, e_2, e_3, e_4, e_5 form a two-loop subgraph on the three carrying edges $\{(e_1, e_3), (e_4, e_5), e_2\}$. Distributing $3 + 3 + 2$ dots in any way over these three carrying edges leaves for example the two-loop subgraphs like $\{(e_1, e_3), (e_4, e_5), e_6\}$, containing e_6 , divergent.

So there is no primitive three-loop kernel, and eleven or more markings are needed to render the three-loop graphs finite. This argument continues in a straightforward manner and hence we will not find a primitive in $d_{\omega(\Gamma)}$ for any graph Γ beyond first loop order. We hence have proven for any graph Γ

Theorem 3. *The set $d_{\omega(\Gamma)}$ contains no primitive element.*

This is strikingly different in comparison to a generic non-renormalizable theory like QED_6 which we discussed in the previous section.

3.2. Relations between Green functions. The double infinity, in loops and legs, of a non-renormalizable theory has turned for quantum gravity into a single infinity parameterized by the different number of external legs. While for a renormalizable theory, the degree of divergence is constant over the number of loops and varies with the number of external legs, we have the dual situation for gravity: it is constant over the number of external legs, but varies with the loop number.

The question we ask is if there is a map which connects the two situations of quantum gravity and a renormalizable field theory, using the structure of their respective DSEs? Is quantum gravity, by such a map, non-pertubatively renormalizable whilst being perturbatively non-renormalizable? This thought, actively pursued in the context of the Wilsonian renormalization group since several years [12], indeed emerges also from our study of the DSE of quantum gravity.

We can work out how such a map should look. What we need is a map which gives an infinite number of relations between the infinite number of amplitudes needing renormalization such that only a finite number of amplitudes remain undetermined, possibly on the expense of increasing the number of primitives infinitely.

Our input is the structure of the DSE written in terms of their Hochschild one-cocycles, combined with the request for multiplicative renormalization. In [7], in the context of a non-abelian gauge theory, it was shown how this request leads to the Slavnov–Taylor identities for the couplings. There, this request led, for such a gauge theory as a renormalizable theory with a finite set $\mathcal{R}$, to a finite number of relations ensuring that there is a unique renormalized coupling.

The request [7]

(19)
$$X_\gamma = X_{\gamma'}$$

leads to a Hopf ideal in the pure gluon sector given by [7]

$$(20) \quad \frac{\Gamma^{gl_4}}{\Gamma^{gl_3}} = \frac{\Gamma^{gl_3}}{\Gamma^{gl_2}},$$

with

$$(21) \quad X_\gamma = \frac{\prod_{v \in \gamma^{[0]}} \Gamma^v}{\prod_{e \in \gamma_{\text{int}}^{[1]}} \Gamma^e},$$

where $\gamma^{[0]}$ is the set of vertices of γ and $\gamma_{\text{int}}^{[1]}$ the set of internal edges. This simplifies in the pure gluon sector to a well-known Slavnov–Taylor identity for the renormalization of the two-, three- and four-point gluon amplitudes:

$$(22) \quad \frac{Z^{gl_4}}{Z^{gl_3}} = \frac{Z^{gl_3}}{Z^{gl_2}}.$$

Here, the superscript gl_n indicates an n -gluon amplitude. Note that this identity is also in accordance with unitarity and cutting symmetries, as it exhibits that the contributions to any amputated 4-gluon amplitude renormalize consistently.

Correspondingly, we will denote by a superscript gr_n an n -graviton amplitude. These amplitudes span an infinite set $\mathcal{R}$. In contrast to the situation in gauge theory, if we have any number of graviton self-couplings gr_n , we now need an infinite number of renormalization constants Z^{gr_n} if we are to maintain our theory unitary.

The above Slavnov–Taylor identity was derived in [7] by the requirement that the sum of all graphs contributing to a given amplitude at a fixed order of perturbation theory furnishes a generator of a sub-Hopf algebra. The same requirement delivers now an infinite sequence of identities,

$$(23) \quad \frac{\Gamma^{gr_{n+1}}}{\Gamma^{gr_n}} = \frac{\Gamma^{gr_n}}{\Gamma^{gr_{n-1}}},$$

which is indeed an infinite number of identities leaving only $\Gamma^{gr_2}, \Gamma^{gr_3}$ undetermined. In the case of non-abelian gauge theory the corresponding ideal is respected by the counterterms regarded as an element of $\text{Spec}(G)$, which hence become an algebra map on the quotient of the Hopf algebra by this Hopf ideal. This implies non-trivial relations between Hochschild one-cocycles beyond one loop, which iterate in the DSE to provide the desired identities for the counterterms. In the gravity case, the situation is simpler from the viewpoint of absence of one-cocycles beyond one loop. This is compensated though by the necessity to construct elements in $\text{Spec}(G)$ in accordance with the desired ideal.

To see this in some detail, we identify a dressed primitive one-loop graph with r external legs with an ordered partition ω_r of an integer r into integers $n_i > 0$. We consider partitions up to cyclic permutations. Let $\{\omega_r\}$ be the set of such partitions of r . For $\omega \in \{\omega_r\}$ let $\|\omega\|$ be the size of the partition, and let us consider partitions of size greater than one. We write sometimes $r = |\omega|$.

The identification between such a partition and a one-loop graph proceeds as follows. A one loop graph provides say m internal propagators $\phi_R([\Gamma^{gr_2}]^{-1})$, and m internal vertices which are dressed by vertex functions $\phi_R(\Gamma^{gr_{s_i}})$, $s_i > 2$. We have $r = \sum_{i=1}^m (s_i - 2)$, $n_i = s_i - 2$ and identify such a graph with the partition $\{s_1 - 2, s_2 - 2, \dots, s_m - 2\}$, as for example for $r = 4 = |\omega|$, $\omega = \{2, 1, 1\}$, $\|\omega\| = 3$:

$$(24) \quad \begin{array}{c} \begin{array}{c} \text{Diagram of a one-loop graph with 4 external legs and 3 internal vertices. The vertices are labeled } \Gamma^3, \Gamma^3, \text{ and } \Gamma^4. \text{ The internal edges are labeled } [\Gamma^2]^{-1}. \end{array} \end{array}.$$

We write $\gamma(\omega)$ for a one-loop graph which as a dressed graph can be identified with the partition ω and we regard partitions up to cyclic permutations.

Proposition 4. *For all ω, ω' with $|\omega| = |\omega'|$, the relations*

$$(25) \quad \prod_{j \in \omega} \Gamma^{j+2} / [\Gamma^2]^{\|\omega\|} = \prod_{j \in \omega'} \Gamma^{j+2} / [\Gamma^2]^{\|\omega'\|}$$

define a sub-Hopf algebra with Hochschild closed one-cocycles $B_+^{1,|\omega|} = \sum_{\omega \in \{|\omega|_{|\omega|}\}} B_+^{\gamma(\omega)}$.

Proof: By construction, the above relations define invariant charges $\phi_R(Q)$ for the $|\omega|$ -point scattering amplitudes, as

$$(26) \quad X_\gamma = X_{\gamma'} \Leftrightarrow \left(\frac{\prod_{v \in \gamma'^{[0]}} \Gamma^v}{\prod_{e \in \gamma'^{[1]}} \Gamma^e} \right) = \left(\frac{\prod_{v \in \gamma^{[0]}} \Gamma^v}{\prod_{e \in \gamma^{[1]}} \Gamma^e} \right) \Leftrightarrow X_\gamma = \Gamma^{gr|\omega|} Q^2,$$

for all $\gamma = \gamma(\omega)$, $\gamma' = \gamma'(\omega')$ and

$$(27) \quad Q = \frac{\Gamma^{gr_3}}{[\Gamma^{gr_2}]^{3/2}} = \frac{[\Gamma^{gr_4}]^{1/2}}{[\Gamma^{gr_2}]} = \dots = \frac{[\Gamma^{gr_n}]^{1/n-2}}{[\Gamma^{gr_2}]^{n/(2n-4)}},$$

by construction. Hence one-cocycles B_+^γ lead to sub-Hopf algebras as in [7]. $\square$

To determine these Hopf algebras explicitly it suffices to give their linearized coproduct

$$(28) \quad \Delta'_{\text{lin}}(c_m^{gr_n}) = \sum_{j=1}^{m-1} c_j^{gr_n} \otimes c_{m-j}^{gr_n} + \sum_{j=1}^{m-1} 2j c_{m-j}^Q \otimes c_j^{gr_n},$$

where $c_m^{gr_n}$ is the sum of all m -loop graphs with n external legs, and c_m^Q is the sum of all 1PI contributions to Q in any of the representations (27) above.

3.3. KLT relations. What we expect from the above is the ability to define a sequence of renormalization conditions on Green functions G^{gr_n} , $n > 3$, such that suitably defined counterterms give an element in $\text{Spec}(G)$ in accordance with the above identities.

This is comparable to the situation in non-abelian gauge theory, where the renormalization condition of the four-gluon vertex is determined by the renormalization of the gluon propagator and three-gluon vertex such that the Slavnov Taylor identity holds [13]. Note though that the residue of a one-loop amplitude is independent of the choice of renormalization scheme.

If we go back to a non-abelian gauge theory, the identity (22) leads then to identities between residues of one-loop graphs which are straightforward to check in particular in a symmetric renormalization scheme, where s, t, u channels renormalize in the same way, and kinematics becomes combinatorics.

Similarly, for gravity we need an identity which connects the residues of $(n+1)$ -graviton amplitudes to the n - and $(n-1)$ -graviton amplitudes. It suffices to consider a new external leg at zero momentum transfer as we are only interested in the residue.

At zero momentum transfer, with $V(k)_{\mu_1\mu_2;\nu_1\nu_2;\rho_1\rho_2} \equiv V(k)_{\underline{\mu\nu\rho}}$ the tree-level vertex for a zero-momentum graviton coupling to a free propagating graviton with momentum k , $\Pi(k)_{\alpha_1\alpha_2;\beta_1\beta_2} \equiv \Pi(k)_{\underline{\alpha\beta}}$ the comparison of the residue in a $n+1$ and n graviton one-loop amplitude reduces to a comparison

$$(29) \quad \Pi(k)_{\underline{\alpha\beta}}(k) \leftrightarrow \Pi(k)_{\underline{\alpha\mu}} V(k)_{\underline{\mu\nu\rho}} \Pi(k)_{\underline{\rho\beta}}.$$

In general, this has no particular structure which allows to proceed [14]. The situation improves considerably though when one uses Feynman rules as suggested by the KLT relations [14], which were put to use profitably in recent years relating perturbative gravity to non-abelian gauge theory [14]. For our analysis of DSE they lead to the identity

$$(30) \quad \Pi(k)_{\underline{\alpha\mu}} V(k)_{\underline{\mu\nu\rho}} \Pi(k)_{\underline{\rho\beta}} = \Pi(k)_{\underline{\alpha\beta}} \frac{k_{\nu_1} k_{\nu_2}}{k^2},$$

using that for KLT Feynman rules one has

$$(31) \quad k^2 \Pi(k)_{\underline{\mu\alpha}} \Pi(k)_{\underline{\alpha\nu}} = \Pi_{\underline{\mu\nu}},$$

$$(32) \quad V(k)_{\underline{\mu\nu\rho}} = V(k)_{\mu_1\nu_1\rho_1} V(k)_{\mu_2\nu_2\rho_2}.$$

One hence relates those residues by insertion of the scale-invariant tensor $\frac{k_{\nu_1} k_{\nu_2}}{k^2}$ into the n -graviton one-loop integral, which relates the residues, actually the full Mellin transforms, by contraction with a metric tensor. This immediately reduces relations between residues to combinatorics which determines the choice of a character on the Hopf algebra so that the relations (23) are fulfilled. The fact that there are no further primitives at higher loop order allows these relations to iterate into the DSE. Explicit computations and an extension to ghost and matter fields is beyond the scope of this paper.

4. REMARKS AND CONCLUSIONS

We finish this short paper with a few remarks concerning the structure of theories with a power-counting as above such that propagators and vertices cancel in their contributions to the superficial degree of divergence. We call theories with such a powercounting leg-renormalizable, to contrast them from the ordinary (loop)-renormalizable theories.

4.1. Diffeomorphism invariance and residues. We might accept that local geometry influences the renormalization conditions for Green functions, but the diffeomorphism invariance of residues leaves their relation invariant in accordance with (23), and sits well with a the conceptual set-up of a gravity theory.

4.2. Duality. The loop-to-leg duality which we observe here deserves further investigation. Extending the first steps done here by incorporating matter and ghost fields, and hence combining the Hopf algebra structure above with the dual one of renormalizable fields hints towards a pairing between the loop- and leg-renormalizable theories which deserves clarification.

4.3. Free Theory. There is a natural resource to find other leg-renormalizable theories: start from a loop-renormalizable theory with coupling constant g say and impose a non-linear field transformation. Even setting the coupling constant g to zero after the transformation leaves us with an interacting theory which indeed has vertices whose power-counting is inverse to the contributions of propagators, by construction. This should be a good starting point to come to a better algebraic understanding of loop- and leg-renormalizable theories, and their interplay.

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