

SKETCH OF A PROGRAM FOR UNIVERSAL AUTOMORPHIC FUNCTIONS TO CAPTURE MONSTROUS MOONSHINE

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ABSTRACT. We review and reformulate old and prove new results about the triad $\mathrm{PPSL}_2(\mathbb{Z}) \subseteq \mathrm{PPSL}_2(\mathbb{R}) \supset \mathrm{ppsl}_2(\mathbb{R})$, which provides a universal generalization of the classical automorphic triad $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R}) \supset \mathrm{psl}_2(\mathbb{R})$. The leading P or p in the universal setting stands for *piecewise*, and the group $\mathrm{PPSL}_2(\mathbb{Z})$ plays at once the role of universal modular group, universal mapping class group, Thompson group T and Ptolemy group. In particular, we construct and study new framed holographic coordinates on the universal Teichmüller space and its symmetry group $\mathrm{PPSL}_2(\mathbb{R})$, the group of piecewise $\mathrm{PSL}_2(\mathbb{R})$ homeomorphisms of the circle with finitely many pieces, which is dense in the group of orientation-preserving homeomorphisms of the circle. We produce a new basis of its Lie algebra $\mathrm{ppsl}_2(\mathbb{R})$ and compute the structure constants of the Lie bracket in this basis. We define a central extension of $\mathrm{ppsl}_2(\mathbb{R})$ and compare it with the Weil-Petersson form. Finally, we construct a $\mathrm{PPSL}_2(\mathbb{Z})$ -invariant 1-form on the universal Teichmüller space formally as the Maurer-Cartan form of $\mathrm{ppsl}_2(\mathbb{R})$, which suggests the full program for developing the theory of automorphic functions for the universal triad which is analogous, as much as possible, to the classical triad. In the last section we discuss the representation theory of the Lie algebra $\mathrm{ppsl}_2(\mathbb{R})$ and then pursue the universal analogy for the invariant 1-form $E_2(z)dz$, which gives rise to the spin 1 representation of $\mathrm{psl}_2(\mathbb{R})$ extended by the trivial representation. We conjecture that the corresponding automorphic representation of $\mathrm{ppsl}_2(\mathbb{R})$ yields the bosonic CFT_2 . Relaxing the automorphic condition from $\mathrm{PSL}_2(\mathbb{Z})$ to its commutant allows the increase of the space of 1-forms six-fold additively in the classical case and twelve-fold multiplicatively in our universal case. This leads to our ultimate conjecture that we can realize the Monster CFT_2 via the automorphic representation for the universal triad. This conjecture is also bolstered by the links of both the universal Teichmüller and the Monster CFT_2 theories to the three-dimensional quantum gravity.

Keywords: universal Teichmüller space, Thompson group T , loop algebra for $\mathfrak{sl}_2$, Weil-Petersson form, Maurer-Cartan form, Monstrous Moonshine, three dimensional quantum gravity

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INTRODUCTION

The idea of studying a universal Teichmüller space that contains the union of images of all classical Teichmüller spaces goes back to Lipman Bers [1], who considered the group of all quasimetric mappings of the circle $\mathbb{S}^1$ to itself modulo the Möbius group $\mathrm{PSL}_2(\mathbb{R})$.

Further development of the classical Teichmüller spaces [23] and their applications to string theory led the second-named author to a new model [24] of the universal Teichmüller space based on the group, denoted $\mathrm{Homeo}_+(\mathbb{S}^1)$, of all orientation-preserving self-homeomorphisms of $\mathbb{S}^1$ with the compact-open topology modulo the Möbius group. The problem of providing a parametrization for the new larger model of universal Teichmüller space and the corresponding group of homeomorphisms of $\mathbb{S}^1$ was also resolved in [24] by identification of the latter group with the space $\mathcal{Tess}'$ of all ideal tessellations of the hyperbolic plane together with a choice of distinguished oriented edge, or simply *doe*, that is,

$$(0.1) \quad \mathrm{Homeo}_+(\mathbb{S}^1) \approx \mathcal{Tess}'.$$

In particular, the identity element in $\mathrm{Homeo}_+(\mathbb{S}^1)$ corresponds to a special tessellation called the Farey tessellation. It is obtained by applying the modular group $\mathrm{PSL}_2(\mathbb{Z})$ to the *doe* running from 0 to ∞ . In view of the isomorphism (0.1), one can consider coordinates on $\mathcal{Tess} = \mathcal{Tess}'/\mathrm{PSL}_2(\mathbb{R})$ as a measure of the distortion of a given tessellation from the specified Farey tessellation by so called shearing coordinates associated to every nonoriented edge of the Farey tessellation, or equivalently to the elements of $\mathrm{PSL}_2(\mathbb{Z})/(\mathbb{Z}/2)$, where the $\mathbb{Z}/2$ subgroup reverses the orientation of edges. The shearing coordinates can be realized as logarithms of cross-ratios of certain hyperbolic lengths and are invariant under the action of the Möbius group. As a result we obtain an injection

$$(0.2) \quad \mathrm{Homeo}_+(\mathbb{S}^1)/\mathrm{PSL}_2(\mathbb{R}) \rightarrow \prod_{e \in \{\text{edges}\}} \mathbb{R}_+^e,$$

where $\mathbb{R}_+^e$ is a copy of $\mathbb{R}_+$ associated to the edge e .

Two questions arise in relation to the parametrization (0.2): the first is how to characterize its image, and the second is how to circumvent the factorization by the Möbius group and obtain directly the coordinates of $\mathrm{Homeo}_+(\mathbb{S}^1)$ itself. To answer these questions, we introduce new

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holographic coordinates and framings in Sections 2 and 3 the paper. These new coordinates are actually based upon elaborations of the shearing coordinates defined on a decorated bundle from [23] over the universal Teichmüller space which was central also in [24].

The shearing coordinates as well as our holographic coordinates admit especially simple transformation under the dense subgroup $\mathrm{PPSL}_2(\mathbb{R}) \subseteq \mathrm{Homeo}_+(\mathbb{S}^1)$. This subgroup in its turn contains a discrete subgroup $\mathrm{PPSL}_2(\mathbb{Z}) \subseteq \mathrm{PPSL}_2(\mathbb{R})$ of piecewise $\mathrm{PSL}_2(\mathbb{Z})$ homeomorphisms with rational breakpoints between pieces. Elements of $\mathrm{PPSL}_2(\mathbb{Z})$ turn out automatically to be once-continuously differentiable on the circle. This pair of groups contains the classical pair $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R})$, which is the first hint towards the new extended theory of automorphic forms.

Another discrete group that is indispensable in the Teichmüller theory is the mapping class group or Teichmüller modular group. In our universal context it is realized by the group of the Farey-type tessellations which coincide with the Farey tessellation outside of a finite polygon and known as the Ptolemy group Pt , also introduced in [24].

More precisely, the *flip* on an edge e in a tessellation is defined by replacing it by the other diagonal of the quadrilateral complementary to $\cup(\tau - \{e\})$ in $\mathbb{D}$; if e is the *doe*, then the flip on it is enhanced by inducing the orientation coming from the counter-clockwise rotation of e . The *Ptolemy group(oid)* Pt has objects given by tessellations with *doe* of $\mathbb{D}$ which coincide with the Farey tessellation outside of a finite polygon and morphisms given by finite compositions of flips. Triangulations with *doe* are combinatorially rigid, and this allows flips to be labeled by edges of a fixed tessellation, so words in these labels render Pt in fact a group. Furthermore, $\mathrm{PSL}_2(\mathbb{Z})$ sits inside Pt as those tessellations which are identical to the Farey tessellation except perhaps for the location of the *doe*.

The remarkable fact of the universal setting is that under the isomorphism (0.1), these two discrete subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ coincide $\mathrm{PPSL}_2(\mathbb{Z}) \approx \mathrm{Pt}$. Furthermore, this universal mapping class group is also isomorphic to the celebrated Thompson group T , so

$$(0.3) \quad \text{Thompson } \mathrm{T} \approx \mathrm{PPSL}_2(\mathbb{Z}) \approx \text{Ptolemy } \mathrm{Pt}.$$

The rich combinatorial structure of T is studied in [5] and numerous sequels. In particular, T admits a presentation by means of two generators and certain relations similar to those of the modular group.

The classical theory of automorphic forms on $\mathrm{PSL}_2(\mathbb{R})$ involves, besides the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$, also the large class of discrete

subgroups $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ commensurable with the modular group. In our universal setting, one considers a similar class of infinite discrete groups $P\Gamma$ associated to such subgroups $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$. By definition, $P\Gamma$ is the subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ consisting of piecewise Γ homeomorphisms with finitely many rational breakpoints. To study these groups, one can consider again the paving, i.e., decomposition into finite-sided congruent ideal polygons, determined by the action of Γ on the disc from 0 to ∞ . In Section 1, we consider the special case when $\Gamma = \mathrm{PSL}_2(\mathbb{Z})'$ is the commutant of the modular group. We also review the results of [19] about the class of finitely generated such groups $P\Gamma$, which turn out to be precisely the groups of genus zero. This is a hint towards Monstrous Moonshine, and we conclude Section 1 with the problem of characterizing those $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ that occur in Monstrous Moonshine in these terms.

In the classical theory of automorphic functions, besides the pair of groups $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R})$, the Lie algebra $\mathfrak{psl}_2 = \mathfrak{psl}_2(\mathbb{R})$ plays a pivotal role; we employ this notation $\mathfrak{psl}_2$ in the current discussion instead of the more standard notation $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{R})$ simply to emphasize the relationship with the associated Lie group. To develop the universal counterpart of the classical theory, one requires a suitable Lie algebra for the topological group $\mathrm{Homeo}_+(\mathbb{S}^1)$. It has been argued in [20] that this infinite-dimensional counterpart is precisely the algebra of piecewise $\mathfrak{sl}_2$ vector fields on the circle with finitely many pieces and rational breakpoints between them. This Lie algebra was denoted $\mathfrak{psl}_2$, where the p stood for *piecewise*. In the present paper, this Lie algebra will be denoted $\mathfrak{ppsl}_2 = \mathfrak{ppsl}_2(\mathbb{R})$ to emphasize its relationship with the group $\mathrm{PPSL}_2(\mathbb{R})$. In Sections 4 and 5, we continue the study of this Lie algebra $\mathfrak{ppsl}_2$. In particular following [20], we find a basis parametrized by the edges of the Farey tessellation, or equivalently by $\mathrm{PSL}_2(\mathbb{Z})/(\mathbb{Z}/2)$, and explicitly derive the commutation relations in this basis. In the next Section 6, we define the central extension of the Lie algebra $\mathfrak{ppsl}_2$ viewed as a loop algebra and compare it with the central extension given by the universal Weil-Petersson 2-form first studied in [24], which naturally extends the classical Weil-Petersson Kähler form computed in [25].

All three structures of our universal triad

$$(0.4) \quad \mathrm{PPSL}_2(\mathbb{Z}) \subseteq \mathrm{PPSL}_2(\mathbb{R}) \hookrightarrow \mathfrak{ppsl}_2(\mathbb{R})$$

are combined in our construction of an automorphic 1-form on the universal Teichmüller space in Section 7. This construction is one of the main results of the paper, and it is the first step towards our program

of developing the theory of automorphic forms for the triad (0.4). Since $\mathrm{PPSL}_2(\mathbb{R})$ allows for irrational breakpoints, the triad (0.4) should more properly be written $\mathrm{PPSL}_2(\mathbb{R}) \supseteq \mathrm{PPSL}_2(\mathbb{Z}) \supsetneq \mathrm{ppsl}_2(\mathbb{R})$. One might simply restrict to rational breakpoints for $\mathrm{PPSL}_2(\mathbb{R})$ as well in order to ensure (0.4), but let us not dwell on this perhaps interesting detail.

Though the proposed new theory is expected to be of a higher level of complexity than the usual theory for the classical triad

$$\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R}) \supsetneq \mathrm{psl}_2(\mathbb{R}),$$

one can pursue the analogy with the classical case whenever possible. In particular, we argue that our automorphic 1-form on the universal Teichmüller space has its classical counterpart in the 1-form $E_2(z)dz$ on the hyperbolic plane, the universal cover of the classical modular curve, where $E_2(z)$ is the non-holomorphic weight two covariant Eisenstein series.

In Appendix B, we study its lift to an automorphic function on $\mathrm{PSL}_2(\mathbb{R})$ and show that the resulting lift generates an indecomposable representation of $\mathrm{psl}_2(\mathbb{R})$ with a one-dimensional sub-representation. This indecomposable representation together with its conjugate, and common one-dimensional representation, can be characterized as the harmonic subspace of automorphic functions, i.e., it is annihilated by the Laplace operator. Though the realization of the holomorphic and antiholomorphic discrete series of weights $4, 6, 8, \dots$ by the automorphic functions is well known, the case of weight 2 has not appeared in the literature to the best of our knowledge. Yet it is exactly an analogue of this special automorphic form which arises in our construction of a $\mathrm{PPSL}_2(\mathbb{Z})$ -invariant 1-form on universal Teichmüller space in the formal guise of the Maurer-Cartan form of $\mathrm{PPSL}_2(\mathbb{R})$.

Our example of this universal automorphic form opens a new program of research in this subject for the triad (0.4), as discussed in Section 8. One of the first challenges is to find an analogue of the indecomposable representation generated by the lift of the 1-form $E_2(z)dz$ in our universal setting. Our conjecture explained in Section 8 is that this is the infinite symmetric power $S^\infty V$ of the bosonic space that appears in the classical theory, so that $S^\infty V$ is the free bosonic field and is one of the simplest CFT_2 .

The theory of automorphic forms instantly enlarges various spaces by relaxing the automorphic property to smaller discrete groups. Thus, the passage from $\mathrm{PSL}_2(\mathbb{Z})$ to its commutant $\mathrm{PSL}_2(\mathbb{Z})'$ of index six increases the space V additively 6-fold. Similarly, we expect that

replacing $\mathrm{PPSL}_2(\mathbb{Z})$ by $\mathrm{PPSL}_2(\mathbb{Z})'$ will yield a correspondingly multiplicative increase of $S^\infty V$. Thus, we expect that this mechanism will allow the construction of the Monstrous CFT_2 of [12], and in other words capture the Monster as in the title of this paper.

In fact at the end of Section 8, we explain this from the other end of the theory by recalling that Monstrous Moonshine allows one to attach a genus zero subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ to every conjugacy class of the Monster group. Then one can construct as above a Γ -invariant 1-form $E_2^\Gamma(z)dz$, where $E_2^\Gamma(z)$ is again the non-holomorphic weight two covariant Eisenstein series. This can be lifted to a representation V^Γ of $\mathrm{PSL}_2(\mathbb{R})$ isomorphic to V but realized by a different space of automorphic functions. Then the universal analogue is conjectured to yield a twisted Monster representation Λ^Γ as in [10], and the comparison of Λ and Λ^Γ should reveal the Monster in the universal automorphic realization. It will be interesting to see the correspondence between the Thompson-like groups $\mathrm{PT} \subseteq \mathrm{PPSL}_2(\mathbb{R})$ introduced in Section 1 and the Monster Moonshine groups $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, both of which are genus zero and satisfy certain additional properties [6, 7]

Our conjecture on the relationship between automorphic forms on the universal Teichmüller space and the Monster CFT_2 is strongly supported by the links of both subjects to yet another: three-dimensional quantum gravity. In fact, the link with universal Teichmüller theory has a long history in the physics literature going back to [33]. More recently, a rigorous definition of the universal phase space of AdS_3 gravity was given in [29], and it was proven there that it can be identified with the cotangent bundle over the universal Teichmüller space in its Bers formulation. Thus, one could expect that the space of states of AdS_3 quantum gravity can be realized in some class of functions on the universal Teichmüller space. However, it was also understood in the physics literature that the Bers version of universal Teichmüller space must be enlarged to account for all states related to black holes. Indeed, such enlargement was an original motivation for [24].

It was argued in [35] that the general philosophy of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence suggests that the space of states of the simplest pure quantum gravity is precisely the Monster CFT_2 constructed in [12]. This correspondence was further supported by the explanation of Monstrous Moonshine, and in particular by the mysterious genus zero property, from the point of view of twisted state sums of three-dimensional quantum gravity [10]. However in spite of all these tantalizing observations, a rigorous mathematical theory of three-dimensional quantum gravity

is still missing. Thus, we believe that the development of the theory of universal automorphic representations can help to build a Monstertrap for all the Moonshine and at the same time help us to understand the true nature of three-dimensional quantum gravity.

1. FAREY TESSELATION AND MODULAR GROUP

Let $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ denote the integers, rational, real and complex numbers, respectively, with $\hat{\mathbb{Z}} \subseteq \hat{\mathbb{Q}} \subseteq \hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$ denoting their respective one-point compactifications by the point ∞ at infinity. Set $i = \sqrt{-1}$.

The open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is identified with the Poincaré disk model of the hyperbolic plane in the standard way, where the boundary unit circle $\mathbb{S}^1$ is identified with the circle at infinity. Also consider the upper half-plane $\mathcal{U} = \{z = x + iy \in \mathbb{C} : y > 0\}$. The *Cayley transform*

$$C : (\mathcal{U}, \hat{\mathbb{R}}) \rightarrow (\mathbb{D}, \mathbb{S}^1)$$

$$s \mapsto \frac{s - i}{s + i}$$

induces an isomorphism of pairs.

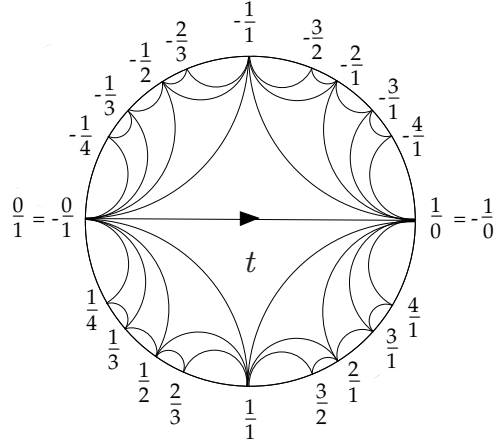


Figure 1. The first several generations of the Farey tessellation τ_* of the Poincaré disk $\mathbb{D}$ with its distinguished oriented edge.

Let t denote the ideal hyperbolic triangle with vertices $\pm 1, -i \in \mathbb{S}^1$ as in Figure 1, and consider the group $\mathcal{R}$ generated by reflections in the sides of t . Define the *Farey tessellation* τ_* to be the full $\mathcal{R}$ -orbit of the

frontier of t . τ_* has a distinguished oriented edge given by the interval from -1 to $+1$. A direct Euclidean construction of τ_* with a discussion of its history and number theoretic significance is given in [26, §3.1].

More generally, an arbitrary *tessellation* of $\mathbb{D}$ is a locally finite collection τ of hyperbolic geodesics decomposing $\mathbb{D}$ into complementary ideal triangles. Geodesics in τ are called its *edges*, and τ itself is regarded as a set of edges. Let $\tilde{\tau}$ denote the set of all oriented edges of τ , and if $e \in \tilde{\tau}$, then let $|e| \in \tau$ denote the unoriented edge underlying e . A *distinguished oriented edge* or *doe* on τ is the specification of an element $e \in \tilde{\tau}$. Let $\tau^0 \subseteq \mathbb{S}^1$ denote the set of all *vertices* of τ , namely, the collection of all endpoints of all edges in τ .

As is well-known, $\hat{\mathbb{Q}} \approx \tau_*^0$ under the Cayley transform, namely,

$$C(p/q) = \frac{p - iq}{p + iq} = \frac{p^2 - q^2}{p^2 + q^2} - i \frac{2pq}{p^2 + q^2} \in \mathbb{S}^1,$$

as indicated in Figure 1, and we shall refer to these points as the *rational points* of $\mathbb{S}^1$. We abuse notation slightly and sometimes let $\frac{p}{q} \in \hat{\mathbb{Q}}$ denote the point $C(p/q) \in \mathbb{S}^1$. $\frac{p}{q} \in \hat{\mathbb{Q}}$ is said to be of *generation* g if the radial arc in $\mathbb{D}$ from the origin to $\frac{p}{q}$ meets the interior of $g \geq 0$ distinct ideal triangles complementary to τ_* . The standard doe of τ_* runs between the two rational points of generation zero, from $\frac{0}{1}$ to $\frac{1}{0}$.

Another canonical tessellation of $\mathbb{D}$ with doe is the *dyadic tessellation* τ_d , which has the same doe as the Farey tessellation, and indeed the same generation one vertices as well, and which is recursively characterized by the property that one vertex of each triangle complementary to τ_d bisects the angle between its other two vertices. Thus, one has $\tau_d^0 = \{e^{2\pi i(1-k)} : k \in \mathbb{Z}\}$, the points in the circle with dyadic rational arguments. In contrast, τ_*^0 consists of points in the circle with rational rectilinear coordinates.

The *modular group*

$$\mathrm{PSL}_2 = \mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathcal{R}$$

of integral fractional linear transformations is the subgroup of $\mathcal{R}$ consisting of compositions of an even number of reflections, or in other words, the group of two-by-two integral matrices A of unit determinant modulo the equivalence relation generated by identifying A with $-A$. More generally, the *Möbius group*

$$\mathrm{Möb} = \mathrm{PSL}_2(\mathbb{R}) \supseteq \mathrm{PSL}_2$$

consists of the two-by-two unimodular matrices over $\mathbb{R}$ modulo the same equivalence relation. $Möb$ is the group of orientation-preserving hyperbolic isometries of $\mathbb{D}$.

In particular, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2$ acts on the right (here following Gauss) on the rational points by

$$A : \frac{p}{q} \mapsto \frac{pd - qb}{qa - pc},$$

so the edge $e_A = (doe).A$ has initial point $-\frac{b}{a} = \frac{b+ia}{b-ia}$ and terminal point $-\frac{d}{c} = \frac{d+ic}{d-ic}$.

The *Thompson group* T is the collection of all orientation-preserving piecewise homeomorphisms of $\mathbb{S}^1$ with finitely many breakpoints among τ_d^0 which are affine in the coordinate θ on each piece. Recall from the Introduction the group $\mathrm{PPSL}_2(\mathbb{Z})$ of all piecewise $\mathrm{PSL}_2(\mathbb{Z})$ homeomorphisms of $\mathbb{S}^1$ with finitely many breakpoints among the Farey rationals τ_*^0 . In fact, the Thompson group T is conjugate in $\mathrm{Homeo}(\mathbb{S}^1)_+$ to $\mathrm{PPSL}_2(\mathbb{Z})$, where the conjugating homeomorphism fixes the endpoints of the doe and maps the Farey tessellation to the dyadic tessellation of $\mathbb{D}$ in the natural way; this conjugating homeomorphism was first studied by Minkowski in [21] for its remarkable analytic properties.

Here are two standard propositions which are the starting points of our discussion:

Proposition 1.1. *The modular group PSL_2 leaves setwise invariant the Farey tessellation τ_* , mapping $\cup\tau_*$ onto $\cup\tau_*$. Any orientation-preserving homeomorphism of the circle leaving invariant τ_* in this manner lies in PSL_2 . The modular group acts simply transitively on $\tilde{\tau}_*$. A fundamental domain for the action of the modular group on $\mathcal{U}$ is given by $\{x + iy \in \mathcal{U} : x^2 + y^2 > 1 \text{ and } |x| < \frac{1}{2}\}$. $\square$*

Proposition 1.2. *A generating set for PSL_2 is given by any pair of*

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and $S^2 = 1 = R^3$ is a complete set of relations in the generators $R = T^{-1}U$ and $S = TU^{-1}T$, so $\mathrm{PSL}_2 \approx \mathbb{Z}/2 * \mathbb{Z}/3$. In fact, $T^{-1} = R^2S$ and $U = SR^2$ are each of infinite order and are conjugate in PSL_2 . $\square$

Complete relations in the generators U and T are given by $T^{-1}UT^{-1} = TU^{-1}T = UT^{-1}U = U^{-1}TU^{-1}$, with these so-called braiding relations reflecting the fact that PSL_2 is also the mapping class group of the once-punctured torus with U and T representing Dehn twists.

Geometrically, the elliptic element S setwise fixes $|e_I|$ and reverses the orientation of the doe e_I , R is the elliptic transformation cyclically permuting the vertices of the triangle to the right of the doe, and U (respectively T) is the parabolic transformation with the fixed point $\frac{0}{1}$ (respectively $\frac{1}{0}$) which cyclically permutes the incident edges of τ_* in the counter-clockwise sense about $\frac{0}{1}$ (respectively the clockwise sense about $\frac{1}{0}$). Typical aspects of our enumeration of oriented edges by elements of PSL_2 are illustrated in Figure 2.

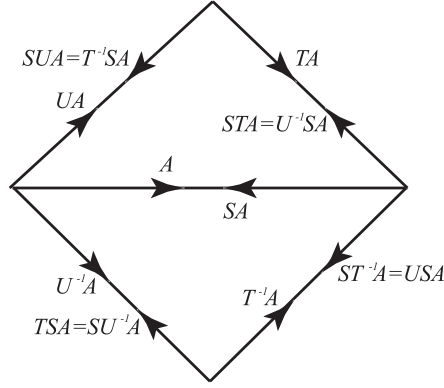


Figure 2. Enumeration of oriented edges near $|e_A| \in \tau_*$.

We turn our attention now to the *commutant* PSL'_2 of PSL_2 , namely, the subgroup of PSL_2 generated by the group commutators $aba^{-1}b^{-1}$, for $a, b \in \mathrm{PSL}_2$. As follows immediately from Proposition 1.2, the quotient $\mathrm{PSL}_2/\mathrm{PSL}'_2 \approx \mathbb{Z}/6$.

Proposition 1.3. *Consider an element of PSL_2 written as a word $W = W(U, T)$ in the generators U and T . Then the coset of the element W in $\mathrm{PSL}_2/\mathrm{PSL}'_2$ is given by the residue modulo six of the total exponent sum of W in the letters U and T^{-1} .*

Proof. Each of U and T^{-1} is infinite order and maps to the generator of $\mathbb{Z}/6 \approx \mathrm{PSL}_2/\mathrm{PSL}'_2$. Since U and T^{-1} are conjugate in PSL_2 , the word W may be written as a product of conjugates of U , which thus abelianizes to the exponent sum of U in this expression of W times the generator of $\mathbb{Z}/6$. $\square$

Corollary 1.4. *Fix any triangle complementary to $\cup \tau_*$ and consider the six possible orientations on its frontier edges. The labels in PSL_2 of these six oriented edges span the six commutant cosets. Furthermore, PSL'_2 corresponds to the collection $\mathcal{C} \subseteq \tilde{\tau}_*$ of oriented edges determined by the following conditions: the doe lies in $\mathcal{C}$; suppose an oriented edge*

$e \in \tilde{\tau}_*$ lying in $\mathcal{C}$ has ideal endpoint $v \in \mathbb{S}^1$, then every third edge of τ_* incident on v also lies in $\mathcal{C}$ with alternating orientations around v .

Proof. For the first part, consideration of Figure 2 shows that for the triangle to the right of the doe, we must show that the PSL_2' -cosets of $I, S, US, T^{-1} = T^{-1}, TS, U^{-1}$ are distinct. These have respective expressions in U, T given by $I, UT^{-1}U, U^2T^{-1}U, T^{-1}, TUT^{-1}U, U^{-1}$ with respective total exponent sums in U and T^{-1} given by $0, 3, 4, 1, 2, -1 \equiv 5$. The general case follows upon conjugation.

For the second part, we likewise argue for the doe with the general result then following by conjugation. For the doe we must prove that $SU^{\pm 1}, SU^{\pm 2} \notin \mathcal{C}$ and $SU^{\pm 3} \in \mathcal{C}$ (since obviously $U^{\pm 1}, U^{\pm 2} \notin \mathcal{C}$) and likewise for T instead of U . To this end, we write $SU = UT^{-1}U^2$, $SU^{-1} = UT^{-1}$, $SU^2 = UT^{-1}U^3$, $SU^{-2} = UT^{-1}U^{-1}$, $SU^3 = UT^{-1}U^4$ and $SU^{-3} = UT^{-1}U^{-2}$ with respective total exponent sums in U and T^{-1} given by $4, 2, 5, -1, 6 \equiv 0, 0$ as required. $\square$

The theory of the Ptolemy-Thompson group can be compared in richness only with the modular group. It is natural to ask what is the wider class of similar groups that have similar properties as $\mathrm{PPSL}_2(\mathbb{Z})$. In the case of the modular group, such a class is embodied into so-called arithmetic groups, namely, subgroups of $\mathrm{PSL}_2(\mathbb{R})$ which are commensurable with $\mathrm{PSL}_2(\mathbb{Z})$. They have an important common property with the modular group: their ideal compactification points in $\mathbb{D}/\Gamma$ are again $\hat{\mathbb{Q}}$. Thus, the piecewise Γ subgroups of $\mathrm{PPSL}_2(\mathbb{R})$, denoted by $\mathrm{P}\Gamma$, for Γ arithmetic, might serve as a natural generalization of $\mathrm{PPSL}_2(\mathbb{Z})$.

The first question one might ask about these groups $\mathrm{P}\Gamma$ is when they are finitely generated. The answer is contained in the dissertation of Laget [19]: if and only if Γ is of genus zero. But this is a hint towards a possible connection with Monstrous Moonshine, where the genus zero property also plays a key role. The groups Γ that appear in Monstrous Moonshine also have the width 1 property, namely the Farey-type tessellation or paving obtained by the action of Γ on the edge $(0, \infty)$ in $\mathbb{D}$ is 1-periodic.

These three conditions on Γ of arithmeticity, genus zero property and width 1 restrict the number of possible Γ to the large but finite number 6486 according to [8]. On the other hand, there are 194 conjugacy classes in the Monster, which give rise to groups Γ with the same three properties as above, with some identifications that reduce the number of groups Γ involved in Monster Moonshine to 171 [6]. The problem that we want to address in this conclusion to the section is:

How to characterize the Monster Moonshine Γ among arithmetic, genus 0, width 1 groups from the properties of the piecewise Γ groups PT .

So the Monster began to show up already in Thompson-like groups. But to capture it, we shall need other aspects of our universal automorphic triad: The universal Teichmüller space and the corresponding Lie algebra which we shall review and study in the next sections.

2. FRAMED HOLOGRAPHIC COORDINATES

We shall regard $ppsl_2$ as an appropriate limit of copies of sl_2 , one copy at each Farey rational point in $\mathbb{S}^1$. To take the limit, we require suitable coordinates on ideal polygons in which to calculate representations, and this section is dedicated to this end. Let us begin with the basic “lambda length” coordinates (sometimes called Penner coordinates) which we recall from [23, 26].

A *decorated ideal n -gon* is an ideal polygon of n sides in hyperbolic space together with n horocycles, one centered at each of its ideal vertices. One such coordinate is associated to each of the $2n - 3$ edges in an ideal triangulation of the n -gon including its frontier edges. As in the Introduction, there is a basic move, called a *flip*, on the interior edge of such a triangulation, where one removes the edge so as to produce a complimentary ideal quadrilateral and then replaces it with the other diagonal of this quadrilateral to finally produce another ideal triangulation.

Theorem 2.1. *Fix some $n \geq 3$ and consider a decorated ideal n -gon P in $\mathbb{D}$. Suppose that the frontier edges of P are labeled and choose an ideal triangulation Δ of P . Then the moduli space of such decorated polygons up to the natural action of $Möb$ is given by the assignment of one real lambda length $\lambda = \sqrt{\exp \ell}$ to each unoriented edge $e \in \Delta$, where ℓ is the signed hyperbolic distance along e between the horocycles centered at its endpoints, taken with a positive sign if and only if these horocycles are disjoint. Moreover, the lambda lengths are governed by the Ptolemy equation*

$$ef = ac + bd$$

whenever f arises from a flip on e in the quadrilateral bounded by a, b, c, d in this counter-clockwise cyclic order, here identifying an edge with its lambda length for convenience. $\square$

Beyond the lambda lengths, other basic coordinates are the *h-lengths* assigned to any ideal vertex v of a decorated polygon P by taking the hyperbolic distance along the horocycle centered at v between the incident frontier edges of P . A fundamental formula illustrated in Figure 3 relates lambda lengths and h-lengths in a triangle, where $\frac{\lambda_i}{\lambda_j \lambda_k}$ is the h-length opposite the edge λ_i , for $\{i, j, k\} = \{0, 1, 2\}$, and we here and hereafter again often conflate an edge with its lambda length for convenience. It follows that the product of h-lengths of consecutive vertices is the reciprocal square of the lambda length of the edge they span, i.e., $(\frac{\lambda_i}{\lambda_j \lambda_k})(\frac{\lambda_j}{\lambda_i \lambda_k}) = \frac{1}{\lambda_k^2}$, so either the triple of lambda lengths λ_i , for $i = 0, 1, 2$ or the triple of h-lengths $\frac{\lambda_i}{\lambda_j \lambda_k}$, for $\{i, j, k\} = \{0, 1, 2\}$, give coordinates on the moduli space of *Möb*-orbits of decorated ideal triangles by the previous result.

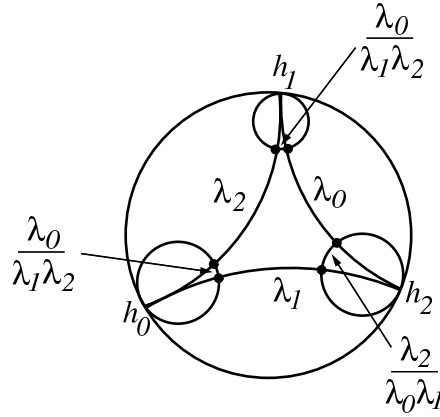


Figure 3. Three horocycles h_i whose centers span a decorated ideal triangle with opposite lambda lengths λ_i and adjacent h-lengths $\frac{\lambda_i}{\lambda_j \lambda_k}$ for $\{i, j, k\} = \{0, 1, 2\}$.

Corollary 2.2. [Holographic parameters on a decorated polygon]: *For any $n \geq 3$, the moduli space of decorated ideal n -gons P up to the action of *Möb* is parametrized by the assignment of the lambda length of each frontier edge together with the assignment of the h-length of each vertex of P . These parameters are constrained by three rational equations.*

Proof. The proof is by induction on n , and the basis step $n = 3$ was just discussed. For the induction, choose consecutive edges a, b of P with intermediate vertex w , let T be the triangular convex hull of $\{a, b\}$, α, β the respective h-lengths in T opposite the endpoints of a, b distinct from w , and let γ denote the h-length at w of T . Consider also the h-lengths α', β' of the same vertices but in P rather than T .

Now consider the closure Q of the complement of T in P . Thus Q is an ideal $(n-1)$ -gon which inherits all the moduli of P except that the respective h-lengths in Q at u and v are taken to be $\alpha' - \alpha$ and $\beta' - \beta$. The inductive hypothesis therefore applies to Q with these parameters, and hence also to P with parameters α', β' at u, v . Notice the constraint that $\alpha\beta$ equals the reciprocal square of the lambda length of $T \cap Q$. This constraint in T accounts for dimension reduction to confirm that the three rational constraints in Q persist in P . $\square$

The significance of this parametrization arises upon consideration of the *Ptolemy groupoid* $\text{Pt}(P)$ of an ideal polygon P whose objects are the ideal triangulations of P with interior *doe*, and whose morphisms are the flips along interior edges, where the flip on the *doe* e induces an orientation on the resulting edge and is of order four, whereas the flip on any other edge is of order two; these are the *face relations*. The flips on any two edges which do not lie in the frontier of a common triangle commute, and these are the *commutativity relations*. The *pentagon relations* arise from the serial sequence of flips alternating between two edges lying in the frontier of a common triangle, which has order five, unless one of these edges is the *doe*, in which case the order is ten. See [23, 24, 26] for further details.

According to Proposition 4.5 in [24] and remarks in [23, 24], we have

Theorem 2.3. *Finite sequences of flips supported on the interior of an ideal polygon P act transitively on the collection of all its triangulations with interior *doe*. It follows that flips generate $\text{Pt}(P)$. Furthermore, a complete set of relations are given by the face, commutativity and pentagon relations.* $\square$

So here finally is the significance of the holographic parameters: Taking the quotient by flips evidently renders meaningless the notion of lambda lengths of interior edges, these edges being precisely the data are obliterated. However, the lambda lengths of its frontier edges and the h-lengths of its vertices survive flips on interior edges to give parameters on the quotient. This explains our mechanism of holography.

We must go a bit further still in order to derive useful coordinates. An ordered triple of pairwise distinct points in the circle is a member of the configuration space

$$C_3 = \{(u, v, w) \in (\mathbb{S}^1)^3 : u, v, w \text{ are pairwise distinct}\},$$

on which *Möb* acts simply transitively according to Proposition 1.1. A *framing on an ideal polygon* is the specification of an arbitrary element

of C_3 , whose members need not be among the vertices of the polygon. A framed ideal polygon does not necessarily contain any doe, though an interior doe in a triangulation evidently determines an *associated framing* given by the respective initial and terminal points of the doe followed by the third vertex of the triangle immediately to the right of the doe.

An ordered triple $(u, v, w) \in C_3$ uniquely determines a unit tangent vector in $\mathbb{D}$ as follows. There is a unique geodesic g asymptotic to u and v , and the orthogonal projection of w onto g provides a point $p \in \mathbb{D}$. The unit tangent vector to g at p , where g is oriented with w on its right, finally provides the asserted unit vector associated to (u, v, w) . In these coordinates on the unit tangent bundle to $\mathbb{D}$, the almost complex structure is furthermore conveniently described by $(u, v, w) \mapsto (w, x, v)$, where $\mathbb{S}^1 \ni x \neq w$ is the endpoint of the geodesic through p which is asymptotic to w .

Remark 2.4. The term “framing” is motivated by instantons, where the incorporation of framing dramatically simplifies the formulas and accounts for the gauge group, cf. [3]. The same phenomena occur here: the framing kills both the *Möb* action and the three relations on holographic parameters in order to produce finally useful coordinates.

To explain the utility of the new coordinates, let us consider λ and h -lengths in the upper half space model $\mathcal{U}$. A decorated ideal n -gon P is uniquely determined by a collection of pairwise disjoint points $s_i \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, for $i = 1, \dots, n$, together with a collection of Euclidean diameters δ_i of corresponding horocycles and perhaps the y -coordinate δ_∞ of a horocycle about ∞ . Evidently these are actually coordinates, not just parameters, as they satisfy no relations other than pairwise inequality among the $\{s_i\}_1^n$.

The collection of pairs (s_i, δ_i) , for $i = 1, \dots, n$, are called *framed holographic coordinates* on the space of all decorated ideal polygons, and these are our desired coordinates. To explain the relationship with framed polygons, take a fixed copy of $\mathbb{D}$ and choose a fixed Cayley map $C^{-1} : \mathbb{D} \rightarrow \mathcal{U}$. Thus, given a *Möb*-orbit of decorated ideal polygon in $\mathbb{D}$ with framing $(u, v, w) \in C_3$, there is a well-defined decorated ideal polygon in $\mathcal{U}$ gotten by post-composing C with the unique Möbius transformation mapping $C^{-1}(u), C^{-1}(v), C^{-1}(w)$ to the respective points $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$.

We have proved

Theorem 2.5. *The moduli space of all framed decorated ideal n -gons modulo the action of $M\ddot{o}b$ is given by the pairwise distinct n -tuples $\{s_i\}_1^n$ of centers of horocycles in $\hat{\mathbb{R}}$ with Euclidean diameters $\{\delta_i\}_1^n$ called the framed holographic coordinates $\{(s_i, \delta_i) : 1 \leq i \leq n\}$. The action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M\ddot{o}b$ on these coordinates is the usual right fractional linear action $s_i \mapsto \frac{ds_i - b}{-cs_i + a}$ on the $\{s_i\}_1^n$, while the $\{\delta_i\}_1^n$ scale by the derivative of A at s , so $\delta_i \mapsto \frac{\delta_i}{(a - cs_i)^2}$. $\square$*

Proof. The only point requiring clarification is the transformation law for δ coordinates, which follows from the fact [23, 26] that Euclidean diameters of horocycles scale by the derivative under a Möbius transformation. $\square$

Remark 2.6. It is unsurprising that our framed holographic coordinates are reminiscent of the dihedral coordinates of [4] since the latter, which pertain only to the case of planar surfaces, are effectively related to the antecedent lambda lengths from [23] for any punctured surface, which are elaborated in the new coordinates.

Turning finally to stabilization given a framing $\mathcal{F} = (u, v, w) \in C_3$, define the matrix

$$L_{\mathcal{F}} = \frac{1}{(w-v)(u-w)(v-u)} \begin{pmatrix} v(u-w) & u(w-v) \\ u-w & w-v \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

which maps $(u, v, w) \mapsto (\frac{0}{1}, \frac{1}{0}, \frac{1}{1})$ under the right action, and given a second framing $\bar{\mathcal{F}}$, define

$$L_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}} = L_{\mathcal{F}} \circ L_{\bar{\mathcal{F}}}^{-1} = \begin{pmatrix} a_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}} & b_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}} \\ c_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}} & d_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Now, given holographic coordinates $\{(s_i, \delta_i)\}_1^n$ with framing $\mathcal{F} = (u, v, w)$ and given (s, δ) with $s \notin \{s_i\}_1^n$, there is a unique $i^* \in \{1, \dots, n\}$ with $s_{i^*} < s < s_{i^*+1}$. For $j = 1, \dots, n+1$, define

$$(\bar{s}_j, \bar{\delta}_j) = \begin{cases} (s_j, \delta_j), & \text{if } j \leq i^*; \\ (s, \delta), & \text{if } j = i^* + 1; \\ (s_{j+1}, \delta_{j+1}), & \text{if } j > i^*, \end{cases}$$

and given a framing $\bar{\mathcal{F}} = (\bar{u}, \bar{v}, \bar{w})$ on $\{(\bar{s}_j, \bar{\delta}_j)\}_1^{n+1}$, finally let

$$\begin{aligned} s'_k &= (\bar{s}_k) L_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}}, \\ \delta'_k &= (\bar{\delta}_k) \frac{d}{ds} \left((\bar{s}_k) L_{\bar{\mathcal{F}}}^{\bar{\mathcal{F}}} \right), \end{aligned}$$

for $k = 1, \dots, n+1$.

Corollary 2.7. *In the notation above, the mapping*

$$\{(s_i, \delta_i)\}_1^n \mapsto \{(s'_i, \delta'_i)\}_1^{n+1}$$

describes the stablization of framed holographic coordinates. In particular, if the framing $(u, v, w) = (\bar{u}, \bar{v}, \bar{w})$ is constant, then the stablization is given by inclusion and re-indexing. $\square$

Remark 2.8. Furthermore according to [23, 26], the lambda length between (s, δ) and (s', δ') for $s, s' \neq \infty$ is given by $(\delta\delta')^{-\frac{1}{2}}|s - s'|$ and between (s, δ) and (∞, δ_∞) is given by $(\delta_\infty/\delta)^{\frac{1}{2}}$. On the level of the holographic parameters in Proposition 2.2 and in the foregoing notation, let c denote the lambda length of the edge between s_{i*} and s_{i*+1} , and let h_{i*} and h_{i*+1} denote the respective nearby h-lengths. If a, b denote the respective lambda lengths of the decorated edges between s and s_{i*} , s_{i*+1} , then after stabilization, the h-lengths at s_{i*} , s and s_{i*+1} are respectively given by $h_{i*} + \frac{b}{ac}$, $\frac{c}{ab}$ and $h_{i*+1} + \frac{a}{bc}$.

3. FRAMED AND DECORATED HOMEMORPHISMS OF $\mathbb{S}^1$

Having developed framings on polygons as an alternative to the specification of a doe in the previous section, let us revisit from this framed point of view a basic result, Theorem 2.3 of [24], whose proof we first recall.

Theorem 3.1. *The space*

$$\mathcal{Tess}' = \{\text{tesselations of } \mathbb{D} \text{ with doe}\}$$

with topology induced by the Hausdorff topology on $\cup \tau \subset \mathbb{D}$ is naturally homeomorphic to the space

$$\text{Homeo}_+ \approx \mathcal{Tess}'$$

with the compact-open topology.

Proof. Given $f \in \text{Homeo}_+$, the image $\tau = f(\tau_*)$ of the Farey tessellation is another tessellation of $\mathbb{D}$, and the canonical doe $e_I \in \tau_*$ maps to the doe $e = f(e_I)$ of $\tau' = (\tau, e) \in \mathcal{Tess}'$ by definition. For the inverse map given a tessellation $\tau' = (\tau, e)$ with doe $e \in \tau$, begin by mapping the respective initial and terminal points of e_I to the initial and terminal points of e . Continue by mapping the further points in τ_*^0 of the respective triangles containing $|e_I|$ complementary to τ_* to the left and right of e_I in τ_* to those to the left and right of e in τ . Proceed in this way, essentially relying on the combinatorial rigidity of a tessellation with doe,

in order to define $f^0 : \tau_*^0 \rightarrow \tau^0$, an order-preserving injection between dense subsets by construction. Using that a tessellation is locally-finite in $\mathbb{D}$ by definition, f^0 is seen to be a surjection as well, which thus interpolates a homeomorphism $f_{\tau'} \in \text{Homeo}_+$ called the *characteristic map* of τ' . That the bijective assignment $\tau' \leftrightarrow f_{\tau'}$ is bicontinuous for the stated topologies is clear, completing the proof. $\square$

A *framing on a tessellation* τ of $\mathbb{D}$ is the specification of an element of the configuration space C_3 of distinct triples in $\mathbb{S}^1$, whose members are not required to lie in τ^0 , where the framed version of the basic result is

Corollary 3.2. *The space*

$$\mathcal{Tess}^f = \{\text{tessellations of } \mathbb{D}\}$$

is naturally homeomorphic to $\mathcal{Tess}' \approx \text{Homeo}_+$.

Proof. As before for polygons, a doe e for τ determines an associated framing, and this gives an inclusion $\mathcal{Tess}' \subset \mathcal{Tess}^f$. Conversely given a framing (u, v, w) on the tessellation τ , choose a doe e_1 on τ with associated framing (u_1, v_1, w_1) , let $L_1 = L_{u,v,w}^{u_1,v_1,w_1} \in \text{Möb}$ as in the previous section, and define

$$\tau'(\tau, u, v, w, e_1) = L_1^{-1} \circ f_{\tau'}(\tau_*) \in \mathcal{Tess}'$$

with its doe induced from e_I on τ_* .

We claim that $\tau'_1 = \tau'(\tau, u, v, w, e_1)$ is independent of the choice of doe e_1 and to this end choose another doe e_2 on τ with associated framing (u_2, v_2, w_2) , let $\tau'_2 = \tau'(\tau, u, v, w, e_2)$ and $L_2 = L_{u,v,w}^{u_2,v_2,w_2}$. The composition $(L_2^{-1} \circ f_{\tau'_2})^{-1} \circ (L_1^{-1} \circ f_{\tau'_1})$ lies in PSL_2 by Lemma 1.1 since it leaves τ_* invariant, and in fact, it must be the identity since it furthermore pointwise fixes u, v, w . It follows that τ'_1 and τ'_2 have the same characteristic maps, and hence $\tau'_1 = \tau'_2$ as tessellations with doe.

The assignment $(\tau, u, v, w) \mapsto L_1^{-1} \circ f_{\tau'_1}$ is thus a well-defined inverse to the inclusion $\mathcal{Tess}' \subseteq \mathcal{Tess}^f$ induced by associated framings, which is thus a homeomorphism as required. $\square$

Let us emphasize that framings thus replace distinguished oriented edges in all regards: from determining global affine coordinates in this section, to the identification of $\mathcal{Tess}^f \approx \mathcal{Tess}'$ with Homeo_+ , to the isomorphisms $\text{Pt} \approx \text{T} \approx \text{PPSL}_2(\mathbb{Z})$ as well as their actions on tessellations.

It is straight-forward to describe the group structure on $\mathcal{Tess}^f$ induced by composition of homeomorphisms in Homeo_+ but difficult to visualize on the level of tessellations, except in special cases.

Proposition 3.3. *Consider framings $\mathcal{F}_X$ on the Farey tessellation τ_* associated to does e_X for $X \in PSL_2$. Then*

$$(\tau_*, \mathcal{F}_A) \circ (\tau_*, \mathcal{F}_B) = (\tau_*, \mathcal{F}_{BA}),$$

with the analogous statement holding for any tessellation. $\square$

Next we introduce decorated versions of the foregoing spaces and begin by defining

$$\widetilde{\mathcal{T}ess}^f = \{\text{decorated and framed tessellations of } \mathbb{D}\},$$

where a *decoration* on a tessellation τ is the specification of one horocycle centered at each point of τ^0 , so there is a natural forgetful mapping to $\widetilde{\mathcal{T}ess}^f \rightarrow \mathcal{T}ess^f$, whose fiber can be identified with $\mathbb{R}^\omega$.

Taking the quotient by the Möb-action on framings, we have

$$\begin{aligned} \mathcal{T}ess^f / \text{Möb} &\approx \mathcal{T}ess^n = \{\text{tessellations } \tau : \frac{0}{1}, \frac{1}{1}, \frac{1}{0} \in \tau^0\}, \\ \widetilde{\mathcal{T}ess}^f / \text{Möb} &\approx \widetilde{\mathcal{T}ess}^n = \{\text{decorated tessellations } \tau : \frac{0}{1}, \frac{1}{1}, \frac{1}{0} \in \tau^0\} \end{aligned}$$

where the superscript n stands for *normalized*. Notice that the Farey tessellation τ_* is itself already normalized.

As for the analogous elaboration of circle homomorphisms, first define spaces

$$\text{Homeo}_+^n = \{f \in \text{Homeo}_+ : f(t) = t \text{ for } t = \frac{0}{1}, \frac{1}{1}, \frac{1}{0}\}$$

of *normalized* homeomorphisms, so $\mathcal{T}ess^n \approx \text{Homeo}_+^n$ as Homeo_+^n -spaces. Next define the decorated version

$$\widetilde{\text{Homeo}}_+^n = \{(\tilde{f}, f) : f \in \text{Homeo}_+^n \text{ covered by } \tilde{f} : \widetilde{\mathcal{T}ess}^n \rightarrow \widetilde{\mathcal{T}ess}^n\}$$

with the natural group structure, making $(\tilde{f}, f) \mapsto f$ a group homomorphism $\widetilde{\text{Homeo}}_+^n \rightarrow \text{Homeo}_+^n$. There is analogously an isomorphism $\widetilde{\text{Homeo}}_+^n \approx \widetilde{\mathcal{T}ess}^n$ of Homeo_+^n -spaces, where (identity, identity) corresponds to the Farey tessellation with its canonical decoration $\tilde{\tau}_*$, namely, the one determined by the condition that the horocycles at endpoints of any common geodesic are taken to be osculating.

This leads to the commutative cube

$$\begin{array}{ccccc}
\widetilde{\mathcal{T}ess^f} & \xrightarrow{\approx} & \widetilde{\text{Homeo}_+} & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
& \mathcal{T}ess^f & \xrightarrow{\approx} & \text{Homeo}_+ & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\widetilde{\mathcal{T}ess^n} & \xrightarrow{\approx} & \widetilde{\text{Homeo}_+^n} & & \\
\downarrow & \downarrow & \downarrow & \searrow & \\
& \mathcal{T}ess^n & \xrightarrow{\approx} & \text{Homeo}_+^n &
\end{array}$$

where $\widetilde{\text{Homeo}_+}$ is defined by pull-back, the vertical maps are principal Möb-bundles, and the maps out of the plane of the page are $\mathbb{R}^\omega$ -bundles.

Theorem 3.4. *Lambda lengths on edges of τ_* give global affine coordinates on $\widetilde{\mathcal{T}ess^n} \subseteq \mathbb{R}_+^*$. Indeed, there is an explicit construction of a Möb-orbit of tessellations of $\mathbb{D}$ with doe from the assignment of a putative lambda length to each edge of τ_* . $\square$*

The proof of this result as Theorem 3.1 in [24] provides the explicit recursive construction of a Möb-orbit of decorated tessellations of $\mathbb{D}$ with doe from the assignment of a putative lambda length to each edge of τ_* , in analogy to the proof of Theorem 2.1. The canonical decoration $\tilde{\tau}_*$ on τ_* , or in other words the identity element of the group $\widetilde{\text{Homeo}_+^n}$, corresponds to taking all these lambda length coordinates equal to unity. In the classical setting of a suitably decorated punctured surface of finite topological type uniformized by a torsion free subgroup of finite-index in PSL_2 , or a so-called *punctured arithmetic surface*, the classical lambda length coordinates on decorated Teichmüller space from [23] are likewise all unity, so this identity element in $\widetilde{\text{Homeo}_+^n}$ may be regarded as the universal punctured arithmetic surface.

Remark 3.5. It is not difficult to descend these lambda length coordinates on the bottom-back of the previous commutative cube to its bottom-front by assigning cross ratios $\frac{ac}{bd}$, or shear coordinates $\ln \frac{ac}{bd}$, instead of lambda lengths e to each edge, where e is the diagonal of the quadrilateral with frontier edges a, b, c, d in this cyclic order; see [24].

We next discuss framed tessellations as limits of framed polygons. To this end, notice that there is the natural linear ordering on $\hat{\mathbb{Q}}$ arising from the lexicographic ordering on pairs given by Farey generation

followed by the counter-clockwise order in $\mathbb{S}^1$ starting from $\frac{0}{1}$, and there is thus an induced bijective enumeration $\mathbb{Z}_{\geq 0} \rightarrow \hat{\mathbb{Q}}$.

Remark 3.6. This is the *raison d'être* for the Farey construction. Indeed, the mineralogist Farey published without proof this solution to the long-standing open problem of producing an explicit bijective enumeration of the rational numbers, and the proof was essentially immediately supplied by Cauchy.

Construction 3.7. Suppose that $S \subseteq \mathbb{S}^1$ is a countable dense subset enumerated by the bijection $\mu : \mathbb{Z}_{\geq 0} \rightarrow S$. Construct a function $f_\mu : \hat{\mathbb{Q}} \rightarrow S$ as follows. For the basis step of our recursive construction, define $f_\mu(\frac{0}{1}) = \mu(0)$ and $f_\mu(\frac{1}{0}) = \mu(1) \in \mathbb{S}^1$. Recursively suppose that the images of the Farey points of generation at most $g \geq 0$ have been defined. This collection of image points decomposes $\mathbb{S}^1$ into 2^{g+1} open circular intervals. Each such interval with endpoints $x = f_\mu(\frac{p}{q}), y = f_\mu(\frac{r}{s})$ contains a point $z \in S$ of least index $\mu^{-1}(z)$, and we define $f_\mu(\frac{p+r}{q+s}) = z$ in this case, thereby extending the function f_μ to generation $g+1$. We furthermore derive a collection $\tau_\mu = \{f_\mu(e) : e \in \tau_*\}$ of geodesics with doe $f_\mu(e_I)$, where $f_\mu(e)$ denotes the geodesic in $\mathbb{D}$ with endpoints the f_μ -image of the endpoints of $e \in \tau_*$.

Suppose that $\mu : \mathbb{Z}_{\geq 0} \rightarrow S$ is a bijective enumeration of a countable dense subset $S \subseteq \mathbb{S}^1$. A circular interval $I \subseteq \mathbb{S}^1$ with endpoints $x, y \in S$ is said to be *solid* provided $\mu^{-1}(z) > \max\{\mu^{-1}(x), \mu^{-1}(y)\}$ for every z in the interior of I . The enumeration μ is said to be *convergent* if every infinite proper nested family $I_0 \supsetneq I_1 \supsetneq \cdots$ of solid open intervals is disjoint from S , that is, $S \cap \bigcap_{j \geq 0} I_j = \emptyset$.

For any $e \in \tau_*$ other than the doe, exactly one of the circular intervals in $\mathbb{S}^1$ complementary to its endpoints is solid, and the Farey enumeration of $\hat{\mathbb{Q}}$ is convergent by definition of the Farey ordering. We may always assume that an enumeration of a countable dense subset is indexed by the Farey tessellation in its canonical linear ordering.

Proposition 3.8. *For any convergent bijective enumeration $\hat{\mathbb{Q}} \rightarrow S$ of a countable dense subset $S \subset \mathbb{S}^1$, Construction 3.7 yields a bijection $f_\mu : \hat{\mathbb{Q}} \rightarrow S$ and a tessellation $\tau_\mu = f_\mu(\tau_*)$ with doe. Conversely, if $f_\mu : \hat{\mathbb{Q}} \rightarrow S$ is a bijection, then τ_μ is a tessellation, and μ must be convergent.*

Proof. Suppose that μ is convergent and $z \in S$. For $g \geq 0$, let I_g denote the component of $\mathbb{S}^1 - \{f_\mu(x) : x \in \hat{\mathbb{Q}} \text{ is of generation at most } g\}$ which contains z . It follows that $I_0 \supsetneq I_1 \supsetneq \cdots$ is a nested family

of solid intervals. This sequence must terminate, for otherwise $z \in S \cap \bigcap_{g \geq 0} I_g \neq \emptyset$ contradicts convergence of μ . Thus, f_μ maps onto S and is injective by construction, and so is a bijection.

To see that $\tau_\mu = f_\mu(\tau_*)$ is locally finite, suppose in order to derive a contradiction that a sequence of points in $e_i = f_\mu(e'_i)$, for $i \geq 1$ and $e'_i \in \tau_*$, accumulates at some point in $\mathbb{D}$. It follows that e_i limits to some geodesic e_∞ in $\mathbb{D}$. Since the $\{e_i\}$ are pairwise disjoint in $\mathbb{D}$ by construction, we may assume that they all lie on one side of e_∞ . By density of S in $\mathbb{S}^1$, there is some $z \in S$ on the other side of e_∞ , and it lies in the f_μ -image of some generation g point of $\hat{\mathbb{Q}}$ by the already established surjectivity. Taking a further subsequence if necessary, the complementary intervals to the endpoints of e_i that contains z are solid. The generation of these endpoints is therefore bounded above by g , so that $\{e_i\}$ is a finite set, as required. The same argument shows that each component of $\mathbb{D} - \cup \tau_\mu$ is an ideal triangle, so τ_μ is indeed a tessellation.

Conversely, suppose that $f_\mu : \mathbb{Q} \rightarrow S \subseteq \mathbb{S}^1$ is surjective and $z \in S \cap \bigcap_{j \geq 1} I_j$ for some nested sequence $I_0 \supsetneq I_1 \supsetneq \dots$ of solid intervals with endpoints in S . The point $z \in S$ must a fortiori have some fixed generation, which bounds above the generations of the endpoints of the intervals. Since there are only finitely many points of any given generation, the sequence must terminate, as required. $\square$

It is an easy matter now to stabilize nested, decorated and framed ideal polygons:

Theorem 3.9. *Consider a nested sequence*

$$P_0 \subseteq P_1 \subseteq \dots \subseteq P_g \subseteq P_{g+1} \subseteq \dots$$

of ideal polygons with a common framing $\mathcal{F} \in C_3$, where P_0 is a geodesic, P_g has 2^{g+1} sides for $g \geq 1$, and $P_{g+1} - P_g$ consists of 2^g ideal triangles, for all $g \geq 0$. If the union of the ideal vertices of all the polygons is dense, then there is a well-defined limiting tessellation also framed by $\mathcal{F}$, and any such is conversely given by such a limit. The analogous statement holds for decorated framed polygons and decorated framed tessellations. Moreover, the tuple of framed holographic coordinates

$$(s_{\frac{0}{1}}, \delta_{\frac{0}{1}}), (s_{\frac{1}{0}}, \delta_{\frac{1}{0}}), (s_{\frac{1}{1}}, \delta_{\frac{1}{1}}), (s_{-\frac{1}{1}}, \delta_{-\frac{1}{1}}), \dots$$

in $(\hat{\mathbb{R}} \times \mathbb{R}_{>0})^\omega$ of decorated polygons in the weak topology given in their natural Farey ordering, where the $\{s_i\}$ are required to be pairwise distinct and $\cup\{s_i\}$ to be dense in $\hat{\mathbb{R}}$, provide global coordinates on $\widetilde{\mathcal{Tess}^f} \approx \widetilde{\text{Homeo}_+}$. $\square$

4. THE LIE ALGEBRA $p\mathfrak{psl}_2$

Let $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{R})$ denote the usual Lie algebra of traceless two-by-two real matrices with generators $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$ and Lie brackets $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. (Context will distinguish the notational inconvenience of determining between $e \in \mathfrak{sl}_2$ or $e \in \tilde{\tau}$.)

Exponentiating $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}_2$ yields a one-parameter family of diffeomorphisms giving rise to a vector field on the circle which is given by

$$\{(\gamma + \beta) \cos \theta + 2\alpha \sin \theta + (\gamma - \beta)\} \frac{\partial}{\partial \theta},$$

where $\frac{\partial}{\partial \theta}$ denotes the constant unit vector field on $\mathbb{S}^1$ and θ its usual angular coordinate. A vector field A on $\mathbb{S}^1$ arising in this way is called a (*global*) $\mathfrak{sl}_2$ vector field, and we write $A \in \mathfrak{sl}_2$ in this case. The values of an $\mathfrak{sl}_2$ vector field at any three distinct points in $\mathbb{S}^1$ determine it uniquely. In case two vector fields ϑ_1 and ϑ_2 on $\mathbb{S}^1$ differ by a global $\mathfrak{sl}_2$ vector field, then we shall write $\vartheta_1 \doteq \vartheta_2$.

More generally, a vector field ϑ on $\mathbb{S}^1$ is a *piecewise $\mathfrak{sl}_2$ vector field* if $\mathbb{S}^1$ decomposes into *finitely many* open connected circular intervals with pairwise disjoint interiors whose endpoints are among the rational points of $\mathbb{S}^1$ so that ϑ restricts on the interior of each such interval to some global $\mathfrak{sl}_2$ vector field. $p\mathfrak{psl}_2$ denotes the collection of all such vector fields including the possibility of no breakpoints, namely $\mathfrak{sl}_2 \subseteq p\mathfrak{psl}_2$ itself. The endpoints of the *maximal such intervals* are called the *breakpoints* of ϑ itself. There are no restrictions on the behavior of ϑ at its breakpoints (except that the breakpoints, if any, are rational), and indeed $\vartheta \in p\mathfrak{psl}_2$ may not even be defined at its breakpoints in general, in which case its value is implicitly given as the average of the two one-sided limits.

Remark 4.1. Let us parenthetically recall a seminal result of Dirichlet, which he proved in Berlin at age 24 thereby besting the top mathematicians of his day: the Fourier series of a piecewise smooth function with

finitely many pieces converges pointwise to the function itself except at the breakpoints where it is the average of its two one-sided limits.

A bracket $[\vartheta_1, \vartheta_2] \in ppsl_2$ of two $\vartheta_1, \vartheta_2 \in ppsl_2$ is defined in the natural way: the resulting vector field has preliminary breakpoints given by the union of those of ϑ_1 and those of ϑ_2 , and on each complementary component in $\mathbb{S}^1$ of this union, the bracket is given by the usual bracket on sl_2 . It may happen that the actual breakpoints of $[\vartheta_1, \vartheta_2]$ form a proper subset of the preliminary breakpoints since we demand maximality of intervals complementary to the actual breakpoints.

Consider breakpoints $\pm 1, \pm i \in \mathbb{S}^1 \subseteq \mathbb{C}$, so the complementary intervals lie in respective quadrants I-IV in the complex plane enumerated as usual in the counterclockwise sense beginning with quadrant I where both coordinates are non-negative, and define

$$\Lambda(s) = \begin{cases} \begin{pmatrix} s & s-s^{-1} \\ 0 & s^{-1} \end{pmatrix}, & \text{on quadrant I;} \\ \begin{pmatrix} s^{-1} & 0 \\ s-s^{-1} & s \end{pmatrix}, & \text{on quadrant II;} \\ \begin{pmatrix} s^{-1} & 0 \\ s^{-1}-s & s \end{pmatrix}, & \text{on quadrant III;} \\ \begin{pmatrix} s & s^{-1}-s \\ 0 & s^{-1} \end{pmatrix}, & \text{on quadrant IV,} \end{cases}$$

for $s \in \mathbb{R} - \{0\}$, a one-parameter family in $\mathrm{P}\mathrm{P}\mathrm{S}\mathrm{L}_2(\mathbb{R})$. It is not difficult to check that each $\Lambda(s)$ is moreover once-continuously differentiable on $\mathbb{S}^1$ including at its breakpoints.

Since lambda lengths are *Möb*-invariant, it is clear that $\Lambda(s)$ leaves invariant all lambda lengths on τ_* with any decoration, except the lambda length of the doe, which it scales by a factor s . Thus, $\Lambda(s)$ is a one-parameter family in $\mathrm{P}\mathrm{P}\mathrm{S}\mathrm{L}_2(\mathbb{R})$ scaling just one affine lambda length coordinate, or in other words a multiplicative coordinate deformation.

More specifically, $\Lambda(s) \in \mathrm{P}\mathrm{P}\mathrm{S}\mathrm{L}_2(\mathbb{R})$ is a one-parameter family which on each circular interval determined by the intersection of a quadrant with $\mathbb{S}^1$ is a hyperbolic transformation whose axis is spanned by the endpoints of the interval. In the parlance of the Thurston school, an *earthquake* is an element of $\mathrm{P}\mathrm{P}\mathrm{S}\mathrm{L}_2(\mathbb{R})$ with two pieces, one of which is the identity and the other is a hyperbolic transformation whose axis is asymptotic to the breakpoints. Thus, $\Lambda(s)$ is a special one-parameter family of compositions of four earthquakes, where the four hyperbolic translations are chosen in order to produce homeomorphisms $\Lambda(s)$ that are once-continuously differentiable on $\mathbb{S}^1$.

The derivative of $\Lambda(s)$ with respect to s at $s = 1$ is the extremely special element $\vartheta \in ppsl_2$ illustrated in Figure 4 and called the *mother*

wavelet. It is the basic building block of $p\text{psl}_2$. Justification for the appellation “wavelet” used here is given in [27].

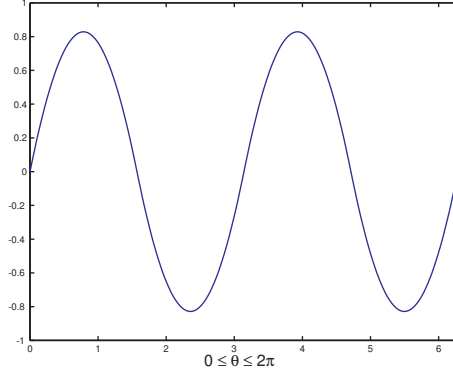


Figure 4. The mother wavelet ϑ .

Direct computation confirms that the mother wavelet ϑ is given by

$$\vartheta = \begin{cases} +h + 2e, & \text{in quadrant I;} \\ -h + 2f, & \text{in quadrant II;} \\ -h - 2f, & \text{in quadrant III;} \\ +h - 2e, & \text{in quadrant IV} \end{cases}$$

and vanishes at each of its breakpoints, where it is once-continuously differentiable. Notice that in general if an element of $p\text{psl}_2$ is *twice* continuously differentiable at a breakpoint, then it is not actually a breakpoint at all, that is, if two elements of $M\ddot{o}b$ agree to second order at a point, then they must coincide.

Let us employ the adjoint action on each piece and define the (*arithmetic*) *wavelets*

$$\vartheta_A(\theta) = A^{-1}\vartheta_A(\theta.A)A, \text{ for } A \in \text{PSL}_2,$$

where the right A -action on $\theta \in \mathbb{S}^1$ is the natural one. A short calculation shows that $\vartheta_S = \vartheta_I = \vartheta$, and therefore if A_+, A_- correspond to the two different orientations on a common edge, i.e., if $A_{\mp} = SA_{\pm}$ then $\vartheta_{A_+} = \vartheta_{A_-}$. It follows that for any unoriented edge we have $|e_A| = |e_{SA}|$, and there is a corresponding vector field $\vartheta_{|e_A|} = \vartheta_A = \vartheta_{SA}$.

Now for each $A \in \text{PSL}_2$, define a corresponding $X_A \in \mathfrak{sl}_2$ where X_A and ϑ_A take the same values at the three points $\pm 1, -i \in \mathbb{S}^1$, or in other words at Farey points $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$, and define the *normalization*

$$\bar{\vartheta}_A = \vartheta_A - X_A, \text{ for } A \in \text{PSL}_2.$$

It is not difficult to compute X_A explicitly for $A \in \text{PSL}_2$.

Here and herein we shall deviate slightly from the notation of previous works, which we first recall.

Following [20, 26] for each $A \in \mathrm{PSL}_2$, define for each $A \in \mathrm{PSL}_2$ the infinite sums

$$\bar{\phi}_A = \sum_{n \geq 0} \bar{\vartheta}_{U^n A} \text{ and } \bar{\phi}_A^* = \sum_{n \leq 0} \bar{\vartheta}_{U^n A}$$

respectively called *normalized left and right fans* and the further infinite sums

$$\bar{\psi}_A = \sum_{n \geq 0} n \bar{\vartheta}_{U^n A} \text{ and } \bar{\psi}_A^* = \sum_{n \leq 0} n \bar{\vartheta}_{U^n A}$$

respectively called *normalized left and right hyperfans*. We have added bars to the notation for normalized (hyper)fans from earlier work and keep the undecorated symbol for more natural normalizations to be introduced in the next section. It is not difficult to compute that $\sum_{n \geq 0} X_{U^n A}$ diverges thus explaining the need for normalization.

Assuming for the moment that these sums converge, as we shall discuss presently, there is the following prescribed consequence of this “hyperfan formalism” given by infinite sums of infinite sums in this way.

Proposition 4.2. *For each $A \in \mathrm{PSL}_2$, we have*

$$\bar{\psi}_A = \sum_{n \geq 1} \bar{\phi}_{U^n A} \text{ and } \bar{\psi}_A^* = \sum_{n < 0} \bar{\phi}_{U^n A}^*,$$

as well as

$$\begin{aligned} \bar{\psi}_A - \bar{\psi}_{UA} &= \bar{\phi}_{UA} \text{ and } \bar{\phi}_A - \bar{\phi}_{UA} = \vartheta_A, \\ \bar{\psi}_{UA}^* - \bar{\psi}_A^* &= \bar{\phi}_A^* \text{ and } \bar{\phi}_{UA}^* - \bar{\phi}_A^* = \vartheta_A^* \end{aligned}$$

□

Corollary 4.3. *For each $A \in \mathrm{PSL}_2$, equating $\bar{\vartheta}_A = \bar{\vartheta}_{SA}$ gives the so-called USA relation*

$$\begin{aligned} \bar{\psi}_{UA} - 2\bar{\psi}_A + \bar{\psi}_{U^{-1}A} &= \bar{\psi}_{USA} - 2\bar{\psi}_{SA} + \bar{\psi}_{U^{-1}SA}, \\ \bar{\psi}_{UA}^* - 2\bar{\psi}_A^* + \bar{\psi}_{U^{-1}A}^* &= \bar{\psi}_{USA}^* - 2\bar{\psi}_{SA}^* + \bar{\psi}_{U^{-1}SA}^*. \end{aligned}$$

□

The formulas in Proposition 4.2 and Corollary 4.3 thus follow immediately from the hyperfan formalism and the symmetry $\bar{\vartheta}_A = \bar{\vartheta}_{SA}$. The catch is showing that the putative (hyper)fans converge. However, notice that convergence of $\sum nx_n$ implies that of $\sum x_n$, so the hyperfan formalism follows from convergence of hyperfans alone.

In the remainder of this section, we shall recall results from [20] which in particular give sense to the *normalized* (hyper)fans as elements of $ppsl_2$, and indeed converge pointwise uniformly on compacta. The harmonic analysis of normalized hyperfans is discussed in Appendix A.

The **first big surprise** is that in fact normalized fans and hyperfans have only finitely many breakpoints and lie in $ppsl_2$, even though they are defined by infinite sums. Indeed, a fan always has exactly three breakpoints and is described by a continuous but not differentiable function on the circle, and a hyperfan always has exactly two breakpoints described by a function on the circle which is discontinuous at exactly one point. Such is the nature of telescoping for normalized (hyper)fans, which are *not enjoyed* by unnormalized (hyper)fans.

For example, $\bar{\phi}_U$ takes values $-2e$ on quadrant I, $2(h - f)$ on quadrant II and vanishes on quadrants III and IV, while $\bar{\psi}_I$ takes values $-2e$ on quadrants I and II and vanishes on quadrants III and IV. In fact, one finds that $\bar{\psi}_I + \bar{\psi}_I^*$ is the global sl_2 vector field $-2e$, and together with the fact that $\psi_A \doteq A^{-1}\psi_I A$ by construction, it follows that the span of the left hyperfans together with sl_2 contains the right hyperfans. We shall therefore henceforth restrict our attention to the former and drop the appellation “left” tacitly taking only left fans and hyperfans. There is an entirely parallel discussion using right fans and hyperfans.

Just as one might suspect, bracketing destroys one degree of smoothness. Brackets of normalized wavelets are similarly expressed as finite sums of normalized fans while brackets of normalized fans are expressed in terms of normalized hyperfans.

The **second big surprise** is that here the algebra closes with the additive basis of normalized left hyperfans, i.e., brackets of hyperfans are finite linear combinations of hyperfans.

Summarizing several of the main achievements in [20], to which we refer the reader for proofs of the first two big surprises, we have

Theorem 4.4. *The set of normalized left hyperfans together with the generators $e, f, h \in sl_2$ give an additive spanning set for the vector space $ppsl_2$ which is closed under bracketing. Moreover, the collection of USA relations in Corollary 4.3, one for each edge of τ_* , gives a complete set of relations among normalized left hyperfans. $\square$*

In fact, there is an error in the proof in [20] that normalized left hyperfans span $ppsl_2$ which is corrected in Theorem 2.1 of [27] as follows.

Theorem 4.5. *Define orientations on the edges of τ_* with the orientation from $\frac{0}{1}$ to $\frac{1}{0}$ as usual and otherwise always pointing from lower to*

higher generation in the Farey enumeration, and denote this set of oriented edges $\mathcal{O} \subset \tilde{\tau}_*$. Then $\mathcal{B} = \{\bar{\psi}_A : A \in \mathcal{O}\}$ together with $e, f, h \in sl_2$ is an additive basis for $p\mathfrak{psl}_2$. $\square$

5. THE NEW FORMALISM FOR $p\mathfrak{psl}_2$

The normalization of vector fields in the previous section was necessary in order to guarantee their pointwise convergence and assure the hyperfan formalism. However from the point of view of representation theory, the failure of exact equality in favor of equality $\bar{\psi}_A \doteq A^{-1}\bar{\psi}_I A$ up to global sl_2 vector fields introduces untoward complications. Furthermore from the point of view of physics, the treatment of sl_2 in the previous section represents merely an additional copy of sl_2 essentially decoupled from the higher Fourier modes, not the creation/annihilation and energy operators one might anticipate.

Let us remedy both of these deficiencies by defining (*unnormalized*) (or really differently normalized) *hyperfans* as follows:

$$\psi_I = \begin{cases} e, & \text{on quadrants I and II;} \\ 0, & \text{on quadrants III and IV,} \end{cases}$$

thus dropping the pre-factor -2 from before and furthermore guaranteeing the desired conjugacy formula by defining

$$\psi_A(\theta) = A^{-1}\psi_I(\theta.A)A, \text{ for } A \in \mathrm{PSL}_2.$$

It thus follows that $B^{-1}\psi_A B = \psi_{AB}$.

The **third big surprise**, which is new to this paper, is that sl_2 is actually in the finitely supported span of these new hyperfans, as we next prove.

Proposition 5.1. *For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2$, define*

$$\Psi_A = \psi_{STA} - 2\psi_{SA} + \psi_{ST^{-1}A} - \{\psi_{UA} - 2\psi_A + \psi_{U^{-1}A}\}.$$

Then we have the identity

$$\Psi_A = \{c(d+b) + a(d-b)\}h + (d^2 - b^2 + 2bd)e + (a^2 - c^2 - 2ac)f.$$

In particular

$$\Psi_I = h + e + f, \quad \Psi_T = 2e + f, \quad \Psi_{U^{-1}} = e + 2f,$$

whence

$$\begin{pmatrix} h \\ e \\ f \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \Psi_I \\ \Psi_T \\ \Psi_{U^{-1}} \end{pmatrix}$$

Proof. The reader will recognize Ψ_A as the difference of the two sides of the USA relation which holds for normalized hyperfans and fails by a global sl_2 for unnormalized hyperfans. The proposition follows from direct computation using that $\gamma - 2\beta = 2\alpha - \delta$, where

$$\begin{aligned} \alpha &= A^{-1}eA = cdh + d^2e - c^2f, \quad \beta = (SA)^{-1}eSA = abh + b^2e - a^2f, \\ \gamma &= (UA)^{-1}e(UA) = (c+a)(d+b)h + (d+b)^2e - (c+a)^2f \\ &= (STA)^{-1}e(STA), \\ \delta &= (U^{-1}A)^{-1}e(U^{-1}A) = (c-a)(d-b)h + (d-b)^2e - (c-a)^2f \\ &= (ST^{-1}A)^{-1}e(ST^{-1}A). \end{aligned}$$

□

We shall refer to the expression for Ψ_A in terms of e, f, h given in Proposition 5.1 as the *USA identity* for $A \in \mathrm{PSL}_2$.

Theorem 5.2. *The hyperfans $\{\psi_A : A \in \mathrm{PSL}_2\}$ span $p\mathrm{psl}_2$, a complete set of relations on them is given by the USA identities as A varies over PSL_2 , and $\{\psi_A : A \in \mathcal{O}\}$ provides a basis for $p\mathrm{psl}_2$, where $\mathcal{O} \subset \tilde{\tau}$ is given in Theorem 4.5.*

Proof. In the old formalism, the span of normalized hyperfans together with e, f, h was shown to be all of $p\mathrm{psl}_2$. This contains all unnormalized hyperfans by definition; indeed, $\psi_A \doteq \bar{\psi}_A$, and the reverse inclusion follows from the last part of the previous proposition. For the second assertion, suppose that $\sum \alpha_A \psi_A = 0$ is a finite linear relation among the unnormalized hyperfans. It follows that then $\sum \alpha_A \bar{\psi}_A \doteq 0$, and since both sides of this equation vanish at $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$, it must be that in fact $\sum \alpha_A \bar{\psi}_A = 0$, whence this relation is a consequence of the USA relations by Theorem 4.4. A non-trivial finite linear relation among $\{\psi_A : A \in \mathcal{O}\}$ likewise gives such a relation on $\{\bar{\psi}_A : A \in \mathcal{O}\}$, which would contradict Theorem 4.5. □

Another favorable aspect of the new formalism is that the structure constants of $p\mathrm{psl}_2$ admit an explicit if not entirely trivial expression, and it is to this end that we dedicate the remainder of this section. The next two results follow from direct and elementary computation, which are left to the reader.

Proposition 5.3. *In the basis e, f, h for sl_2 , the adjoint $x \mapsto A^{-1}xA$ of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl_2$ is given by the matrix*

$$M_A = \begin{pmatrix} d^2 & -b^2 & 2bd \\ -c^2 & a^2 & -2ac \\ cd & -ab & ad + bc \end{pmatrix},$$

and so

$$M_A^{-1} = M_{A^{-1}} = \begin{pmatrix} a^2 & -b^2 & -2ab \\ -c^2 & d^2 & 2cd \\ -ac & bd & ad + bc \end{pmatrix}.$$

□

Just as ψ_I takes values $e \in sl_2$ on quadrants I and II and vanishes on quadrants III and IV, we have the following analogous result for $f, h \in sl_2$.

Proposition 5.4. *$\psi_S + f$ and $\psi_I + \psi_{US} - \psi_S - \psi_{U^{-1}} - f$ each have support on quadrants I and II with the former taking there the value f and the latter the value h .* □

Now in order to compute the bracket $[\psi_B, \psi_A]$, for $A, B \in \mathrm{PSL}_2$, it suffices to compute simply $[\psi_I, \psi_A]$, since we have

$$(*) \quad [\psi_B, \psi_A] = [B^{-1}\psi_I B, \psi_A] = B^{-1}[\psi_I, \psi_{AB^{-1}}]B.$$

To this end with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there are four essential cases:

Case 1: $0 \leq -\frac{d}{c} < -\frac{b}{a}$, so e_A lies in the bottom half plane oriented from right to left.

Case 2: $0 \leq -\frac{b}{a} < -\frac{d}{c}$, so e_A lies in the bottom half plane oriented from left to right.

Case 3: $-\frac{d}{c} < -\frac{b}{a} \leq 0$, so e_A lies in the top half plane oriented from left to right.

Case 4: $-\frac{b}{a} < -\frac{d}{c} \leq 0$, so e_A lies in the top half plane oriented from right to left.

In each case, the support of ψ_A lies to the left of e_A by definition. In Case 1, the supports are therefore disjoint, and so $[\psi_I, \psi_A] = 0$. In Case 2, the support of ψ_I is contained in the support of ψ_A , and the

bracket is supported on quadrants I and II taking value $-c^2h + 2cde$, which is given according to Proposition 5.4 by

$$\begin{aligned} & -c^2(\psi_I + \psi_{US} - \psi_S - \psi_{U^{-1}} - f) + 2cd\psi_I \\ & = c(2d - c)\psi_I + c^2(\psi_S + \psi_{U^{-1}} - \psi_{US} + f). \end{aligned}$$

Case 3 is more challenging requiring Proposition 5.3. We compute

$$M_A^{-1} \begin{pmatrix} 2cd \\ 0 \\ -c^2 \end{pmatrix} = \begin{pmatrix} 2ac(ad + bc) \\ -4c^3d \\ -c^2(3ad + bc) \end{pmatrix},$$

whence

$$\begin{aligned} [\psi_I, \psi_A] &= 2ac(ad + bc)\psi_A - 4c^3d(\psi_{SA} + A^{-1}fA) \\ &\quad - c^2(3ad + bc)\{\psi_A + \psi_{USA} - \psi_{SA} - \psi_{U^{-1}} - A^{-1}fA\}. \end{aligned}$$

Case 4 is still a bit more involved since $[\psi_I, \psi_A]$ perhaps has two components and can be expressed as the difference $\mu - \nu$, where μ is supported on quadrants I and II taking value $2cde - c^2h$, and ν is supported on the region to the left of e_{SA} and taking the same value there. Meanwhile, Proposition 5.4 gives the expression

$$\mu = c(2d - c)\psi_I + c^2(\psi_S + \psi_{U^{-1}} - \psi_{US} + f)$$

as in Case 2. For the other term ν , we compute

$$\begin{aligned} M_{SA}^{-1} \begin{pmatrix} 2cd \\ 0 \\ -c^2 \end{pmatrix} &= \begin{pmatrix} c^2 & -d^2 & -2cd \\ -a^2 & b^2 & 2ab \\ ac & -bd & -(ad + bc) \end{pmatrix} \begin{pmatrix} 2cd \\ 0 \\ -c^2 \end{pmatrix} \\ &= \begin{pmatrix} 4c^3d \\ -2ac(ad + bc) \\ c^2(3ad + bc) \end{pmatrix}, \end{aligned}$$

whence

$$\begin{aligned} \nu &= 4c^3d\psi_{SA} - 2ac(ad + bc)(\psi_A + (SA)^{-1}fSA) \\ &\quad + c^2(3ad + bc)\{\psi_{SA} + \psi_{UA} - \psi_A - \psi_{U^{-1}SA} - (SA)^{-1}fSA\}. \end{aligned}$$

Putting all this together, a little further computation finally proves

Theorem 5.5. *For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2$, we have*

$$[\psi_I, \psi_A] = \begin{cases} 0, & \text{if } 0 \leq -\frac{d}{c} < -\frac{b}{a}; \\ c(2d - c)\psi_I + c^2(\psi_S + \psi_{U^{-1}} - \psi_{US} + f), \\ & \text{if } 0 \leq -\frac{b}{a} < -\frac{d}{c}; \\ 2ac(ad + bc)\psi_A - 4c^3d(\psi_{SA} + A^{-1}fA) \\ - c^2(3ad + bc)\{\psi_A + \psi_{USA} - \psi_{SA} - \psi_{U^{-1}} - A^{-1}fA\}, \\ & \text{if } -\frac{d}{c} < -\frac{b}{a} \leq 0; \\ c(2d - c)\psi_I + c^2(\psi_S + \psi_{U^{-1}} - \psi_{US} + f) \\ - 4c^3d\psi_{SA} + 2ac(ad + bc)(\psi_A + (SA)^{-1}fAS) \\ - c^2(3ad + bc)\{\psi_{SA} + \psi_{UA} - \psi_A - \psi_{U^{-1}SA} - (SA)^{-1}fAS\}, \\ & \text{if } -\frac{b}{a} < -\frac{d}{c} \leq 0. \end{cases}$$

□

It is worth noting that $[\psi_I, \psi_A] = 0$ if $c = 0$ and that to complete the calculation of brackets in the basis $\{\psi_A : A \in \mathcal{O}\}$, one must still conjugate as in equation (*) and finally use the constructive proof of our Theorem 5.2 given in [20] to express several of the resulting hyperfans in terms of this basis. The point, however, is that brackets are explicitly computable in the new formalism in contrast to the old.

6. THREE 2-FORMS

As before, let $\frac{\partial}{\partial\theta}$ denote the counter-clockwise unit vector field on $\mathbb{S}^1$ in its usual angular coordinate θ and $L_n = ie^{in\theta}\frac{\partial}{\partial\theta}$, for $n \in \mathbb{Z}$, denote the usual generators for the *Witt algebra*, satisfying for $m, n \in \mathbb{Z}$ the bracket identity $[L_m, L_n] = (m - n)L_{m+n}$. The Witt algebra is naturally regarded [30] as the tangent space at the identity to the manifold $\mathrm{Diff}_+ = \mathrm{Diff}_+(\mathbb{S}^1)$ of real-analytic orientation-preserving diffeomorphisms of $\mathbb{S}^1$.

As the tangent space to $\mathrm{Homeo}_+ \supseteq \mathrm{Diff}_+$ as in Section 3, $\mathfrak{ppsl}_2$ contains the Witt algebra; this inclusion is explicitly computed in Theorem A.1. Conversely for $A \in \mathrm{PSL}_2$, the Fourier expansion of the normalized

wavelets $\bar{\vartheta}_A$ (given in Theorem A.2) or hyperfans ψ_A (given in Proposition A.4), describes how $ppsl_2$ lies inside the topological closure of the Witt algebra.

The quotient manifold $\text{Diff}_+/M\ddot{o}b$, has tangent space at the identity given by the span of L_n , for $n^2 > 1$, that is, the span of $e, f, h \in sl_2$ is naturally identified with the span of L_{-1}, L_0, L_{+1} . As likewise follows from Section 3, the quotient $\text{Homeo}_+^n \approx \text{Homeo}_+/M\ddot{o}b$ can be identified with the space $\mathcal{T}ess^n$ of all normalized tessellations of $\mathbb{D}$, and the natural bundle $\widetilde{\mathcal{T}ess}^n \rightarrow \mathcal{T}ess^n$ over this space admits global affine lambda length coordinates.

There are the following several 2-forms defined on these spaces, which are compared in this section:

- The (pull-back from $\mathcal{T}ess^n$ of the) universal Weil-Petersson (WP) Kähler 2-form [25, 24] to $\widetilde{\mathcal{T}ess}^n \approx \text{Homeo}_+^n$ is given by

$$\omega = -2 \sum d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a,$$

where the sum is over the set of all triangles complementary to τ_* in $\mathbb{D}$ and the triangle has edges, here conflated with lambda lengths as usual, in the clockwise order a, b, c in the orientation on the underlying surface. This sum converges provided the homeomorphism is $C^{\frac{3}{2}+\epsilon}$ smooth on $\mathbb{S}^1$, cf. [32].

- The Kirillov-Kostant (KK) form [18, 32] is defined on $\text{Diff}_+/M\ddot{o}b$ by

$$\kappa_a(L_m, L_n) = a(m^3 - m)\delta_{m,-n},$$

where $a \in \mathbb{C}$ and δ is the Kronecker delta function.

- The natural loop-algebra (LA) 2-cocycle [13] is defined by

$$\gamma(\psi^1, \psi^2) = \frac{1}{2\pi} \int_0^{2\pi} \text{tr} \left(\psi^1(\theta) \cdot d\psi^2(\theta) \right),$$

where $\psi_1, \psi_2 \in sl_2^{\mathbb{S}^1}$ and tr denotes the trace. Our algebra $ppsl_2$ is a sub-algebra of a certain completion of $sl_2^{\mathbb{S}^1}$ on which this 2-cocycle still makes sense, and indeed integration by parts provides in this case the explicit expression

$$\gamma(\psi^1, \psi^2) = \frac{1}{2} \sum_{\theta \in \Pi(\psi^2)} \text{tr} \left\{ [\psi^1(\theta^+) + \psi^1(\theta^-)] [\psi^2(\theta^+) - \psi^2(\theta^-)] \right\}$$

for the 2-cocycle γ on $ppsl_2$, where $\Pi(\psi) \subset \hat{\mathbb{Q}} \subset \mathbb{S}^1$ denotes the set of breakpoints of ψ and $\theta^\pm$ denotes a point slightly $\frac{\text{beyond}}{\text{before}}$ the point

$\theta \in \Pi(\psi)$ in the counter-clockwise orientation on $\mathbb{S}^1$. One can verify directly that γ is a 2-cocycle by checking skew-symmetry and the 2-cocycle property, which follows from the 2-cocycle property for the first expression in this paragraph for γ on $sl_2^{\mathbb{S}^1}$.

In fact, these three 2-forms are pairwise identical up to overall constants, as we shall discuss. The constant for WP and LA will be computed here, and the constant for KK and WP was already calculated in Theorem 5.5 of [24], namely, we have

Theorem 6.1. *The WP Kähler 2-form ω and the KK form κ_a are related by*

$$\omega = \kappa_a, \text{ for } a = 2\pi i.$$

The computational proof in [24] using the basis of normalized arithmetic wavelets is involved and delicate depending upon the constraints on small Fourier modes described in Remark A.3.

Theorem 6.2. *The WP Kähler 2-form ω and the LA cocycle γ are related by*

$$\gamma = -4\omega.$$

Proof. Recall that $\Pi(\psi) \subset \hat{\mathbb{Q}} \subset \mathbb{S}^1$ denotes the (finite) set of break-points of $\psi \in ppsl_2$ and let $\bar{\vartheta}_A = \vartheta_A - X_A$, i.e., $\bar{\vartheta}_A$ is the normalized arithmetic wavelet and ϑ_A the unnormalized one. In particular, we have

$$\Pi(\bar{\vartheta}_A) = \Pi(\vartheta_A) = \Pi(\vartheta).A, \text{ for } A \in \text{PSL}_2,$$

and moreover

$$\gamma(\bar{\vartheta}_A, \bar{\vartheta}_B) = \gamma(\bar{\vartheta}_A, \vartheta_B), \text{ for } A, B \in sl_2,$$

since X_B is both added and subtracted in a difference of $\bar{\vartheta}_B$ -values in their contribution to γ . Thus

$$\gamma(\bar{\vartheta}_A, \bar{\vartheta}_B) = \gamma(\vartheta_A, \vartheta_B), \text{ for } A, B \in sl_2$$

as well by skew symmetry of γ .

We first claim that $\gamma(\bar{\vartheta}_A, \bar{\vartheta}_B) = 0$ if $\bar{\vartheta}_A$ takes a common value $Y_A \in sl_2$ at each point of $\Pi(\bar{\vartheta}_B)$. To see this, we compute

$$\begin{aligned}
\gamma(\bar{\vartheta}_A, \bar{\vartheta}_B) &= \gamma(\bar{\vartheta}_A, \vartheta_B) = \gamma(Y_A, \vartheta_B) \\
&= \sum_{\theta \in \Pi(\vartheta_B)} \text{tr} \left\{ Y_A \cdot [\vartheta_B(\theta^+) - \vartheta_B(\theta^-)] \right\} \\
&= \sum_{\theta \in \Pi(\vartheta)} \text{tr} \left\{ Y_A \cdot [\vartheta_B(\theta^+.B) - \vartheta_B(\theta^-.B)] \right\} \\
&= \text{tr} \sum_{\theta \in \Pi(\vartheta)} Y_A \left\{ B^{-1} [\vartheta(\theta^+) - \vartheta(\theta^-)] B \right\} \\
&= \text{tr} \left\{ Y_A B^{-1} \left(\sum_{\theta \in \Pi(\vartheta)} [\vartheta(\theta^+) - \vartheta(\theta^-)] \right) B \right\} \\
&= \text{tr} \left\{ Y_A B^{-1} [4e + 2(f - e - h) - 4f + 2(h + f - e)] B \right\} \\
&= \text{tr} Y_A B^{-1} 0 B \\
&= 0.
\end{aligned}$$

It remains to consider $\gamma(\bar{\vartheta}_A, \bar{\vartheta}_{UA})$ and $\gamma(\bar{\vartheta}_A, \bar{\vartheta}_{TA})$, and we begin with the former. Let ξ_B, η_B denote the respective initial and terminal point of $e_B = e_I.B$, where e_I is the doe of τ_* as usual, so

$$\begin{aligned}
\Pi(\bar{\vartheta}_A) &= \{\xi_A, \eta_A, \eta_{U^{-1}A}, \eta_{UA}\}, \\
\Pi(\bar{\vartheta}_{UA}) &= \{\xi_A, \eta_A, \eta_{UA}, \eta_{U^2A}\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\gamma &= \gamma(\bar{\vartheta}_A, \bar{\vartheta}_{UA}) = \gamma(\vartheta_A, \vartheta_{UA}) \\
&= \frac{1}{2} \sum_{\theta \in \Pi(\vartheta_{UA})} \text{tr} \{ [\vartheta_A(\theta^+) + \vartheta_A(\theta^-)] [\vartheta_A(\theta^+) - \vartheta_A(\theta^-)] \},
\end{aligned}$$

so

$$\begin{aligned}
2\gamma &= \text{tr} \left\{ \begin{aligned} &[\vartheta_A(\xi^+.A) + \vartheta_A(\xi^-.A)] [\vartheta_{UA}(\xi^+.A) - \vartheta_{UA}(\xi^-.A)] \\ &+ [\vartheta_A(\eta^+.A) + \vartheta_A(\eta^-.A)] [\vartheta_{UA}(\eta^+.A) - \vartheta_{UA}(\eta^-.A)] \\ &+ [\vartheta_A(\eta^+.UA) + \vartheta_A(\eta^-.UA)] [\vartheta_{UA}(\eta^+.UA) - \vartheta_{UA}(\eta^-.UA)] \\ &+ 2[\vartheta_A(\eta^+.U^2A)] [\vartheta_{UA}(\eta^+.U^2A) - \vartheta_{UA}(\eta^-.U^2A)] \end{aligned} \right\} \\
&= \text{tr} \left\{ \begin{aligned} &[\vartheta(\xi^+) + \vartheta(\xi^-)] [(U)^{-1}\vartheta(\xi^+.U^{-1})U - (U)^{-1}\vartheta(\xi^-.U^{-1})U] \\ &+ [\vartheta(\eta^+) + \vartheta(\eta^-)] [(U)^{-1}\vartheta(\eta^+.U^{-1})U - (U)^{-1}\vartheta(\eta^-.U^{-1})U] \\ &+ [\vartheta(\eta^+.U) + A^{-1}\vartheta(\eta^-.U)] [(U)^{-1}\vartheta(\eta^+.U) - (U)^{-1}\vartheta(\eta^-.U)] \\ &+ 2[\vartheta(\eta^+.U^2)] [(U)^{-1}\vartheta(\eta^+.U)U - (U)^{-1}\vartheta(\eta^-.U)U] \end{aligned} \right\} \\
&= \text{tr} \left\{ \begin{aligned} &\left[\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \right] \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \left[\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &+ \left[\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &+ \left[\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &+ 2 \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \left[\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned} \right\} \\
&= \text{tr} \left\{ \begin{pmatrix} 0 & 0 \\ -8 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ -8 & 0 \end{pmatrix} + \begin{pmatrix} -8 & -8 \\ 8 & 8 \end{pmatrix} + 2 \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} \right\} \\
&= 8,
\end{aligned}$$

and accounting for the orientation and -2 in the expression above for the WP form yields the asserted constant $\gamma = -4\omega$.

Finally in the remaining case to compute $\gamma(\bar{\vartheta}_A, \bar{\vartheta}_{TA})$, replace $A \in \text{PSL}_2$ by SUA , so that

$$\gamma(\bar{\vartheta}_{SUA}, \bar{\vartheta}_{TSUA}) = \gamma(\bar{\vartheta}_{UA}, \bar{\vartheta}_A) = +8$$

by skew symmetry since $\bar{\vartheta}_{SB} = \bar{\vartheta}_B$, for $B \in \text{PSL}_2$, and

$$TSUA = TT^{-1}UT^{-1}UA = UT^{-1}UA = SA.$$

□

7. UNIVERSAL AUTOMORPHIC 1-FORM ON $\mathrm{PPSL}_2(\mathbb{R})$

Given a decoration $\tilde{\tau}$ on a tessellation τ with framing $\mathcal{F}$, define

$$\xi_{\mathcal{F}} = \frac{1}{2} \sum_{A \in \mathrm{PSL}_2} \bar{\vartheta}_A \, d\log \lambda_{e_A} \in \Omega^1(\widetilde{\mathrm{Homeo}}_+^n, \mathfrak{psl}_2),$$

where $\bar{\vartheta}_A$ is the wavelet ϑ_A normalized with respect to $\mathcal{F}$, and λ_e denotes the lambda length of $e = e_A$. Alternatively, we may write $\xi_{\mathcal{F}} = \sum_{e \in \tau_*} \bar{\vartheta}_e \, d\log \lambda_e$ since $\vartheta_A = \vartheta_{SA}$ and $\lambda_e = \lambda_{e_A} = \lambda_{e_{SA}}$.

$\xi_{\mathcal{F}}$ is to be apprehended as a 1-form on the group $\widetilde{\mathrm{Homeo}}_+^n$ taking values in its Lie algebra $\mathfrak{psl}_2$. It evidently could be interpreted as the Maurer-Cartan form since the result of applying it to a multiplicative deformation of the lambda length, or in other words an additive deformation of the logarithm of the lambda length, is precisely the corresponding normalized vector field on $\mathbb{S}^1$ in its Lie algebra $\mathfrak{psl}_2$, as was discussed in Section 3.

More precisely, we have the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Homeo}}_+^n & \approx & \widetilde{\mathrm{Homeo}}_+^n / \mathrm{Möb} \\ \downarrow \pi & & \downarrow \\ \mathrm{Homeo}_+ & \xrightleftharpoons[s_{\mathcal{F}}]{p} & \mathrm{Homeo}_+^n \approx \mathrm{Homeo}_+ / \mathrm{Möb}, \end{array}$$

where π is the forgetful map, p is the projection given by precomposition with $L_{f(\frac{0}{1}), f(\frac{1}{0}), f(\frac{1}{1})}^{\frac{0}{1}, \frac{1}{0}, \frac{1}{1}}$, and for any framing $\mathcal{F} = (u, v, w)$, the section $s_{\mathcal{F}}$ of p is given by precomposition with $L_{\frac{u}{1}, \frac{v}{0}, \frac{w}{1}}^{\frac{u}{1}, \frac{v}{0}, \frac{w}{1}}$. It is a tautology that if ω is the Maurer-Cartan form of Homeo_+ , then $\xi_{\mathcal{F}} = \pi^* \circ s_{\mathcal{F}}^* \omega$.

Theorem 7.1. *For any fixed framing $\mathcal{F} \in C_3$, the Lie-algebra valued 1-form $\xi_{\mathcal{F}} \in \Omega^1(\widetilde{\mathrm{Homeo}}_+^n, \mathfrak{psl}_2)$ is invariant under flips.*

Proof. As a Lie algebra valued 1-form on a group, it is described by right-translating from the cotangent plane of the identity, and our computations will take place there.

Recall that the Farey tessellation τ_* admits a canonical decoration $\tilde{\tau}_*$ determined by the property that all the lambda lengths are constant equal to unity, and that this is the identity element of the group $\widetilde{\mathrm{Homeo}}_+^n$ as discussed in Section 3.

To establish notation, refer to Figure 1 and consider the four generation-zero and -one Farey points $\frac{1}{0}, -\frac{1}{1}, \frac{0}{1}, \frac{1}{1}$ decomposing $\mathbb{S}^1$ into four circular intervals which are conflated with the respective quadrants I, II, III and

IV containing them. The convex hull of these points is an ideal quadrilateral with oriented frontier edges $a = e_{ST}, b = e_{SU}, c = e_{U^{-1}}$ and $d = e_{T^{-1}}$ in this counter-clockwise order starting from $\frac{1}{0}$. The edge e_I will be denoted simply $e = e_I$, and the edge arising from its flip, with respective initial and terminal points $\frac{1}{1}$ and $-\frac{1}{1}$, will be denoted f .

The second-generation $-\frac{2}{1}, -\frac{1}{2}, \frac{1}{2}, \frac{2}{1}$ Farey points further decompose the circular intervals $I = I_- \cup I_+, \dots, IV = IV_- \cup IV_+$ occurring in this counter-clockwise order starting from $\frac{1}{0}$. The Farey points of generation at most two together span an ideal octagon with oriented frontier edges $e_{ST^2}, e_{SUT}, e_{TSU}, e_{SU^2}, e_{U^{-2}}, e_{T^{-1}U^{-1}}, e_{U^{-1}T^{-1}}$, and $e_{T^{-2}}$ in this counter-clockwise order starting from $\frac{1}{0}$, as indicated in Figure 2.

The hyperbolic transformations which are primitive in PSL_2 along the respective axes e and f are given by the exponentials of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and along each of the other edges e_A above by the exponential of $X_A \in \mathfrak{sl}_2$ with

$$X_{ST} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad X_{SU} = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}, \quad X_{U^{-1}} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, \quad X_{T^{-1}} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} X_{ST^2} &= \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}, \quad X_{SUT} = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}, \\ X_{TSU} &= \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}, \quad X_{SU^2} = \begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix}, \\ X_{U^{-2}} &= \begin{pmatrix} -1 & 0 \\ -4 & 1 \end{pmatrix}, \quad X_{T^{-1}U^{-1}} = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}, \\ X_{U^{-1}T^{-1}} &= \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}, \quad X_{U^{-1}T^{-1}} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The proof is a direct computation and is omitted.

These are the logarithms of the component earthquakes for the wavelets ϑ_x , for $x \in \{a, b, c, d, e, f\}$, so it is not difficult to combine them four at a time and normalize with respect to the the standard framing $\mathcal{F} = (\frac{0}{1}, \frac{1}{0}, \frac{1}{1})$ on τ_* . One finds

$$\bar{\vartheta}_e = \vartheta_e = \begin{cases} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, & \text{on I;} \\ \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}, & \text{on II;} \\ \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, & \text{on III;} \\ \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, & \text{on IV,} \end{cases}$$

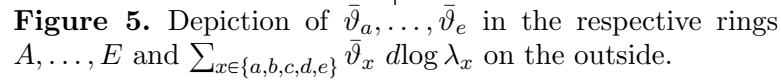
$$\vartheta_a = \begin{cases} \begin{pmatrix} -1 & -4 \\ 0 & 1 \end{pmatrix}, & \text{on I}_-; \\ \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}, & \text{on I}_+; \\ \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, & \text{on II;} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{on III and IV,} \end{cases} \quad \text{so } \bar{\vartheta}_a = \begin{cases} \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}, & \text{on I}_-; \\ \begin{pmatrix} 4 & 4 \\ -2 & -4 \end{pmatrix}, & \text{on I}_+; \\ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, & \text{on II;} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{on III and IV,} \end{cases}$$

$$\vartheta_b = \begin{cases} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, & \text{on I;} \\ \begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix}, & \text{on II}_-; \\ \begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix}, & \text{on II}_+; \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{on III and IV,} \end{cases} \quad \text{so } \bar{\vartheta}_b = \begin{cases} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, & \text{on I;} \\ \begin{pmatrix} -4 & -2 \\ 4 & 4 \end{pmatrix}, & \text{on II}_-; \\ \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, & \text{on II}_+; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{on III and IV,} \end{cases}$$

$$\bar{\vartheta}_c = \vartheta_c = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{on I and II;} \\ \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}, & \text{on III}_-; \\ \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}, & \text{on III}_+; \\ \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}, & \text{on IV,} \end{cases} \quad \text{and } \bar{\vartheta}_d = \vartheta_d = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{on I and II;} \\ \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, & \text{on III;} \\ \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}, & \text{on IV}_-; \\ \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, & \text{on IV}_+ \end{cases}$$

As illustrated in Figure 5, letting $\tilde{x} = d\log \lambda_x$, for $x \in \{a, b, c, d, e, f\}$, and putting all of this together, we find that

$$\begin{aligned} & \sum_{x \in \{a, b, c, d, e\}} \bar{\vartheta}_x d\log \lambda_x \\ &= \begin{cases} \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \tilde{a} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tilde{b} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \tilde{e}, & \text{on I}_-; \\ \begin{pmatrix} 4 & 4 \\ -2 & -4 \end{pmatrix} \tilde{a} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tilde{b} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \tilde{e}, & \text{on I}_+; \\ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tilde{a} + \begin{pmatrix} -4 & -2 \\ 4 & 4 \end{pmatrix} \tilde{b} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \tilde{e}, & \text{on II}_-; \\ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tilde{a} + \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} \tilde{b} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \tilde{e}, & \text{on II}_+; \\ \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \tilde{e}, & \text{on III}_-; \\ \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \tilde{e}, & \text{on III}_+; \\ \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \tilde{e}, & \text{on IV}_-; \\ \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \tilde{c} + \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \tilde{e}, & \text{on IV}_+. \end{cases} \end{aligned}$$


$$\bar{\vartheta}'_f = \vartheta'_f = \bar{\vartheta}_e,$$

and a further consequence of the primitives for logarithms of hyperbolic transformations computed before is that

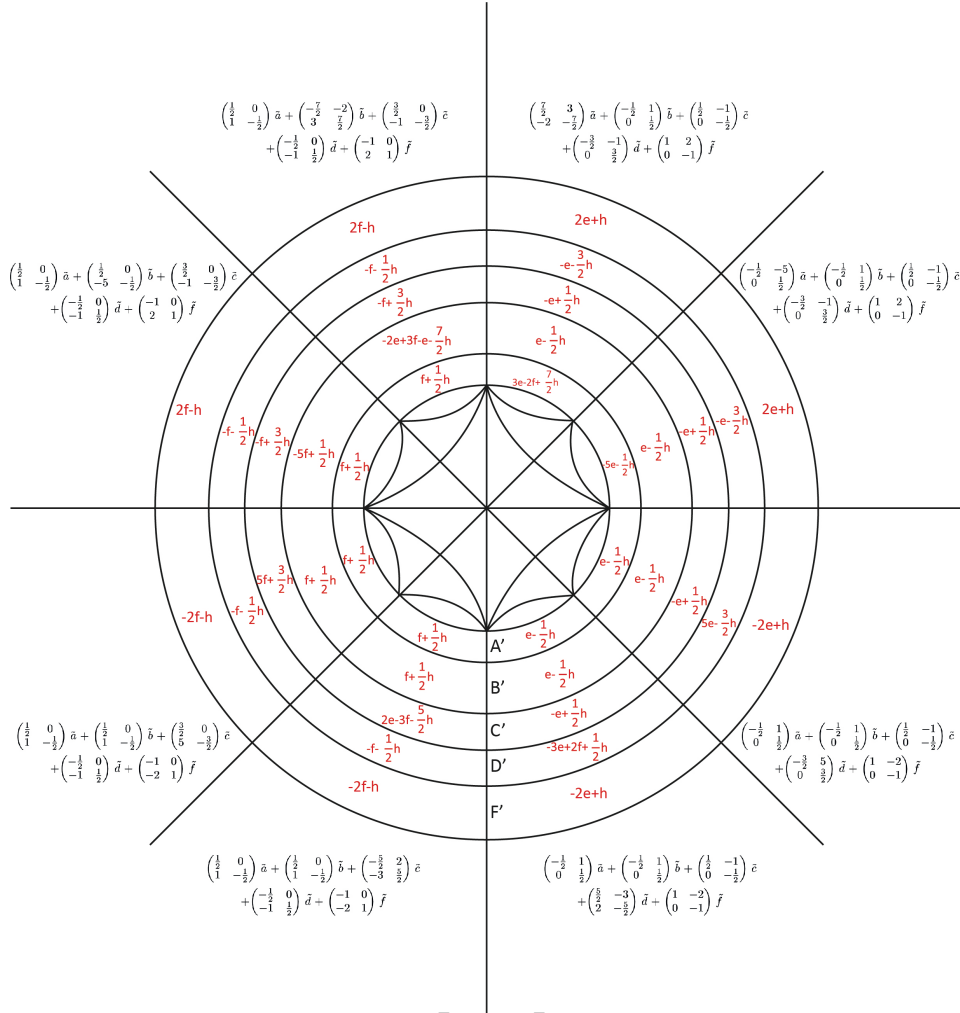


Figure 6. Depiction of $\bar{\vartheta}'_a, \dots, \bar{\vartheta}'_f$ in the respective rings $A', \dots, D', F'$ and $\sum_{x \in \{a, b, c, d, f\}} \bar{\vartheta}'_x d\log \lambda_x$ on the outside.

$$\bar{\vartheta}'_a = \vartheta'_a + \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} \end{pmatrix}, \text{ where } \vartheta'_a = \begin{cases} \begin{pmatrix} -1 & -4 \\ 0 & 1 \end{pmatrix}, & \text{on } I_-; \\ \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}, & \text{on } I_+; \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{on II and III}; \\ \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, & \text{on IV}, \end{cases}$$

$$\bar{\vartheta}'_b = \vartheta'_b + \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix}, \text{ where } \vartheta'_b = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{on I and IV}; \\ \begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix}, & \text{on II}_-; \\ \begin{pmatrix} -1 & 0 \\ -4 & -1 \end{pmatrix}, & \text{on II}_+; \\ \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, & \text{on III}, \end{cases}$$

$$\bar{\vartheta}'_c = \vartheta'_c + \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}, \text{ where } \vartheta'_c = \begin{cases} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & \text{on I and IV;} \\ \begin{pmatrix} -\frac{1}{2} & 0 \\ -2 & -1 \end{pmatrix}, & \text{on II;} \\ \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}, & \text{on III}_-; \\ \begin{pmatrix} -3 & 2 \\ -4 & -1 \end{pmatrix}, & \text{on III}_+, \end{cases}$$

$$\bar{\vartheta}'_d = \vartheta'_d + \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \text{ where } \vartheta'_d = \begin{cases} \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, & \text{on I;} \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & \text{on II and III;} \\ \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}, & \text{on IV}_-; \\ \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, & \text{on IV}_+. \end{cases}$$

Again putting all this together as illustrated in Figure 6, we find

$$\sum_{x \in \{a, b, c, d, f\}} \bar{\vartheta}'_x d\log \lambda_x = \begin{cases} \begin{pmatrix} -\frac{1}{2} & -5 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{3}{2} & -1 \\ 0 & \frac{3}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \tilde{f}, & \text{on I}_-; \\ \begin{pmatrix} \frac{7}{2} & 3 \\ -2 & -\frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{3}{2} & -1 \\ 0 & \frac{3}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \tilde{f}, & \text{on I}_+; \\ \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} -\frac{7}{3} & -2 \\ 3 & \frac{7}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{3}{2} & 0 \\ -1 & -\frac{3}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \tilde{f}, & \text{on II}_-; \\ \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} \frac{1}{2} & 0 \\ -5 & -\frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{3}{2} & 0 \\ -1 & -\frac{3}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \tilde{f}, & \text{on II}_+; \\ \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{3}{2} & 0 \\ 5 & -\frac{3}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \tilde{f}, & \text{on III}_-; \\ \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} -\frac{5}{2} & 2 \\ -3 & \frac{5}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \tilde{f}, & \text{on III}_+; \\ \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} \frac{5}{2} & -3 \\ 2 & -\frac{5}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \tilde{f}, & \text{on IV}_-; \\ \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{a} + \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \tilde{b} + \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} \end{pmatrix} \tilde{c} + \begin{pmatrix} -\frac{3}{2} & 5 \\ 0 & \frac{3}{2} \end{pmatrix} \tilde{d} + \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \tilde{f}, & \text{on IV}_+; \end{cases}$$

Using the fact that

$$\tilde{e} + \tilde{f} = \frac{1}{ac + bd}(\tilde{a} + \tilde{b} + \tilde{c} + \tilde{d}) = \frac{1}{2}(\tilde{a} + \tilde{b} + \tilde{c} + \tilde{d})$$

by the Ptolemy relation $ef = ac + bd$, since $a = b = c = d = 1$ for the identity element $\tilde{\tau}_*$, a computation with rather miraculous cancellations as illustrated in Figure 7 confirms that

$$\sum_{x \in \{a, b, c, d, e\}} \bar{\vartheta}_x d\log \lambda_x = \sum_{x \in \{a, b, c, d, e\}} \bar{\vartheta}'_x d\log \lambda_x,$$

as required to check invariance under the flip along the doe for the standard framing $\mathcal{F} = (\frac{0}{1}, \frac{1}{0}, \frac{1}{1})$.

$$\begin{array}{c}
 \begin{array}{l}
 \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix} \right) a + \left(\begin{smallmatrix} -4 & -2 \\ 4 & 4 \end{smallmatrix} \right) b + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) c + \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) d + \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix} \right) e \\
 - \left\{ \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{smallmatrix} \right) a + \left(\begin{smallmatrix} -\frac{1}{2} & -2 \\ 3 & \frac{1}{2} \end{smallmatrix} \right) b + \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ -1 & -\frac{1}{2} \end{smallmatrix} \right) c \right. \\
 \left. + \left(\begin{smallmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{smallmatrix} \right) d + \left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix} \right) [e + \frac{1}{2}(a+b+c+d)] \right\}
 \end{array}
 \quad
 \begin{array}{l}
 \left(\begin{smallmatrix} 4 & 4 \\ -2 & -4 \end{smallmatrix} \right) a + \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix} \right) b + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) c + \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) d + \left(\begin{smallmatrix} -1 & -2 \\ 0 & -1 \end{smallmatrix} \right) e \\
 - \left\{ \left(\begin{smallmatrix} \frac{1}{2} & 3 \\ -2 & -\frac{1}{2} \end{smallmatrix} \right) a + \left(\begin{smallmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{smallmatrix} \right) b + \left(\begin{smallmatrix} \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} \end{smallmatrix} \right) c \right. \\
 \left. + \left(\begin{smallmatrix} -\frac{3}{2} & -1 \\ 0 & \frac{1}{2} \end{smallmatrix} \right) d + \left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix} \right) [e + \frac{1}{2}(a+b+c+d)] \right\}
 \end{array}
 \end{array}$$

Figure 7. Difference between expressions in Figures 5 and 6 vanishes for all lambda lengths equal to unity, i.e., at the universal punctured arithmetic surface.

This establishes the invariance for a flip on the doe with the associated framing. In the general case first of all, the 1-form is invariant under push-forward by the Möbius group since lambda lengths are *Möb*-invariant, so we may assume that the framing is one that is associated to a doe. Likewise without loss of generality, we may assume that the flip is performed on the edge with endpoints $\frac{0}{1}, \frac{1}{0} \in \mathbb{S}^1$, with the doe however potentially located elsewhere.

Thus, the asserted invariance must be checked for the various possibilities of relative position of doe and the edge with endpoints $\frac{0}{1}, \frac{1}{0}$ to be flipped. In the presented computation, these two edges coincide, and there are four further cases to check depending upon which quadrant in $\mathbb{C} \supseteq \mathbb{D}$ contains the doe, thereby determining the normalization. These four similar but simpler calculations, each corresponding to a different normalization of the common unnormalized formulae already presented in the proof, are left as an exercise for the reader. $\square$

Consider an arbitrary framing $\mathcal{F}$ and let

$$\eta_{\mathcal{F}} = s_{\mathcal{F}}^* \omega \in \Omega^1(\text{Homeo}_+/M\ddot{o}b),$$

which by general principles should satisfy

Maurer – Cartan equation : $0 = d\eta_{\mathcal{F}} + \frac{1}{2}[\eta_{\mathcal{F}}, \eta_{\mathcal{F}}]$, for any framing $\mathcal{F}$;

Compatibility Equation : $\eta_{\mathcal{G}} = \text{Ad}(h_{\mathcal{FG}}^{-1}) \eta_{\mathcal{F}} + h_{\mathcal{FG}}^* \omega_{M\ddot{o}b}$, for framings $\mathcal{F}, \mathcal{G}$, where $h_{\mathcal{FG}} = s_{\mathcal{G}} \circ s_{\mathcal{F}}^{-1}$, Ad denotes the adjoint and $\omega_{M\ddot{o}b}$ is the Maurer-Cartan form on $M\ddot{o}b$.

Note that the Compatibility Equation together with the result for the flip on the doe with its associated framing would give an alternative proof of Theorem 7.1. This putative proof is a bit of a swindle because *there is no Maurer-Cartan form on the topological group Homeo_+ except for the version here, and hence no a priori Compatibility Equation.*

Remark 7.2. The expression for $\xi_{\mathcal{F}}$ as a Poincaré series in framed holographic coordinates is easily derived from Remark 2.8.

As we have noted, the form $\xi_{\mathcal{F}}$ is morally the Maurer-Cartan form in the sense that its value on the tangent vector $\frac{\partial}{\partial \ln \lambda_A}$ at the identity of $\text{PPSL}_2(\mathbb{R})$ is the corresponding normalized element $\bar{\vartheta}_A$ of the Lie algebra $\mathfrak{ppsl}_2$. However, $\xi_{\mathcal{F}}$ has been shown here to be invariant under the lattice $\text{PPSL}_2(\mathbb{Z})$ rather than under the full group $\text{PPSL}_2(\mathbb{R})$, whose invariance we may conjecture. On the other hand, one might therefore more squarely regard $\xi_{\mathcal{F}}$ solely as an automorphic form, and this is the viewpoint of the next section.

8. FUTURE PERSPECTIVES

We conclude with a heuristic discussion of constructing the automorphic representation for our universal triad corresponding to the automorphic 1-form obtained in Section 7. To this end, we combine the

original constructions of [34] and of Vaughan Jones [16, 17], also described in [22].

We recall from Appendix B that we have constructed an indecomposable automorphic representation V of $\mathrm{PSL}_2(\mathbb{R})$ by lifting the weight 2 Eisenstein series E_2 . Viewed as the Lie algebra $\mathfrak{psl}_2 = \mathfrak{psl}_2(\mathbb{R})$ representation, it can be identified with the Laurent polynomial ring

$$V \approx \mathcal{O}(\mathbb{C}^\times) = \mathbb{C}[z^{\pm 1}].$$

with the $\mathfrak{psl}_2$ action given by

$$e = \frac{d}{dz}, \quad h = 2z \frac{d}{dz}, \quad f = z^2 \frac{d}{dz}.$$

The space V has a natural multiplication map

$$V \rightarrow V \otimes V$$

which commutes with the action of $\mathfrak{psl}_2$. One can also identify the restricted dual space with Laurent 1-forms

$$V' \approx \Omega^1(\mathbb{C}^\times) = \mathbb{C}[z^{\pm 1}]dz,$$

with the pairing of V and V' given by the residue of the product at 0. This gives the dual map

$$(8.1) \quad V' \rightarrow V' \otimes V',$$

which can be used for the inductive limit construction as in [16, 17, 22].

Namely for any polygon P_γ with frontier γ , we assign a tensor product

$$H_\gamma = \bigotimes_{j=1}^n V',$$

where n is the number of edges in γ . There is a partial ordering on the set of polygons given by inclusion $P_\gamma \subseteq P_{\gamma'}$, which induces the partial ordering $\gamma \leq \gamma'$ on frontiers. Clearly the Lie subalgebra

$$\mathfrak{ppsl}_2(\gamma) \subseteq \mathfrak{ppsl}_2$$

of piecewise $\mathfrak{psl}_2$ maps with breakpoints at the vertices of γ acts on H_γ in the natural way, where $\mathfrak{psl}_2$ corresponding to the j^{th} edge acts on the j^{th} factor of H_γ . Then the map (8.1) induces a family of intertwining operators

$$T_\gamma^{\gamma'} : H_\gamma \rightarrow H_{\gamma'},$$

so that $T_\gamma^\gamma = 1$ and the consistency condition $T_{\gamma''}^{\gamma'} = T_{\gamma''}^{\gamma'} T_\gamma^{\gamma'}$ holds whenever $\gamma \leq \gamma' \leq \gamma''$. The direct limit

$$(8.2) \quad H = \varinjlim H_\gamma$$

yields a representation of $p\mathfrak{psl}_2$. It can be made unitary by means of a 1-cocycle determined by the one-dimensional subrepresentation $V_0 \subseteq V$ as in [34] or alternatively using a modification of the construction above based upon the approximation of the representation V' of $\mathfrak{psl}_2$ by the representations of the complementary series $V(\lambda)$, for $0 < \lambda < 2$ with $\lambda \neq 1$, so that $V(\lambda) \approx V(2-\lambda)$ with $\lim_{\lambda \rightarrow 2} V(\lambda) = V$ and $\lim_{\lambda \rightarrow 0} V(\lambda) = V'$; see [34] for details.

Remark 8.1. The partial ordering above on ideal polygons occurring as fundamental domains for punctured surfaces leads one to consider the so-called punctured solenoid $\mathcal{H}$, which is an initial object for the category of punctured surfaces with morphisms given by finite covers branched only over the punctures. The unpunctured version was introduced and studied by Dennis Sullivan in [31]. The decorated Teichmüller space of $\mathcal{H}$ was studied in [28] and parametrized by certain coordinates on the Farey tessellation τ_* , namely, lambda length *functions*, one such continuous and $\mathrm{PSL}_2(\mathbb{Z})$ -equivariant function from the profinite completion of $\mathrm{PSL}_2(\mathbb{Z})$ to $\mathbb{R}_{>0}$ for each edge of τ_* . Flip-like generators for the mapping class group of $\mathcal{H}$ are derived in [28], and a complete set of relations in these generators is finally given in [2].

To implement our program, we want to realize H , or heuristically $(V')^{\otimes \infty}$, as an automorphic representation on $\mathrm{P}\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{P}\mathrm{PSL}_2(\mathbb{R})$ of $p\mathfrak{psl}_2$. To this end, we notice first that the space of harmonic functions on $\mathrm{PSL}_2(\mathbb{R})$, cf. (B.3), can be identified with the space of functions on the boundary of $\mathrm{PSL}_2(\mathbb{R})$, which is dual to the space of functions of the holographic coordinates $\{(s, \delta)\}$ introduced in Section 2. Furthermore for both spaces, we can also identify the corresponding $\mathrm{PSL}_2(\mathbb{R})$ actions, and therefore, we can realize V' by imposing modular invariance on the space of functions depending on a single pair (s, δ) of holographic coordinates. This realization clearly extends to finite tensor products $(V')^{\otimes n}$ associated to n -gons P_γ . We conjecture that the inductive limit H also admits a realization by the automorphic functions in holographic coordinates, with $\mathrm{P}\mathrm{PSL}_2(\mathbb{Z})$ playing the role of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

On the other hand, the analogy between $\mathrm{P}\mathrm{PSL}_2(\mathbb{Z})$ and the mapping class group suggests an extension of the former by an infinite symmetric group, or more generally by an infinite braid group, which is still a discrete subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$, cf. [14] and the references therein. The reduction of H by the additional symmetry yields the space Λ , or heuristically $S^\infty V'$, which is naturally identified with the free bosonic field, one of the simplest CFT_2 .

As discussed in Appendix B, the theory of automorphic forms allows the enlargement of various spaces by relaxing the automorphic property to smaller discrete subgroups; for example, the restriction to invariance by the commutant $\mathrm{PSL}_2(\mathbb{Z})' \subseteq \mathrm{PSL}_2(\mathbb{Z})$ enlarges V sixfold additively. Similarly, we expect that replacing $\mathrm{PPSL}_2(\mathbb{Z})$ by $\mathrm{P}(\mathrm{PSL}_2(\mathbb{Z}))'$ will correspondingly yield a multiplicative increase of Λ and thereby allow the construction of a CFT_2 of size comparable to the Monster CFT_2 .

Finally we come to the question of how we can effectively capture the Monster using the new universal automorphic theory, and in particular how we can derive the Monstrous Moonshine. At this moment, we do not know the complete answer, but we can begin on the opposite side, which will help to clarify the problem and capture the beast. Namely, we consider all 194 conjugacy classes of the Monster and the Thompson series of the representations of these classes in the the Monster representation of [12]. Then we know from [1] that they are the canonical Hauptmoduln $J_\Gamma(z)$ for some genus-zero discrete subgroups Γ of $\mathrm{PSL}_2(\mathbb{R})$ of Moonshine type, i.e., groups commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ which contain $\Gamma_\infty = \{\pm(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix})\}$. Then one has

$$J_\Gamma(z) = q^{-1} + \sum_{n=1}^{\infty} c_\Gamma(n) q^n,$$

for some integral coefficients $c_\Gamma(n)$. One can also construct the corresponding Eisenstein series using a logarithmic derivative of $J_\Gamma(z)$, or as usual an analytic continuation to $s = 0$ of the series

$$E_2^\Gamma(z, s) = \sum_{(c,d) \in \Gamma/\Gamma_\infty} \frac{(\mathrm{Im} z)^s}{(cz + d)^2 |cz + d|^{2s}}.$$

As in the classical example $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, the resulting Eisenstein series after multiplication by dz is Γ -invariant though not holomorphic. Then we can repeat the lift of a Γ -invariant 1-form recalled in Appendix B to the automorphic function

$$f = E_2^\Gamma \rightsquigarrow \phi_f(g) = (ci + d)^{-2} f(g \cdot i),$$

where $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{PSL}_2(\mathbb{R})$, and we again obtain an indecomposable representation of $\mathrm{PSL}_2(\mathbb{R})$ that we shall denote V^Γ . In fact, we just get another model of the same indecomposable representation of $\mathrm{PSL}_2(\mathbb{R})$. We can also expect a similar construction of the space of universal automorphic forms for $\mathrm{P}\Gamma$ denoted Λ^Γ . It is natural to conjecture that we find a twisted conformal field theory associated to Λ^Γ and the Monster conjugacy class corresponding to Γ . The comparison of Λ and Λ^Γ should yield the Monster element in this conjugacy class, and they

together will allow realization of the Monster group via automorphisms of Λ . At this point, the trap will close upon the Monster, and the Moonshine will follow.

APPENDIX A. HARMONIC ANALYSIS

To begin, we express the generators of the Witt algebra in terms of the normalized arithmetic wavelets as computed in Theorem 4.13 of [26] and going back in essence to [20].

Theorem A.1. *For each $n \in \mathbb{Z}$, we have the wavelet expansion*

$$L_n = e^{in\theta} \frac{\partial}{\partial \theta} = (b_0^n + b_{+1}^n e^{i\theta} + b_{-1}^n e^{-i\theta}) \frac{\partial}{\partial \theta} + \frac{i}{4} \sum_{e \in \tau_*} \left\{ n(\xi^n + \eta^n) + \frac{\eta + \xi}{\eta - \xi} (\xi^n - \eta^n) \right\} \bar{\vartheta}_e(\theta),$$

where $e \in \tau_*$ has ideal points $\xi, \eta \in \mathbb{S}^1$ and

$$b_0^n = \begin{cases} +1, n \equiv 0(4); \\ 0, n \equiv 1(4); \\ +1, n \equiv 2(4); \\ 0, n \equiv 3(4); \end{cases} \quad b_{+1}^n = \begin{cases} 0, n \equiv 0(4); \\ +1, n \equiv 1(4); \\ -i, n \equiv 2(4); \\ 0, n \equiv 3(4); \end{cases} \quad b_{-1}^n = \begin{cases} 0, n \equiv 0(4); \\ 0, n \equiv 1(4); \\ +i, n \equiv 2(4); \\ +1, n \equiv 3(4). \end{cases}$$

This is not the simplest expression $e^{in\theta} \doteq \sum_{|e| \in \tau_*} g_n^e \bar{\vartheta}_{|e|}$, rather these particular coefficients g_n^e are specially chosen in Theorem A.1 to guarantee their suitable decay in n , cf. Theorem 6.4 of [27].

The next result, Theorem 4.11 of [26], which goes back to [24], is of basic utility and provides the Fourier expansion of the normalized arithmetic wavelets.

Theorem A.2. *If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2$, then the Fourier expansion $\bar{\vartheta}_A \sim \sum_{n \in \mathbb{Z}} c_n e^{in\theta} \frac{\partial}{\partial \theta}$ for $n^2 > 1$ is given by*

$$\begin{aligned} \pi i(n^3 - n) c_n &= -[(c - a)^2 + (b - d)^2] \left[\frac{(b - d) - i(a - c)}{(b - d) + i(a - c)} \right]^n \\ &\quad + 2(c^2 + d^2) \left[\frac{d - ic}{d + ic} \right]^n + 2(a^2 + b^2) \left[\frac{b - ia}{b + ia} \right]^n \\ &\quad - [(c + a)^2 + (b + d)^2] \left[\frac{(b + d) - i(a + c)}{(b + d) + i(a + c)} \right]^n, \end{aligned}$$

where the Fourier modes $c_0, c_{\pm 1}$ are chosen to guarantee that the expansion is normalized. In particular for the mother wavelet, we have $\bar{\vartheta} = \vartheta \sim \frac{8}{\pi i} \sum_{n \equiv 2(4)} \frac{1}{n^3 - n} e^{in\theta} \frac{\partial}{\partial \theta}$.

Remark A.3. A sketch of the proof is that since $\bar{\vartheta}_A$ is once continuously differentiable, we can twice integrate by parts the standard expression for the Fourier coefficients c_n with $n^2 > 1$ and derive these expressions without difficulty. For $n^2 \leq 1$ the corresponding equations give certain constraining equations on the zeroth Fourier modes, which are crucial for the computation of the Kirillov-Kostant form, cf. Section 6

It is not difficult to compute directly the Fourier expansion of the (unnormalized) hyperfans as follows.

Proposition A.4. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2$, let

$$\zeta_- = \frac{b - ia}{b + ia}, \quad \zeta_+ = \frac{d - ic}{d + ic} \in \mathbb{S}^1.$$

Then the hyperfan ψ_A has Fourier expansion $\psi_A \sim \sum c_n e^{in\theta}$ with

$$\begin{aligned} 2\pi i \, c_n = & \zeta_-^n \left[\frac{(d - ic)^2}{2(n+1)} \zeta_- + \frac{(d + ic)^2}{2(n-1)} \zeta_-^{-1} - \frac{c^2 + d^2}{n} \right] \\ & - \zeta_+^n \left[\frac{(d - ic)^2}{2(n+1)} \zeta_+ + \frac{(d + ic)^2}{2(n-1)} \zeta_+^{-1} - \frac{c^2 + d^2}{n} \right], \end{aligned}$$

for $n^2 > 2$, and

$$\begin{aligned} 2\pi i \, c_0 = & \frac{(d - ic)^2}{2} [\zeta_+^{-1} - \zeta_-^{-1}] - \frac{(d + ic)^2}{2} [\zeta_+ - \zeta_-] \\ & + i(c^2 + d^2)[\theta_+ - \theta_-], \\ 2\pi i \, c_{\pm 1} = & \mp \frac{(d \pm ic)^2}{2} [\zeta_+^{\pm 2} - \zeta_-^{\pm 2}] \pm (c^2 + d^2)[\zeta_+^{\pm 1} - \zeta_-^{\pm 1}] \\ & - \frac{i(d \mp ic)^2}{2} [\theta_+ - \theta_-], \end{aligned}$$

where $\theta_{\mp} = \arctan \zeta_{\pm}$.

Proof. ψ_A is supported on an interval with respective initial and terminal points given by ζ_-^{-1} and ζ_+^{-1} , and takes there the values

$$A^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A = \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}$$

or in other words $\frac{(d-ic)^2}{2} e^{i\theta} + \frac{(d+ic)^2}{2} e^{-i\theta} - (c^2 + d^2)$. The standard expression for Fourier coefficients yields the results. $\square$

Notice that the normalized hyperfans $\bar{\psi}_A$ have the same higher Fourier coefficients as ψ_A since $\psi_A \doteq \bar{\psi}_A$. The small Fourier modes for $\bar{\psi}_A$, and likewise those of $\bar{\theta}_A$, can be computed with some difficulty by calculating the values of these vector fields at $\frac{0}{1}, \frac{1}{1}, \frac{1}{0} \in \mathbb{S}^1$. As we do not require them, we omit the details.

APPENDIX B. THE E_2 AUTOMORPHIC REPRESENTATION

The automorphic form that we have constructed on the universal Teichmüller space has a simple analogue on the hyperbolic plane, namely,

$$\xi(z) = E_2(z)dz,$$

where $E_2(z)$ is an almost holomorphic modular form of weight 2. Since dz transforms with respect to $\mathrm{PSL}_2(\mathbb{R})$ with the factor balancing that of $E_2(z)$, we obtain a $\mathrm{PSL}_2(\mathbb{Z})$ -invariant 1-form that can be viewed as the classical counterpart of the automorphic form constructed in Section 7.

To state our results unambiguously, let us recall the following explicit formulae, c.f. [9]. Set

$$\mathbb{E}_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n,$$

where $\sigma(n)$ is the sum of all the positive divisors of n and we set $q = \exp(2\pi iz)$. Then $\mathbb{E}_2(z)$ is holomorphic but not quite modular invariant since

$$\mathbb{E}_2\left(-\frac{1}{z}\right) = z^2 \mathbb{E}_2(z) + \frac{12}{2\pi i} z.$$

To get an honest weight 2 form, we need a correction

$$E(z) = \mathbb{E}_2(z) - \frac{3}{\pi \operatorname{Im}(z)},$$

but we thereby lose the holomorphicity of $\mathbb{E}_2(z)$, since

$$\frac{d}{d\bar{z}} E(z) = \frac{3i}{2\pi} \frac{1}{(\operatorname{Im}(z))^2}.$$

In the classical theory of automorphic forms, one can lift the holomorphic modular forms from the hyperbolic plane to $\mathrm{PSL}_2(\mathbb{R})$ so that they become the lowest weight vectors of the holomorphic series representations. Specifically for a weight $2k$ holomorphic form $f : \mathcal{U} \rightarrow \mathbb{C}$, where $k \geq 2$, we define

$$\phi_f(g) = (ci + d)^{-2k} f(g \cdot i),$$

as in [15], where $g = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. It follows that ϕ_f is invariant under the left action

$$(B.1) \quad \phi_f(\gamma g) = \phi_f(g), \text{ for } \gamma \in \mathrm{PSL}_2(\mathbb{Z})$$

of $\mathrm{PSL}_2(\mathbb{Z})$. One can then define the action of $\mathrm{PSL}_2(\mathbb{R})$ on a certain space of automorphic functions generated by ϕ_f by the right action

$$(\pi(g)\phi)(h) = \phi(hg), \text{ for } h, g \in \mathrm{PSL}_2(\mathbb{R}),$$

so that ϕ_f becomes the lowest weight vector for the Lie algebra $\mathfrak{psl}_2(\mathbb{R})$ in the appropriate basis E, F, H .

To state the explicit formulae, we shall rely on explicit calculations from [11]. Let h, e, f be the standard basis of $\mathfrak{psl}_2(\mathbb{R})$ and consider the Cayley transform of this basis

$$H = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, E = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, F = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Using the standard parametrization

$$(B.2) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

of $\mathrm{PSL}_2(\mathbb{R})$, one can find an explicit action of the Lie algebra basis as follows

$$\begin{aligned} H &= -\partial_\theta, \\ E &= 2ie^{2i\theta}(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta), \\ F &= -2ie^{-2i\theta}(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta). \end{aligned}$$

Thus, one has

$$\begin{aligned} F\phi_f &= 0, \\ H\phi_f &= 2k\phi_f. \end{aligned}$$

In the case that $k = 1$, the first of these two equations no longer holds, and instead one has

$$F\phi_f = \frac{3}{\pi}.$$

It follows that ϕ_f is no longer a highest weight vector, but it also generates a one-dimensional sub-representation. Only after factorization with respect to this sub-representation does one obtain the weight 2 irreducible representation of the holomorphic series, denoted V_+ .

Similarly using the complex conjugation, one gets the weight 2 representation of the antiholomorphic series denoted V_- . The one-dimensional

sub-representation, denoted V_0 , is common to both the weight 2 indecomposable representations, and we get a larger indecomposable representation, denoted V , which has a simple composition series

$$0 \rightarrow V_0 \rightarrow V \rightarrow V_+ \otimes V_- \rightarrow 0.$$

In order to characterize the resulting three component representation, one can use the Casimir operator, which in the explicit coordinates (B.2) has the form

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta}.$$

It annihilates the weight 2 automorphic representation generated by Φ_f and its conjugate, where $f = E_2$, to wit

$$(B.3) \quad \Delta \phi_{E_2} = 0.$$

One can check that this harmonic property (B.3) together with the automorphicity condition (B.1) and the usual growth condition of automorphic functions characterizes the three component representation V that we have obtained by lifting the E_2 modular 1-form. The problem of fundamental importance in our new theory is to construct a representation of the Lie algebra $ppsl_2(\mathbb{R})$ using a lift of our automorphic 1-form in Section 7.

The automorphic representation V that we have constructed is well known in the representation theory of $\mathrm{PSL}_2(\mathbb{R})$ as a limit of complementary series [34]. The representations V_+ and V_- are the spin 1 irreducible representations of the holomorphic and antiholomorphic discrete series. Note that the lift of Eisenstein series E_{2k} , for $k = 2, 3, \dots$, yields spin k irreducible representations of the same series. The representation V plays a pivotal role and somehow seems to have been missed in the theory of automorphic functions for the pair $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R})$. Thus it is expected that the counterpart of the representation V in our new theory will play an equally fundamental role, and it is one of our main open problems to construct it explicitly.

One can ask what is an advantage of a realization of this representation as an automorphic representation. One such benefit is that we can naturally increase this representation by relaxing the automorphicity condition (B.1) from the modular group $\mathrm{PSL}_2(\mathbb{Z})$ to various subgroups including its commutant $\mathrm{PSL}_2(\mathbb{Z})'$ considered in Section 1. This immediately yields six copies of the weight 2 representation.

Finally, we note that we could also consider the weight 1 representations which correspond to the limit of the holomorphic discrete series of

$\mathrm{SL}_2(\mathbb{R})$. The index of the commutant is $[\mathrm{SL}_2(\mathbb{R}) : \mathrm{SL}_2(\mathbb{R})'] = 12$ in this case, and we get correspondingly 12 copies upon relaxing the automorphy condition.

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