Remarks on some locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$ in the crystalline case

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Contents

1 Introduction and notations

Let p be a prime number, F a finite extension of $\mathbb{Q}_p$, $\overline{\mathbb{Q}_p}$ an algebraic closure of F and pick a finite extension E of $\mathbb{Q}_p$ containing the Galois closure of F. This note fits into the local p-adic Langlands programme, whose aim is to attach and study locally $\mathbb{Q}_p$-analytic or continuous p-adic representations of $\mathrm{GL}_n(F)$ over E to n-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over E. One of the most important cases is when the Galois representation is crystalline with distinct Hodge-Tate weights. When $n = 2$ and $F = \mathbb{Q}_p$, we completely understand the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations both from a p-adic and from a locally analytic point of view ([6], [1], [14]). When $n = 2$ but $F \neq \mathbb{Q}_p$, several complications occur, all more or less related to the fact that one has to deal with the "mixture" of several embeddings of the base field F into the coefficient field E. This note focusses on the locally analytic point of view when $n = 2$. To most of the 2-dimensional crystalline representations V of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over E with distinct Hodge-Tate weights for each embedding $F \hookrightarrow E$, we attach and study a locally $\mathbb{Q}_p$-analytic representation $\Pi(V)$ of $\mathrm{GL}_2(F)$. Before we sum up the results of this note, let us mention right away that we *don't expect* $\Pi(V)$ to be the "complete" locally $\mathbb{Q}_p$-analytic representation associated to V when $F \neq \mathbb{Q}_p$, but only a subrepresentation of it. For instance, one can't recover V from $\Pi(V)$ in general when $F \neq \mathbb{Q}_p$. However, the study of $\Pi(V)$ is not hard and already reveals interesting features. Moreover it already seems a non-trivial task to prove that $\Pi(V) \otimes_E V$ occurs for instance inside the completed H^1 of Hilbert Shimura curves.

We now explain the main results of this note. Let f be the residual index of F. To any rank 2 filtered φ-module D (not necessarily weakly admissible) with distinct Hodge-Tate weights and such that φ^f has two distinct eigenvalues, we first associate a locally $\mathbb{Q}_p$-analytic representation $\Pi(D)$ of $\mathrm{GL}_2(F)$. The underlying idea for the definition of $\Pi(D)$ is the following. Since we are in dimension 2 with distinct Hodge-Tate weights, the Hodge filtration on the rank 2 module D_F is just the datum of a rank one submodule. If this submodule is completely generic, then $\Pi(D)$ is just the amalgamated sum of two natural locally $\mathbb{Q}_p$-analytic parabolic inductions associated to D relative to their common locally algebraic vectors. But it can happen that the Hodge filtration is in a special position (with respect to the eigenvectors of φ^f). In that case, one replaces each locally $\mathbb{Q}_p$-analytic parabolic induction in the previous amalgamated sum by a certain direct sum of some of its subquotients (depending on the position of the Hodge filtration) so that the final representation has the same Jordan-Hölder constituents. One uses here results of Frommer and Schraen (and others) on the Jordan-Hölder filtration of locally $\mathbb{Q}_p$-analytic parabolic inductions of characters (which are themselves based on foundational results of Schneider, Teitelbaum and Morita).

Let $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ be the direct sum of the topologically irreducible subrepresentations of $\Pi(D)$. We then prove the following statements, all giving evidence that $\Pi(D)$ is (a piece of) the right representation to consider.

(i) If $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ has a p-adic norm which is invariant under the action of $\mathrm{GL}_2(F)$ (for instance if $\Pi(D)$ itself has such an invariant norm), then D is weakly admissible (§5).

(ii) If $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ has a p-adic invariant norm, then the completion of $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ with respect to this norm automatically contains a larger locally $\mathbb{Q}_p$-analytic representation $\Pi(D)^{\mathrm{Amice}}$ of $\mathrm{GL}_2(F)$ which is such that $\Pi(D)^{\mathrm{Amice}} \subseteq \Pi(D)$ (§7).

(iii) If D is weakly admissible and corresponds to a reducible crystalline Galois representation $V = \begin{pmatrix} \chi_2\varepsilon & * \\ 0 & \chi_1 \end{pmatrix}$ (where ε is the p-adic cyclotomic character and χ_1, χ_2 are crystalline characters), then $\Pi(D)$ has a natural increasing filtration by $\mathrm{GL}_2(F)$-subrepresentations:

$$0 = \mathrm{Fil}^0\Pi(D) \subsetneq \mathrm{Fil}^1\Pi(D) \subsetneq \cdots \subsetneq \mathrm{Fil}^{[F:\mathbb{Q}_p]}\Pi(D) \subsetneq \mathrm{Fil}^{[F:\mathbb{Q}_p]+1}\Pi(D) = \Pi(D)$$

such that, when $\chi_1\chi_2^{-1} \notin \{1, \varepsilon^2\}$, the graded pieces:

$$\Pi(D)_j := \mathrm{Fil}^{j+1}\Pi(D)/\mathrm{Fil}^j\Pi(D)$$

satisfy $\Pi(D)_0 = \big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\,\chi_1\otimes\chi_2\big)^{\mathbb{Q}_p-\mathrm{an}}$, $\Pi(D)_{[F:\mathbb{Q}_p]} = \big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\,\chi_2\varepsilon\otimes\chi_1\varepsilon^{-1}\big)^{\mathbb{Q}_p-\mathrm{an}}$ (where $B(F)$ is the upper parabolic) and such that $\Pi(D) \simeq \oplus_{j=0}^{[F:\mathbb{Q}_p]}\Pi(D)_j$ if and only if V is split (§9).

(i) follows from an easy necessary condition for a parabolic induction which is locally analytic in some "directions" and locally algebraic in the others to admit a p-adic invariant norm. (ii) is based on well-known techniques of Amice-Vélu and Vishik which give that, if a unitary Banach space representation of $\mathrm{GL}_2(F)$ contains $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$, then p-adic analysis forces it to contain a larger representation $\Pi(D)^{\mathrm{Amice}}$, which turns out to be a subrepresentation of $\Pi(D)$. Note that one knows examples of unitary Banach spaces representations of $\mathrm{GL}_2(F)$ containing $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ (e.g. completed cohomology groups when $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ is the locally algebraic vectors of $\Pi(D)$). The results of (i) and (ii) might be (very) special cases of some of the results of [9], [10]. Finally, (iii) is consistent with results in characteristic p when F is unramified ([3]) giving evidence that the smooth representation(s) of $\mathrm{GL}_2(F)$ in characteristic p corresponding to a reducible non-split (resp. split) 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ should generically be a successive extension (resp. a direct sum) of $[F:\mathbb{Q}_p]$ irreducible representations, the first and the last being two principal series analogous to the two parabolic inductions in (iii) above. We refer to the body of the text for more detailed and more precise statements.

For any finite extension L of $\mathbb{Q}_p$, we denote by $\mathcal{O}_L$ its ring of integers, ϖ_L a uniformizer and $k_L := \mathcal{O}_L/(\varpi_L)$ its residue field.

Throughout the text, F and E are two fixed finite extensions of $\mathbb{Q}_p$ such that the set S of field embeddings of F into E has cardinality $[F : \mathbb{Q}_p]$ (F is the base field, E the coefficient field). We denote by F_0 the maximal unramified subfield in F, $f := [F_0 : \mathbb{Q}_p]$, $e := [F : F_0]$, $q := p^f$ and S_0 the set of embeddings of F_0 into E. We let φ be the arithmetic Frobenius on F_0 inducing $x \mapsto x^p$ on k_F.

The p-adic valuation val_F on F or on E is normalized by $\mathrm{val}_F(p) := [F : \mathbb{Q}_p]$ and we set $|x|_F := p^{-\mathrm{val}_F(x)}$ if $x \in F$ or $x \in E$. If $\lambda \in E^\times$, $\mathrm{unr}_F(\lambda) : F^\times \to E^\times$ is by definition the unramified character sending $x \in F^\times$ to $\lambda^{\mathrm{val}_F(x)}$.

We normalize local class field theory such that uniformizers are sent to geometric Frobeniuses. We view without comment a character of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ as a character of $F^\times$. We denote by ε the p-adic cyclotomic character. It corresponds to the character of $F^\times$ given by $x \mapsto |x|_F \prod_{\sigma \in S} \sigma(x)$.

A p-adic norm on an E-vector space V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ such that $\|v\| = 0$ if and only if $v = 0$, $\|\lambda v\| = |\lambda|_F \|v\|$ ($\lambda \in E$, $v \in V$) and $\|v + w\| \leq \sup(\|v\|, \|w\|)$ ($v, w \in V$). A p-adic Banach space over E is an E-vector space endowed with a topology coming from a p-adic norm and such that the underlying metric space is complete. An invariant norm on an E-vector space V endowed with an E-linear action of a group G is a p-adic norm $\|\cdot\|$ such that $\|gv\| = \|v\|$ for all $v \in V$ and $g \in G$. A unitary Banach space representation of a topological group G over E is a p-adic Banach space B over E endowed with an E-linear action of G such that the map $G \times B \to B$ is continuous and such that the topology on B can be defined by an invariant norm.

By (topologically) irreducible for a (continuous) representation of a (topological) group on an E-vector space, we always mean (topologically) absolutely irreducible.

If R_0 and R_1 are objects in an abelian category, we denote by $R_0 \text{ ------ } R_1$ an arbitrary non-split extension of R_1 by R_0. If R and $(R_j)_j$ are objects of this category, $R \simeq R_0 \text{ ------ } R_1 \text{ ------ } R_2 \text{ ------ } \cdots$ means that R contains a non-split extension of R_1 by R_0 such that the quotient R/R_0 contains a non-split extension of R_2 by R_1 etc.

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2 Quick review of the $\mathrm{GL}_2(\mathbb{Q}_p)$-case

We review the locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E with distinct Hodge-Tate weights.

Recall that, when $F=\mathbb{Q}_p$, a filtered φ-module $(D,\varphi,\mathrm{Fil}^{\cdot} D)$ is a finite dimensional E-vector space equipped with a bijective automorphism φ and with a decreasing filtration by subvector spaces $\mathrm{Fil}^i D$, $i\in\mathbb{Z}$ which is exhaustive ($\mathrm{Fil}^i D=D$ for $i\ll 0$) and separated ($\mathrm{Fil}^i D=0$ for $i\gg 0$).

Let V be a 2-dimensional crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E with distinct Hodge-Tate weights. Twisting V if necessary, we can assume that its Hodge-Tate weights are $(0,k-1)$ where $k\in\mathbb{Z}_{\geq 2}$. Then by [12] we have $V=V_{\mathrm{cris}}(D):=\mathrm{Fil}^0(\mathrm{B}_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}D)^{\varphi=1}$ for a filtered φ-module which can be written as follows:

(i) If φ has distinct eigenvalues, then $D=Ee\oplus E\widetilde{e}$, $\varphi(e)=\alpha^{-1}e$, $\varphi(\widetilde{e})=\widetilde{\alpha}^{-1}\widetilde{e}$ (with $\alpha,\widetilde{\alpha}\in E^\times$, $\alpha\neq\widetilde{\alpha}$), $\mathrm{Fil}^i D=D$ if $i\leq -(k-1)$, $\mathrm{Fil}^i D=E(ae+\widetilde{a}\widetilde{e})$ if $-(k-2)\leq i\leq 0$ (with $(a,\widetilde{a})\in E\times E\backslash\{(0,0)\}$) and $\mathrm{Fil}^i D=0$ if $1\leq i$.

(ii) If the eigenvalues of φ are the same, then $D=Ee\oplus E\widetilde{e}$, $\varphi(e)=\alpha^{-1}e$, $\varphi(\widetilde{e})=\alpha^{-1}(e+\widetilde{e})$ (with $\alpha\in E^\times$), $\mathrm{Fil}^i D=D$ if $i\leq -(k-1)$, $\mathrm{Fil}^i D=E(e+\widetilde{e})$ if $-(k-2)\leq i\leq 0$ and $\mathrm{Fil}^i D=0$ if $1\leq i$.

Moreover, the so-called weak admissibility conditions ([12]) imply in (i): $\mathrm{val}_{\mathbb{Q}_p}(\alpha)+\mathrm{val}_{\mathbb{Q}_p}(\widetilde{\alpha})=k-1$ with $0\leq\mathrm{val}_{\mathbb{Q}_p}(\alpha)\leq k-1$ and $\mathrm{val}_{\mathbb{Q}_p}(\alpha)=k-1$ (resp. $\mathrm{val}_{\mathbb{Q}_p}(\widetilde{\alpha})=k-1$) if $a=0$ (resp. $\widetilde{a}=0$), and in (ii): $\mathrm{val}_{\mathbb{Q}_p}(\alpha)=\frac{k-1}{2}$. Note that φ can never be scalar when $F=\mathbb{Q}_p$.

Let $B(\mathbb{Q}_p)\subset\mathrm{GL}_2(\mathbb{Q}_p)$ be the subgroup of upper triangular matrices and $\chi_1,\chi_2:\mathbb{Q}_p^\times\to E^\times$ two locally analytic characters. We define the locally analytic parabolic induction:

$$\begin{aligned}\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1\otimes\chi_2\right)^{\mathrm{an}}:=\{f:\ &\mathrm{GL}_2(\mathbb{Q}_p)\longrightarrow E\ ,f\text{ is locally analytic and}\\ &f(bg)=(\chi_1\otimes\chi_2)(b)f(g),\ b\in B(\mathbb{Q}_p),\ g\in\mathrm{GL}_2(\mathbb{Q}_p)\}\end{aligned}$$

where $\chi_1\otimes\chi_2$ maps $\begin{pmatrix}a&*\\0&d\end{pmatrix}\in B(\mathbb{Q}_p)$ to $\chi_1(a)\chi_2(d)\in E^\times$. We endow this parabolic induction with a left E-linear action of $\mathrm{GL}_2(\mathbb{Q}_p)$ via $(g\cdot f)(g'):=f(g'g)$. This makes $\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1\otimes\chi_2\right)^{\mathrm{an}}$ into a locally analytic admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ in the sense of [17], [18].

Let D be a filtered module as in (i) above such that φ has distinct eigenvalues.

We define:

$$\begin{aligned}
\pi_D &:= \left(\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}_{\mathbb{Q}_p}(\alpha^{-1}) \otimes \mathrm{unr}_{\mathbb{Q}_p}(p\widetilde{\alpha}^{-1}) d^{k-2}\right)^{\mathrm{an}} \qquad (1)\\
\widetilde{\pi}_D &:= \left(\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}_{\mathbb{Q}_p}(\widetilde{\alpha}^{-1}) \otimes \mathrm{unr}_{\mathbb{Q}_p}(p\alpha^{-1}) d^{k-2}\right)^{\mathrm{an}}\\
\pi_D^\infty &:= \mathrm{Sym}^{k-2} E^2 \otimes_E \left(\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}_{\mathbb{Q}_p}(\alpha^{-1}) \otimes \mathrm{unr}_{\mathbb{Q}_p}(p\widetilde{\alpha}^{-1})\right)^{\infty}\\
\widetilde{\pi}_D^\infty &:= \mathrm{Sym}^{k-2} E^2 \otimes_E \left(\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}_{\mathbb{Q}_p}(\widetilde{\alpha}^{-1}) \otimes \mathrm{unr}_{\mathbb{Q}_p}(p\alpha^{-1})\right)^{\infty}
\end{aligned}$$

where the parabolic inductions in the last two tensor products are the classical smooth parabolic inductions. Note that we have inclusions $\pi_D^\infty \subset \pi_D$ and $\widetilde{\pi}_D^\infty \subset \widetilde{\pi}_D$. If $\alpha\widetilde{\alpha}^{-1} \neq p^{\pm 1}$ we have the classical intertwining $\pi_D^\infty \simeq \widetilde{\pi}_D^\infty$. If $\alpha\widetilde{\alpha}^{-1} = p^{-1}$ (resp. $\alpha\widetilde{\alpha}^{-1} = p$), we let $F_D \subset \pi_D^\infty$ (resp. $\widetilde{F}_D \subset \widetilde{\pi}_D^\infty$) be the unique non-zero finite dimensional subrepresentation. Otherwise, we let $F_D := 0$ (resp. $\widetilde{F}_D := 0$). In all cases, we denote by $\pi(D)$ the unique non-zero irreducible subrepresentation of both π_D^∞/F_D and $\widetilde{\pi}_D^\infty/\widetilde{F}_D$.

If D is a filtered module as in (ii) above such that φ has twice the same eigenvalue, we just define π_D as in (1) with $\widetilde{\alpha} = \alpha$.

To any 2-dimensional continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E, the local p-adic Langlands correspondence as in [7] associates a unitary Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. The following theorem describes the locally analytic vectors $B(V)^{\mathrm{an}}$ inside $B(V)$. It was conjectured (in the case $\alpha \neq \widetilde{\alpha}$) in [1] and proved independently by Liu ([14]) and Colmez:

Theorem 2.1. *We keep all of the above notations.*

(i) Assume φ has distinct eigenvalues, we have:

$$\begin{aligned}
B(V)^{\mathrm{an}} &= (\pi_D/F_D) \oplus_{\pi(D)} (\widetilde{\pi}_D/\widetilde{F}_D) \text{ if } a\widetilde{a} \neq 0\\
B(V)^{\mathrm{an}} &= (\pi_D/\pi_D^\infty) \oplus ((\pi_D^\infty/F_D) \oplus_{\pi(D)} (\widetilde{\pi}_D/\widetilde{F}_D)) \text{ if } a = 0,\ \widetilde{a} \neq 0\\
B(V)^{\mathrm{an}} &= ((\pi_D/F_D) \oplus_{\pi(D)} (\widetilde{\pi}_D^\infty/\widetilde{F}_D)) \oplus (\widetilde{\pi}_D/\widetilde{\pi}_D^\infty) \text{ if } a \neq 0,\ \widetilde{a} = 0.
\end{aligned}$$

(ii) Assume the eigenvalues of φ are the same, we have $B(V)^{\mathrm{an}} = \pi_D$.

Let us rename $B(V)^{\mathrm{an}}$ as $\Pi(D)$, and note that, by the same formulas as those in Theorem 2.1, one can define $\Pi(D)$ for any filtered φ-module as in (i) or (ii) before which is not necessarily weakly admissible. When $F \neq \mathbb{Q}_p$, it is not known at present how to define a reasonable $B(V)$, but one can easily extend and study the definition of $\Pi(D)$, as we will see.

Remark 2.2. When $\alpha\widetilde{\alpha}^{-1} \neq p^{\pm 1}$, one can rewrite $\Pi(D)$ in (i) of Theorem 2.1 in a simpler way as $\pi_D \oplus_{\pi(D)} \widetilde{\pi}_D$ if $a\widetilde{a} \neq 0$, $(\pi_D/\pi(D)) \oplus \widetilde{\pi}_D$ if $a = 0$, $\pi_D \oplus (\widetilde{\pi}_D/\pi(D))$ if $\widetilde{a} = 0$.

3 Quick review of weakly admissible filtered φ-modules

We list weakly admissible filtered φ-modules of rank 2 with distinct Hodge-Tate weights and such that φ^f has distinct eigenvalues.

When F is not necessarily $\mathbb{Q}_p$, a filtered φ-module $(D, \varphi, \mathrm{Fil}^{\cdot} D_F)$ is a free $F_0 \otimes_{\mathbb{Q}_p} E$-module D of finite rank equipped with a bijective F_0-semi-linear and E-linear endomorphism φ such that $D_F := F \otimes_{F_0} D$ is equipped with a decreasing exhaustive separated filtration by $F \otimes_{\mathbb{Q}_p} E$-submodules $\mathrm{Fil}^i D_F$, $i \in \mathbb{Z}$ (not necessarily free over $F \otimes_{\mathbb{Q}_p} E$). Using the isomorphism:

$$\begin{aligned} F_0 \otimes_{\mathbb{Q}_p} E &\xrightarrow{\sim} \textstyle\prod_{\sigma \in S_0} E \\ x \otimes y &\longmapsto (\sigma(x)y)_{\sigma \in S_0} \end{aligned}$$

one can write D as $\prod_{\sigma_0 \in S_0} D_\sigma$ where $D_{\sigma_0} := (0, \cdots, 0, 1, 0, \cdots, 0)D$ (1 being "at σ_0"). Likewise, one has $D_F = \prod_{\sigma \in S} D_\sigma$ and:

$$F \otimes_{F_0, \sigma_0} D_{\sigma_0} = \prod_{\substack{\sigma \in S \\ \sigma|_{F_0} = \sigma_0}} D_\sigma$$

(viewing D_{σ_0} as an F_0-vector space via $\sigma_0 : F_0 \hookrightarrow E$).

In the rest of the text, we consider rank 2 filtered φ-modules:

$$D = D(\alpha, \widetilde{\alpha}, (k_\sigma, a_\sigma, \widetilde{a}_\sigma)_{\sigma \in S})$$

with $\alpha, \widetilde{\alpha} \in E^\times$, $\alpha^f \neq \widetilde{\alpha}^f$, $k_\sigma \in \mathbb{Z}_{>1}$, $(a_\sigma, \widetilde{a}_\sigma) \in E \times E \backslash \{(0,0)\}$ $(\forall \sigma \in S)$ and with:

$$\begin{cases} D_{\sigma_0} &= & Ee_{\sigma_0} \oplus E\widetilde{e}_{\sigma_0} & (\sigma_0 \in S_0) \\ \varphi(e_{\sigma_0}) &= & \alpha^{-1} e_{\sigma_0 \circ \varphi^{-1}} & \\ \varphi(\widetilde{e}_{\sigma_0}) &= & \widetilde{\alpha}^{-1} \widetilde{e}_{\sigma_0 \circ \varphi^{-1}} & \end{cases}$$

$$\begin{cases} D_\sigma &= & Ee_\sigma \oplus E\widetilde{e}_\sigma & (\sigma \in S) \\ \mathrm{Fil}^i D_\sigma &= & D_\sigma & i \leq -(k_\sigma - 1) \\ \mathrm{Fil}^i D_\sigma &= & E(a_\sigma e_\sigma + \widetilde{a}_\sigma \widetilde{e}_\sigma) & -(k_\sigma - 2) \leq i \leq 0 \\ \mathrm{Fil}^i D_\sigma &= & 0 & 1 \leq i \end{cases}$$

where $1 \otimes e_{\sigma_0} = (e_\sigma)$ and $1 \otimes \widetilde{e}_{\sigma_0} = (\widetilde{e}_\sigma)$ in $F \otimes_{F_0, \sigma_0} D_{\sigma_0} = \prod_{\sigma|_{F_0} = \sigma_0} D_\sigma$.

The following lemma is straightforward and left to the reader.

Lemma 3.1. *One has $D(\alpha, \widetilde{\alpha}, (k_\sigma, a_\sigma, \widetilde{a}_\sigma)_{\sigma \in S}) \simeq D(\alpha', \widetilde{\alpha}', (k'_\sigma, a'_\sigma, \widetilde{a}'_\sigma)_{\sigma \in S})$ if and only if $k_\sigma = k'_\sigma$ for all $\sigma \in S$ and there exists $(\lambda_{\sigma_0}, \widetilde{\lambda}_{\sigma_0})_{\sigma_0 \in S_0} \in (E^\times \times E^\times)^{|S_0|}$ such* *that either:*

$$\begin{cases} \alpha' & = \alpha\frac{\lambda_{\sigma|_{F_0}}}{\lambda_{\sigma|_{F_0}\circ\varphi}} \\ \widetilde{\alpha}' & = \widetilde{\alpha}\frac{\widetilde{\lambda}_{\sigma|_{F_0}}}{\widetilde{\lambda}_{\sigma|_{F_0}\circ\varphi}} \\ (a'_\sigma,\widetilde{a}'_\sigma) & = (\lambda_{\sigma|_{F_0}}a_\sigma,\widetilde{\lambda}_{\sigma|_{F_0}}\widetilde{a}_\sigma) \text{ in } \mathbb{P}^1(E) \end{cases} \quad \textit{for all } \sigma\in S$$

or:

$$\begin{cases} \widetilde{\alpha}' & = \alpha\frac{\lambda_{\sigma|_{F_0}}}{\lambda_{\sigma|_{F_0}\circ\varphi}} \\ \alpha' & = \widetilde{\alpha}\frac{\widetilde{\lambda}_{\sigma|_{F_0}}}{\widetilde{\lambda}_{\sigma|_{F_0}\circ\varphi}} \\ (\widetilde{a}'_\sigma,a'_\sigma) & = (\lambda_{\sigma|_{F_0}}a_\sigma,\widetilde{\lambda}_{\sigma|_{F_0}}\widetilde{a}_\sigma) \text{ in } \mathbb{P}^1(E) \end{cases} \quad \textit{for all } \sigma\in S.$$

Remark 3.2. Note that if $D(\alpha,\widetilde{\alpha},(k_\sigma,a_\sigma,\widetilde{a}_\sigma)_{\sigma\in S}) \simeq D(\alpha',\widetilde{\alpha}',(k'_\sigma,a'_\sigma,\widetilde{a}'_\sigma)_{\sigma\in S})$ then one has $\{\alpha^f,\widetilde{\alpha}^f\}=\{\alpha'^f,\widetilde{\alpha}'^f\}$.

For $D=D(\alpha,\widetilde{\alpha},(k_\sigma,a_\sigma,\widetilde{a}_\sigma)_{\sigma\in S})$ we let:

$$Z_D:=\{\sigma\in S, a_\sigma=0\},\qquad \widetilde{Z}_D:=\{\sigma\in S,\widetilde{a}_\sigma=0\}. \tag{2}$$

One obviously always has $Z_D\cap\widetilde{Z}_D=\emptyset$.

Lemma 3.3. *The filtered φ-module D is weakly admissible (in the sense of [12]) if and only if the following hold:*

$$\mathrm{val}_F(\alpha)+\mathrm{val}_F(\widetilde{\alpha})=\sum_{\sigma\in S}(k_\sigma-1) \tag{3}$$

$$\sum_{\sigma\in Z_D}(k_\sigma-1)\le \mathrm{val}_F(\alpha)\le\sum_{\sigma\notin\widetilde{Z}_D}(k_\sigma-1). \tag{4}$$

The proof is straightforward and omitted.

Remark 3.4. (i) In the presence of (3), (4) is equivalent to:

$$\sum_{\sigma\in\widetilde{Z}_D}(k_\sigma-1)\le \mathrm{val}_F(\widetilde{\alpha})\le\sum_{\sigma\notin Z_D}(k_\sigma-1). \tag{5}$$

(ii) One could also consider the case $\alpha^f=\widetilde{\alpha}^f$. When $F=\mathbb{Q}_p$, the weak admissibility condition forces φ to be non-semi-simple (see §2). But this breaks down when $F\neq\mathbb{Q}_p$, that is, there exist plenty of weakly admissible filtered φ-modules D with Hodge-Tate weights $(0,k_\sigma-1)_{\sigma\in S}$ as above such that φ^f is scalar (i.e. is the multiplication by $\alpha^{-f}=\widetilde{\alpha}^{-f}$). As I am not sure how to define a reasonable $\Pi(D)$ if φ^f is scalar (see §4 below for $\Pi(D)$ when $\alpha^f\neq\widetilde{\alpha}^f$), I prefer to ignore here this case.

By the main result of [8], when D runs along the weakly admissible modules $D(\alpha, \widetilde{\alpha}, (k_\sigma, a_\sigma, \widetilde{a}_\sigma)_{\sigma\in S})$ of Lemma 3.3, the E-vector spaces:

$$V_{\mathrm{cris}}(D) := (\mathrm{B}_{\mathrm{cris}} \otimes_{F_0} D)^{\varphi=1} \bigcap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}} \otimes_F D_F)$$

endowed with the continuous E-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ induced by that on $\mathrm{B}_{\mathrm{cris}}$ and B_{dR} exhaust the 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over E with Hodge-Tate weights $(0, k_\sigma - 1)_{\sigma\in S}$ such that the crystalline Frobenius has distinct eigenvalues.

One easily checks that $V_{\mathrm{cris}}(D)$ is reducible if and only if either $\mathrm{val}_F(\alpha) = \sum_{\sigma\in Z_D}(k_\sigma - 1)$ or $\mathrm{val}_F(\widetilde{\alpha}) = \sum_{\sigma\in\widetilde{Z}_D}(k_\sigma - 1)$, and that $V_{\mathrm{cris}}(D)$ is reducible split if and only if both equalities hold, which, granting (3), (4) and (5), is equivalent to just $Z_D \amalg \widetilde{Z}_D = S$.

4 Some locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$

To a filtered φ-module as in §3 (not necessarily weakly admissible), we associate a locally $\mathbb{Q}_p$-analytic representation $\Pi(D)$ of $\mathrm{GL}_2(F)$ over E.

For every p-adic analytic group G, we have the E-vector space $C^{\mathbb{Q}_p-\mathrm{an}}(G,E)$ of locally $\mathbb{Q}_p$-analytic functions $f : G \to E$. Let $\mathfrak{g}$ be the Lie algebra of G and for $\mathfrak{x} \in \mathfrak{g}$ and $f \in C^{\mathbb{Q}_p-\mathrm{an}}(G,E)$, define as usual $\mathfrak{x}\cdot f : G \to E$ by:

$$(\mathfrak{x}\cdot f)(g) := \frac{d}{dt} f\big(g\exp(t\mathfrak{x})\big)|_{t=0}. \tag{6}$$

This endows $C^{\mathbb{Q}_p-\mathrm{an}}(G,E)$ with a $\mathbb{Q}_p$-linear action of $\mathfrak{g}$ which extends linearly to an E-linear action of $\mathfrak{g}\otimes_{\mathbb{Q}_p} E$. If G is F-analytic, then $\mathfrak{g}$ is an F-vector space and we have the usual decomposition induced by $F \otimes_{\mathbb{Q}_p} E \simeq \prod_{\sigma\in S} E$:

$$\mathfrak{g}\otimes_{\mathbb{Q}_p} E \simeq \prod_{\sigma\in S} \mathfrak{g}\otimes_{F,\sigma} E.$$

Let J be any subset of S. Following [20, §1.3.1] we say that $f \in C^{\mathbb{Q}_p-\mathrm{an}}(G,E)$ is locally J-analytic if the action of $\mathfrak{g}\otimes_{\mathbb{Q}_p} E$ on f in (6) factors through $\prod_{\sigma\in J} \mathfrak{g}\otimes_{F,\sigma} E$. Note the two extreme cases: when $J = S$, we rather say that f is locally $\mathbb{Q}_p$-analytic (instead of S-analytic) and write "$\mathbb{Q}_p - \mathrm{an}$" (instead of "$S - \mathrm{an}$") and when $J = \emptyset$, we rather say that f is locally constant or smooth (instead of $\emptyset$-analytic).

Let $J \subseteq S$, $\chi_1, \chi_2 : F^\times \to E^\times$ be two locally J-analytic multiplicative characters and $B(F) \subset \mathrm{GL}_2(F)$ the Borel subgroup of upper triangular matrices. We set:

$$\begin{aligned}\chi_1\otimes\chi_2 \quad : \quad & B(F) \longrightarrow E^\times\\ & \begin{pmatrix} a & * \\ 0 & d\end{pmatrix} \longmapsto \chi_1(a)\chi_2(d)\end{aligned}$$

and define as in §2 the locally J-analytic parabolic induction:

$$\begin{aligned}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\right)^{J-\mathrm{an}} := \{f:\ & \mathrm{GL}_2(F)\longrightarrow E\ , f \text{ is locally } J-\text{analytic and}\\ & f(bg)=(\chi_1\otimes\chi_2)(b)f(g),\ b\in B(F),\ g\in\mathrm{GL}_2(F)\}.\end{aligned}$$

As in §2 we endow this parabolic induction with a left E-linear action of $\mathrm{GL}_2(F)$ by $(g\cdot f)(g'):=f(g'g)$. This makes $\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\right)^{J-\mathrm{an}}$ into a locally $\mathbb{Q}_p$-analytic admissible representation of $\mathrm{GL}_2(F)$ in the sense of [17], [18].

For the rest of this section, we fix $D=D(\alpha,\widetilde{\alpha},(k_\sigma,a_\sigma,\widetilde{a}_\sigma)_{\sigma\in S})$ a rank 2 filtered φ-module as in §3 (not necessarily weakly admissible). For $r_\sigma\in\mathbb{Z}_{\geq 0}$ $(\sigma\in S)$, denote by $(\mathrm{Sym}^{r_\sigma}E^2)^\sigma$ the r_σ-symmetric product of the representation E^2 on which $\mathrm{GL}_2(F)$ acts via the embedding σ. For $J\subseteq S$ and $r_\sigma\in\mathbb{Z}$ $(\sigma\in J)$, denote by $\prod_{\sigma\in J}\sigma^{r_\sigma}: F^\times\rightarrow E^\times$ the locally J-analytic character sending x to $\prod_{\sigma\in J}\sigma(x)^{r_\sigma}$ (it is in fact "J-algebraic").

For $J_1\subseteq J_2\subseteq S$, we first define the following locally J_2-analytic parabolic inductions:

$$I_D(J_1,J_2):=\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\mathrm{unr}_F(\alpha^{-1})\prod_{\sigma\in J_1}\sigma^{k_\sigma-1}\otimes\mathrm{unr}_F(p\widetilde{\alpha}^{-1})\prod_{\sigma\in J_1}\sigma^{-1}\prod_{\sigma\in J_2\backslash J_1}\sigma^{k_\sigma-2}\right)^{J_2-\mathrm{an}}.$$

Note that by definition the unramified characters $\mathrm{unr}_F(\alpha^{-1})$ and $\mathrm{unr}_F(p\widetilde{\alpha}^{-1})$ only depend on α^f and $\widetilde{\alpha}^f$. Note also that $I_D(\emptyset,\emptyset)$ is a smooth unramified parabolic induction which is irreducible unless $(\alpha\widetilde{\alpha}^{-1})^f=q$ (resp. $(\alpha\widetilde{\alpha}^{-1})^f=q^{-1}$) in which case it is the twist by $\mathrm{unr}_F(\widetilde{\alpha}^{-1})\circ\det$ (resp. by $\mathrm{unr}_F(\alpha^{-1})\circ\det$) of the unique non-split extension of the trivial representation by the Steinberg representation (resp. of the Steinberg representation by the trivial representation).

For $J_1\subseteq J_2\subseteq S$, we then define the following locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$:

$$\pi_D(J_1,J_2)\ :=\ \left(\otimes_{\sigma\notin J_2}(\mathrm{Sym}^{k_\sigma-2}E^2)^\sigma\right)\otimes_E I_D(J_1,J_2). \tag{7}$$

Theorem 4.1 ([20]). *(i) The $\pi_D(J_1',J_2')$ for $J_1\subseteq J_1'\subseteq J_2'\subseteq J_2\subseteq S$ are all distinct and are subquotients of $\pi_D(J_1,J_2)$. Moreover, if $J_1'=J_1$ (resp. $J_2'=J_2$) then $\pi_D(J_1',J_2')$ is a subrepresentation (resp. a quotient) of $\pi_D(J_1,J_2)$.*

(ii) If $(\alpha\widetilde{\alpha}^{-1})^f \neq q^{\pm 1}$ or $J_1 \neq \emptyset$, the representations $\pi_D(J,J)$ for $J_1 \subseteq J \subseteq J_2$ are all topologically irreducible and exhaust the irreducible constituents of $\pi_D(J_1,J_2)$.

(iii) If $|J_2\backslash J_1| = 1$, $\pi_D(J_1,J_2)$ is the unique non-split extension of $\pi_D(J_2,J_2)$ by $\pi_D(J_1,J_1)$ (in the abelian category of admissible locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$ over E [18]).

Theorem 4.1 is proved in details by Schraen in [20, §1.3.3] (the proof relies on work of Schneider-Teitelbaum ([17]), Frommer ([13]) and Orlik-Strauch ([15])). It tells us that the position of the constituents $\pi_D(J,J)$ inside $\pi_D(J_1,J_2)$ form a "hypercube" with $\pi_D(\emptyset,\emptyset)$ as "first vertex" and $\pi_D(S,S)$ as "last vertex". Note that $\pi_D(\emptyset,\emptyset)$ is a locally algebraic representation of $\mathrm{GL}_2(F)$.

We define $\widetilde{I}_D(J_1,J_2)$ and $\widetilde{\pi}_D(J_1,J_2)$ exactly as $I_D(J_1,J_2)$ and $\pi_D(J_1,J_2)$ by exchanging α and $\widetilde{\alpha}$.

As in §2, if $(\alpha\widetilde{\alpha}^{-1})^f \neq q^{\pm 1}$, there is a $\mathrm{GL}_2(F)$-equivariant isomorphism $I_D(\emptyset,\emptyset) \simeq \widetilde{I}_D(\emptyset,\emptyset)$ which induces a $\mathrm{GL}_2(F)$-equivariant isomorphism $\pi_D(\emptyset,\emptyset) \simeq \widetilde{\pi}_D(\emptyset,\emptyset)$. When $(\alpha\widetilde{\alpha}^{-1})^f = q^{-1}$ (resp. $(\alpha\widetilde{\alpha}^{-1})^f = q$), we let $F_D \subset \pi_D(\emptyset,\emptyset)$ (resp. $\widetilde{F}_D \subset \widetilde{\pi}_D(\emptyset,\emptyset)$) be the unique non-zero finite dimensional subrepresentation. Otherwise, we let $F_D := 0$ (resp. $\widetilde{F}_D := 0$). We denote by $\pi(D)$ the unique non-zero irreducible subrepresentation of both $\pi_D(\emptyset,\emptyset)/F_D$ and $\widetilde{\pi}_D(\emptyset,\emptyset)/\widetilde{F}_D$ (note that $\pi(D)$ is $\pi_D(\emptyset,\emptyset)/F_D$ or $\widetilde{\pi}_D(\emptyset,\emptyset)/\widetilde{F}_D$ or both).

We define:

$$\begin{aligned}\Pi(D) \;:=\; & \Big(\pi_D(\emptyset,S\backslash Z_D)/F_D \oplus_{\pi(D)} \widetilde{\pi}_D(\emptyset,S\backslash \widetilde{Z}_D)/\widetilde{F}_D\Big) \\ & \bigoplus \big(\oplus_{\emptyset\subsetneq J\subseteq Z_D} \pi_D(J, J \amalg (S\backslash Z_D))\big) \bigoplus \big(\oplus_{\emptyset\subsetneq J\subseteq \widetilde{Z}_D} \widetilde{\pi}_D(J, J \amalg (S\backslash \widetilde{Z}_D))\big).\end{aligned} \tag{8}$$

When $(\alpha\widetilde{\alpha}^{-1})^f \neq q^{\pm 1}$, we can rewrite it more simply as:

$$\Pi(D) = \big(\oplus_{\emptyset\subseteq J\subseteq Z_D} \pi_D(J, J \amalg (S\backslash Z_D))\big) \bigoplus_{\pi(D)} \big(\oplus_{\emptyset\subseteq J\subseteq \widetilde{Z}_D} \widetilde{\pi}_D(J, J \amalg (S\backslash \widetilde{Z}_D))\big).$$

The representation $\Pi(D)$ is locally $\mathbb{Q}_p$-analytic and admissible. If $(\alpha\widetilde{\alpha}^{-1})^f \neq q^{\pm 1}$, (ii) of Theorem 4.1 implies that it has exactly $2^{|Z_D|} + 2^{|\widetilde{Z}_D|} - 1$ topologically irreducible constituents and that its socle $\mathrm{soc}_{\mathrm{GL}_2(F)}\,\Pi(D)$ is exactly:

$$\pi(D) \oplus \bigoplus \big(\oplus_{\emptyset\subsetneq J\subseteq Z_D} \pi_D(J,J)\big) \bigoplus \big(\oplus_{\emptyset\subsetneq J\subseteq \widetilde{Z}_D} \widetilde{\pi}_D(J,J)\big).$$

For later use, we also define:

$$\begin{aligned}\mathrm{soc}'_{\mathrm{GL}_2(F)}\,\Pi(D) \;:=\; & \Big(\pi_D(\emptyset,\emptyset)/F_D \oplus_{\pi(D)} \widetilde{\pi}_D(\emptyset,\emptyset)/\widetilde{F}_D\Big) \\ & \bigoplus \big(\oplus_{\emptyset\subsetneq J\subseteq Z_D} \pi_D(J,J)\big) \bigoplus \big(\oplus_{\emptyset\subsetneq J\subseteq \widetilde{Z}_D} \widetilde{\pi}_D(J,J)\big)\end{aligned} \tag{9}$$

which coincides with the above socle if $(\alpha\widetilde{\alpha}^{-1})^f \neq q^{\pm 1}$ (note that $\pi_D(\emptyset,\emptyset)/F_D \oplus_{\pi(D)} \widetilde{\pi}_D(\emptyset,\emptyset)/\widetilde{F}_D$ is the locally agebraic vectors of $\Pi(D)$).

Basically, what we do in (8) is that we decompose each "hypercube" $\pi_D(\emptyset, S)$ and $\widetilde{\pi}_D(\emptyset, S)$ into a direct sum of smaller "hypercubes" of the same size according to where the parameters a_σ and $\widetilde{a}_\sigma$ (defining the Hodge filtration) vanish. Note also that if $F = \mathbb{Q}_p$ and D is weakly admissible, we exactly recover the locally analytic representation in (i) of Theorem 2.1 (we leave this as an exercise). One big difference if $F \neq \mathbb{Q}_p$ is that one obviously can't recover D from $\Pi(D)$ as we miss the exact values of the a_σ.

Remark 4.2. By twisting by a suitable crystalline character, one can extend in an obvious way the definition of $\Pi(D)$ to any filtered φ-module with distinct Hodge-Tate weights for each $\sigma \in S$ and such that φ^f has distinct eigenvalues. This can be useful as the natural filtered φ-modules coming from, e.g., Hilbert eigenforms when $F \neq \mathbb{Q}_p$ only have Hodge-Tate weights $(0, k_\sigma - 1)_{\sigma\in S}$ after such a twist (see §8).

5 Weak admissibility and $\mathrm{GL}_2(F)$-unitarity I

The most interesting locally $\mathbb{Q}_p$-analytic representations of a p-adic analytic group are those which occur inside continuous unitary Banach spaces representations of this group. Assuming that $\Pi(D)$ occurs inside such a unitary representation of $\mathrm{GL}_2(F)$, we show that D is weakly admissible.

Recall that an invariant lattice on a locally $\mathbb{Q}_p$-analytic representation over E of a p-adic analytic group G is a closed $\mathcal{O}_E$-submodule that generates the underlying E-vector space of the representation, that doesn't contain non-zero E-lines and that is preserved by G. A locally $\mathbb{Q}_p$-analytic representation of G contains an invariant lattice if and only if it is continuously contained in a unitary Banach space representation of G over E (the p-adic completion of a lattice is a unit ball).

Proposition 5.1. *Let $J \subseteq S$, $r_\sigma \in \mathbb{Z}_{\geq 0}$ $(\sigma \in S\backslash J)$ and $\chi_1, \chi_2 : F^\times \to E^\times$ two locally J-analytic multiplicative characters. If:*

$$\big(\otimes_{\sigma\notin J} (\mathrm{Sym}^{r_\sigma} E^2)^\sigma\big) \otimes_E \Big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\Big)^{J-\mathrm{an}} \qquad (10)$$

is contained in a unitary Banach space representation of $\mathrm{GL}_2(F)$ *then one has:*

$$\mathrm{val}_{\mathbb{Q}_p}(\chi_1(p)) + \mathrm{val}_{\mathbb{Q}_p}(\chi_2(p)) + \sum_{\sigma\notin J} r_\sigma \;=\; 0 \qquad (11)$$

$$\mathrm{val}_{\mathbb{Q}_p}(\chi_2(p)) + \sum_{\sigma\notin J} r_\sigma \;\geq\; 0. \qquad (12)$$

Proof. The equality (11) is just the integrality of the central character, so we are left to prove (12). Viewing the representation (10) inside $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\prod_{\sigma\notin J}\sigma(z)^{r_\sigma}\big)^{\mathbb{Q}_p-\mathrm{an}}$, we see that it contains the functions $\mathbf{1}_{\mathcal{O}_F}:\mathrm{GL}_2(F)\to E$ (resp. $\mathbf{1}_{x+p\mathcal{O}_F}:\mathrm{GL}_2(F)\to E$ for $x\in\mathcal{O}_F$) defined by $\mathbf{1}_{\mathcal{O}_F}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right):=\chi_1(ad-bc)\chi_2\chi_1^{-1}(d)\prod_{\sigma\notin J}\sigma(d)^{r_\sigma}$ if $c/d\in\mathcal{O}_F$ (resp. $\mathbf{1}_{x+p\mathcal{O}_F}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right):=\chi_1(ad-bc)\chi_2\chi_1^{-1}(d)\prod_{\sigma\notin J}\sigma(d)^{r_\sigma}$ if $c/d\in x+p\mathcal{O}_F$) and $\mathbf{1}_{\mathcal{O}_F}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=0$ (resp. $\mathbf{1}_{x+p\mathcal{O}_F}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=0$) otherwise. It is straightforward to check that for $x\in\mathcal{O}_F$:

$$\begin{pmatrix}1&0\\x&p\end{pmatrix}\mathbf{1}_{\mathcal{O}_F}=\chi_2(p)p^{\sum_{\sigma\notin J}r_\sigma}\mathbf{1}_{x+p\mathcal{O}_F}$$

which implies:

$$\|\mathbf{1}_{x+p\mathcal{O}_F}\|=|\chi_2(p^{-1})p^{-\sum_{\sigma\notin J}r_\sigma}|_F\|\mathbf{1}_{\mathcal{O}_F}\| \tag{13}$$

where $\|\cdot\|$ is any invariant norm on (10) (induced by a unitary Banach space representation). Taking a set $(x_i)_{i\in I}$ of representatives of $\mathcal{O}_F/p\mathcal{O}_F$ in $\mathcal{O}_F$, we obviously have $\mathbf{1}_{\mathcal{O}_F}=\sum_{i\in I}\mathbf{1}_{x_i+p\mathcal{O}_F}$ which implies:

$$\|\mathbf{1}_{\mathcal{O}_F}\|\le\sup_{i\in I}\|\mathbf{1}_{x_i+p\mathcal{O}_F}\|=|\chi_2(p^{-1})p^{-\sum_{\sigma\notin J}r_\sigma}|_F\|\mathbf{1}_{\mathcal{O}_F}\|.$$

Since $\|\mathbf{1}_{\mathcal{O}_F}\|\neq0$, we deduce $|\chi_2(p^{-1})p^{-\sum_{\sigma\notin J}r_\sigma}|_F\ge1$ which is just what we want. □

Corollary 5.2. *Let $D=D(\alpha,\widetilde{\alpha},(k_\sigma,a_\sigma,\widetilde{a}_\sigma)_{\sigma\in S})$ be a rank 2 filtered φ-module as in §3 (not necessarily weakly admissible) and let $\Pi(D)$ be the locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F)$ associated to D in §4. If $\Pi(D)$ is contained in a unitary Banach space representation of $\mathrm{GL}_2(F)$ then D is weakly admissible.*

Proof. The central character of $\Pi(D)$ is also the one of $\pi_D(\emptyset,S)$ and the assumption implies that it sends p to an element of valuation zero inside $E^\times$, which immediately gives (3). Assume first $(\alpha\widetilde{\alpha}^{-1})^f\neq q^{\pm1}$. The locally $\mathbb{Q}_p$-analytic representations $\pi_D(Z_D,S)$ and $\widetilde{\pi}_D(\widetilde{Z}_D,S)$ both appear as subrepresentations of $\Pi(D)$, hence are contained in a unitary Banach space representation if $\Pi(D)$ is. Applying (12) to $\pi_D(Z_D,S)$ yields:

$$[F:\mathbb{Q}_p]-\mathrm{val}_F(\widetilde{\alpha})-|Z_D|+\sum_{\sigma\in S\setminus Z_D}(k_\sigma-2)\ge0$$

which can be rewritten as:

$$\mathrm{val}_F(\widetilde{\alpha}) \leq \sum_{\sigma \in S \setminus Z_D} (k_\sigma - 1)$$

which, combined with (3), is equivalent to:

$$\sum_{\sigma \in Z_D} (k_\sigma - 1) \leq \mathrm{val}_F(\alpha). \tag{14}$$

Applying (12) to $\widetilde{\pi}_D(\widetilde{Z}_D, S)$ yields:

$$[F : \mathbb{Q}_p] - \mathrm{val}_F(\alpha) - |\widetilde{Z}_D| + \sum_{\sigma \in S \setminus \widetilde{Z}_D} (k_\sigma - 2) \geq 0$$

which can be rewritten as:

$$\mathrm{val}_F(\alpha) \leq \sum_{\sigma \in S \setminus \widetilde{Z}_D} (k_\sigma - 1). \tag{15}$$

We see that (14) and (15) are just (4), and by Lemma 3.3 this finishes the proof in the case $(\alpha\widetilde{\alpha}^{-1})^f \neq q^{\pm 1}$. Assume now $(\alpha\widetilde{\alpha}^{-1})^f = q$. As $\pi_D(Z_D, S)$ is a subrepresentation of $\Pi(D)$, the first part of the above proof gives (14). If $\widetilde{Z}_D \neq \emptyset$ then $\widetilde{\pi}_D(\widetilde{Z}_D, S)$ is still a subrepresentation of $\Pi(D)$ and the second part of the above proof gives (15) and hence the result by Lemma 3.3. Assume $\widetilde{Z}_D = \emptyset$. The equality $\mathrm{val}_F(\alpha\widetilde{\alpha}^{-1}) = [F : \mathbb{Q}_p]$ combined with (3) gives:

$$\mathrm{val}_F(\alpha) = \frac{1}{2} \sum_{\sigma \in S} k_\sigma \quad \text{and} \quad \mathrm{val}_F(\widetilde{\alpha}) = \frac{1}{2} \sum_{\sigma \in S} (k_\sigma - 2).$$

We thus have (15) since $\widetilde{Z}_D = \emptyset$ and $k_\sigma \geq 2$ for all σ and we are done by Lemma 3.3. The case $(\alpha\widetilde{\alpha}^{-1})^f = q^{-1}$ is symmetric by exchanging α and $\widetilde{\alpha}$. □

Remark 5.3. (i) One can expect that the converse statement of Corollary 5.2 holds, namely that if D is weakly admissible, then there always exists an invariant norm (or lattice) on $\Pi(D)$. This holds for instance when $F = \mathbb{Q}_p$ but is non-trivial and ultimately rests on the construction of $\Pi(D)$ via (φ, Γ)-modules ([6], [1]).

(ii) Corollary 5.2 still holds replacing $\Pi(D)$ by $\mathrm{soc}'_{\mathrm{GL}_2(F)}\Pi(D)$ (and thus yielding a stronger statement). Indeed, the proof is the same by applying Proposition 5.1 to $\pi_D(Z_D, Z_D)$ and $\widetilde{\pi}_D(\widetilde{Z}_D, \widetilde{Z}_D)$ instead of $\pi_D(Z_D, S)$ and $\widetilde{\pi}_D(\widetilde{Z}_D, S)$.

6 Amice-Vélu and Vishik revisited

We state and (re)prove a slight generalization of a well-known result of Amice-Vélu and Vishik.

Let $U \subseteq \mathcal{O}_F$ an open subset, $J \subseteq S$ and $r_\sigma \in \mathbb{Z}_{\geq 0}$ for $\sigma \in S\backslash J$. Denote by $\mathcal{F}(U, J, (r_\sigma)_{\sigma \in S\backslash J})$ the E-vector space of functions $f : U \to E$ such that there exists an open (disjoint) cover $(a_i + \varpi_F^{n_i}\mathcal{O}_F)_{i\in I}$ of U such that, for each i, one has an expansion:

$$f(z)|_{a_i+\varpi_F^{n_i}\mathcal{O}_F} = \sum_{\substack{\underline{m}=(m_\sigma)_{\sigma\in S}\in\mathbb{Z}_{\geq 0}^{[F:\mathbb{Q}_p]} \\ m_\sigma \leq r_\sigma \text{ if } \sigma\notin J}} a_{\underline{m}} \prod_{\sigma\in S} \sigma(z-a_i)^{m_\sigma} \tag{16}$$

with $|a_{\underline{m}}|_F q^{-n_i(\sum_{\sigma\in S} m_\sigma)} \to 0$ when $\sum_{\sigma\in S} m_\sigma \to +\infty$ ($a_{\underline{m}} \in E$). Recall that $\mathcal{F}(U, J, (r_\sigma)_{\sigma \in S\backslash J})$ is an inductive limit of Banach spaces with injective and compact transition maps ([16, §16]), namely the Banach spaces of functions as in (16) with norm:

$$\sup_{\underline{m}} \Big(|a_{\underline{m}}|_F \, q^{-n_i(\sum_{\sigma\in S} m_\sigma)} \Big).$$

Note that $\mathcal{F}(U, J, (r_\sigma)_{\sigma \in S\backslash J}) \subseteq \mathcal{F}(U, J', (r_\sigma)_{\sigma \in S\backslash J'})$ for any $J \subseteq J'$.

The technical but key lemma that follows is essentially due to Amice-Vélu and Vishik.

Lemma 6.1. *Let B be a p-adic Banach space over E and ι be an E-linear map $\mathcal{F}(U, J, (r_\sigma)_{\sigma \in S\backslash J}) \to B$. Let $\|\cdot\|$ be a norm on B defining its topology and assume that there exist $C \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{\geq 0}$ such that, for any $a \in \mathcal{O}_F$, any $n \in \mathbb{Z}_{\geq 0}$ and any $(m_\sigma)_{\sigma\in S} \in \mathbb{Z}_{\geq 0}^{[F:\mathbb{Q}_p]}$ with $m_\sigma \leq r_\sigma$ if $\sigma \notin J$, one has:*

$$\Big\| \iota\Big(\mathbf{1}_{a+\varpi_F^n\mathcal{O}_F}(z) \prod_{\sigma\in S} \sigma(z-a)^{m_\sigma} \Big) \Big\| \leq C q^{-n(\sum_{\sigma\in S} m_\sigma - c)} \tag{17}$$

where $\mathbf{1}_{a+\varpi_F^n\mathcal{O}_F}$ is the characteristic function of $a+\varpi_F^n\mathcal{O}_F$. Let:

$$J' := J \amalg \{\tau \in S\backslash J, c < r_\tau + 1\}.$$

Then ι uniquely extends to an E-linear map $\iota' : \mathcal{F}(U, J', (r_\sigma)_{\sigma \in S\backslash J'}) \to B$ such that the diagram:

$$\begin{tikzcd} \mathcal{F}(U, J, (r_\sigma)_{\sigma \in S\backslash J}) \arrow[r, "\iota"] \arrow[d, hook] & B \\ \mathcal{F}(U, J', (r_\sigma)_{\sigma \in S\backslash J'}) \arrow[ur, "\iota'"'] & \end{tikzcd}$$

commutes and such that (17) holds for all $(m_\sigma)_{\sigma\in S} \in \mathbb{Z}_{\geq 0}^{[F:\mathbb{Q}_p]}$ with $m_\sigma \leq r_\sigma$ if $\sigma \notin J'$ (possibly up to increasing C). Moreover, ι and ι' are continuous.

Proof. First, the map ι is automatically continuous. Indeed, for any $a \in \mathcal{O}_F$ and any $n \in \mathbb{Z}_{\geq 0}$, the inequality (17) gives that ι is continuous upon restriction to the Banach subspace of analytic functions on $a + \varpi_F^n \mathcal{O}_F$ as in (16). Since the topology on $\mathcal{F}(U, J, (r_\sigma)_{\sigma \in S \backslash J})$ is the locally convex topology with respect to these Banach subspaces, this implies ι is continuous. Let $\tau \in J' \backslash J$. By induction it is enough to prove the statement replacing J' by $J \amalg \{\tau\}$. By E-linearity, it is then enough to prove that ι uniquely extends to each function of the form:

$$\mathbf{1}_{a+\varpi_F^n \mathcal{O}_F}(z) \tau(z-a)^{m_\tau} \prod_{\sigma \in S \backslash \tau} \sigma(z-a)^{m_\sigma} \tag{18}$$

with $m_\sigma \leq r_\sigma$ if $\sigma \notin J \amalg \{\tau\}$ and $m_\tau \geq r_\tau + 1$ so that (17) still holds (maybe up to modifying C). Let f be a function as in (18) and let:

$$\begin{aligned} D^- &:= \{\underline{d} = (d_\sigma)_{\sigma \in S} \in \mathbb{Z}_{\geq 0}^{[F:\mathbb{Q}_p]}, d_\sigma \leq m_\sigma \text{ and } d_\tau \leq r_\tau\} \\ D^+ &:= \{\underline{d} = (d_\sigma)_{\sigma \in S} \in \mathbb{Z}_{\geq 0}^{[F:\mathbb{Q}_p]}, d_\sigma \leq m_\sigma \text{ and } r_\tau + 1 \leq d_\tau\}. \end{aligned}$$

Since for any function h on U:

$$\mathbf{1}_{a+\varpi_F^n \mathcal{O}_F}(z) h(z-a) = \sum_{a' \in a + \varpi_F^n [k_F]} \mathbf{1}_{a'+\varpi_F^{n+1} \mathcal{O}_F}(z) h((z-a') + (a'-a)),$$

an easy computation shows we can rewrite f as $f_n^+ + f_n^-$ where:

$$\begin{aligned} f_n^+ &:= \sum_{a' \in a+\varpi_F^n [k_F]} \mathbf{1}_{a'+\varpi_F^{n+1}\mathcal{O}_F}(z) \Big(\sum_{\underline{d} \in D^+} \Big(a_{\underline{d}}^+ \prod_{\sigma \in S} \sigma(a-a')^{m_\sigma - d_\sigma} \prod_{\sigma \in S} \sigma(z-a')^{d_\sigma} \Big) \Big) \\ f_n^- &:= \sum_{a' \in a+\varpi_F^n [k_F]} \mathbf{1}_{a'+\varpi_F^{n+1}\mathcal{O}_F}(z) \Big(\sum_{\underline{d} \in D^-} \Big(a_{\underline{d}}^- \prod_{\sigma \in S} \sigma(a-a')^{m_\sigma - d_\sigma} \prod_{\sigma \in S} \sigma(z-a')^{d_\sigma} \Big) \Big) \end{aligned}$$

for some $a_{\underline{d}}^+, a_{\underline{d}}^- \in \mathcal{O}_E$. Since:

$$\left| a_{\underline{d}}^- \prod_{\sigma \in S} \sigma(a-a')^{m_\sigma - d_\sigma} \right|_F \leq q^{-n(\sum_{\sigma \in S} m_\sigma - d_\sigma)},$$

and since one has by (17) and the definition of D^-:

$$\left\| \iota\Big(\mathbf{1}_{a'+\varpi_F^{n+1}\mathcal{O}_F}(z) \prod_{\sigma \in S} \sigma(z-a')^{d_\sigma} \Big) \right\| \leq C q^{-(n+1)(\sum_{\sigma \in S} d_\sigma - c)},$$

we see that:

$$\left\| \iota(f_n^-) \right\| \leq C q^c q^{-n(\sum_{\sigma \in S} m_\sigma - c)}.$$

One can start again and write f_n^+ as $f_{n+1}^+ + f_{n+1}^-$ where f_{n+1}^+, f_{n+1}^- are finite linear combinations over $\mathcal{O}_E$ of functions $\mathbf{1}_{a''+\varpi_F^{n+2}\mathcal{O}_F}(z) \prod_{\sigma \in S} \sigma(z-a'')^{d_\sigma}$ with

$r_\tau + 1 \le d_\tau \le m_\tau$ for f^+_{n+1} and where $\|\iota(f^-_{n+1})\| \le Cq^c q^{-(n+1)(\sum_{\sigma\in S} m_\sigma - c)}$ by the same proof as before. Iterating this process, we see that for any integer $M \ge n$ the function f in (18) can be written:

$$f = f^+_M + \sum_{i=n}^{M} f^-_i \tag{19}$$

where $\iota(f^-_i)$ is defined and satisfies $\|\iota(f^-_i)\| \le Cq^c q^{-i(\sum_{\sigma\in S} m_\sigma - c)}$ and where f^+_M is a finite linear combination over $\mathcal{O}_E$ of functions $\mathbf{1}_{a''+\varpi_F^M\mathcal{O}_F}(z)\prod_{\sigma\in S}\sigma(z-a'')^{d_\sigma}$ with $d_\sigma \le m_\sigma$ and $r_\tau + 1 \le d_\tau$. As $c < r_\tau + 1 \le \sum_{\sigma\in S} m_\sigma$ (recall $r_\tau + 1 \le m_\tau$), we see that $\iota(f^-_i) \to 0$ in B when $i \to +\infty$. If ι extends to $\mathcal{F}(U, J', (r_\sigma)_{\sigma\in S\setminus J'})$ in such a way that (17) is satisfied (up to modifying C), we see that we have in particular $\|\iota(f^+_M)\| \le Cq^{-M(r_\tau+1-c)}$ and hence $\iota(f^+_M) \to 0$ when $M \to +\infty$. So we must have $\iota(f) = \sum_{i=n}^{+\infty}\iota(f^-_i)$. Conversely, setting $\iota(f) := \sum_{i=n}^{+\infty}\iota(f^-_i)$ implies:

$$\|\iota(f)\| \le \sup_{i\ge n}\|\iota(f^-_i)\| = Cq^c\sup_{i\ge n} q^{-i(\sum_{\sigma\in S} m_\sigma - c)} = Cq^c q^{-n(\sum_{\sigma\in S} m_\sigma - c)}$$

and hence (17) is still satisfied replacing C by Cq^c. The continuity of ι' is checked as for ι. $\square$

7 Weak admissibility and $\mathrm{GL}_2(F)$-unitarity II

Using Lemma 6.1, we show that if a continuous unitary Banach space representation of $\mathrm{GL}_2(F)$ over E contains $\mathrm{soc}'_{\mathrm{GL}_2(F)}\Pi(D)$ (see (9)), then it automatically contains a larger locally $\mathbb{Q}_p$-analytic representation $\Pi(D)^{\mathrm{Amice}}$ which is included in $\Pi(D)$.

We start with the following theorem.

Theorem 7.1. *Let $J \subseteq S$, $r_\sigma \in \mathbb{Z}_{\ge 0}$ ($\sigma \in S\setminus J$) and $\chi_1, \chi_2 : F^\times \to E^\times$ two locally J-analytic multiplicative characters. Define:*

$$J' := J \amalg \{\tau \in S\setminus J, -\mathrm{val}_{\mathbb{Q}_p}(\chi_1(p)) < r_\tau + 1\}. \tag{20}$$

Then any continuous E-linear equivariant injection:

$$\big(\otimes_{\sigma\notin J}(\mathrm{Sym}^{r_\sigma}E^2)^\sigma\big)\otimes_E\Big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\Big)^{J-\mathrm{an}}\hookrightarrow B \tag{21}$$

where B is a unitary Banach space representation of $\mathrm{GL}_2(F)$ over E canonically extends to an E-linear continuous equivariant injection:

$$\big(\otimes_{\sigma\notin J'}(\mathrm{Sym}^{r_\sigma}E^2)^\sigma\big)\otimes_E\Big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\textstyle\prod_{\sigma\in J'\setminus J}\sigma^{r_\sigma}\Big)^{J'-\mathrm{an}}\hookrightarrow B\,. \tag{22}$$

Proof. Any element F of:

$$\big(\otimes_{\sigma\notin J}(\mathrm{Sym}^{r_\sigma}E^2)^\sigma\big)\otimes_E\Big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\Big)^{J-\mathrm{an}} \tag{23}$$

can be seen as a pair of functions $(f_1:\mathcal{O}_F\to E,\ f_2:\varpi_F\mathcal{O}_F\to E)$ by setting:

$$f_1(z):=F\Big(\begin{pmatrix}0&1\\-1&z\end{pmatrix}\Big)\quad\text{and}\quad f_2(z):=\chi_2\chi_1^{-1}(z)\Big(\prod_{\sigma\notin J}\sigma(z)^{r_\sigma}\Big)f_1\Big(\frac{1}{z}\Big) \tag{24}$$

where f_2 is the only continuous function on $\varpi_F\mathcal{O}_F$ agreeing with the right hand side of (24) on $\varpi_F\mathcal{O}_F\backslash\{0\}$. The map $F\mapsto f_1\oplus f_2$ yields an isomorphism between (23) and $\mathcal{F}(\mathcal{O}_F,J,(r_\sigma)_{\sigma\in S\backslash J})\oplus\mathcal{F}(\varpi_F\mathcal{O}_F,J,(r_\sigma)_{\sigma\in S\backslash J})$ and the action of $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(F)$ is given by:

$$(gF)_1(z)=\chi_1(\det g)\chi_2\chi_1^{-1}(-cz+a)\prod_{\sigma\notin J}\sigma(-cz+a)^{r_\sigma}f_1\Big(\frac{dz-b}{-cz+a}\Big) \tag{25}$$

if $\frac{dz-b}{-cz+a}\in\mathcal{O}_F$ and:

$$(gF)_1(z)=\chi_1(\det g)\chi_2\chi_1^{-1}(dz-b)\prod_{\sigma\notin J}\sigma(dz-b)^{r_\sigma}f_2\Big(\frac{-cz+a}{dz-b}\Big)$$

if $\frac{dz-b}{-cz+a}\in F\backslash\mathcal{O}_F$, and symmetric formulas for $(gF)_2(z)$. Let ι be a continuous injection as in (21) and let $\|\cdot\|$ be an invariant norm on B (which exists by assumption). Let $F:=f_1\oplus 0=f_1$ where $f_1(z):=\prod_{\sigma\in S}\sigma(z)^{m_\sigma}$ for $z\in\mathcal{O}_F$ and some $(m_\sigma)_{\sigma\in S}\in\mathbb{Z}_{\geq 0}^{[F:\mathbb{Q}_p]}$ such that $m_\sigma\leq r_\sigma$ if $\sigma\notin J$. By continuity of ι, there is $C\in\mathbb{R}_{>0}$ such that $\|\iota(F)\|\leq C$ for all such F. Using $\|\iota(gF)\|=\|\iota(F)\|$ and then applying (25) with $g=\left(\begin{smallmatrix}1&a/\varpi_F^n\\0&1/\varpi_F^n\end{smallmatrix}\right)$ ($a\in\mathcal{O}_F$, $n\in\mathbb{Z}_{\geq 0}$) gives:

$$\begin{aligned}\Big\|\iota\Big(\mathbf{1}_{a+\varpi_F^n\mathcal{O}_F}(z)\prod_{\sigma\in S}\sigma(z-a)^{m_\sigma}\Big)\Big\| &= \Big|\chi_1(\varpi_F^n)\prod_{\sigma\in S}\sigma(\varpi_F^n)^{m_\sigma}\Big|_F\|\iota(F)\|\\ &\leq q^{-n\big(\sum_{\sigma\in S}m_\sigma+\mathrm{val}_{\mathbb{Q}_p}(\chi_1(p))\big)}C.\end{aligned}$$

Lemma 6.1 applied with $c:=-\mathrm{val}_{\mathbb{Q}_p}(\chi_1(p))$ and the norm induced by B (via ι) gives that $\mathcal{F}(\mathcal{O}_F,J,(r_\sigma)_{\sigma\in S\backslash J})\oplus 0\hookrightarrow B$ canonically extends to a continuous map $\mathcal{F}(\mathcal{O}_F,J',(r_\sigma)_{\sigma\in S\backslash J'})\oplus 0\to B$. Let $F:=0\oplus f_2=f_2$ where $f_2(z):=\prod_{\sigma\in S}\sigma(z)^{m_\sigma}$ with the m_σ as before. Applying (25) (more precisely its symmetric version for $(gF)_2(z)$) with $g:=\left(\begin{smallmatrix}1/\varpi_F^n&0\\a/\varpi_F^n&1\end{smallmatrix}\right)\in\mathrm{GL}_2(F)$ gives by an analogous proof that $0\oplus\mathcal{F}(\varpi_F\mathcal{O}_F,J,(r_\sigma)_{\sigma\in S\backslash J})\hookrightarrow B$ canonically extends to a continuous map $0\oplus\mathcal{F}(\varpi_F\mathcal{O}_F,J',(r_\sigma)_{\sigma\in S\backslash J'})\to B$. Via the isomorphism:

$$\begin{aligned}&\big(\otimes_{\sigma\notin J'}(\mathrm{Sym}^{r_\sigma}E^2)^\sigma\big)\otimes_E\Big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\prod_{\sigma\in J'\backslash J}\sigma^{r_\sigma}\Big)^{J'-\mathrm{an}}\simeq\\ &\qquad\mathcal{F}(\mathcal{O}_F,J',(r_\sigma)_{\sigma\in S\backslash J'})\oplus\mathcal{F}(\varpi_F\mathcal{O}_F,J',(r_\sigma)_{\sigma\in S\backslash J'})\end{aligned} \tag{26}$$

(defined as previously) we have that ι extends to an E-linear continuous map as in (22) (still denoted ι) except that it remains to prove that it is equivariant and injective. Let's first indicate how equivariance can be checked. By linearity and symmetry, it is enough to prove $\iota(g(f_1 \oplus 0)) = g\iota(f_1 \oplus 0)$ for $f_1 = \mathbf{1}_{a+\varpi_F^n\mathcal{O}_F}(z)\prod_{\sigma\in S}\sigma(z-a)^{m_\sigma} \in \mathcal{F}(\mathcal{O}_F, J', (r_\sigma)_{\sigma\in S\backslash J'})$ and $g \in \mathrm{GL}_2(F)$. Going back to the proof of Lemma 6.1, one writes $f_1 = f_M^+ + \sum_{i=n}^M f_i^-$ for all $M \geq n$ as in (19). Using that ι is equivariant upon restriction to $\mathcal{F}(\mathcal{O}_F, J, (r_\sigma)_{\sigma\in S\backslash J})$ and that $\|\cdot\|$ is invariant, it is enough to check that $\|\iota(gf_M^+)\| \to 0$ when $M \to +\infty$. This follows again from the bounds (17) and an explicit computation of gf_M^+ using (25). Now injectivity follows from continuity and equivariance. Indeed, if (22) is not injective, then its non-zero kernel must be a closed invariant subspace. But by Theorem 4.1, this closed invariant subspace must contain (23) (which is a topologically irreducible $\mathrm{GL}_2(F)$-representation). However, ι in (21) being injective, this is impossible. □

Let D be a rank 2 filtered φ-module with Hodge-Tate weights $(0, k_\sigma - 1)_{\sigma\in S}$ as in §3 (not necessarily weakly admissible). For $J \subseteq S$ define:

$$\begin{aligned} Z_D(J) &:= J \amalg \left\{\tau \in S\backslash J, \mathrm{val}_F(\alpha) \geq k_\tau - 1 + \sum_{\sigma\in J}(k_\sigma - 1)\right\} \\ \widetilde{Z}_D(J) &:= J \amalg \left\{\tau \in S\backslash J, \mathrm{val}_F(\widetilde{\alpha}) \geq k_\tau - 1 + \sum_{\sigma\in J}(k_\sigma - 1)\right\}. \end{aligned}$$

We set:

$$\begin{aligned} \Pi(D)^{\mathrm{Amice}} &:= \Big(\pi_D(\emptyset, S\backslash Z_D(\emptyset))/F_D \oplus_{\pi(D)} \widetilde{\pi}_D(\emptyset, S\backslash \widetilde{Z}_D(\emptyset))/\widetilde{F}_D\Big) \\ &\quad \bigoplus \oplus_{\emptyset\subsetneq J\subseteq Z_D} \pi_D\big(J, J \amalg (S\backslash Z_D(J))\big) \\ &\quad \bigoplus \oplus_{\emptyset\subsetneq J\subseteq \widetilde{Z}_D} \widetilde{\pi}_D\big(J, J \amalg (S\backslash \widetilde{Z}_D(J))\big). \end{aligned} \tag{27}$$

We have $\mathrm{soc}'_{\mathrm{GL}_2(F)}\,\Pi(D) \subseteq \Pi(D)^{\mathrm{Amice}}$ (see (9) for $\mathrm{soc}'_{\mathrm{GL}_2(F)}\,\Pi(D)$).

Corollary 7.2. *Let D be a rank 2 filtered φ-module as above.*

(i) If a unitary Banach space representation of $\mathrm{GL}_2(F)$ over E continuously contains $\mathrm{soc}'_{\mathrm{GL}_2(F)}\,\Pi(D)$ then it continuously contains $\Pi(D)^{\mathrm{Amice}}$ and D is weakly admissible.

(ii) If D is weakly admissible, then $\Pi(D)^{\mathrm{Amice}} \subseteq \Pi(D)$.

Proof. (i) The last statement is (ii) of Remark 5.3. Let $J \subseteq S$, $J \neq \emptyset$. Rewriting (20) as $J' = J \amalg S\backslash Z(J)$ where $Z(J) := J \amalg \{\tau \in S\backslash J, -\mathrm{val}_{\mathbb{Q}_p}(\chi_1(p)) \geq r_\tau + 1\}$,

Theorem 7.1 implies that a unitary Banach space representation B of $\mathrm{GL}_2(F)$ (continuously) contains $\pi_D(J,J)$ (resp. $\widetilde{\pi}_D(J,J)$) if and only if it (continuously) contains $\pi_D(J, J \amalg (S\backslash Z_D(J)))$ (resp. $\widetilde{\pi}_D(J, J \amalg (S\backslash \widetilde{Z}_D(J)))$). Likewise, if B contains $\pi_D(\emptyset,\emptyset)/F_D$ (resp. $\widetilde{\pi}_D(\emptyset,\emptyset)/\widetilde{F}_D$), the same proof as for Theorem 7.1 gives an equivariant continuous map $\pi_D(\emptyset, S\backslash Z_D(\emptyset)) \to B$ with kernel containing F_D (resp. with tildes). If this kernel were strictly bigger than F_D, then it would also contain $\pi_D(\emptyset,\emptyset)$ which is impossible. We conclude using that all embeddings $\pi_D(J,J) \hookrightarrow \pi_D(J, J \amalg (S\backslash Z_D(J)))$ etc. are essential (Theorem 4.1), that is, any subrepresentation of the right hand side intersects non-trivially the left hand side.
(ii) If $J \subseteq Z_D$ (resp. $J \subseteq \widetilde{Z}_D$), the first inequality in (4) (resp. in (5)) implies $Z_D \subseteq Z_D(J)$ (resp. $\widetilde{Z}_D \subseteq \widetilde{Z}_D(J)$) and thus we have:

$$J \amalg S\backslash Z_D(J) \subseteq J \amalg S\backslash Z_D \quad \text{(resp. } J \amalg S\backslash \widetilde{Z}_D(J) \subseteq J \amalg S\backslash \widetilde{Z}_D\text{)}.$$

Comparing (8) and (27), we see that $\Pi(D)^{\mathrm{Amice}} \subseteq \Pi(D)$. □

We say that a continuous 2-dimensional E-linear representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ is ordinary if its semi-simplification has restriction to inertia isomorphic to $\varepsilon^* \oplus 1$ for some integer $*$. It would have been nice to have in many cases $\Pi(D)^{\mathrm{Amice}} = \Pi(D)$, however this turns out to be quite rare.

Proposition 7.3. *We have $\Pi(D)^{\mathrm{Amice}} = \Pi(D)$ if and only if either $V_{\mathrm{cris}}(D)$ is split ordinary or $V_{\mathrm{cris}}(D)$ is irreducible and $F = \mathbb{Q}_p$.*

Proof. Clearly the statement holds if and only if $Z_D(J) = Z_D$ for all $J \subseteq Z_D$ and $\widetilde{Z}_D(J) = \widetilde{Z}_D$ for all $J \subseteq \widetilde{Z}_D$. But $Z_D(J) = Z_D$ if and only if:

$$\mathrm{val}_F(\alpha) < k_\tau - 1 + \sum_{\sigma \in J}(k_\sigma - 1)$$

for all $\tau \in S\backslash Z_D$, so $Z_D(J) = Z_D$ for all $J \subseteq Z_D$ if and only if :

$$\tau \notin Z_D \Rightarrow \mathrm{val}_F(\alpha) < k_\tau - 1 \tag{28}$$

and likewise with tildes everywhere. Assume first that there are $\tau, \widetilde{\tau} \in S$ with $\tau \neq \widetilde{\tau}$ such that $\tau \notin Z_D$ and $\widetilde{\tau} \notin \widetilde{Z}_D$. By (3), the inequality (28) and its tilde analogue imply $\sum_{\sigma\in S}(k_\sigma - 1) < k_\tau - 1 + k_{\widetilde{\tau}} - 1$ which is impossible. Since $Z_D \cap \widetilde{Z}_D = \emptyset$, we thus have either $Z_D = S$ and $\widetilde{Z}_D = \emptyset$, or $Z_D = \emptyset$ and $\widetilde{Z}_D = S$, or $Z_D = \widetilde{Z}_D = \emptyset$ and $|S| = 1$. The first two cases correspond to $V_{\mathrm{cris}}(D)$ being split ordinary and the last to $F = \mathbb{Q}_p$ and $V_{\mathrm{cris}}(D)$ being indecomposable. Finally, it is straightforward to check that (28) and its tilde analogue hold in the first two cases, and hold in the last if and only if $V_{\mathrm{cris}}(D)$ is moreover irreducible. □

8 Local-global considerations

We briefly put the previous considerations within a global setting.

Let L be a totally real finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_L$. Assume for simplicity that there is a unique prime ideal $\mathfrak{p}$ in $\mathcal{O}_L$ above p and let $L_{\mathfrak{p}}$ denote the completion of L at $\mathfrak{p}$ and $L_{\mathfrak{p},0}$ its maximal absolutely unramified subfield. Denote by $\mathbb{A}_{L,f}^p$ the finite adèles of L outside p. To any quaternion algebra D over L which splits at only one of the infinite places and which splits at $\mathfrak{p}$ and to any compact open subgroup $K_f^p \subset (D \otimes_L \mathbb{A}_{L,f}^p)^\times$, one can associate a tower of Shimura algebraic curves $(S(K_f^p K_{f,p}))_{K_{f,p}}$ over L where $K_{f,p}$ runs over the compact open subgroups of $(D \otimes_L L_{\mathfrak{p}})^\times \simeq \mathrm{GL}_2(L_{\mathfrak{p}})$ (see e.g. [4]). Consider:

$$\widehat{H}^1(K_f^p) \ := \ \Big(\varprojlim_n \varinjlim_{K_{f,p}} H^1_{\text{ét}}\big(S(K_f^p K_{f,p}) \times_L \overline{\mathbb{Q}}, \mathcal{O}_E/p^n\mathcal{O}_E\big)\Big) \otimes_{\mathcal{O}_E} E$$

which is a p-adic Banach space over E (an open unit ball being the $\mathcal{O}_E$-module $\varprojlim \varinjlim H^1_{\text{ét}}\big(S(K_f^p K_{f,p}) \times_L \overline{\mathbb{Q}}, \mathcal{O}_E/p^n\mathcal{O}_E\big)$) endowed with a linear continuous unitary action of $\mathrm{GL}_2(L_{\mathfrak{p}}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ ([5]).

Let g be a parabolic Hilbert eigenform of level prime to p, E a finite extension of $\mathbb{Q}_p$ containing the Galois closure of $L_{\mathfrak{p}}$ and the Hecke eigenvalues associated to g, and $k_\sigma \geq 2$ for $\sigma \in S := \mathrm{Hom}(L_{\mathfrak{p}}, E)$ the various weights of g. We denote by:

$$\rho_g : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to \mathrm{GL}_2(E)$$

the continuous totally odd Galois representation associated to g ([21]). We normalize ρ_g so that the traces of arithmetic Frobeniuses at unramified places are the Hecke eigenvalues. We let :

$$(D_g, \varphi, \mathrm{Fil}^\cdot D_{g,L_{\mathfrak{p}}}) := \Big((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \rho_g)^{\mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}})}, \varphi \otimes \mathrm{Id}, (\mathrm{Fil}^\cdot B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \rho_g)^{\mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}})}\Big).$$

Choose $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}}) \to E^\times$ a crystalline character such that $\rho_g|_{\mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}})} \otimes \eta$ has Hodge-Tate weights $(0, k_\sigma - 1)_{\sigma \in S}$ (such a character always exists) and define the filtered module $D_{g,\eta}$ as D_g but replacing $\rho_g|_{\mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}})}$ by $\rho_g|_{\mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}})} \otimes \eta$. If the eigenvalues of $\varphi^{[L_{\mathfrak{p},0}:\mathbb{Q}_p]}$ on $D_{g,\eta}$ (or equivalently on D_g) are distinct, then the locally $\mathbb{Q}_p$-analytic representation $\Pi(D_{g,\eta})$ is well-defined (§4). We set:

$$\Pi(D_g) := \Pi(D_{g,\eta}) \otimes \eta^{-1} \circ \det.$$

The locally $\mathbb{Q}_p$-analytic representation $\Pi(D_g)$ of $\mathrm{GL}_2(L_{\mathfrak{p}})$ is easily checked to be independent of the choice of the crystalline character η as above (note that the ratio of two such η is an unramified character of $\mathrm{Gal}(\overline{\mathbb{Q}}/L_{\mathfrak{p}})$).

Conjecture 8.1. *Assume that the eigenvalues of $\varphi^{[L_{\mathfrak{p},0}:\mathbb{Q}_p]}$ on D_g are distinct. If $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(\rho_g^\vee, \widehat{H}^1(K_f^p)) \neq 0$ then there is an integer $n > 0$ (depending on g and K_f^p) such that:*

$$\Pi(D_g)^n \subseteq \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(\rho_g^\vee, \widehat{H}^1(K_f^p)) \tag{29}$$

and such that any $\mathrm{GL}_2(L_\mathfrak{p})$-subrepresentation of $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(\rho_g^\vee, \widehat{H}^1(K_f^p))$ intersects $\Pi(D_g)^n$ non-trivially.

This conjecture is known so far only for $L = \mathbb{Q}$ ([2], [11]). If one knows that $(\mathrm{soc}'_{\mathrm{GL}_2(F)} \Pi(D_{g,\eta}) \otimes \eta^{-1} \circ \det)^n$ embeds into the right hand side of (29), then (i) of Corollary 7.2 gives that $(\Pi(D_{g,\eta})^{\mathrm{Amice}} \otimes \eta^{-1} \circ \det)^n$ also embeds. This should allow one to check Conjecture 8.1 if $L_\mathfrak{p} = \mathbb{Q}_p$ and $\rho_g|_{\mathrm{Gal}(\overline{\mathbb{Q}}/L_\mathfrak{p})}$ is irreducible using the last case of Proposition 7.3 (as the locally algebraic representation $(\pi_{D_{g,\eta}}(\emptyset, \emptyset) \otimes \eta^{-1} \circ \det)^n$ should embed). But $(\Pi(D_{g,\eta})^{\mathrm{Amice}} \otimes \eta^{-1} \circ \det)^n$ is the "maximum that p-adic analysis will give you". Going from $(\Pi(D_{g,\eta})^{\mathrm{Amice}} \otimes \eta^{-1} \circ \det)^n$ to $(\Pi(D_{g,\eta}) \otimes \eta^{-1} \circ \det)^n = \Pi(D_g)^n$ will require some non-trivial arithmetic geometry.

Finally, if $L_\mathfrak{p} \neq \mathbb{Q}_p$, I never expect (29) to be a topological isomorphism for any n as $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(\rho_g^\vee, \widehat{H}^1(K_f^p))$ should determine $\rho_g|_{\mathrm{Gal}(\overline{\mathbb{Q}}/L_\mathfrak{p})}$ (which is not the case of $\Pi(D_g)^n$).

9 The case where the Galois representation is reducible

We examine more closely the structure of $\Pi(D)$ when $V_{\mathrm{cris}}(D)$ is reducible and relate it to considerations of [3].

Let us first make a detour via the modulo p theory. Let $\overline{V} \simeq \begin{pmatrix} \overline{\chi}_2\omega & * \\ 0 & \overline{\chi}_1 \end{pmatrix}$ be a continuous 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E where ω is the reduction modulo p of ε and where $\overline{\chi}_1, \overline{\chi}_2 : F^\times \to k_E^\times$ are smooth characters. The results of [3, §19] (in the case F is unramified) suggest that, when $\overline{V}$ is non-split, the corresponding "good" representation(s) π of $\mathrm{GL}_2(F)$ over k_E (e.g. the representation(s) $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(\overline{V}_g, \varinjlim H^1_{\text{ét}}(S(K_f^pK_{f,p}) \times_L \overline{\mathbb{Q}}, k_E))$ when $\overline{V}$ globalizes to a representation $\overline{V}_g$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$, see §8) should (generically) have the form:

$$\pi_0 \;\text{——}\; \pi_1 \;\text{——}\; \cdots \;\text{——}\; \pi_{[F:\mathbb{Q}_p]-1} \;\text{——}\; \pi_{[F:\mathbb{Q}_p]}$$

where $\pi_0 = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \overline{\chi}_1 \otimes \overline{\chi}_2$, $\pi_{[F:\mathbb{Q}_p]} = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \overline{\chi}_2\omega \otimes \overline{\chi}_1\omega^{-1}$ (smooth parabolic inductions) and where the π_j for $1 \leq j \leq [F : \mathbb{Q}_p] - 1$ are irreducible and are not subquotients of parabolic inductions (extensions are taken in the category of smooth representations of $\mathrm{GL}_2(F)$ over k_E). When $\overline{V}$ is split, one should have $\oplus_{j=0}^{[F:\mathbb{Q}_p]}\pi_j$. The representations π_j, $1 \le j \le [F:\mathbb{Q}_p]-1$ are still quite mysterious, although one knows a few things about them (such as their $\mathrm{GL}_2(\mathcal{O}_F)$-socle, see [3]).

Now let D be a rank 2 filtered φ-module with Hodge-Tate weights $(0, k_\sigma - 1)_{\sigma\in S}$ as in §3 (not necessarily weakly admissible). For $j \in \{0, \cdots, [F:\mathbb{Q}_p]\}$ let us first define two series $(\Pi(D)_j)_j$ and $(\widetilde{\Pi}(D)_j)_j$ of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$:

$$\begin{aligned}
\Pi(D)_j &:= \bigoplus_{\substack{J\subseteq Z_D\\ |J|=|Z_D|-j}} \pi_D(J, J \amalg S\backslash Z_D) && \text{if } 0 \le j \le |Z_D|-1\\
\Pi(D)_j &:= \pi_D(\emptyset, S\backslash Z_D)/F_D \oplus_{\pi(D)} \widetilde{\pi}_D(\emptyset, Z_D)/\widetilde{F}_D && \text{if } j = |Z_D|\\
\Pi(D)_j &:= \bigoplus_{\substack{J\subseteq S\backslash Z_D\\ |J|=j-|Z_D|}} \widetilde{\pi}_D(J, J \amalg Z_D) && \text{if } |Z_D|+1 \le j \le [F:\mathbb{Q}_p]
\end{aligned}$$

and likewise for $\widetilde{\Pi}(D)_j$ replacing Z_D by $\widetilde{Z}_D$, $\pi_D(J, J\amalg S\backslash Z_D)$ by $\widetilde{\pi}_D(J, J\amalg S\backslash \widetilde{Z}_D)$ and $\widetilde{\pi}_D(J, J\amalg Z_D)$ by $\pi_D(J, J\amalg \widetilde{Z}_D)$. Since $Z_D \subseteq S\backslash\widetilde{Z}_D$ and $\widetilde{Z}_D \subseteq S\backslash Z_D$, the $\Pi(D)_j$ and $\widetilde{\Pi}(D)_j$ are easily checked to be subquotients of $\Pi(D)$ (we leave this to the reader). Note that the above two series coincide (up to numbering) if and only if $Z_D \amalg \widetilde{Z}_D = S$ in which case one has $\Pi(D)_j = \widetilde{\Pi}(D)_{[F:\mathbb{Q}_p]-j}$.

If D is weakly admissible, recall from §3 that $V_{\mathrm{cris}}(D)$ is reducible if and only if either $\mathrm{val}_F(\alpha) = \sum_{\sigma\in Z_D}(k_\sigma - 1)$ or $\mathrm{val}_F(\widetilde{\alpha}) = \sum_{\sigma\in\widetilde{Z}_D}(k_\sigma-1)$, and that $V_{\mathrm{cris}}(D)$ is reducible split if and only if both equalities hold if and only if $Z_D \amalg \widetilde{Z}_D = S$.

We consider below extensions in the abelian category of admissible locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$ over E ([18]).

Theorem 9.1. *Let D be a weakly admissible rank 2 filtered φ-module as in §3.*

(i) *$V_{\mathrm{cris}}(D)$ is indecomposable if and only if:*

$$\Pi(D) \simeq \oplus_{j=0}^{|Z_D|-1}\Pi(D)_j \bigoplus \Pi(D)_{|Z_D|} \text{ ---- } \Pi(D)_{|Z_D|+1} \text{ ---- } \cdots \text{ ---- } \Pi(D)_{[F:\mathbb{Q}_p]}$$

if and only if:

$$\Pi(D) \simeq \oplus_{j=0}^{|\widetilde{Z}_D|-1}\widetilde{\Pi}(D)_j \bigoplus \widetilde{\Pi}(D)_{|\widetilde{Z}_D|} \text{ ---- } \widetilde{\Pi}(D)_{|\widetilde{Z}_D|+1} \text{ ---- } \cdots \text{ ---- } \widetilde{\Pi}(D)_{[F:\mathbb{Q}_p]}.$$

(ii) $V_{\mathrm{cris}}(D)$ *is reducible split if and only if* $\Pi(D) \simeq \oplus_{j=0}^{[F:\mathbb{Q}_p]}\Pi(D)_j$ *if and only if* $\Pi(D) \simeq \oplus_{j=0}^{[F:\mathbb{Q}_p]}\widetilde{\Pi}(D)_j$.

(iii) *Let* $\chi_1, \chi_2 : F^\times \to \mathcal{O}_E^\times$ *be two locally* $\mathbb{Q}_p$*-analytic integral characters. The following are equivalent:*

- $V_{\mathrm{cris}}(D)$ *is reducible and isomorphic to* $\begin{pmatrix} \chi_2\varepsilon & * \\ 0 & \chi_1 \end{pmatrix}$
- $\Pi(D)$ *contains* $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\big)^{\mathbb{Q}_p-\mathrm{an}}$ *if* $\chi_1 \neq \chi_2$ *or* $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_1\big)^{\mathbb{Q}_p-\mathrm{an}}/\chi_1 \circ \det$ *if* $\chi_1 = \chi_2$.

(iv) *Assume* $V_{\mathrm{cris}}(D)$ *is reducible and isomorphic to* $\begin{pmatrix} \chi_2\varepsilon & * \\ 0 & \chi_1 \end{pmatrix}$ *with* $\chi_1\chi_2^{-1} \notin \{1, \varepsilon^2\}$.

- *If* $\mathrm{val}_F(\alpha) = \sum_{\sigma\in Z_D}(k_\sigma - 1)$, *then* $\Pi(D)_0 \simeq \big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\big)^{\mathbb{Q}_p-\mathrm{an}}$ *and* $\Pi(D)_{[F:\mathbb{Q}_p]} \simeq \big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2\varepsilon \otimes \chi_1\varepsilon^{-1}\big)^{\mathbb{Q}_p-\mathrm{an}}$.
- *If* $\mathrm{val}_F(\widetilde{\alpha}) = \sum_{\sigma\in \widetilde{Z}_D}(k_\sigma - 1)$, *then* $\widetilde{\Pi}(D)_0 \simeq \big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\big)^{\mathbb{Q}_p-\mathrm{an}}$ *and* $\widetilde{\Pi}(D)_{[F:\mathbb{Q}_p]} \simeq \big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2\varepsilon \otimes \chi_1\varepsilon^{-1}\big)^{\mathbb{Q}_p-\mathrm{an}}$.

Proof. (ii) is straightforward using $Z_D = S\backslash\widetilde{Z}_D$ and (i) is a consequence of some easy combinatorics using $Z_D \subsetneq S\backslash\widetilde{Z}_D$ and Theorem 4.1 that we leave to the reader.

We prove one implication in (iii). Assume first that $\Pi(D)$ contains $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1\otimes \chi_2\big)^{\mathbb{Q}_p-\mathrm{an}}$ for some integral characters $\chi_1 \neq \chi_2$. From (8), we see that this parabolic induction must be either $\pi_D(Z_D, S)$ or $\widetilde{\pi}_D(\widetilde{Z}_D, S)$. Going back to (7), this implies either:

$$\chi_1 = \mathrm{unr}_F(\alpha^{-1}) \prod_{\sigma\in Z_D} \sigma^{k_\sigma-1} \text{ and } \chi_2 = \mathrm{unr}_F(\widetilde{\alpha}^{-1})|\cdot|_F^{-1} \prod_{\sigma\in Z_D}\sigma^{-1}\prod_{\sigma\notin Z_D}\sigma^{k_\sigma-2} \tag{30}$$

or:

$$\chi_1 = \mathrm{unr}_F(\widetilde{\alpha}^{-1}) \prod_{\sigma\in \widetilde{Z}_D} \sigma^{k_\sigma-1} \text{ and } \chi_2 = \mathrm{unr}_F(\alpha^{-1})|\cdot|_F^{-1} \prod_{\sigma\in \widetilde{Z}_D}\sigma^{-1}\prod_{\sigma\notin \widetilde{Z}_D}\sigma^{k_\sigma-2}.$$

The integrality of χ_1 implies either $\mathrm{val}_F(\alpha) = \sum_{\sigma\in Z_D}(k_\sigma - 1)$ or $\mathrm{val}_F(\widetilde{\alpha}) = \sum_{\sigma\in\widetilde{Z}_D}(k_\sigma - 1)$. In the first case, we have from the definition (2) of Z_D that $\widetilde{e}_\sigma \in \mathrm{Fil}^0 D_\sigma$ if $\sigma \in Z_D$ and $\widetilde{e}_\sigma \in \mathrm{Fil}^{-(k_\sigma-1)} D_\sigma$ if $\sigma \notin Z_D$. Hence the crystalline Galois character $V_{\mathrm{cris}}(\prod_{\sigma_0\in S_0} E\widetilde{e}_{\sigma_0})$ sends p to $\widetilde{\alpha}^{-[F:\mathbb{Q}_p]}$ and has Hodge-Tate weights $((0)_{\sigma\in Z_D}, (k_\sigma - 1)_{\sigma\notin Z_D})$, hence is exactly $\chi_2\varepsilon$ (using $\varepsilon = |\cdot|_F\prod_{\sigma\in S}\sigma$). Likewise, we have $V_{\mathrm{cris}}(D/\prod_{\sigma_0\in S_0} E\widetilde{e}_{\sigma_0}) = \chi_1$, and thus $V_{\mathrm{cris}}(D) \simeq \begin{pmatrix} \chi_2\varepsilon & * \\ 0 & \chi_1 \end{pmatrix}$.

The second case is symmetric. Assume now that $\Pi(D)$ contains $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_1\big)^{\mathbb{Q}_p-\mathrm{an}}/\chi_1\circ\det$, which from (8) must be either $\pi_D(\emptyset,S)/F_D$ (with $F_D\neq 0$) or $\widetilde{\pi}_D(\emptyset,S)/\widetilde{F}_D$ (with $\widetilde{F}_D\neq 0$). Thus $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_1\big)^{\mathbb{Q}_p-\mathrm{an}}$ is either $\pi_D(\emptyset,S)$ or $\widetilde{\pi}_D(\emptyset,S)$ and the proof is the same as previously.
Now let us prove (iv) and the other implication in (iii). Assume $\mathrm{val}_F(\alpha)=\sum_{\sigma\in Z_D}(k_\sigma-1)$. By computing $V_{\mathrm{cris}}(\prod_{\sigma_0\in S_0}E\widetilde{e}_{\sigma_0})$ as before, we see that we can assume that χ_1 and χ_2 are as in (30) (replacing if necessary χ_1 by $\chi_2\varepsilon$ and χ_2 by $\chi_1\varepsilon^{-1}$ when $V_{\mathrm{cris}}(D)$ is split). If $Z_D\neq\emptyset$ or $(\alpha\widetilde{\alpha}^{-1})^f\notin q^{\pm 1}$, we have $\Pi(D)_0=\pi_D(Z_D,S)=\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_2\big)^{\mathbb{Q}_p-\mathrm{an}}$ and $\Pi(D)$ contains $\Pi(D)_0$ from (i). If $Z_D\neq S$ or $(\alpha\widetilde{\alpha}^{-1})^f\notin q^{\pm 1}$, we have $\Pi(D)_{[F:\mathbb{Q}_p]}=\widetilde{\pi}_D(S\backslash Z_D,S)$ which is easily checked to be $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_2\varepsilon\otimes\chi_1\varepsilon^{-1}\big)^{\mathbb{Q}_p-\mathrm{an}}$. This proves the first part of (iv). If $Z_D=\emptyset$ and $(\alpha\widetilde{\alpha}^{-1})^f=q^{\pm 1}$, the above equality for $\mathrm{val}_F(\alpha)$ together with (3) imply $(\alpha\widetilde{\alpha}^{-1})^f=q^{-1}$ and $k_\sigma=2$ for all $\sigma\in S$, and we see that $\Pi(D)$ contains $\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\chi_1\otimes\chi_1\big)^{\mathbb{Q}_p-\mathrm{an}}/\chi_1\circ\det=\pi_D(\emptyset,S)/F_D$. The proof for $\mathrm{val}_F(\widetilde{\alpha})=\sum_{\sigma\in\widetilde{Z}_D}(k_\sigma-1)$ is completely symmetric and left to the reader. This finishes the proofs of (iv) and (iii). □

Remark 9.2. There is a variant of (iv) in Theorem 9.1 when $\chi_1\chi_2^{-1}\in\{1,\varepsilon^2\}$ that we leave to the reader.

Let us finish the paper with some free speculations. Assume $V_{\mathrm{cris}}(D)$ is reducible as in (iv) above and consider the case, say, $\mathrm{val}_F(\alpha)=\sum_{\sigma\in Z_D}(k_\sigma-1)$ (the other case being symmetric). Then $\Pi(D)_0$ (resp. $\Pi(D)_{[F:\mathbb{Q}_p]}$) has a unique (admissible) unitary completion $\widehat{\Pi}(D)_0$ (resp. $\widehat{\Pi}(D)_{[F:\mathbb{Q}_p]}$) which is just the continuous parabolic induction of $\chi_1\otimes\chi_2$ (resp. $\chi_2\varepsilon\otimes\chi_1\varepsilon^{-1}$) and which is topologically irreducible if $(k_\sigma)_{\sigma\in S}\neq(2,\cdots,2)$ or $(\alpha\widetilde{\alpha}^{-1})^f\neq q$ (resp. or $(\alpha\widetilde{\alpha}^{-1})^f\neq q^{-1}$). Theorem 9.1 together with the characteristic p considerations at the beginning of this section strongly suggest that there should also exist topologically irreducible admissible unitary completions $\widehat{\Pi}(D)_j$ of $\Pi(D)_j$ for $1\leq j\leq[F:\mathbb{Q}_p]-1$, containing $\Pi(D)_j$ and which may - or may not - only depend on D, together with an admissible unitary Banach space representation B of $\mathrm{GL}_2(F)$ of the form:

$$B\simeq\ \widehat{\Pi}(D)_0 \text{ —— } \widehat{\Pi}(D)_1 \text{ —— } \cdots \text{ —— } \widehat{\Pi}(D)_{[F:\mathbb{Q}_p]-1} \text{ —— } \widehat{\Pi}(D)_{[F:\mathbb{Q}_p]}$$

when $V_{\mathrm{cris}}(D)$ is non-split and of the form $B\simeq\oplus_{j=0}^{[F:\mathbb{Q}_p]}\widehat{\Pi}(D)_j$ when $V_{\mathrm{cris}}(D)$ is split, and such that B completely determines D. Here, admissibility for Banach space representations is as in [19].

For simplicity (and because it is speculative) we just focus on the case $[F:\mathbb{Q}_p]=2$ (and keep the same assumptions as above). We write $S=\{\sigma,\sigma'\}$. Assuming a unitary Banach space representation as $\widehat{\Pi}(D)_1$ exists, one can consider its locally $\mathbb{Q}_p$-analytic vectors $\widehat{\Pi}(D)_1^{\mathbb{Q}_p-\mathrm{an}}$. One has an exact sequence (in the category of admissible locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F)$):

$$0 \longrightarrow \Pi(D)_1 \longrightarrow \widehat{\Pi}(D)_1^{\mathbb{Q}_p-\mathrm{an}} \longrightarrow \Pi(D)_1^? \longrightarrow 0$$

where:

$$\begin{aligned}
\Pi(D)_1 &= \widetilde{\pi}_D(\{\sigma\},\{\sigma\}) \oplus \widetilde{\pi}_D(\{\sigma'\},\{\sigma'\}) \quad \text{if} \quad Z_D = \emptyset \\
\Pi(D)_1 &= \pi_D(\emptyset, S\backslash Z_D) \oplus_{\pi(D)} \widetilde{\pi}_D(\emptyset, Z_D) \quad \text{if} \quad |Z_D| = 1 \\
\Pi(D)_1 &= \pi_D(\{\sigma\},\{\sigma\}) \oplus \pi_D(\{\sigma'\},\{\sigma'\}) \quad \text{if} \quad Z_D = S.
\end{aligned}$$

Granting the existence of such an "extra-constituent" $\Pi(D)_1^?$, we can speculate that the "complete" locally $\mathbb{Q}_p$-analytic representation(s) $\Pi(D)^?$ of $\mathrm{GL}_2(F)$ associated to D should be a non-split extension:

$$0 \longrightarrow \Pi(D) \longrightarrow \Pi(D)^? \longrightarrow \Pi(D)_1^? \longrightarrow 0$$

(even if $V_{\mathrm{cris}}(D)$ is irreducible) and that the "parameters" giving the isomorphism class of this extension should be related to (and determine) the values of a_σ, $a_{\sigma'}$, $\widetilde{a}_\sigma$ and $\widetilde{a}_{\sigma'}$ (up to modifications as in Lemma 3.1).

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