NON-DIFFERENTIABLE DEFORMATIONS OF $\mathbb{R}^n$

by

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Abstract. — Many problems of physics or biology involve very irregular objects like the rugged surface of a malignant cell nucleus or the structure of space-time at the atomic scale. We define and study non-differentiable deformations of the classical Cartesian space $\mathbb{R}^n$ which can be viewed as the basic bricks to construct irregular objets. They are obtain by taking the topological product of n-graphs of nowhere differentiable real valued functions. Our point of view is to replace the study of a non-differentiable function by the dynamical study of a one-parameter family of smooth regularization of this function. In particular, this allows us to construct a one parameter family of smooth coordinates systems on non-differentiable deformations of $\mathbb{R}^n$ which depend on the smoothing parameter via an explicit differential equation called a scale-law. Deformations of $\mathbb{R}^n$ are examples of a new class of geometrical objects called scale manifolds which are defined in this paper. As an application, we derive rigorously the main results of the scale relativity theory developed by L. Nottale in the framework of a Scale space-time manifold.

1. Introduction

Many problems of physics and biology involve very irregular objects like the rugged surface of a malignant breast cell nucleus [**5**] or the structure of space-time at the atomic scale ([**18**],p.151 and [**7**],p.131). Such objects are characterized by the fact that their local geometry is not diffeomorphic to $\mathbb{R}^n$. The main phenomenon is that new structures appear at all scales, in contrary to what happens for differentiable manifolds. The basic idea in order to construct an analysis of irregular objects is then to define the basic bricks which can be used to understand their local geometry.

In this paper, we propose to use non-differentiable deformation of $\mathbb{R}^n$ as fundamental geometric bricks, i.e. the simplest object displaying the main features of general irregular objects. A non-differentiable deformation of $\mathbb{R}^n$ is a topological product of n-graphs of everywhere non-differentiable real valued functions. Such deformations have a very complicated geometrical structure. The non-differentiability of the underlying functions prevents us to use tools from classical analysis and differential geometry. Indeed, a non-differentiable deformation of $\mathbb{R}^n$ has a natural structure of topological manifold but not of differentiable manifold.

We have at least two different point of view on these objects:

First, we can try to develop an analysis on topological manifolds by extending the ordinary differential calculus. A preliminary attempt is done in [**2**] using the local-fractional calculus [**3**]. A forthcoming paper will explore this problem in the context of the scale calculus developed in ([**4**],[**11**],[**12**]).

In this paper, we follow a different strategy. In many cases, we have not a direct access to the object itself, but to a sequence of smooth approximations of it, obtained at different scales. This leads us to perform a scale-analysis of a non-differentiable deformation of $\mathbb{R}^n$. On each regularized curves one can defined a canonical coordinate depending on the smoothing parameter. As a consequence, the analysis on the non-differentiable deformation of $\mathbb{R}^n$ is replaced by the study of the dynamics of the associated one-parameter smooth approximation. This dynamical study leads us to introduce several notions, the main one being scale laws. A scale law permits to relate quantities at different scales, i.e. degree of smoothing, using a canonical coordinates system at each scale. As each smooth regularization of a non-differentiable deformation of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$, most of the geometry is captured by the data of the one-parameter coordinates system coupled with a scale law.

However, non-differentiable deformations of $\mathbb{R}^n$ can not be used to develop an intrinsic theory of irregular objects. The main difficulty is that deformations depend on several choices, like the underlying non-differentiable functions. The same is true when choosing a particular smoothing technique.

In order to define a universal or canonical geometrical object, we introduce an abstract framework which mimics the main features of the scale analysis for non-differentiable deformations of $\mathbb{R}^n$. The main notion is that of scale-coordinates systems which is the data of a one-parameter coordinates system with a scale-law and the associated notion of scale manifolds.

This abstract framework can appear as a formal construction. As a consequence, we provide an application to microphysics, precisely the scale-relativity theory developed by L. Nottale [**26**]. Assuming that at the scale of microphysics space-time has a scale-manifold structure, we derive rigorously the main results of the special scale-relativity theory.

The paper is organized as follow :

In section 2, we define non-differentiable deformations of the Cartesian space $\mathbb{R}^n$. Using smooth-regularization, we construct a one-parameter family of smooth coordinates system. The dependence of this coordinates system on the smoothing parameter is controlled by a differential equation called a scale law, which is explicit.

In section 3.1, using our approach of non-differentiable deformations of $\mathbb{R}^n$, we define the notion of scale-coordinates systems and discuss its main properties. Using this notion, we define scale-manifolds.

In section 4, we derive the main results of the spatial special scale relativity theory [**26**] assuming that space-time in microphysics is given by a scale-manifold. In particular, we prove rigorously that there exists an *horizon* for lengths, i.e. a limit which can not be overreached, as the speed of light for velocities in Einstein's special relativity theory. This horizon for lengths can be identified with the *Planck length*. These results are based on the approach of Lorentz transformations as performed by Levy-Leblond [**23**] and refined by Nottale [**27**].

2. Non-differentiable deformations of $\mathbb{R}^n$

Let $n \in \mathbb{N}$, $n \geq 1$, we denote by $\mathbb{R}^n$ the classical n-dimensional Cartesian space. We refer to ([**14**], Chap.1) for basic notions of geometry.

2.1. The space $\mathbb{R}^n_\xi$. — We define the notion of *deformation* of $\mathbb{R}^n$ in order to generalize the classical notion of manifold.

Definition 2.1. — *Let* $\xi = (\xi_i(t))_{i=1,\dots,n}$*, be a finite family of continuous real valued functions defined on* $\mathbb{R}$*, and* Γ_i *their associated graphs. A* ξ*-deformation of* $\mathbb{R}^n$*, denoted by* $\mathbb{R}^n_\xi$ *is the topological manifold* $\Gamma_1 \times \cdots \times \Gamma_n$.

Of course, a ξ-deformation is non-trivial if and only if at least one of the underlying functions is nowhere differentiable. Indeed, we have:

Gluing property. *When all the* ξ_i *functions are differentiable, we obtain a manifold which is globally diffeomorphic to* $\mathbb{R}^n$.

In the following, a ξ-deformation of $\mathbb{R}^n$ is called *non-differentiable* if all the ξ_i, $i = 1,\dots,n$ are nowhere differentiable functions.

We already know that the set of nowhere differentiable functions is topologically "generic" in the space of continuous functions of real variable. As a consequence, a ξ-deformation of $\mathbb{R}^n$ is generically non-differentiable. However, using the concept of *prevalence* introduced in [**19**], which give a measure-theoretic definition of "almost every" on infinite dimensional spaces, we can go further:

Theorem 2.1. — *Almost every* ξ*-deformation of* $\mathbb{R}^n$ *is non-differentiable.*

Proof. — We use the result of B. Hunt ([**20**], Theorem 1) on the prevalence of continuous nowhere differentiable functions in the Banach space of continuous functions from $[a, b]$ to $\mathbb{R}$, for all $a, b \in \mathbb{R}$, $a < b$. □

For more results on prevalence of some known typical properties of functions spaces, we refer to [**32**].

As a particular example of non-differentiable deformation of $\mathbb{R}^n$, we can use as deformation the Knopp (or Takagi) function (see [**34**], §.13.1): Let $0 < \alpha < 1$, and $g(t)$ be the function of period 1 defined on $[0, 1]$ by

$$g(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1/2, \\ 2 - 2t, & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

The Knopp function is defined by

$$K(t) = \sum_{n=0}^{\infty} 2^{-n\alpha} g(2^n t), \tag{1}$$

and is a continuous everywhere non differentiable function.

2.2. The space $\mathbb{R}^n_{\xi_\epsilon}$. — As graphs of continuous functions are *manifolds*, we can find local coordinates systems on each Γ_i, i.e. for each $p \in \Gamma_i$, there exists an open neighborhood U_p of p in Γ_i and an homeomorphism ϕ_i from U_p into an open neighborhood U_0 of $0 \in \mathbb{R}$. However, as these local coordinates are of *topological* nature, there are not suitable in order to develop *analysis*.

On differentiable curves, we can find a special parametrization, called *normal parametrization* (see [**22**]), by choosing a point o on the curve, and taking as coordinate $s(p)$ of a given point p, the length of the curve between o and p. This coordinates system is known as *curvilinear coordinates system*. In the non-differentiable case, due to the Lebesgue theorem, which asserts that every pieces of the graph of an everywhere non-differentiable continuous function has an *infinite* length (see [**34**],p.82), such normal parametrization does not exist.

A way to capture the non-differentiable character of this set is then to introduce suitable *smoothing* of these curves, on which we can define useful coordinates systems.

Definition 2.2. — *Let $\xi = (\xi_i)_{i=1,\dots,n}$ be a finite family of continuous functions as in definition 2.1. For each $\epsilon > 0$, we associate the finite family of smooth functions denoted by $\xi_\epsilon = (\xi_{i,\epsilon})_{i=1,\dots,n}$, and defined by*

$$\xi_{i,\epsilon}(t) = \int \Phi_\epsilon(t,y)\xi_i(y)dy, \tag{2}$$

with $\int \Phi_\epsilon(t,y)dy = 1$, and Φ_ϵ a differentiable function with respect to t (except at a finite number of points).

We denote by $\Gamma_{i,\epsilon}$ the graph of $\xi_{i,\epsilon}$. We denote the manifold $\Gamma_{1,\epsilon} \times \dots \Gamma_{n,\epsilon}$ by $\mathbb{R}^n_{\xi_\epsilon}(\Phi)$.

For example, we can take $\Phi_\epsilon(t,y) = 1_{[t-\epsilon,t+\epsilon]}(y)$, where $1_{[t-\epsilon,t+\epsilon]}(y)$ denotes the function defined by

$$1_{[t-\epsilon,t+\epsilon]}(y) = \begin{cases} 1 \text{ if } y \in [t-\epsilon, t+\epsilon], \\ 0 \text{ otherwise.} \end{cases}$$

We then obtain the classical ϵ-*mean functions*

$$\xi_{i,\epsilon}(t) = \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \xi_i(y)dy,\ i = 1,\dots,n. \tag{3}$$

On these curves, it is possible to define a normal parametrization, by considering the curvilinear coordinates.

We now define a coordinates system on $\mathbb{R}^n_\xi$ as follow :

For each $p \in \mathbb{R}^n_\xi$ there exists a sequences of variables t_i, $i = 1, \dots, n$, such that $p = (p_1, \dots, p_n)$, $p_i \in \Gamma_i$, $p_i = (t_i, \xi_i(t_i))$. The mapping $T : \Gamma \to \mathbb{R}^n$, associating to each $p \in \mathbb{R}^n_\xi$ its sequence $t = (t_1, \dots, t_n)$ is bijective and continuous. We denote by o the point of $\mathbb{R}^n_\xi$ defined by $o = T^{-1}(0)$.

For all $\epsilon > 0$, we can associate to each $p \in \mathbb{R}^n_\xi$ a point $p_\epsilon \in \mathbb{R}^n_{\xi_\epsilon}$ as follow :

For all $p \in \mathbb{R}^n_\xi$, $t = T(p)$, we associate the point $p_\epsilon = ((t_1, \xi_{1,\epsilon}(t_1)), \dots, (t_n, \xi_{n,\epsilon}(t_n)))$. We denote this mapping by π_ϵ, i.e.

$$\begin{array}{rccl} \pi_\epsilon : & \mathbb{R}^n_\xi & \longrightarrow & \mathbb{R}^n_{\xi_\epsilon}, \\ & p & \longmapsto & p_\epsilon = \pi_\epsilon(p). \end{array}$$

The mapping π_ϵ is bijective and continuous. We denote by $o_\epsilon = \pi_\epsilon(o)$.

Definition 2.3. — *An ϵ-coordinates system on $\mathbb{R}^n_\xi$ is defined for all $p \in \mathbb{R}^n_\xi$ by $p = (x_1(\epsilon)(p), \dots, x_n(\epsilon)(p)) \in \mathbb{R}^n$ where*

$$x_i(\epsilon)(p) = L(o_\epsilon, p_\epsilon), \tag{4}$$

where $o_\epsilon = \pi_\epsilon(o)$, $p_\epsilon = \pi_\epsilon(p)$ and L is the length between o_ϵ and p_ϵ on the curve $\Gamma_{i,\epsilon}$.

Let $\Lambda \in \mathbb{R}^{+*}$ be a fixed positive real number. To each $\epsilon \in \mathbb{R}^{+*}$, we associate the number

$$e_\Lambda(\epsilon) = \ln(\Lambda/\epsilon). \tag{5}$$

When ϵ plays the role of a measure precision (i.e. of a resolution), the ratio Λ/ϵ is a scale.

Remark 2.1. — *The necessity to introduce the variable Λ, and its counterpart e_Λ comes from physics : only ratio of resolutions have sense.*

In the following, we fix a real number $\Lambda \in \mathbb{R}^{+*}$, and we denote by $x_i(e_\Lambda)$ the coordinates system (4) in the scale (5).

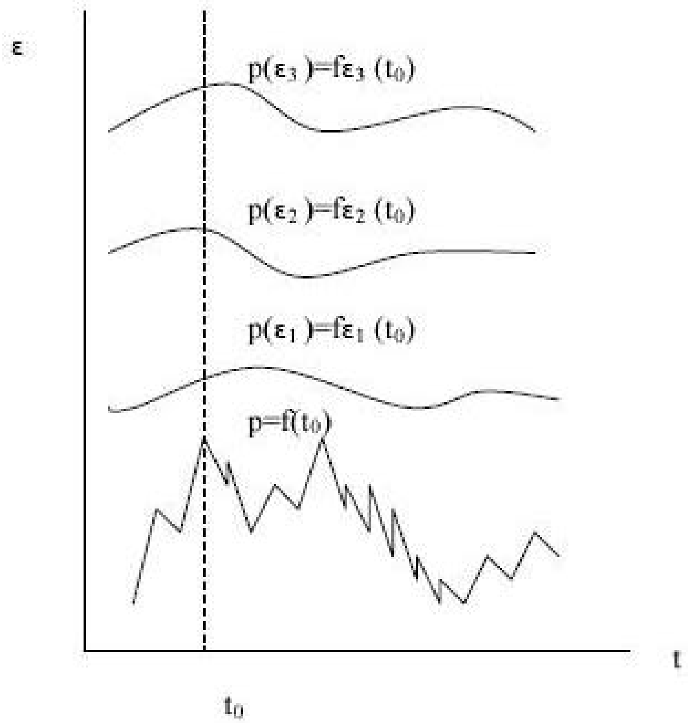

FIGURE 1. ϵ-coordinates system

2.3. Scale laws. — The one-parameter e_Λ family of coordinates systems $x(e_\Lambda)$ is useful in order to understand $\mathbb{R}^n_\xi$ if we can precise the *dynamics* of these functions with respect to the scale e_Λ. We are lead to the notion of *scale law*.

In the following, without loss of generality, we fix $\Lambda = 1$ and we denote e for e_1.

Let f and g be two real valued functions defined on $\mathbb{R}$. We denote by $f \sim g$ if there exist two constants c_1 and c_2 such that $0 < c_1 <\mid f(t)/g(t) \mid < c_2$ for all $t \in \mathbb{R}$.

Definition 2.4. — *Let $e \in \mathbb{R}$, and $x(e) : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}, \\ t & \mapsto & x(e)(t), \end{array}$ be a one parameter family of continuous functions. We say that $x(e)$ satisfy a scale law if there exists two functions $x_-(e)$ and $x_+(e)$ such that :*

i) we have for all $t \in \mathbb{R}$, $x_-(e) \leq x(e) \leq x_+(e)$, satisfying $x(e) \sim x_-(e)$ and $x(e) \sim x_+(e)$,

ii) there exists a function $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{dx_\pm(e)}{de} = E(x_\pm(e), t). \tag{6}$$

In this case, we call the function E a scale law.

Remark 2.2. — *The first idea is to define scale law as the quantity $dx(e)/de$. However, the functions $x(e)$ are not, in general, differentiable with respect to e.*

Remark 2.3. — *A scale law is not unique.*

We denote by $H^\alpha(c, C)$ the set of continuous real valued functions $f(t)$, defined on $\mathbb{R}$ such that there exists two constants c, $C \in \mathbb{R}^+$, such that

$$ch^\alpha \leq \mid f(t+h) - f(t) \mid \leq Ch^\alpha, \tag{7}$$

for all $h > 0$.

The set H^α corresponds to *Hölderian* functions, which are *inverse-Hölder* (see [**34**]).

Remark 2.4. — *The Knopp function (1) belongs to H^α (see* [**34**]*,§.13.1).*

For functions in H^α, it is easy to characterize a scale law for the associated one parameter family of ϵ-mean functions (see equation 3).

Theorem 2.2. — *Let $0 < \alpha < 1$, we consider $\xi \in H^{\alpha}(c, C)$ and we denote by Γ its graph. For all $1 > \epsilon > 0$, we denote by ξ_{ϵ} the ϵ-mean function defined as (3) and Γ_{ϵ} its graph. For all $p \in \Gamma$, $p = (t, \xi(t))$, we define $x(e)(t) = L(o_{\epsilon}, p_{\epsilon})$, where $e = \ln(1/\epsilon)$, $o_{\epsilon} = (0, \xi_{\epsilon}(0))$, $p_{\epsilon} = (t, \xi_{\epsilon}(t))$, and L is the length between o_{ϵ} and p_{ϵ} on Γ_{ϵ}.*

The one parameter function $x(e)$ satisfy a scale law define by

$$E(y, t) = (1 - \alpha)y. \tag{8}$$

Moreover, we have $x(e)^{-}(t) = t\epsilon^{\alpha-1} \mid c \mid$ and $x(e)^{+}(t) = t\epsilon^{\alpha-1}\sqrt{1 + C^2}$.

Proof. — We have

$$x(\epsilon)(t) = \frac{1}{2\epsilon}\int_0^t \sqrt{4\epsilon^2 + (f(x+\epsilon) - f(x-\epsilon))^2}dx.$$

As $f \in H^{\alpha}$, we have

$$4c^2\epsilon^{2\alpha} \leq (f(x+\epsilon) - f(x-\epsilon))^2 \leq 4C^2\epsilon^{2\alpha}.$$

As a consequence, we obtain

$$\epsilon^{\alpha-1}t\sqrt{\epsilon^{2(1-\alpha)} + c^2} \leq x(\epsilon)(t) \leq t\epsilon^{\alpha-1}\sqrt{\epsilon^{2(1-\alpha)} + C^2}.$$

We deduce $x^{-}(\epsilon)(t) \leq x(\epsilon)(t) \leq x^{+}(\epsilon)(t)$, $\forall t \in \mathbb{R}$, and $1 > \epsilon > 0$.

By differentiating $x^{\sigma}(\epsilon)(t)$, $\sigma = \pm$, with respect to ϵ, we obtain

$$\frac{dx^{\sigma}(\epsilon)(t)}{d\epsilon} = \frac{\alpha - 1}{\epsilon}x^{\sigma}(\epsilon)(t).$$

Using

$$\frac{dx^{\sigma}(\epsilon)(t)}{d\ln(1/\epsilon)} = -\epsilon\frac{dx^{\sigma}(\epsilon)}{d\epsilon},$$

we obtain the scale function $E(y, t) = (1 - \alpha)y$. □

3. Scale-coordinates systems and scale-manifolds

The previous construction of a one-parameter coordinates system governed by a scale law for non-differentiable deformations of $\mathbb{R}^n$ depends on the given family ξ of non-differentiable functions, as well as the smoothing function Φ_{ϵ}. In the following, in order to avoid these arbitrary choices and to define a more universal geometric object, we formalize this construction by introducing the notion of *scale-coordinates systems* on $\mathbb{R}^n$.

3.1. Definition. —

Definition 3.1. — *A scale-coordinates system on $\mathbb{R}^n$, is a one-parameter family of coordinates system on $\mathbb{R}^n$ $x(e) = (x^1(e), \ldots, x^n(e))$, where $e \in \mathbb{R}$, called scale, satisfying the scale law*

$$\frac{dx(e)}{de} = \delta x(e), \tag{9}$$

where $\delta > 0$ is a parameter.

In the rest of the text, a scale-coordinates system being fixed, we use the following terminology :

- For all $e \in \mathbb{R}$ fixed, we call the variables $x(e)$ *space variables*. A translation in space is then a translation of the coordinates system $x(e)$ where e is fixed.

- The parameter δ is called the *djinn* parameter(1).

In the definition 3.1, the djinn variable δ is assumed to be strictly positive. This condition is fundamental if one want to use the term of "scale" coordinates system. Indeed, if $\delta = 0$, then

$$\frac{dx(e)}{de} = 0,$$

which implies the *scale independence* of the coordinates system, i.e.

$$x(e) = x.$$

We then obtain a classical coordinates system on $\mathbb{R}^n$.

Remark 3.1. — *In the case of non-differentiable deformations of $\mathbb{R}^n$, the djinn variable is connected to the Hölder regularity of the curve Γ_i.*

One can also understand the fundamental nature of scale-coordinates systems from the *metric* point of view.

Let P and Q, be two points of $\mathbb{R}^n$. We denote by $d_e(M, N)$ the ordinary euclidian distance between the points M and N in the coordinates system $x(e)$.

(1)In order to follow Nottale's terminology [**28**].

As in the previous section, we will frequently use the fact that e can be written as $e = \ln(1/\epsilon)$, for some $\epsilon > 0$. In that case, we denote by

$$S_{e'/e} = \ln(\epsilon/\epsilon'). \tag{10}$$

the scale translation defined by $e' = e + S_{e'/e}$.

Lemma 3.1. — *For all e, $e' \in \mathbb{R}$, we have $d_{e'} = d_e \exp(\delta S_{e'/e})$.*

Proof. — Let $(x_i(e))_{i=1,\dots,n}$ and $(y_i(e))_{i=1,\dots,n}$ (resp. $(x_i(e'))_{i=1,\dots,n}$ and $(y_i(e'))_{i=1,\dots,n}$) be the coordinate of M and N at the scale e (resp. e'). We have

$$d_e(M,N) = \sqrt{\sum_{i=1}^{n} (x_i(e) - y_i(e))^2}.$$

Using the scale law, we obtain for all reference scale $e_0 \in \mathbb{R}$, $x_i(e) = x_i(e_0)\exp(\delta(e - e_0))$. As a consequence, we deduce $x_i(e') = x_i(e_0)\exp(\delta(e - e_0))\exp(\delta S_{e'/e})$ and $x_i(e') = x_i(e)\exp(\delta S_{e'/e})$. We have the same expression for $y_i(e)$. Replacing in the formula for $d_{e'}(M,N)$, we obtain the result. □

If $e' \to \infty$, then $S_{e'/e} \to \infty$ and we have $d_{e'} \to \infty$ for all points $M, N \in \mathbb{R}^n$. This property mimics the Lebesgue theorem concerning the infinite length of the graph of a non-differentiable function.

A natural generalization of the notion of scale-coordinates system is to allow more general scale laws for each coordinates, in particular

$$\frac{dx_i(e)}{de} = E_i(x_i(e)), \quad i = 1, \dots, n, \tag{11}$$

with $E_i : \mathbb{R} \to \mathbb{R}$ a real valued function such that $x_i(e) \to \infty$ for $e \to \infty$.

The choice of the scale law E_i depends on the structure of the problem (physics, biology, ...). In our definition, we have taken the simplest case (linear!).

3.2. Properties of scale-coordinates systems. — A scale-coordinates system being fixed, we will study the effect of two basic transformations: translation in scale and translation in space.

3.2.1. Translation is scale. — During the proof of lemma 3.1, we have obtained the following result:

Lemma 3.2. — *Let $x(e)$ be a scale-coordinates system. We denote by $S_{e'/e}$ the scale-translation satisfying $e' = e + S_{e'/e}$. We have $x(e') = x(e)\exp(\delta S_{e'/e})$.*

Translations in scale take a better form when one use logarithmic variables (see §.3.3).

3.2.2. Translation in space. — A translation in space is, in general, scale dependent :

$$y(e) = x(e) + T(e), \tag{12}$$

where $T(e)$ is subject to the scale law (9).

We prove easily that the new coordinates system satisfies $\dfrac{dy(e)}{de} = \delta y(e)$. As a consequence, translations in space induce no effects.

Remark 3.2. — *In the case of non-differentiable deformations of $\mathbb{R}^n$ studied in section 2, this result is equivalent to the fact that curvilinear coordinates are defined up to translations on the curve.*

3.3. Logarithmic scale-coordinates systems. — The previous properties of scale-coordinates systems take a simpler form when one use *logarithmic* coordinates systems.

Definition 3.2. — *Let $x(e)$ be a scale-coordinates system on $\mathbb{R}^n$. By restriction to $(\mathbb{R}^{+*})^n$, we define a logarithmic scale-coordinates system on $\mathbb{R}^n$ by $X(e) = \ln x(e)$.*

In this coordinates system, the effect of translation in scale takes the following form :

Lemma 3.3. — *For all $e, e' \in \mathbb{R}$, we have $X(e') = X(e) + \delta S_{e'/e}$, where $e' = e + S_{e'/e}$.*

This result is a consequence of the scale independence of the djinn variable, i.e. $\delta(e') = \delta(e) = \delta$, for all e, $e' \in \mathbb{R}$.

This remark leads us to consider the djinn variable as a scale dependent variable, i.e. $\delta = \delta(e)$ with a trivial behavior.

Lemma 3.4. — *Let $X(e)$ be a logarithmic scale-coordinates system, then it satisfies*

$$\frac{dX(e)}{de} = \delta, \quad \frac{d\delta}{de} = 0. \tag{13}$$

The proof is immediate.

3.4. Scale manifolds. — We define the notion of *scale manifolds*. The main purpose is to obtain a tractable definition to scale dependant objects. The definition must be sufficiently universal in order to cover a large class of examples and sufficiently simple in its formulation to allow explicit computations.

3.4.1. Definition. — For basic notions about *differential geometry*, we refer to Spivak [**33**].

Definition 3.3. — *Let $n \geq 1$, a n-dimensional scale-manifold is a one parameter e, family of sets V_e, endowed with a one-parameter differentiable structure, defined as follow :*

i) For all $e \in \mathbb{R}$, there exists an atlas $(\Omega_{i,e}, \phi_{i,e})_{i\in J_e}$ of V_e, where J_e is a set of indices, i.e. $\Omega_{i,e}$ are open neighborhoods of V_e, $\bigcup_{i\in J_e} \Omega_{i,e} = V_e$, $\phi_{i,e} : \Omega_{i,e} \to \mathbb{R}^n$ are homeomorphisms;

ii) if $\Omega_{i,e} \cap \Omega_{j,e} \neq \emptyset$, the change of charts $\phi_{i,e} \circ \phi_{j,e}^{-1}$ is (at least) C^1,

iii) for all $i \in J_e$, the local coordinates system $x_i(e) = \phi_{i,e}(x)$ is a scale-coordinates system.

iv) there exists a diffeomorphism $T_{e'/e} : V_e \to V_{e'}$, called scale-map.

The main point is that scale manifolds are not a priori *topological manifolds* when $e \to \infty$. Indeed, we have not, a priori, a well defined limiting function for the charts functions $\phi_{i,e}$ when $e \to \infty$. As a consequence, we have a more general *structure* than classical differentiable manifolds on which one can do analysis.

Remark 3.3. — *A convenient way to consider scale manifolds is to follow the Elie Cartan approach* [**6**] *to the geometric definition of space-time in Einstein's relativity theory (see* [**25**]*,p.8-9). We must consider the fiber bundle*

$$V = \bigcup_{e\in\mathbb{R}} V_e,$$

with the canonical projection $\pi : V \to \mathbb{R}$, defined by $\pi(x) = e$ if $x \in V_e$. Of course the dynamics on this fiber bundle is characterized by the scale-map. A precise study of these structures will be done in a forthcoming paper.

3.4.2. Examples. — As a trivial example, we can consider the one parameter family of spaces denoted $\mathbb{R}^n(e)$ and called the scale-Cartesian space: We have $V_e = \mathbb{R}^n$ for all $e \in \mathbb{R}$.

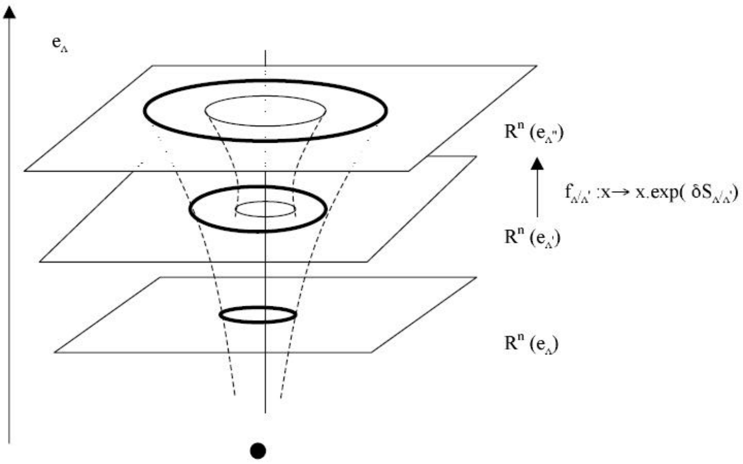

FIGURE 2. Example of a scale manifold : $\mathbb{R}^n(e)$.

We then endow $\mathbb{R}^n$ with a scale-coordinates system. The scale map $T_{e'/e}$ is given by

$$T_{e'/e}(x) = x\exp(\delta S_{e'/e}),$$

keeping the notations of equation (10).

- A direct consequence of theorem 2.2 is that non-differentiable deformations of $\mathbb{R}^n$ using H^α functions give rise to a large class of scale-manifolds:

Theorem 3.1. — *Let $\xi = (\xi_1, \dots, \xi_n)$ be a family of n continuous real valued functions of $H^\alpha(c, C)$, $0 < \alpha < 1$. The family $\mathbb{R}^n_{\xi_\epsilon}$, $\epsilon > 0$, is a scale-manifold.*

It must be noted that the scale map is the same as for the scale-Cartesian space.

4. Application: Special scale relativity

4.1. Scale relativity principle. — We are going to state the special scale relativity principle. We must precise the geometry of the model space where physical processes must be described, that we call "Physical" space.

"Physical" space geometry. *A physical process can be described in a scale space-time-djinn manifold, i.e. the space* $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+$*, provided with a scale-coordinates system-*$(X(e), t(e), \delta(e))$*, logarithmic scale in* X*, for all* $e \in \mathbb{R}$*.*

Following Levy-Leblond [**23**], a very general form of the Einstein's *principle of relativity* is : *there exists an infinite continuous class of reference frames in space-time which are physically equivalent.*

"The abstract principle of relativity is thus open to many realizations as concrete *theories of relativity* which tell us how to relate two expressions of the same physical quantity as refered to two of these equivalent frames. Such a theory may then be expressed exactly by *the transformation formulas connecting equivalent frames*" (see [**23**]).

Let $\mathcal{R}_e$ and $\mathcal{R}_{e'}$ be two scale-frames of $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+$, and denote by Id the identity map, Id : $(\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+, \mathcal{R}_e) \to (\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+, \mathcal{R}_{e'})$. For all point $M \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+$, of coordinates $(X(e), t(e), \delta(e))$ in $\mathcal{R}_e$ and $(X(e'), t(e'), \delta(e'))$ in $\mathcal{R}_{e'}$, we obtain a transformation formula

$$(X(e'), t(e'), \delta(e')) = T(X(e), t(e), \delta(e)).$$

The parameter $e \in \mathbb{R}$ is called, following Nottale's terminology [**26**], the *scale-state* of the coordinates system.

In order to simplify our approach, we will consider transformations satisfying

$$t(e) = t(e'), \tag{14}$$

for all e, $e' \in \mathbb{R}$.

As a consequence, the time is here assumed to be *absolute*.

Remark 4.1. — *The absolute character of time implies that we do not look for consequences of Einstein's motion relativity on the scale behavior. Indeed, the mixing of scale and motion relativity can not be done in special (meaning via linear transformations) relativity theory, but demands nonlinear transformations of coordinates systems (due to the fact that the scale behavior is linear for a logarithmic space variable, but not the motion one).*

Following Levey-Leblond [**23**], we can characterize the admissible transformations by imposing natural constraints on the set of scale-transformations.

We consider a one-parameter family of transformations denoted $\mathcal{T} = (T_v)$, $v \in A \subset \mathbb{R}$, $A =]a, b[$, $a < 0 < b$, where A is an open interval to be determined. An element of $\mathcal{T}$ is called a scale-transformation in the following. The action of T_v on a given point $(X(e), \delta(e))$ is denoted by

$$(X(e'), \delta(e')) = T_v(X(e), \delta(e)).$$

The fact that we consider only one-parameter transformations is based on the discussion of Levy-Leblond ([**23**],p.272).

We assume that scale-transformations satisfy the following scale relativity principle:

Spatial special scale relativity principle. *The laws of physics take the same form under scale transformations satisfying :*

i) spatial scale-transformations are linear;
ii) spatial scale transformations preserve isotropy of space;
iii) the set of spatial scale-transformations $(\mathcal{T}, \circ)$*, where* $\circ$ *is the classical composition law, is a group,*
iv) non-compactness of the group.

Remark 4.2. — *- The word "spatial" reflects the absolute character of time.*
- The word "special" reflects the fact that we will only consider linear transformations of coordinates systems.

The fact that scale-transformations preserve the isotropy of space can be stated as follow:

***Definition 4.1* (Isotropy)**. — *The set* $\mathcal{T} = (T_v)_{v \in A}$ *of scale-transformations preserves the isotropy of space if for all points* $(X(e), \delta(e'))$, $(X(e'), \delta(e'))$ *such that* $(X(e'), \delta(e')) = T_v(X(e), \delta(e))$, *there exists* $u \in \mathbb{R}$ *which satisfies*

$$(-X(e'), \delta(e')) = T_u(-X(e), \delta(e)).$$

In ([**27**],p.4906, condition (2)) Nottale imposes global isotropy (i.e. isotropy of space-time) but uses only space isotropy in his computations ([**27**],p.4907, equation (4.6)).

4.2. About Lorentz transformations. — Before stating our main result, we recall the following definition of Lorentz transformations which will be used in all the paper.

Definition 4.2. — *For all $k \in \mathbb{R}$, we denote by $A_k \subset \mathbb{R}$ the set of real numbers $v \in \mathbb{R}$ such that $1-kv^2 > 0$. We call k-Lorentz transformation of parameter $v \in A_k$, the transformation denoted $L_k(v) : \mathbb{R}^2 \to \mathbb{R}^2$, $(X,\delta) \mapsto (X',\delta') = L_k(v)(X,\delta)$ and defined by*

$$\begin{array}{rcl} X' & = & (1-kv^2)^{-1/2}(X - v\delta), \\ \delta' & = & (1-kv^2)^{-1/2}(-kvX + \delta). \end{array} \tag{15}$$

For all $k \in \mathbb{R}$, we denote by $\mathcal{L}_k$ the one-parameter family of k-Lorentz transformations.

Note that we allow the parameter k to be in $\mathbb{R}$. The classical group of Lorentz transformations corresponds to $k \in \mathbb{R}^+$. For $k = 0$, we obtain the Galilean group and for $k = 1/c^2$, $c \neq 0$, we obtain the Lorentz group used in special relativity where c is the speed of light.

Of course for all $k \in \mathbb{R}$, the set $\mathcal{L}_k$ is a group under composition law. Precisely, we have for all $v, u \in A_k$, $T_u \circ T_v = T_{u\star v}$, where $u \star v$ is given by

$$u \star v = \frac{u+v}{1+kuv}. \tag{16}$$

We easily prove that if $u \in A_k$ and $v \in A_k$ then $u \star v \in A_k$. We call $(\mathcal{L}_k, \circ)$ the Lorentz group of parameter k.

We have the following theorem, which is a direct consequences of Levy-Leblond computations [**23**]:

Theorem 4.1. — *The set of scale-transformations satisfying conditions i),ii) and iii) corresponds to Lorentz group of parameter k, $k \in \mathbb{R}$.*

As pointed out by Nottale [**27**], the set of condition i),ii) and iii) is over-determined. Indeed, using isotropy preservation we can weakened condition iii) by imposing only that $(\mathcal{T}, \circ)$ is a *free magma*, i.e. that for $u \in A$, $v \in A$, $T_u \circ T_v \in \mathcal{T}$.

iii)' The set $(\mathcal{T}, \circ)$ is a free magma.

Precisely, we have:

Theorem 4.2. — *The set of scale-transformations satisfying conditions i),ii) and iii)' corresponds to the Lorentz group of parameter k, $k \in \mathbb{R}$.*

The proof of theorem 4.2 will be given in section 5, clarifying and correcting earlier proofs given by Levey-Leblond [**23**] and Nottale [**27**]. In particular, isotropy and a structure of free magma is not sufficient to recover the classical Lorentez group ($k \geq 0$), contrary to the main statement of Nottale ([**27**],p.4906).

The classical Lorentz group, i.e. $\mathcal{L}_k$, $k \geq 0$, is obtained by Levy-Leblond imposing a causality constraint which is equivalent to the non-compactness of the group.

In our setting, this condition can be formalized as follow:

For all $(X_0, \delta_0) \in \mathbb{R} \times \mathbb{R}^+$, we denote by $\mathcal{C}^\sigma(X_0, y_0)$, $\sigma = \pm$, the set

$$\mathcal{C}^\sigma = \{(X, \delta) \in \mathbb{R} \times \mathbb{R}^+,\ \sigma(\delta - \delta_0) > 0 \text{ and, for all } v \in A,\ \sigma(\delta_v - \delta_{0,v}) > 0\}, \tag{17}$$

where δ_v and $\delta_{0,v}$ are defined for all $v \in A$ by $(X_v, \delta_v) = T_v(X, \delta)$ and $(X_{0,v}, \delta_{0,v}) = T_v(X_0, \delta_0)$ respectively.

Definition 4.3 **(Coherence)**. — *The set $\mathcal{T}$ of transformations satisfy the coherence property if for all $(X, \delta) \in \mathbb{R} \times \mathbb{R}^+$, the sets $\mathcal{C}^\sigma(X, \delta)$, $\sigma = \pm$, have non-zero (Lebesgue) measure.*

We impose that scale-transformations satisfies:

iv) The set of scale transformations is coherent.

We obtain the main theorem of this section:

Theorem 4.3. — *The set of scale-transformations satisfying condition i),ii), iii)' and iv) corresponds to the classical Lorentz group of parameter k, $k \geq 0$.*

The proof is given in section 6.

4.3. Spatial special scale relativity. — Before stating our main result, we must identify the nature of the parameter v governing scale-transformations. In classical relativity, v stands for the velocity between two inertial frames. In our setting, using the fact that $k = 0$ corresponds to Galilean transformations, we easily identify this parameter with the quantity denoted by $S_{e'/e}$ in section 3.3. Using this remark, we can specify the form of scale-transformation using theorem 4.3:

Theorem 4.4 **(spatial special scale relativity)**. — *Spatial scale-transformations are given by the Lorentz transformations of parameter* $k = 1/L^2$, $L \neq 0$, *defined by*

$$\begin{array}{lcl} X(e') & = & \gamma_{e'/e}(X(e) - S_{e'/e}\delta(e)), \\ \delta(e) & = & \gamma_{e'/e}(\delta(e) - S_{e'/e}(X(e)/L^2)), \end{array} \tag{18}$$

where $\gamma_{e'/e}$ *is defined by*

$$\gamma_{e'/e} = \sqrt{1 - S^2_{e'/e}/L^2}, \tag{19}$$

and L *is a constant having the dimension of a scale velocity.*

In the next section, we precise the physical nature of the constant L.

For $L = \infty$, we obtain the classical *Galilean group*. In this case, concrete examples of possible geometries for the physical space are obtained by considering H^α deformations of $\mathbb{R}^n$, $0 < \alpha < 1$.

4.4. About the existence of a limit scale. — The composition rule for velocities in relativistic mechanics, has a direct analogue for composition of scale velocities.

Let $\lambda \in \mathbb{R}^{+*}$ a reference scale, and $\Lambda \in \mathbb{R}^{+*}$ a second scale. We assume that the frame $\mathcal{R}_\Lambda$ has a scale speed $S_{\Lambda/\lambda}$ with respect to the frame $\mathcal{R}_\lambda$. We consider a particle with a scale speed $S_{e/\lambda}$ in the frame $\mathcal{R}_\lambda$ and $S_{e'/\Lambda}$ in the frame $\mathcal{R}_\Lambda$. We have using the classical law $\star$ defined in equation (16):

$$S_{e'/\Lambda} = \frac{S_{e/\lambda} + S_{\Lambda/\lambda}}{1 + \dfrac{S_{\epsilon/\lambda} S_{\Lambda/\lambda}}{L^2}}. \tag{20}$$

The constant L has the dimension of a scale velocity. As a consequence, we can assume that L has the following form

$$L = S_{l/\Lambda}, \tag{21}$$

where l is a length to precise.

We denote by $I_{\Lambda'/\Lambda}(e)$ the mapping which associates to each e the quantity e' defined by (20).

Lemma 4.1. — *The constant* l *is scale invariant, i.e.* $l = I_{\Lambda'/\Lambda}(l)$ *for all* Λ', $\Lambda \in \mathbb{R}^{+*}$.

The constant l is then an *horizon* for length in the sense of Cohen-Tannoudji [**8**]. The consequence is a limit to the divisibility of space, i.e. space-time has a *quantum structure*.

In ([**8**],p.115), Cohen-Tannoudji suggests, in the context of the existence of horizon lines, to interpret the Planck length $l_P = \sqrt{G\bar{h}/c^3}$, where c is the speed of light, and G is the universal constant of gravitation, $\bar{h} = h/2\pi$ is the reduced Planck constant, as an horizon for length, i.e. a quantity which can not be overreached. The constant l can then be identified with l_P, as already proposed by Nottale in [**26**].

5. Proof of theorem 4.2

We assume that the linear transformation T_v of parameter $v \in A$ is given by a matrix of the form (identifying the linear map with its matrix)

$$T_v = \gamma(v) \begin{pmatrix} 1 & -v \\ \mu(v) & \lambda(v) \end{pmatrix}, \tag{22}$$

where $\gamma(v) \neq 0$ for all $v \in \mathbb{R}$.

i) *Isotropy.* As T_v respects the isotropy of space for all $v \in A$, we have, keeping the notations of definition 4.1, for all $(X, \delta) \in \mathbb{R} \times \mathbb{R}^+$, $(X', \delta') = T_v(X, \delta)$, there exists $u \in A$ such that $(-X', \delta') = T_u(-X, \delta)$. We then have

$$\begin{array}{rcl} X' & = & \gamma(v)X - v\gamma(v)\delta, \\ \delta' & = & \gamma(v)(\mu(v)X + \lambda(v)\delta), \end{array} \tag{23}$$

and

$$\begin{array}{rcl} -X' & = & -\gamma(u)X - u\gamma(u)\delta, \\ \delta' & = & \gamma(u)(-\mu(u)X + \lambda(u)\delta). \end{array} \tag{24}$$

Identifying these quantities for X', we easily deduce that

$$\gamma(u) = \gamma(v), \quad -u\gamma(u) = v\gamma(v). \tag{25}$$

We then obtain $\gamma(v)(v + u) = 0$. As $\gamma(v) \neq 0$, we have $v = -u$ and γ is an even function.

The conditions coming from δ' are

$$\gamma(v)\mu(v) = -\gamma(u)\mu(u), \quad \gamma(v)\lambda(v) = \gamma(u)\lambda(u). \tag{26}$$

As a consequence, we have

$$\gamma(v)(\mu(v) + \mu(-v)) = 0, \quad \gamma(v)(\lambda(v) - \lambda(-v)) = 0. \tag{27}$$

We deduce that μ is odd and λ is even.

ii) *Free magma condition.* For all $u, v \in A$, there exists $w \in A$, that we denote also $w = u \star v$, such that $T_u \circ T_v = T_w$. We then have

$$\gamma(u)\gamma(v)\begin{pmatrix} 1-u\mu(v) & -v-u\lambda(v) \\ \mu(u)+\lambda(u)\mu(v) & -v\mu(u)+\lambda(v)\lambda(u) \end{pmatrix} = \gamma(w)\begin{pmatrix} 1 & -w \\ \mu(w) & \lambda(w) \end{pmatrix}. \tag{28}$$

Putting the term $1 - u\mu(v)$ in factor, we easily deduce that

$$v \star u = w = -\frac{v + u\lambda(v)}{1 - u\mu(v)}. \tag{29}$$

The law $\star$ is such that $v \star 0 = 0 \star v = v$, so that T_0 is a neutral element for the composition law.

We have for the remaining conditions of internal composition

$$-v\mu(u) + \lambda(v)\lambda(u) = \lambda\left(\frac{v + u\lambda(v)}{1 - u\mu(v)}\right)(1 - u\mu(v)), \tag{30}$$

$$\mu(u) + \lambda(u)\mu(v) - \mu\left(\frac{v + u\lambda(v)}{1 - u\mu(v)}\right)(1 - u\mu(v)) = 0, \tag{31}$$

$$\gamma(v)\gamma(u)(1 - u\mu(v)) = \gamma\left(\frac{v + u\lambda(v)}{1 - u\mu(v)}\right). \tag{32}$$

Replacing u by 0, we obtain

$$\lambda(0) = 1, \quad \gamma(0) = 1. \tag{33}$$

As a consequence, the matrix T_0 is the identity.

Using $u = -v/\lambda(v)$, and the fact that μ is odd and λ even, we have

$$v\mu(v/\lambda(v)) + \lambda(v)\lambda(v/\lambda(v)) = 1 + (v\mu(v)/\lambda(v)), \tag{34}$$

$$-\mu(v/\lambda(v)) + \lambda(v/\lambda(v))\mu(v) = 0. \tag{35}$$

Replacing $\mu(v/\lambda(v))$ in the first equation using the second equation, we have

$$(\lambda(v/\lambda(v))\lambda(v) - 1)\left(\frac{v\mu(v)}{\lambda(v)} + 1\right) = 0. \tag{36}$$

We can not have $v\mu(v) = -\lambda(v)$. Indeed, it gives $\lambda(0) = 0$ which is in contradiction with our previous result that $\lambda(0) = 1$. As a consequence, we must have

$$\forall\, v \in A, \quad \lambda(v/\lambda(v))\lambda(v) = 1. \tag{37}$$

The function λ is completely determined by this condition. We have

Lemma 5.1. — *Let $I \subset \mathbb{R}$ be an open interval of $\mathbb{R}$, $0 \in I$. A continuous even real valued function $\lambda : I \to \mathbb{R}$, such that $\lambda(v) \neq 0$ for all $v \in \mathbb{R}$, $\lambda(0) = 1$, and satisfying the functional equation*

$$\forall\, v \in I, \quad \lambda\left(\frac{v}{\lambda(v)}\right)\lambda(v) = 1, \tag{38}$$

is the function $\lambda(v) = 1$ for all $v \in I$.

The proof of this lemma is given in section 7. Note that we only need that λ is continuous. In ([**23**],p.274) the author assumes implicitly that λ is differentiable (see ([**23**], equation (26)).

As a consequence, we have the following simplified formulas for μ and γ:

$$-v\mu(u) = -u\mu(v), \tag{39}$$

$$\mu(u) + \mu(v) - (1 - u\mu(v))\mu\left(\frac{v+u}{1-u\mu(v)}\right) = 0, \tag{40}$$

$$\gamma(v)\gamma(u)(1 - u\mu(v)) = \gamma\left(\frac{v+u}{1-u\mu(v)}\right). \tag{41}$$

The first equation gives, fixing $u \neq 0$, that

$$\mu(v) = -kv,\ \forall v \in A^*, \tag{42}$$

where the minus sign is only here for convenience as $k \in \mathbb{R}$. The function μ extend by continuity to $\mu(0) = 0$, so that $\mu(v) = -kv$ for all $v \in A$, $k \in \mathbb{R}$.

Using $u = -v$ in the third equation, we obtain using the evenness of γ and $\gamma(0) = 1$ that

$$(\gamma(v))^2\,(1 - kv^2) = 1. \tag{43}$$

As long as $1 - kv^2 > 0$, we have

$$\gamma(v) = \frac{1}{\sqrt{1-kv^2}}. \tag{44}$$

We the obtain the following form for the transformations T_v, $v \in A_k$,

$$T_v = \frac{1}{\sqrt{1-kv^2}}\begin{pmatrix} 1 & -v \\ -kv & 1 \end{pmatrix}, \tag{45}$$

where A_k is defined by $A_k = \{v \in \mathbb{R} \mid\ 1 - kv^2 > 0\}$.

6. Proof of theorem 4.3

We study the set $\mathcal{C}_k^+$, the proof being the same for $\mathcal{C}_k^-$. Let $(X_0, \delta_0) \in \mathbb{R} \times \mathbb{R}^+$, we look for the set of $(X_1, \delta_1) \in \mathbb{R} \times \mathbb{R}^+$ such that $\Delta\delta = \delta_1 - \delta_0 > 0$ and $\Delta\delta_v = \delta_{1,v} - \delta_{0,v} > 0$ for all $v \in A_k$, where $\delta_{0,v}$ and $\delta_{1,v}$ are defined by $(X_{i,v}, \delta_{i,v}) = T_v(X_i, \delta_i)$, $i = 0, 1$.

We have for all $v \in A_k$,

$$\Delta\delta_v = \gamma_k(v)(-kv\Delta X + \Delta\delta), \tag{46}$$

where $\Delta X = X_1 - X_0$.

i) If $k < 0$ then we have $A_k = \mathbb{R}$. As a consequence, for all $(X_1, \delta_1) \in \mathbb{R} \times \mathbb{R}^+$, one can always find $v \in \mathbb{R}$ such that $\Delta_v < 0$, unless $\Delta X = 0$. As a consequence, we deduce that

$$\mathcal{C}_k^+(X_0, \delta_0) = \{(X_0, \delta),\ \delta \in \mathbb{R}^+ \ \mid \ \delta > \delta_0\}, \tag{47}$$

which is of zero (Lebesgue) measure. We deduce that Lorentz transformations of parameter k, $k < 0$, do not satisfy the coherence condition.

ii) If $k = 0$, we have $A_k = \mathbb{R}$ and $C_k^+(X_0, \delta_0) = \{(X, \delta) \in \mathbb{R} \times \mathbb{R}^+ \ \mid \ \delta > \delta_0\}$.

iii) If $k > 0$, then we have $A_k \subset \mathbb{R}$. Denoting $k = 1/c^2$, $c > 0$, we have $A_k = \{v \in \mathbb{R}, \mid v \mid < c\}$. The coherence condition is given by $c^2\Delta\delta > v\Delta X$; As a consequence, we have

$$\frac{\mid \Delta X \mid}{\mid \Delta\delta \mid} < \frac{c^2}{\mid v \mid}. \tag{48}$$

As $v \in A_k$, we deduce that the most stringent condition on ΔX is

$$\frac{\mid \Delta X \mid}{\mid \Delta\delta \mid} < c. \tag{49}$$

The set of events (X_1, δ_1) such that (49) is satisfied is a cone, which induces that $\mathcal{C}_k^+(X_0, \delta_0)$ is a set of non-zero (Lebesgue) measure.

7. Proof of lemma 5.1

Following Levy-Leblond [**23**], we denote by ξ the function defined by $\xi(v) = v/\lambda(v)$. The assumption of lemma 5.1 is then equivalent to

$$\xi(\xi(v)) = v \ \text{ for } \ v \neq 0. \tag{50}$$

As $\xi(0) = 0$, this equation is still valid for $v = 0$.

As ξ^{-1} exists for all $v \in I$ and ξ is continuous, we deduce that ξ is strictly monotonic on I.

Assume that ξ is an increasing function. Let $v \in I$, if $\xi(v) \leq v$, then $\xi(\xi(v)) \leq \xi(v) \leq v$. As $\xi(\xi(v) = v$, we deduce that $\xi(v) = v$ for all $v \in I$. The same argument applies if $\xi(v) \geq v$. We deduce that if ξ is increasing then $\xi(v) = v$ for all $v \in I$.

Assume that ξ is decreasing, then $f = -\xi$ is increasing. As ξ is odd, we obtain $f(f(v)) = f(-\xi(v)) = -f(\xi(v)) = \xi(\xi(v)) = v$ and $f(f(v)) = v$ for all $v \in I$. The previous paragraph implies that $f(v) = v$ for all $v \in I$, which gives $\xi(v) = -v$ for all $v \in I$.

These results imply that $\lambda(v) = \pm 1$ for all $v \in I$. As $\lambda(0) = 1$, we deduce that $\xi(v) = v$ and $\lambda(v) = 1$ for all $v \in I$.

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