DISTA-2008, IPHT-T-08/019, IHES-P- , NSF-KITP-09-31

Higher-loop amplitudes in the non-minimal pure spinor formalism

Pietro Antonio Grassi[a,b,e] and Pierre Vanhove[c,d,e]

[a] *DISTA, Università del Piemonte Orientale, via Bellini 25/g, 15100 Alessandria, ITALY*

[b] *INFN, gruppo collegato sezione di Torino, ITALY*

[c] *IHES, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France*

[d] *CEA, DSM, Institut de Physique Théorique, IPhT,*
CNRS, MPPU, URA2306, Saclay, F-91191 Gif-sur-Yvette, France

[e] *Kavli Institute for Theoretical Physics,*
University of California at Santa Barbara, CA 93106-4030, USA

email: pgrassi@cern.ch, pierre.vanhove@cea.fr

Abstract

We analyze the properties of the non-minimal pure spinor formalism. We show that Siegel gauge on massless vertex operators implies the primary field constraint and the level-matching condition in closed string theory by reconstructing the integrated vertex operator representation from the unintegrated ones. The pure spinor integration in the non-minimal formalism needs a regularisation. To this end we introduce a new regulator for the pure spinor integration and an extension of the regulator to allow for the saturation of the fermionic d-zero modes to all orders in perturbation. We conclude with a preliminary analysis of the properties of the four-graviton amplitude to all genus order.

23/3/2009

1. Introduction

The pure spinor formulation of perturbative string theory [1,2] has proved to be a powerful tool for implementing the role of maximally extended $\mathcal{N} = 8$ supersymmetries in various amplitude computations. Because this formalism makes use of a constrained ghost variable it allows to construct superspace invariants over fraction of superspace coordinates that are difficult to construct in conventional superspace approaches. In an extended formulation of the pure spinor formalism, Berkovits was able to avoid the complications associated with the picture changing operators of the original multiloop prescription [2,3] and to obtain a new class of partial superspace integrals [4] giving the leading contribution to the low-energy limit of the four-gravitons amplitude at genus order $g \leq 6$

$$F_g = \int d^{16}\theta d^{16}\bar{\theta}\theta^{12-2g}\bar{\theta}^{12-2g}\,(W_{\alpha\beta})^4 \sim \partial^{2g}R^4 + \text{susy completion} \tag{1.1}$$

Where $W_{\alpha\beta}$ is the Ramond-Ramond spin 1 superfield [5,4]. The fact that these quantities give the leading contribution to the low-energy limit of the four-graviton amplitude, up to genus-six order, confirms the non-renormalisation conditions for the $\partial^{2g}R^4$ contributions with $g \leq 6$ to the ten dimensional low-energy effective action for type IIA and type IIB string derived from string dualities in [6].

Since these superspace integrals arise from the zero mode saturation they give a direct indication of the leading ultra-violet divergence structure of the field theory four-graviton amplitude in $\mathcal{N} = 8$ supergravity. A four-graviton amplitude with the leading low-energy limit given by F_g in (1.1) has the following dimensions by

$$[\mathcal{A}_4^g] = [\partial^{2g}R^4]\,\text{mass}^{(D-4)g-6} \qquad g \leq 6 \tag{1.2}$$

where $[\cdots]$ gives the mass dimension. We used that a g-loop gravity amplitude has mass dimension $[\mathcal{A}_4^g] = \text{mass}^{(D-2)g+2}$, that $[\partial]$ =mass and $[R^4] = \text{mass}^8$. It is remarkable that the explicit four-graviton amplitudes performed in field theory up to and included three loop order in [7,8] can be presented in a form that has the manifest ultra-violet behaviour given by (1.2). This formula indicates that the g-loop four-graviton amplitude in (1.2) develops ultra-violet divergences from

$$D \geq D_c = 4 + \frac{6}{g}; \qquad g \leq 6\ . \tag{1.3}$$

When $g = 6$ the integration in (1.1) is over all the full superspace (all the 32 θ variables) and supersymmetric protection is exhausted. But at precisely this order the amplitudes

are ill-defined because of singularities in the integration over the pure spinor ghosts [3,4] and no firm conclusions could be drawn about the structure of the amplitude at higher-genus order. In this work we discuss an alternative modification of the non-minimal pure spinor formalism leading to well defined amplitude at any genus order. A regularisation of the singularities from the tip of the cone has been given in [3] but the resulting formulation makes very difficult to extract information about the structure of the higher-loop amplitudes. In order to understand the systematics of the higher-loop multigraviton amplitudes we introduce an alternative regulator. With this regulator we give a preliminary analysis of the structure of the four-graviton amplitude at higher-genus. We hope that this analysis is a step toward understanding the systematics of $\mathcal{N} = 8$ supergravity amplitudes and the role of the surprising simplifications occurring the structure of the higher-loop amplitudes [6,8,9,10,11,12,13].

In section 2 we review the basics of the minimal pure spinor formalism and its relation to the non-minimal formalism. In section 3, we discuss the massless vertex operators in the non-minimal formalism. We derive the relation between the integrated and unintegrated representation of the vertex operators. Using a Siegel gauge we derive the physical state condition on massless vertex operators, and the level-matching condition in the case of the closed string. A different analysis of the Siegel gauge condition on vertex operators appeared the recent preprint [14]. In section 4 we analyze the origin of divergences in the pure spinor integration. Because the heat kernel regularisation introduced in [3,14] leads to vanishing four-point massless amplitudes after genus 12 (see the end of section 4.4), which is not compatible with unitarity, we introduce a new regulator that regulates the singularities from the large values and the origin of the pure spinors integral without modifying the non-minimal $b_{\mathbf{nm}}$-ghost. We show that in order to be able to saturate the fermionic zero modes to all orders in perturbation one needs to consider an extension of the regulator with more d-zero mode contributions. In section 5 we turn to multiloop amplitudes and give the form of the integrand of the leading low-energy contribution to the multiloop four-graviton amplitude at all genus order. We conclude by showing that the massless $N < 4$-point amplitudes are vanishing to all order in the non-minimal pure spinor formalism. This implies finiteness of string perturbation in the absence of unphysical singularities in the interior of the moduli space.

2. Pure spinor measure of integration in the minimal and non-minimal formalism

The action for type II superstring in the pure spinor formalism in flat ten-dimensional space is given by [1]

$$S = \int d^2z \left(\frac{1}{2\pi\alpha'} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\theta}^\alpha + \widehat{p}_{\widehat{\alpha}} \partial \widehat{\theta}^{\widehat{\alpha}} + w_\alpha \bar{\partial} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} \right) \tag{2.1}$$

The matter fields are organized into ten bosonic fields of conformal weight zero x^m with $m = 0, \ldots, 9$ and two sets of fermionic fields $(p_\alpha, \theta^\alpha)$ and $(\widetilde{p}_{\widehat{\alpha}}, \widetilde{\theta^{\widehat{\alpha}}})$ of conformal weight one and zero with α in **16** and $\widehat{\alpha}$ in **16** or $\bar{\mathbf{16}}$ of $SO(16)$ depending if one treats the type IIA or type IIB string. In the following we will only mention the left-moving sector, but there are identical contributions from the right-moving sector. The pure spinor ghost λ^α of conformal weight zero is constrained by

$$\lambda \gamma^m \lambda = 0 \tag{2.2}$$

where $(\gamma^m)_{\alpha\beta}$ are the 16×16 gamma matrices of $SO(10)$. The pure spinor space defined by the constraint (2.2) is the non-compact conical space defined by a $\mathbb{C}^\star$ bundle over $SO(10)/U(5)$. The scale of the pure spinor varies between 0 and ∞.

The constraint leaves 11 independent components for the pure spinor λ^α and implies that the conjugated pure spinors w_α of conformal weight one has the following Λ-gauge invariance $\delta_\Lambda w_\alpha = \Lambda_m (\gamma^m \lambda)_\alpha$ with Λ_m a gauge parameter. The physical quantities are described as the cohomology of the pure spinor BRST charge

$$Q_{\mathbf{m}} = \oint \lambda^\alpha \, d_\alpha \tag{2.3}$$

where $d_\alpha = p_a - \frac{1}{2} (\gamma^m \theta)_\alpha \partial x_m - \frac{1}{8} (\theta \gamma_m \partial \theta)(\gamma^m \theta)_\alpha$ is the Green-Schwarz constraint, which satisfies the OPE $d_\alpha(z) \, d_\beta(0) \sim -(\gamma^m)_{\alpha\beta} \, \Pi_m / z$ where $\Pi_m = \partial x_m + (\theta \gamma_m \partial \theta)/2$ is the supersymmetric momentum. Analogously for the right-moving sector.

In the case of the minimal formalism [1] at genus g order, the 11 zero modes of the pure spinor ghost λ^α and $11g$ zero modes for the conjugated ghost w_α are saturated by the insertions of delta-functions $\delta(\lambda^\alpha)$ and $\delta(w_\alpha)$. The BRST-invariant and Λ-gauge invariant version of these delta-functions is given by the picture lowering Y_C and the picture raising Z_B operators

$$Y_C = C_\alpha \theta^\alpha \, \delta(C_\alpha \lambda^\alpha) \, , \qquad Z_B = \Big[Q_{\mathbf{m}}, \Theta([wB\lambda]) \Big] = (dB\lambda) \, \delta(wB\lambda) \ , \tag{2.4}$$

where Θ is the Heaviside step-function, and we have made use of the following notation

$$[wB\lambda] \equiv : w_\alpha \, B^\alpha{}_\beta \, \lambda^\beta := B \, J + \frac{1}{2!} \, B_{mn} \, N^{mn} \tag{2.5}$$

where the gauge-fixing parameters are the constant spinor C_α, and the 46 constants B and B_{mn}. We have as well introduced the currents

$$J =: w_\alpha \lambda^\alpha : \qquad N^{mn} =: w \gamma^{mn} \lambda : \tag{2.6}$$

are conformal weight one Λ-gauge invariant quantities.

The integration over the bosonic moduli is taken care by the picture raised conformal weight two $b_{\mathbf{m}}$-ghost which satisfies $[b_{\mathbf{m}}, Q_{\mathbf{m}}] = Z_B \, T_{\mathbf{m}}$ where $T_{\mathbf{m}}$ is the minimal formalism stress energy tensor. This field is integrated over the Riemann surface Σ_g with the help of the Beltrami differentials $(\mu | b_{\mathbf{m}}) \equiv \int_\Sigma d^2 z \, \mu^z{}_{\bar{z}} \, b_{\mathbf{m} \, zz}$ and the prescription for a genus-g amplitude, with $g \geq 2$, in type IIA/IIB string theory is given by [1]

$$\mathcal{A}^g_N = \int d^{10} x \Big| \int d^{16} \theta \int [d\lambda] \prod_{I=1}^{g} [dw^I] \prod_{i=1}^{3g-3} (\mu_i | b_{\mathbf{m}}) \prod_{j=3g-2}^{11g} Z_{B_j} \prod_{k=1}^{11} Y_{C,k} \Big|^2 \prod_{i=1}^{N} V_i \tag{2.7}$$

V_i are the integrated vertex operators. At tree-level there is no w-zero mode and the amplitude is given by 3 unintegrated vertex operators and no insertions of $b_{\mathbf{m}}$-ghost of picture changing operators Z_B. At genus one there are 11 w-zero mode to be integrated over, there is one insertion of the $b_{\mathbf{m}}$-ghost and one vertex operator is unintegrated. The insertion of the picture changing operators Y_C cuts off the large value of the pure spinor λ_α localizing the integration measure in a point.

The pure spinor measure of integration is defined as

$$[d\lambda] = (\epsilon \mathcal{T}^{-1})^{\alpha\beta\gamma}_{k_1 \cdots k_{11}} d\lambda^{k_1} \cdots d\lambda^{k_{11}} \, \partial_{\lambda^\alpha} \partial_{\lambda^\beta} \partial_{\lambda^\gamma} \tag{2.8}$$

where we have introduced the following tensor totally antisymmetric on the k_i indices and fully symmetric γ-traceless on the $\alpha\beta\gamma$ indices [2]

$$(\epsilon \mathcal{T})^{k_1 \cdots k_{11}}_{\alpha\beta\gamma} = \epsilon_{16}^{k_1 \cdots k_{11} r_1 \cdots r_5} \, (\gamma^m)_{((\alpha | r_1 |} \, (\gamma^n)_{\beta | r_2 |} \, (\gamma^p)_{\gamma)) r_3} \, (\gamma_{mnp})_{r_4 r_5} \ . \tag{2.9}$$

Such a definition of the measure of integration using derivatives is natural from the supergeometry point of view as shown in [15]. This measure satisfies the requirement that the overlap between the vacuum $|0\rangle$ and the highest state in the zero momentum cohomology $|C\rangle = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ is a constant

$$\langle 0|C\rangle = \int d^{16}\theta \int [d\lambda] \prod_{i=1}^{11} \theta^{\alpha_i}\delta(\lambda^{\alpha_i})(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) = 1 \ . \tag{2.10}$$

This gives the rules for computing tree-level amplitudes [1]. We will return to this computation in section 5 when analyzing the effect of the regulator on the non-minimal formalism amplitude prescription.

This minimal formalism with only one set of pure spinor ghost, only a picture raised version of the b-ghost can be constructed which make the analysis of the multiloop amplitude difficult beyond two-loops. As well in this formalism the integration over the pure spinor variables has to be done over patches of the pure spinor space and one needs to analyze the Čech-cohomology on this space for global properties [16]. As well because of the presence of picture changing operators the amplitudes are Lorentz and supersymmetric invariant up to boundary term.

The delta-function insertions provided by the picture changing operators in (2.4) can be exponentiated by introducing extra new variables [17,18,2]. Let start by considering the case a single fermionic variable θ whose BRST transformation is $Q\theta = \lambda$ and then by adding a new doublet r and $\bar\lambda$ and their conjugated ghost $\bar w$ and s so that $[\bar w, \bar\lambda] = 1$ and $\{r, s\} = 1$. In order that physical observables do not depend on these new variables,[1] we introduce a new nilpotent BRST operator $\Delta = \oint \bar w r$ so that $(r, s; \bar\lambda, \bar w)$ is a topological quartet under the total BRST-charge $Q + \Delta$. We can now express the delta-function insertions as follows

$$\theta\delta(\lambda)\, d\delta(w) = \int [dr][d\bar\lambda][ds][d\bar w]\, \mathcal{N} \tag{2.11}$$

where

$$\mathcal{N} = e^{-\lambda\bar\lambda - r\theta - w\bar w - sd} \ . \tag{2.12}$$

[1] The physical vertex operators do not depend on the non-minimal sector because the non-minimal ghost number $\bar J = \bar\lambda\bar w - sr = [Q_{\mathbf{nm}}, s_\alpha\bar\lambda^\alpha]$, and as well $[Q_{\mathbf{nm}}, \bar J] = 0$. And the physical states are eigenvalues of the non-minimal ghost number $\bar J\,\Psi = n\,\Psi$. Since $\bar J$ is Q-exact all states with non-zero non-minimal ghost charge are Q-exact $\Psi = [Q_{\mathbf{nm}}, s\bar\lambda\,\Psi]/n$. Therefore the physical states are in the zero oscillator sector with $n = 0$. This is the so-called quartet mechanism.

The exponent can be rewritten as $\bar{\lambda}\lambda + r\theta + w\bar{w} + sd = [Q, \Psi]$ with the gauge fermion

$$\Psi = \bar{\lambda}\theta + s\bar{w} \ . \tag{2.13}$$

The form of the exponent as BRST-exact quantity ensures that the amplitudes do not depend upon the extract form of the gauge fermion Ψ unless some singularities in the amplitude forbid the decoupling of BRST-exact quantities.

This procedure can be seen as a motivation for the introduction of the non-minimal ghosts by Berkovits in [2] for defining the non-minimal pure spinor formalism. He introduced the complex conjugate extra ghosts $\bar{\lambda}_\alpha$ and r_α satisfy the relations

$$\bar{\lambda}\gamma^m\bar{\lambda} = 0\,, \qquad \bar{\lambda}\gamma^m r = 0 \tag{2.14}$$

In this case and the conjugated variables transform under the gauge symmetry $\delta_{\Lambda+L}\bar{w}^\alpha = \Lambda_m(\gamma^m\bar{\lambda})^\alpha + L_m(\gamma^m r)^\alpha$ and $\delta_L s^\alpha = L_m(\gamma^m\bar{\lambda})^\alpha$ where Λ_m and L_m are the gauge parameters. Therefore the conjugated ghost $\bar{w}^\alpha$ and s^α can only appear through the conformal weight one Λ- and L-gauge invariant quantities

$$\begin{aligned} &\bar{N}_{mn} = \bar{w}\gamma_{mn}\bar{\lambda} - s\gamma_{mn}r; \qquad \bar{J} = \bar{w}\bar{\lambda} - sr \\ &S^{mn} = s\gamma^{mn}\bar{\lambda}; \qquad S = s\bar{\lambda}\ . \end{aligned} \tag{2.15}$$

The non-minimal BRST-charge is

$$Q_{\mathbf{nm}} = \oint \lambda^\alpha d_\alpha + \oint \bar{w}^\alpha\, r_\alpha\,. \tag{2.16}$$

3. Vertex operators in the non-minimal

The physical state vertex operators are in the cohomology of $Q_{\mathbf{nm}}$ defined in (2.16). For the massless sector of the type II superstring the vertex operators come into the integrated and the unintegrated representations

$$V = \int d^2z |V_{open}|^2\ e^{ik\cdot X}; \qquad U = |U_{open}|^2\, e^{ik\cdot X} \tag{3.1}$$

where $U_{open} = \lambda^\alpha A_\alpha$ and

$$V_{open} = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + N^{mn}F_{mn} \tag{3.2}$$

where A_α, A_m, W^α and F_{mn} are the $\mathcal{N}=1$ $D=10$ super-Yang-Mills superfields

$$A_\alpha(x,\theta) = \frac{1}{2}\,(\gamma^m\theta)_\alpha\, a_m + \frac{1}{3}(\chi\gamma_m\theta)\,(\gamma^m\theta)_\alpha - \frac{1}{32}\,f_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \cdots \tag{3.3}$$

and

$$\begin{aligned}(\gamma^m)_{\alpha\beta}A_m &= D_\alpha A_\beta + D_\beta A_\alpha \\ (\gamma_m)_{\alpha\beta}W^\beta &= D_\alpha A_m - \partial_m A_\alpha \\ D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}\end{aligned} \tag{3.4}$$

Acting with $Q_{\mathbf{nm}}$ on V_{open} the computation is the same as in the minimal formalism leading to

$$[Q_{\mathbf{nm}}, V_{open}] = \partial_\sigma(U_{open}) + \text{e.o.m.} \tag{3.5}$$

where e.o.m. are the $\mathcal{N}=1$ $D=10$ super-Yang-Mills equations-of-motion given in (3.4). The vertex operator U_{open} satisfies $[Q_{\mathbf{nm}}, U_{open}]=0$. Notice that, since V_{open} and U_{open} are independent of the non-minimal fields only the minimal part of the BRST charge acts on the vertex operator.

Because $\{Q_{\mathbf{nm}}, b_{\mathbf{nm}}\} = T_{\mathbf{nm}}$, one can use the $b_{\mathbf{nm}}$-ghost to construct the integrated vertex operator from the unintegrated vertex operator. If we denote $b_{-1} = \int d\sigma\, b_{\mathbf{nm}}$, we have that $\{Q_{\mathbf{nm}}, b_{-1}\} = \int d\sigma\, T_{\mathbf{nm}} = \partial_\sigma$. So, acting with b_{-1} on U_{open} we can derive the integrated vertex operator V_{open}.

The non-minimal $b_{\mathbf{nm}}$-ghost takes the form [2,3]

$$b_{\mathbf{nm}} = s\partial\bar{\lambda} + \frac{1}{4}\check{\lambda}_\alpha\,\mathbf{b}^\alpha \tag{3.6}$$

where we have introduced the notations

$$\check{\lambda}_\alpha = \frac{\bar{\lambda}_\alpha}{(\lambda\cdot\bar{\lambda})}; \qquad \check{r}_\alpha = \frac{r_\alpha}{(\lambda\cdot\bar{\lambda})} \tag{3.7}$$

and

$$\mathbf{b}^\alpha \equiv G^\alpha + \check{r}_\beta H^{\alpha\beta} + \check{r}_\beta\check{r}_\gamma K^{\alpha\beta\gamma} + \check{r}_\beta\check{r}_\gamma\check{r}_\delta L^{\alpha\beta\gamma\delta} \tag{3.8}$$

and the operators

$$\begin{aligned}G^\alpha &\equiv 2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J\partial\theta^\alpha - \frac{1}{2}\partial^2\theta^\alpha \\ H^{\alpha\beta} &\equiv \frac{1}{196}(\gamma^{mnp})^{\alpha\beta}\,((d\gamma_{mnp}d) + 4!N_{mn}\Pi_p) \\ K^{\alpha\beta\gamma} &\equiv \frac{1}{16}(\gamma_{mnp})^{[\alpha\beta}(\gamma^m d)^{\gamma]}N^{np} \\ L^{\alpha\beta\gamma\delta} &= \frac{1}{128}(\gamma_{mnp})^{[\alpha\beta}(\gamma^{pqr})^{\gamma\delta]}N^{nm}N_{qr}\ .\end{aligned} \tag{3.9}$$

It was shown in [19] that the non-minimal $b_{\mathbf{nm}}$-ghost and the b_Y-ghost of the Y-formalism are related by

$$b_{\mathbf{nm}} = b_Y + [Q_{\mathbf{nm}}, \Omega_v], \qquad b_Y = \frac{v_\alpha G^\alpha}{v \cdot \lambda} \tag{3.10}$$

where v_α is a constant reference pure spinor so that $v\gamma^m v = 0$ and $v \cdot \lambda \neq 0$. Here Ω_v, which expression can be found in [19], depends on the non-minimal sector and the reference spinor v_α.

We want to derive the integrated vertex operators V_{open} by acting with $b_{\mathbf{nm}\,-1}$ on the unintegrated vertex operators $U_{open} = \lambda^\alpha A_\alpha$. This amounts into taking the first order poles of the OPE between the $b_{\mathbf{nm}}$-ghost and the vertex operator. For doing this computation we will use the relation (3.10) and compute the OPE between the Oda-Tonin b_Y-ghost with the vertex operator.

Using the ten-dimensional identity [19]

$$-\frac{1}{8}(B\gamma^{mn}A)(\gamma_{mn}C)^\alpha - \frac{1}{4}(B_\beta A^\beta)C^\alpha = B_\beta A^\alpha C^\beta - \frac{1}{2}(\gamma^m B)^\alpha (A\gamma_m C) \tag{3.11}$$

where $A^\alpha, B_\gamma, C^\beta$ are three spinors of different chirality. It follows from the usual Fierz identities and the OPEs

$$\begin{aligned} N^{mn}(y)\,\lambda^\alpha(z) &\sim \frac{1}{2\,(y-z)}\,(\gamma^{mn}\lambda)^\alpha(0), \\ J(y)\,\lambda^\alpha(z) &\sim \frac{1}{y-z}\,\lambda^\alpha(z) \end{aligned} \tag{3.12}$$

and the equations of motion given in (3.4) and the Feynman gauge condition $\partial_m A^m = 0$ we get

$$\oint_z b_Y(z)\,(\lambda \cdot A)(0) = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + F_{mn}N^{mn} + [Q_{\mathbf{nm}}, \widehat{\Omega}] \tag{3.13}$$

where (all the pieces should be normal ordered)

$$\widehat{\Omega} = \frac{(v\gamma^m A)}{2(v\cdot\lambda)}\Pi_m - \frac{(v\gamma^m d)}{(v\cdot\lambda)}A_m - \frac{(v\gamma^m \partial_m A)}{(v\cdot\lambda)}\frac{\partial(v\cdot\lambda)}{(v\cdot\lambda)} + N^{mn}\frac{(v\gamma_{mn}W)}{(v\cdot\lambda)} + \frac{1}{2}J\frac{(v\cdot W)}{(v\cdot\lambda)}\,, \tag{3.14}$$

This gives for the action of the full non-minimal $b_{\mathbf{nm}}$-ghost that

$$\oint_z (b_{\mathbf{nm}} U_{open} + [Q_{\mathbf{nm}}, \widehat{\Omega}]) = V_{open} \tag{3.15}$$

where the Q-exact part assures that V_{open} does not depend on the non-minimal sector.

3.1. Siegel Gauge for open string

Within the pure spinor formalism, one can verify that the BRST cohomology at ghost number one gives the superspace equations for $\mathcal{N} = 1$ $D = 10$ the super-Yang-Mills. However, these equations are not enough to impose the primary field constraint on the vertex operator. This situation is well-known for example in String Field theory where the equations of motion are manifestly gauge invariant (see also [20]). In order to impose the primary field condition, we impose the Siegel gauge condition.

For this we use the Oda-Tonin b_Y-field given in (3.10). We define the zero mode of it as $b_{Y\,0} = \oint dz\, z\, b_Y$ and we act on the vertex operator $U_{open} = \lambda^\alpha A_\alpha(x,\theta)$. Computing the contributions of the double poles yield

$$\begin{aligned} b_{Y\,0}(U) &= \frac{v_\alpha \lambda^\beta}{v\cdot\lambda}\,(\gamma^m)^{\alpha\gamma}\partial_m D_\gamma\, A_\beta \\ &= \frac{1}{v\cdot\lambda}\,v_\alpha\lambda^\beta(\gamma^m)^{\alpha\gamma}\Big(-D_\beta\partial_m A_\gamma + (\gamma^p)_{\beta\gamma}\partial_m A_p\Big) \\ &= -\frac{1}{v\cdot\lambda}\lambda^\beta D_\beta\Big(v_\alpha(\gamma^m)^{\alpha\gamma}\partial_m A_\gamma\Big) + \frac{1}{v\cdot\lambda}v_\alpha\,\lambda^\beta(\gamma^m\gamma^p)^\alpha{}_\beta\partial_m A_p \\ &= -Q_{\mathbf{nm}}\left(\frac{v_\alpha(\gamma^m)^{\alpha\gamma}\partial_m A_\gamma}{v\cdot\lambda}\right) + \frac{1}{v\cdot\lambda}\Big(v\gamma^m\gamma^p\lambda\Big)\partial_m A_p \\ &= -Q_{\mathbf{nm}}\left(\frac{v_\alpha(\gamma^m)^{\alpha\gamma}\partial_m A_\gamma}{v\cdot\lambda}\right) + \partial^m A_m + \frac{1}{2\,v\cdot\lambda}\Big(v\gamma^{mp}\lambda\Big)F_{mp} \end{aligned} \tag{3.16}$$

and finally, using again the equation of motion $4\,D_\alpha W^\beta = (\gamma^{mn})_\alpha{}^\beta F_{mn}$ we have

$$b_{Y\,0}(U) = \partial^m A_m - Q_{\mathbf{nm}}\left(\frac{v_\alpha(\gamma^m)^{\alpha\gamma}\partial_m A_\gamma - 2\,v_\alpha W^\alpha}{v\cdot\lambda}\right). \tag{3.17}$$

Thanks to the relation (3.10) between the Oda-Tonin b_Y-ghost and the non-minimal $b_{\mathbf{nm}}$-ghost we deduce that as well $b_{\mathbf{nm}\,0}(U) = \partial^m A_m + Q_{\mathbf{nm}}\Omega$. This leads to the usual gauge fixing for the Maxwell field which has been derived from the Siegel gauge condition. As a consequence of the Siegel gauge fixing, the Virasoro constraints must follow and the vertex operator is primary. Indeed, we act with the BRST charge from the left on (3.17) and we get

$$Q_{\mathbf{nm}}\Big(b_{\mathbf{nm}\,0}(U)\Big) = Q_{\mathbf{nm}}\Big(\partial^m A_m\Big), \tag{3.18}$$

then using the relation $[Q_{\mathbf{nm}}, b_{\mathbf{nm}\,0}] = L_0$ and $Q_{\mathbf{nm}}U = 0$ we finally obtain that

$$L_0(U) = Q_{\mathbf{nm}}\Big(\partial^m A_m\Big) \tag{3.19}$$

Evaluating the right-hand-side

$$\begin{aligned} Q_{\mathbf{nm}}\Big(\partial^m A_m\Big) &= \lambda^\alpha \partial^m \Big(D_\alpha A_m\Big) = \lambda^\alpha \partial^m \Big(\partial_m A_\alpha - (\gamma_m)_{\alpha\beta} W^\beta\Big) \\ &= \lambda^\alpha \partial^2 A_\alpha + \lambda^\alpha (\gamma^m)_{\alpha\beta} \partial_m W^\beta \end{aligned} \tag{3.20}$$

and choosing the gauge $\partial^m A_m = 0$, using $\partial^2 A_\alpha = 0$ and the Dirac equation $\not{\partial} W = 0$, we end up with the Virasoro constraint $L_0(U) = 0$ and the vertex operator is primary. Notice that if it were that $L_0(U) = \rho U$ where ρ is a proportionality constant, then U would not be in the cohomology. In addition, it can be proved that, at least on the vertex U, b_{Y0} is nilpotent.

3.2. Siegel-Zwiebach gauge for closed strings

In the case of closed strings, we have a left- and a right-moving $b_{\mathbf{nm}}$-field that can be used to impose the gauge fixing condition. In that case, on the contrary to the open strings case, the BRST condition does not impose the Virasoro constraints and the level matching condition. The level matching condition is obtained by imposing $b_{0L} - b_{0R}$ on the physical states (where L/R denote the holomorphic and the anti-holomorphic part). See for example [21] for a discussion of these points. In the following we will show that imposing the level matching condition leads also to the Virasoro constraints.

The closed unintegrated vertex operator U is given by the expression

$$U = \lambda^\alpha \lambda^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x, \theta, \hat{\theta})\,, \tag{3.21}$$

where $\lambda^{\hat{\alpha}}$ is the pure spinor for the right-moving part. The superfield $A_{\alpha\hat{\alpha}}$ depends upon the two supercoordinates θ and $\hat{\theta}$. This superfield plays the role of the spinorial connection for the supergravity multiplet. In order to relate this superfield to the conventional superfields A_{mn} (whose lowest component is the combination of the metric and of the NSNS two form) one needs to derive a ladder of differential equations starting from

$$D_{(\alpha} A_{\beta)\hat{\gamma}} = (\gamma^m)_{\alpha\beta} A_{m\hat{\gamma}}\,, \qquad \hat{D}_{(\hat{\alpha}} A_{|\beta|\hat{\gamma})} = (\gamma^m)_{\hat{\alpha}\hat{\gamma}} A_{\alpha m}\,. \tag{3.22}$$

The complete set of equations were derived in [22]. Acting with the left- b_{YL} and right b_{YR} Oda-Tonin b_Y-fields on the vertex operator (3.21), we get

$$\begin{aligned} &(b_{YL,0} \pm b_{YR,0})\Big(\lambda^\alpha \lambda^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x, \theta, \hat{\theta})\Big) \\ &= \lambda^{\hat{\alpha}} \partial^m A_{m\hat{\alpha}} + \lambda^\alpha \partial^m A_{\alpha m} + (Q_{L\mathbf{nm}} + Q_{R\mathbf{nm}})(\Omega) \end{aligned} \tag{3.23}$$

where Ω is a polynomial obtained after Fierz rearrangements. As in the open string case, these exact terms are irrelevant. Notice that since the right-hand-side involves explicitly the ghost field λ^α and $\lambda^{\hat{\alpha}}$, this yields the gauge fixing condition

$$\partial^m A_{m\hat{\alpha}} = 0\,, \qquad \partial^m A_{\alpha m} = 0\,. \tag{3.24}$$

Using the equations

$$\hat{D}_{(\hat{\alpha}} A_{m\hat{\beta})} = (\gamma^n)_{\hat{\alpha}\hat{\beta}} A_{mn}\,, \qquad \hat{D}_{(\alpha} A_{\alpha)m} = (\gamma^n)_{\alpha\beta} A_{nm}\,. \tag{3.25}$$

By separating the symmetric and antisymmetric part of A_{nm} these equations lead to the usual De Donder gauge for the metric and Landau gauge for the NSNS two form

$$\partial^m A_{mn} = 0\,, \qquad \partial^n A_{mn} = 0\,. \tag{3.26}$$

Finally, using

$$D_\alpha A_{mn} - \partial_m A_{\alpha n} = (\gamma_m)_{\alpha\beta} W^\beta{}_n\,, \qquad D_{\hat{\alpha}} A_{mn} - \partial_m A_{n\widehat{\alpha}} = (\gamma_m)_{\hat{\alpha}\hat{\beta}} W^{\widehat{\beta}}{}_n\,, \tag{3.27}$$

where $W^\alpha{}_n$ is the gravitino superfield. This implies the set of equations

$$D_\alpha \partial^m A_{mn} - \partial^2 A_{\alpha n} = \not{\partial} W_n\,, \qquad D_\alpha \partial^n A_{mn} - \partial_m \partial^n A_{\alpha n} = (\gamma_m)_{\alpha\beta} \partial^n W^\beta{}_n\,, \tag{3.28}$$

Using Dirac equation $\not{\partial} W^\alpha{}_m = 0$ and the gauge fixing condition $\partial^m W^\alpha{}_m = 0$ we obtain that $\partial^2 A_{\alpha p} = 0$ and $\partial^2 W^\alpha{}_m = 0$. In the same way, one can derive the gauge fixing condition for the other gaugino. The Dirac equation for the gravitino using the present framework was discussed in [22]. Notice that unlike the case of bosonic string, we naturally impose both conditions on the vertex operator $b_{YL,0}$ and $b_{YR,0}$ since they depends upon the independent left- and right-moving pure spinor ghosts that implies the independence of the left- and right-moving b-fields. This means that besides the Virasoro constraints also the level matching is automatically imposed.

4. Regulating the non-minimal pure spinor amplitudes

Because the non-minimal $b_{\mathbf{nm}}$-ghost has $1/(\lambda\cdot\bar{\lambda})$ pole and measure of integration over the conjugated ghosts bring some inverse powers of λ and $\bar{\lambda}$ (see below for details) the amplitudes can develop singularities [2,3] from the tip of the pure spinor cone $\lambda, \bar{\lambda} \sim 0$.

In order to understand the effect of the choice of the regulator on the amplitudes we analyse the effect of the general regulator

$$\Psi = \bar{\lambda}_\alpha \theta^\alpha f(\lambda\cdot\bar{\lambda}) - \frac{1}{2}\sum_{I=1}^{g} S^I_{mn} \mathcal{O}_I^{mn} + \sum_{I=1}^{g} S^I \mathcal{O}_I \tag{4.1}$$

for f is a real function. And $\mathcal{O}_I^{mn}$ and $\mathcal{O}_I$ are ghost number zero Λ- and L-gauge invariant version of (2.12) that that will depend on the zero-modes conformal weight one fields and will be discussed in section 4.4.

4.1. The vacuum of the pure spinor theory

The normalisation of the vacuum of the pure spinor theory $|0\rangle$ is defined by considering its overlap with the highest ghost number state in the zero momentum cohomology $|C\rangle = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$

$$\langle 0|C\rangle = \int d^{16}\theta \int [d\lambda][d\bar{\lambda}][dr]\,\widehat{\mathcal{N}}\,(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\ . \tag{4.2}$$

with the measures of integrations given in (2.8) and

$$[d\bar{\lambda}][dr] = d\bar{\lambda}_{\alpha_1}\wedge\cdots\wedge d\bar{\lambda}_{\alpha_{11}}\ \times\ \partial_{r_{\alpha_1}}\wedge\cdots\wedge\partial_{r_{\alpha_{11}}}\ , \tag{4.3}$$

The integration over the pure spinor cone requires that one regulates the integral. A generic regulator

$$\widehat{\mathcal{N}} = \exp\left(-(\lambda\cdot\bar{\lambda})\,f(\lambda\cdot\bar{\lambda}) + r_\alpha M^\alpha{}_\beta\theta^\beta\right) \tag{4.4}$$

where $M^\alpha{}_\beta = \delta^\alpha{}_\beta\, f(\lambda\cdot\bar{\lambda}) + \lambda^\alpha\bar{\lambda}_\beta\, f'(\lambda\cdot\bar{\lambda})$. This quantity (4.2) gives the normalization of the amplitudes and the prescription for evaluating the integration over the pure spinor ghosts [1].

Two detailed evaluations of this integral are given in the appendix A. Setting $h(\lambda\cdot\bar{\lambda}) = (\lambda\cdot\bar{\lambda})f(\lambda\cdot\bar{\lambda})$ the amplitude takes the form

$$\langle 0|C\rangle = 11!5!\int\prod_{i=1}^{11} d\lambda^{\alpha_i}d\bar{\lambda}_{\alpha_i}\ (\lambda\cdot\bar{\lambda})^{-10}\,e^{-h(\lambda\cdot\bar{\lambda})}\,h(\lambda\cdot\bar{\lambda})^{10}\,h'(\lambda\cdot\bar{\lambda}) \tag{4.5}$$

This expression is proportional to

$$\begin{aligned}\langle 0|C\rangle &\propto (-\partial_\alpha)^{10}\int_0^\infty dx\,e^{-\alpha h(x)}h'(x)\Big|_{\alpha=1}\\ &\propto (-\partial_\alpha)^{10}\left(-\frac{1}{\alpha}\left(e^{-\alpha\,h(\infty)}-e^{-\alpha\,h(0)}\right)\right)\Big|_{\alpha=1}\end{aligned} \tag{4.6}$$

We see that the value of the amplitude (4.2) is controlled by the value of the regulator at the boundary of the pure spinor space $\lambda\cdot\bar{\lambda}=\infty$ and $\lambda\cdot\bar{\lambda}=0$. Therefore any regulator so that $\lim_{x\to\infty}\exp(-h(x)) = 0$ and $\lim_{x\to 0}\exp(-h(x)) = 0$ is too strong and will lead to a vanishing amplitude trivializing the theory.[2]

[2] We thank Nathan Berkovits for an important discussion concerning this point.

In the rest of this paper we will make the choice of a gauge fermion which is strongly dumped at zero

$$\widehat{\Psi} = \frac{\bar{\lambda}_\alpha \theta^\alpha}{(\lambda \cdot \bar{\lambda})^2} \tag{4.7}$$

and the regulator takes the form

$$\widehat{\mathcal{N}} = \exp\left[-\frac{1}{(\lambda \cdot \bar{\lambda})} - r_\alpha \left(\frac{\delta^\alpha{}_\beta}{(\lambda \cdot \bar{\lambda})^2} - 2\frac{\lambda^\alpha \bar{\lambda}_\beta}{(\lambda \cdot \bar{\lambda})^3} \right) \theta^\beta \right] \tag{4.8}$$

With this regulator any divergences from the tip of the cone $\lambda \cdot \bar{\lambda} \sim 0$ will be regularized by the exponential factor, and the region $\lambda \cdot \bar{\lambda} \to \infty$ will be regulated by the powers of $1/(\lambda \cdot \bar{\lambda})$ coming from the r-zero mode contributions. For this regulator the amplitude in (2.15) is a constant

$$\langle 0|C\rangle = 11!\, 15\, (4\pi)^{15} \,. \tag{4.9}$$

that determines the normalisation of the amplitudes.

4.2. Tree-level Amplitudes

The prescription for N-point tree-level amplitude given in [2,3] is

$$\mathcal{A}_N^{\text{tree}} = \int d^{10}x \int |d^{16}\theta \int [d\lambda][d\bar{\lambda}][dr]\, \widehat{\mathcal{N}}|^2\, U(z_1)U(z_2)U(z_3) \prod_{j=4}^{N} V(z_i) \tag{4.10}$$

with the measures of integrations given in (2.8) and (4.3).

The advantage of using the regulator (4.8) is that the amplitudes are less diverging at for $\bar{\lambda} \sim \infty$ than at $\bar{\lambda} \sim 0$. Because it is possible to generate $1/(\lambda \cdot \bar{\lambda})$-poles of any order by inserting enough $b_{\mathbf{nm}}$-ghost (which happens at higher loop order [2,4]), but by ghost charge conservation because the $b_{\mathbf{nm}}$-ghost has ghost charge -1 and the physical vertex operators appear at ghost charge $+1$ or zero, the integrand of the amplitude divergence at most like $(\lambda \cdot \bar{\lambda})^{11}$ for $\lambda \cdot \bar{\lambda} \to \infty$.

We show that with the regulator (4.8) the amplitudes will converge at the boundary $\lambda \cdot \bar{\lambda} \sim \infty$ of the pure spinor cone.

By computing the tree-level amplitude with 3 unintegrated vertex operators and $N-3$ integrated vertex operators as in (4.10), the 11 r-zero modes must come from the regulator. Therefore the integrand becomes $d\lambda d\bar{\lambda}/(\lambda \cdot \bar{\lambda})^2$ which converges for $\lambda \cdot \bar{\lambda} \to \infty$. Using the

representation with all unintegrated vertex operators and $N-3$ $b_{\mathbf{nm}}$-ghost insertions, from seven point $N \geq 7$ it is possible to saturate the 11 r-zero mode from the $b_{\mathbf{nm}}$-ghost only and the integrand seems to behaves as $d\lambda d\bar{\lambda}/(\lambda \cdot \bar{\lambda})$ which corresponds to a logarithmic singularity at infinity. But as remarked in [3] all the terms in the $b_{\mathbf{nm}}$-ghost commute with the conserved charges

$$q_1 = \oint (r_\alpha s^\alpha - \lambda^\alpha w_\alpha); \qquad q_2 = \oint \bar{\lambda}_\alpha s^\alpha \,, \tag{4.11}$$

which imply that the terms of the $b_{\mathbf{nm}}$-ghost (3.6) have opposite r-charge and λ-charge and are invariant under the shift symmetry $\delta r_\alpha = c\,\bar{\lambda}_\alpha$ where c is a constant. Therefore to saturate all the 11 r-zero modes we need to pick 12 r from the $b_{\mathbf{nm}}$-ghost or 11 r from the $b_{\mathbf{nm}}$-ghost and one r from the regulator and contract the left over r-ghost with s-ghosts. In either case this brings enough powers of $1/(\lambda \cdot \bar{\lambda})$ so that the integral converges for the large values of the pure spinor ghost.

In the non-minimal formalism it is possible to construct the following quantity $\xi = (\bar{\lambda} \cdot \theta)/(\lambda \cdot \bar{\lambda} + r \cdot \theta)$ so that $Q_{\mathbf{nm}}\xi = 1$. If this state is allowed it will trivializes the theory by making all physical state Q-exact, and all amplitude vanishing. By evaluating the amplitude $\langle 0|\xi\rangle$ we see that the contribution with 11 r-zero mode lead to a logarithmic divergence at infinity. Because the terms in the expansion of the ξ do not commute with conserved charges q_1 and q_2 and the divergence is not protected by the symmetry $\delta r_\alpha = c\,\bar{\lambda}_\alpha$. Therefore the state ξ is not allowed in the physical Hilbert space of the theory.

We can compare with the prescription given by Berkovits in [2] where the following gauge fermion and regulator are used

$$\Psi = \bar{\lambda}_\alpha \theta^\alpha; \qquad \mathcal{N} = \exp\Big(- \lambda \cdot \bar{\lambda} - r\theta \Big) \,. \tag{4.12}$$

The regulator (4.8) takes the form given in (2.15) with

$$M^\alpha{}_\beta = \frac{\delta^\alpha{}_\beta}{(\lambda \cdot \bar{\lambda})^2} - 2\frac{\lambda^\alpha \bar{\lambda}_\beta}{(\lambda \cdot \bar{\lambda})^3} \tag{4.13}$$

This matrix satisfies the property that $M^\alpha{}_\beta\, M^\beta{}_\gamma = \delta^\alpha{}_\gamma/(\lambda \cdot \bar{\lambda})^4$ that implies that $(M^{-1})^\alpha{}_\beta = (\lambda \cdot \bar{\lambda})^4\, M^\alpha{}_\beta$.

In the amplitude one can eliminate the dependence on this matrix in the regulator by performing the change of variable $r_\alpha = \widetilde{r}_\beta (M^{-1})^\beta{}_\alpha$. This induces a non-trivial Jacobian factor depending only on the λ and $\bar{\lambda}$ pure spinor ghosts

$$[d\bar{\lambda}][dr] \to d\bar{\lambda}_{\alpha_1} \wedge \cdots \wedge d\bar{\lambda}_{\alpha_{11}} \wedge \partial_{\widetilde{r}_{\beta_1}} \wedge \cdots \wedge \partial_{\widetilde{r}_{\beta_{11}}} \prod_{i=1}^{11} M^{\alpha_i}{}_{\beta_i} \ . \tag{4.14}$$

We should stress here that this transformation preserves the pure spinor conditions since $\widetilde{\bar{\lambda}}\gamma^m\widetilde{\bar{\lambda}} = 0$ and $\widetilde{\bar{\lambda}}\gamma^m\widetilde{r} = 0$. Because $M^\alpha{}_\beta = \partial\widetilde{\bar{\lambda}}_\beta/\partial_{\bar{\lambda}_\alpha}$ this Jacobian factor is exactly the one for the transformation

$$\widetilde{\bar{\lambda}}_\alpha = \frac{\bar{\lambda}_\alpha}{(\lambda\cdot\bar{\lambda})^2} \ , \tag{4.15}$$

therefore the measure of integration over the pure spinor ghost with the regulator (4.8) takes the form

$$\int [d\lambda][d\bar{\lambda}][dr]\, e^{-\frac{1}{(\lambda\cdot\bar{\lambda})} - rM\theta} = \int [d\lambda][d\widetilde{\bar{\lambda}}][d\widetilde{r}]\, e^{-\lambda\cdot\widetilde{\bar{\lambda}} - \widetilde{r}\theta} \tag{4.16}$$

which is the original regulator (4.12) introduced by Berkovits in [2] expressed in terms of the inverted variables. This shows that our regulator is making the pure spinor λ massive using $\widetilde{\bar{\lambda}}$ instead of $\bar{\lambda}$.

The massless vertex operators do not depend on the non-minimal variables. This shows that the tree-level amplitudes defined with only three unintegrated vertex operators (4.10) are the same with the regulator (4.8) and the regulator introduced in (4.12) in [2].

Remarking that

$$\begin{aligned} \bar{\lambda}\gamma^{mnp}M^{-1}\widetilde{r} &= (\lambda\cdot\bar{\lambda})^2\,(\bar{\lambda}\gamma^{mnp}\widetilde{r}) \\ \widetilde{r}_{\beta_1}\widetilde{r}_{\beta_2}\,(M^{-1})^{\beta_1}{}_{[\alpha_1}\,(M^{-1})^{\beta_2}{}_{\alpha_2}\,\bar{\lambda}_{\alpha_3]} &= (\lambda\cdot\bar{\lambda})^4\,\widetilde{r}_{[\alpha_1}\widetilde{r}_{\alpha_2}\,\bar{\lambda}_{\alpha_3]} \end{aligned} \tag{4.17}$$

and introducing $s^\alpha = \widetilde{s}^\beta\, M^\alpha{}_\beta$ the $b_{\mathbf{nm}}$-ghost transforms as the non-minimal $b_{\mathbf{nm}}$-ghost of eq. (3.8) transforms as

$$\begin{aligned} b_{\mathbf{nm}} &= \widetilde{s}\partial\widetilde{\bar{\lambda}} + \frac{1}{4}\,\frac{\widetilde{\bar{\lambda}}_\alpha}{\lambda\cdot\widetilde{\bar{\lambda}}}\,\mathbf{b}^\alpha \\ \mathbf{b}^\alpha &\equiv G^\alpha + \frac{\widetilde{r}_\beta}{(\lambda\cdot\widetilde{\bar{\lambda}})}H^{\alpha\beta} + \frac{\widetilde{r}_\beta\widetilde{r}_\gamma}{(\lambda\cdot\widetilde{\bar{\lambda}})^2}\,K^{\alpha\beta\gamma} + \frac{\widetilde{r}_\beta\widetilde{r}_\gamma\widetilde{r}_\delta}{(\lambda\cdot\widetilde{\bar{\lambda}})^3}\,L^{\alpha\beta\gamma\delta} \end{aligned} \tag{4.18}$$

Since the operators G^α, $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\delta}$ do not depend on the non-minimal sector, this shows that this expression is identical to the one in (3.8) and shows the equivalence

of the amplitudes with the insertion of the $b_{\mathbf{nm}}$-ghost. It is important that the $b_{\mathbf{nm}}$-ghost keeps the same functional dependence in the $\widetilde{r}_\alpha$, $\widetilde{s}^\alpha$ and $\widetilde{\bar{\lambda}}_\alpha$ variables are in the r_α, s^α and $\bar{\lambda}_\alpha$ variables.

Because we are not transforming the conjugated ghost w_α and $\bar{w}^\alpha$ and because the measure of integration over these variables bring inverse powers of the pure spinor ghost, we show that this regulator provides divergence free amplitudes that converge at $\lambda, \bar{\lambda} \sim \infty$.

4.3. Regulating the higher-loop amplitudes

The prescription for a genus-g amplitude in this formalism is given by [2]

$$\mathcal{A}_N^g = \int d^{10}x \int d^{6g-6}\tau \left| \int d^{16}\theta \int [d\lambda][d\bar{\lambda}][dr] \prod_{I=1}^{g} \int [dw^I][d\bar{w}^I][ds^I] \prod_{i=1}^{3(g-1)} (\mu_i|b_{\mathbf{nm}})\,\widehat{\mathcal{N}} \right|^2 \prod_{i=1}^{N} V_i \,. \tag{4.19}$$

The integration over the conjugated ghosts is given by

$$\begin{aligned} [dw^I] &= M^{\alpha_1\cdots\alpha_8}_{m_1n_1\cdots m_{10}n_{10}} dN^{m_1n_1\,I}\cdots dN^{m_{10}n_{10}\,I} dJ^I\, \partial_{\lambda^{\alpha_1}}\cdots\partial_{\lambda^{\alpha_8}} \\ [d\bar{w}^I][ds^I] &= \prod_{i=1}^{10} d\bar{N}^I_{m_in_i} \wedge d\bar{J}^I \wedge \prod_{i=1}^{10} \partial_{S^I_{m_in_i}} \wedge \partial_{S^I} \end{aligned} \tag{4.20}$$

where we set $M^{\alpha_1\cdots\alpha_8}_{m_1n_1\cdots m_{10}n_{10}} = (\gamma_{m_1n_1m_2m_3m_4})^{((\alpha_1\alpha_2}(\gamma_{m_5n_5n_2m_6m_7})^{\alpha_3\alpha_4}(\gamma_{m_8n_8n_3n_6m_9})^{\alpha_5\alpha_6}$ $(\gamma_{m_{10}n_{10}n_4n_7n_9})^{\alpha_7\alpha_8))}$ and $((\cdots))$ means that one considers the symmetrized γ-traceless part.

In order to regulate the integration over the zero mode of the conjugated ghosts we make the following choice

$$\begin{aligned} \mathcal{O}_I^{mn} &= (w_I\gamma^{mn}\lambda) \\ \mathcal{O}_I &= (w_I\lambda)\,. \end{aligned} \tag{4.21}$$

The zero modes are defined by integration over the homology a-cycles $\Phi^I \equiv \oint_{a_I} \Phi$ for $1 \le I \le g$.

The associated regulator

$$
\begin{aligned}
\mathcal{N} &= \exp\left[Q_{\mathbf{nm}}, \Psi\right] \\
&= \exp\Big[-\frac{1}{\lambda\cdot\bar\lambda} - \frac{r\cdot\theta}{(\lambda\cdot\bar\lambda)^2} + 2\frac{(\bar\lambda\cdot\theta)(r\cdot\lambda)}{(\lambda\cdot\bar\lambda)^2}\Big] \\
&\times \exp\Big[-\sum_{I=1}^{g}\Big(\frac{1}{2}N^I_{mn}\bar N^{I\,mn} + J_I\bar J^I\Big)\Big] \\
&\times \exp\Big[-\sum_{I=1}^{g}\frac{1}{4}S^I_{mn}(d^I\gamma^{mn}\lambda) + S^I(\lambda d^I)\Big]
\end{aligned}
\tag{4.22}
$$

The third and fourth line are the Λ and L-gauge invariant version of the regulator $\exp(-w\cdot\bar w - s\cdot d)$.

4.4. Zero mode counting in the non-minimal formalism

For having a non vanishing massless n-point genus g amplitude one needs to satisfy the fermionic zero modes constrains given by the following equations

$$
\begin{aligned}
11g &= n_{ds} + n_{s\partial\bar\lambda} \\
11 &= n_{r\theta} + n_{rd^2} + n_{rd^0} + 2n_{r^2d} + 3n_{r^3d^0} \\
16g &= n_{ds} + n_{d\,vop} + n_{r^0d} + 2n_{rd^2} + n_{r^2d} \\
3(g-1) &= n_{s\partial\bar\lambda} + n_{r^0d} + n_{rd^2} + n_{rd^0} + n_{r^2d} + n_{r^3d^0}
\end{aligned}
\tag{4.23}
$$

where n_{ds} is the number of $S\lambda d$ contributions from the regulator, $n_{r\theta}$ is the number of $r\cdot\theta$ contributions from the regulator, $n_{d\,vop}$ is the number of d contributions from the vertex operators and $n_{s\partial\bar\lambda}$ and $n_{r^id^j}$ with $(i,j)\in\{(0,1),(1,0),(1,2),(2,1),(3,0)\}$ are the various contribution from the $b_{\mathbf{nm}}$-ghost.

The d-zero mode constraint implies that

$$
2g = n_{d\,vop} + n_{rd^2} - n_{r^3d^0} - 3 - n_{s\partial\bar\lambda} \tag{4.24}
$$

Since $n_{rd^2}\leq 11$ and $n_{r^3d^0}\geq 0$ we deduce that this system of equation does not have a solution after genus

$$
g > \frac{1}{2}\,n_{d\,vop} + 4\ . \tag{4.25}
$$

An n-point massless amplitude would vanish for all genus $g\geq 5+n/2$ if there are no singularities in the pure spinor integration.

With the $1/(\lambda\cdot\bar\lambda)$ regulator introduced in the previous section the integration over the pure spinor ghost λ and $\bar\lambda$ behaves as

$$I_{\lambda,\bar\lambda} = \int_0^\infty \frac{d\lambda d\bar\lambda}{\lambda\cdot\bar\lambda}\,\frac{1}{(\lambda\cdot\bar\lambda)^{n_{r\theta}}}\,\left(\frac{\lambda\cdot\partial\bar\lambda}{\lambda\cdot\bar\lambda}\right)^{n_{s\partial\bar\lambda}}\,e^{-\frac{1}{\lambda\cdot\bar\lambda}} \tag{4.26}$$

the q_1 and q_2 invariance of the $b_{\mathbf{nm}}$-ghost implies that $n_{r\theta} + n_{s\partial\bar\lambda} \geq 1$ and these integrals are converging both at $\lambda\cdot\bar\lambda = 0$ and $\lambda\cdot\bar\lambda \sim \infty$.

Therefore unless there are extra sources of d-zero modes the theory cannot be unitary. We will present a solution to this problem in section 4.5 below.

The heat kernel regulator introduced in [3,14] provides extra sources of d- and s-zero mode in the multiloop amplitudes from the contribution (see equations (3.20) and (3.29) of [14])

$$\int d^{11}f d^{11}\bar f d^{11}g d^{11}\bar g\, e^{\sum_{I=1}^{g}(f^\alpha w_{\alpha\,I} + g^\alpha d_{\alpha\,I} + \bar f_\alpha \bar w_I^\alpha + \bar g_\alpha s_I^\alpha)}\ . \tag{4.27}$$

This extension of the regulator can provide $n_{\bar g s}$ extra s-zero mode and n_{gd} extra d-zero modes contributions to the counting in (4.23)

$$\begin{aligned} 11g &= n_{ds} + n_{s\partial\bar\lambda} + n_{\bar g s} \\ 16g &= n_{ds} + n_{d\,vop} + n_{r^0 d} + 2n_{rd^2} + n_{r^2 d} + n_{gd} \end{aligned} \tag{4.28}$$

leading to the d-zero mode counting

$$2g = n_{d\,vop} + n_{rd^2} - n_{r^3 d^0} - 3 - n_{s\partial\bar\lambda} + n_{gs} - n_{\bar g s} \tag{4.29}$$

Since $n_{\bar g s} \geq 0$ and $n_{g\,d} \leq 11$ we deduce that this system of equation does not have a solution after genus

$$g > \frac{1}{2}\,n_{d\,vop} + \frac{19}{2}\ . \tag{4.30}$$

In particular the heat kernel regulated four-graviton amplitudes of [3,14] will be vanishing after genus $g \geq 12$, which is not compatible with unitarity. So we conclude on the basis of zero mode saturation that the heat kernel regularisation does not lead to a consistent framework for the non-minimal pure spinor theory.

4.5. Adding d-zero mode contributions

Of course, one can wonder if the entire formalism is consistent or not, but we would like to stress that passing from the minimal to the non-minimal formalism is not straightforward due to the poles in $\lambda\cdot\bar\lambda = 0$ and because $\bar\lambda$ is treated as the conjugate to λ. So, we believe that a suitable gauge fixing could make the formalism working at any genus.

In order to resolve the issue of the vanishing of the amplitudes because of the impossibility of saturating all the d-zero modes after some genus order, we introduce the following piece to the gauge fermion

$$\widehat{\Psi} = \Psi + \sqrt{\alpha'} \sum_{1\leq I,J\leq g} S^I_{mn}\,(d^I\gamma^{mnp}d^J)\,P_{p\,J} \tag{4.31}$$

which modifies the regulator as

$$\begin{aligned}
&\widehat{\mathcal{N}} = \mathcal{N}\times\mathcal{N}_d\\
&\mathcal{N}_d = \exp\Big[-\sqrt{\alpha'}\sum_{1\leq I,J\leq g}\bar{N}^{mn}_I\,(d^I\gamma_{mnp}d^J)\,P^p_J\Big]\\
&\qquad\times\exp\Big[-\sqrt{\alpha'}\sum_{1\leq I,J\leq g}S^{mn}_I\,\Big(P^I_s\,(\lambda\gamma^s\gamma_{mnp}d^J)\,P^p_J+(d^I\gamma_{mnp}\gamma^s\lambda)\,P^J_sP^p_J\Big)\Big]
\end{aligned} \tag{4.32}$$

With this addition to the regulator the d-zero mode counting in the n-point amplitude at genus order $g > 4 + n/2$ can be satisfied by picking $g - (4 + n/2)$ contributions of $N^{mn}_I\,(d^I\gamma_{mnp}d^J)\,P^p_J$.

Under the change of variables $\bar{\lambda}\to\widetilde{\bar{\lambda}}$ of eq. (4.15), the extension of the gauge fermion in (4.31) transforms as

$$\delta\Psi = -\sqrt{\alpha'}\sum_{1\leq I,J\leq g}\widetilde{S}^I_{mn}\,(d^I\gamma^{mnp}d^J)\,P_{p\,J} \tag{4.33}$$

where $\widetilde{S}^I_{mn} = \widetilde{r}^I\gamma_{mn}\widetilde{\bar{\lambda}}$. But only the second line of the regulator $\mathcal{N}_d$ is invariant. This implies that this extension of the regulator makes a difference between the non-minimal formalism regulated with a mass $\lambda\cdot\bar{\lambda}$ introduced in [2] or $\lambda\cdot\widetilde{\bar{\lambda}}$ used here.

We could not justify this extension by a first principle derivation. The difficulty of saturating d-zero mode at higher loop could be related to a background charge screening constraint which is not immediately visible, except because of vanishing amplitudes, due to the gauge fixed definition of the pure spinor formalism.

5. Multigraviton amplitudes at higher-loop

The closed string massless vertex operators is defined as [2,3]

$$V = \int d^2z\,\Big(G_{MN}(X)\partial X^M\bar{\partial}X^N + W^{\alpha\beta}\,d_\alpha\widehat{d}_\beta + \cdots\Big) \tag{5.1}$$

where $X^M = (x^m, \theta^\alpha, \widehat{\theta^\alpha})$, the symmetric part of $G_{(MN)}$ is the graviton superfield and the antisymmetric part $G_{[MN]}$ is the NS B-field superfield. $W^{\alpha\beta}(x, \theta^\alpha, \widehat{\theta^\alpha})$ is the dimension one gauge-invariant superfield whose lowest component is the Ramond-Ramond field strength.

The zero modes saturation of a n_{grav}-graviton amplitude at genus $g \geq 2$ leads to

$$\begin{aligned}
\mathcal{A}_N^g &= \int d^{10}x \int \Big| d^{16}\theta \int [d\lambda][d\bar\lambda][dr] \prod_{I=1}^{g} [dw^I][d\bar w^I][ds^I] \prod_{i=1}^{3g-3} (\mu_i | b_{\mathbf{nm}})\, \widehat{\mathcal{N}} \Big|^2 V^{n_{grav}} \\
&\sim \int d^{10}x \int \Big| d^{16}\theta \int [d\lambda][d\bar\lambda][dr] \prod_{I=1}^{g} [dw^I][d\bar w^I][ds^I]\, e^{-\frac{1}{\lambda\cdot\bar\lambda} - \sum_{I=1}^{g} N_I \cdot \bar N^I} \\
&\quad \times \Big(\frac{\check r \theta}{\lambda\cdot\bar\lambda}\Big)^{n_{r\theta}} \Big(S\lambda d\Big)^{n_{sd}} \Big(\bar N d^2 P\Big)^{n_{d^2P}} \Big(\partial\theta A + \Pi B + dW + NF\Big)^{n_{grav}} \\
&\quad \times \Big(s\partial\bar\lambda\Big)^{n_s} \Big(\check\lambda \Pi d\Big)^{n_{r^0d}} \Big(\check\lambda\, \check r\, d^2\Big)^{n_{rd^2}} \Big(\check\lambda\, \check r\, N\Pi\Big)^{n_{rd^0}} \Big(\check\lambda \check r^2 dN\Big)^{n_{r^2d}} \Big(\check\lambda \check r^3 N^2\Big)^{n_{r^3d^0}} \Big|^2
\end{aligned} \tag{5.2}$$

where we made use of the variables (3.7). We have schematically written down all possible terms coming from the regulator $\widehat{\mathcal{N}}$ and the $b_{\mathbf{nm}}$-ghost using the notations of eq. (3.7). When $n_{r\theta}$ is non zero the contribution is given by an integrations over a subspace of the θ-superspace but when $n_{r\theta} = 0$ this is a full superspace integral. The various powers in (5.2) satisfy the constraint

$$3g - 3 = n_s + \sum_{i=0}^{4} n_i \,, \tag{5.3}$$

that there are $3g-3$ insertions of the (left-moving) $b_{\mathbf{nm}}$-ghost. The saturation of the $11g$ s^α-zero modes, the $16g$ d_α-zero modes, and the 11 r_α-zero mode gives

$$\begin{aligned}
s: &\quad 11g = n_s + n_{sd} - n_{s,r} \\
d: &\quad 16g = n_{sd} + 2n_{d^2P} + n_{r^0d} + 2n_{rd^2} + n_{r^2d} + n_{grav} \\
r: &\quad 11 = n_{r\theta} + n_{rd^2} + n_{rd^0} + 2n_{r^2d} + 3n_{r^3d^0} - n_{s,r} \,.
\end{aligned} \tag{5.4}$$

where $n_{s,r}$ is the number of contractions between the s-ghost and the r-ghost.

5.1. The four-graviton amplitude at higher-genus $g \leq 6$

For the case of the four-graviton amplitude, with $n_{grav} = 4$, the previous conditions have the following solution valid until genus $g \leq 6$ [4]

$$\begin{aligned}
&n_{r\theta} = 12 - 2g, \quad n_{sd} = 11g, \quad n_{rd^2} = 2g - 1, \quad n_{r^0d} = g - 2 \\
&n_s = n_{rd^0} = n_{r^2d} = n_{r^3d^0} = n_{d^2P} = n_{s,r} = 0 \,,
\end{aligned} \tag{5.5}$$

which corresponds to the partial superspace integral when $n_{r\theta} = 12 - 2g \neq 0$ giving the leading contribution to the low-energy limit of the string amplitude [3,4]

$$\mathcal{A}_4^g \sim \int d^{16}\theta d^{16}\bar{\theta}\theta^{12-2g}\bar{\theta}^{12-2g}(W_{\alpha\beta})^4 \times I^g \sim (\alpha'\partial^2)^g R^4\, I^g + O(\alpha' k^2)\ . \tag{5.6}$$

where I^g is a field theory integral which is the low-energy energy of the expression arising from the integration over the moduli.

For this case the good convergence properties over the spinor variables allowed to perform the change of variables $\lambda \to \widetilde{\widetilde{\lambda}}$ of eq. (4.15) and use the BRST invariance to set $\mathcal{N}_d = 1$. By using the same steps as in section 4.2 we can map our amplitude computation to the one in [4] leading to identical results.

For the solution (5.5) the form of the integrand is given by

$$I^g = \int_{\Sigma_g} \prod_{i=1}^{4} \int d^2 z_i e^{ik^i\cdot x}(z_i) \left| \prod_{i=1}^{3g-3} d^2 y \mu(y_i) \prod_{j=1}^{g-2} \Pi(y_j) \right|^2 \tag{5.7}$$

The expression involves $2(g-2)$ insertions of the supersymmetric loop momenta $\Pi^m \sim \partial x^m + (\theta\gamma^m\partial\theta)/2$ flowing through the loops. The field theory limit of this amplitude in ten dimensions has $3g - 3 + 4 = 3g + 1$ propagators, and $2(g-2)$ are loop momentum contracted between themselves or to external polarisation or some of the explicit external momenta in (5.6). The resulting integral has mass dimension $(D-4)g - 6$ as it should be by dimensional analysis. Such an expression displays the explicit superficial ultra-violet behaviour of the amplitude.

5.2. The four-graviton amplitudes at higher-genus $g \geq 7$

At genus $g \geq 7$ the massless four-point amplitude can develop divergences in the pure spinor integration at the tip of the cone $\lambda \cdot \bar{\lambda} \sim 0$ [4], and the change of variables $\lambda \to \widetilde{\widetilde{\lambda}}$ of eq. (4.15) is not allowed. As well because of the potential divergences in the pure spinor integration we cannot use the BRST invariance to set $\mathcal{N}_d = 1$. We will see that this extra contribution to the $\mathcal{N}_d$ regulator will bring extra d-zero mode allowing the saturate the fermionic zero mode after $g \geq 7$. Because the new contributions to the regulator come with one power of α' we want to minimize the number of terms coming from this modification of the regulator to get the leading contribution to the low-energy limit of the amplitude. This is accomplished by the solution parametrized

$$\begin{aligned} &n_s = 1, \quad n_{r^0 d} = 3g - 14, \quad n_{r d^2} = 12, \quad n_{r d^0} = n_{r^2 d} = n_{r^3 d^0} = 0, \\ &n_{r\theta} = 0, \quad n_{sd} = 11g, \quad n_{d^2 P} = g - 6\ . \end{aligned} \tag{5.8}$$

where we have taken $n_{rd^2} > 11$ r-zero mode from the $b_{\mathbf{nm}}$-ghost as required by the invariance under the charges (4.11).

This expression leads to a low-energy expansion of the four-graviton amplitude in ten dimensions

$$\mathcal{A}_4^g \sim (\alpha')^{g-6} \int d^{16}\theta d^{16}\bar{\theta}(W_{\alpha\beta})^4 \times \tilde{I}^g \sim (\alpha')^g \partial^{12} R^4 \, \tilde{I}^g + O(\alpha' k^2) \ . \tag{5.9}$$

where now I^g is

$$\tilde{I}^g = \int_{\Sigma_g} \prod_{i=1}^{4} \int d^2 z_i e^{ik^i \cdot x}(z_i) \left| \prod_{i=1}^{3g-3} d^2 y \mu(y_i) \prod_{j=1}^{g-2} \Pi(y_j) \prod_{I=1}^{g-6} \oint_{a_I} \Pi \right|^2 \tag{5.10}$$

because this expression contains $2g-8$ powers the supersymmetric loop momenta running in the loop, this expression has mass dimension $(D-2)g-18$ and taking into account the dimension twenty operator $\partial^{12}R^4$ multiplying the amplitude the total amplitude has mass dimension $(D-2)g+2$. This confirms this is the leading contribution to low-energy limit of the four-graviton amplitude in ten dimensions.

In the extreme case that all the $g-6$ powers of loop momenta from the regulators are contracted with plane-wave factors, the amplitude with have an extra factor of $2(g-6)$ powers of external momenta and will behaves as

$$\mathcal{A}_4^g \sim \int_{\Sigma_g} \left| \prod_{i=1}^{3g-3} d^2 y \mu(y_i) \prod_{j=1}^{g-2} \Pi(y_j) \right|^2 \alpha'^g \partial^{2g} R^4 + O(\alpha' k^2) \tag{5.11}$$

For this contribution to be the leading low-energy limit of the g-loop four-graviton amplitude at genus order $g > 6$ many cancellations within the integrals (5.10) beyond the supersymmetric ones must take place. They could be the consequence of the extra cancellations detailed in [11,12] occurring in the on-shell colorless amplitudes.

5.3. Vanishing of $N < 4$-point amplitudes

Since the regulator (4.32) or the regularized b_ϵ-ghost of [3] bring an arbitrary number of d-zero modes one needs to make sure that all massless N-point amplitudes with $N < 4$ vanish to all order in perturbation. The vanishing of the $N < 2$-point amplitudes imply by factorisation and the absence of unphysical singularities in the amplitude, the finiteness of string perturbation [23,24,25,26]. The vanishing of the 3-point amplitude at higher genus

is not necessary for the finiteness of string perturbation but is required for the absence of IR singularities when taking the low-energy limit of string amplitudes.

It was shown in [1] that in the minimal pure spinor formalism all the $N < 4$-point amplitudes vanish to all order in perturbation.

The vanishing of the vacuum diagram is ensured by the integration over the sixteen left-moving and right-moving superspace variables. For the following argument we will assume that all the vertex operators are unintegrated. The vanishing of the 1-point amplitude is a consequence of the on-shell relation. At most the integrand can bring 11 powers of θ and the amplitude takes the form $\int |d^{16}\theta\theta^{11}|V_1$ where $V_1 = |(\lambda\gamma^m\theta)a_m(x) + (\lambda\gamma^m\theta)(\theta\gamma_m\chi) + \cdots|^2$ is a massless vertex operators where the ellipsis are for higher-derivative contributions. But one-point on-shell amplitudes have $k_1 = 0$ and all higher order term in V_1 drops out and the integral vanishes after integration over the θ variables. The vanishing of the two-point amplitude follows the same argument that the integration over the superspace θ-variables leads to contributions that vanish on-shell because there is only one on-shell independent momentum.

For the case of the massless three-point function we find that using the original regulator (4.32) that the zero mode constraint can be satisfied for all genus from $g \geq 3$. But we will show that because all the contribution have more than two-derivative (there is no renormalisation of the Planck mass) the on-shell condition assure the vanishing of these amplitudes. For the massless three-point amplitude momentum conservation $k_1 + k_2 + k_3 = 0$ and the on-shell conditions $k_1^2 = k_2^2 = k_3^2 = 0$ imply that $k_i \cdot k_j = 0$ for all $i, j = 1, 2, 3$. At genus 3 we have the contribution $n_{rd^2} = 6$, $n_{r\theta} = 5$ $n_{sd} = 33$ and all the other integers being zero and three $d_\alpha W^\alpha$ from the vertex operators. In the case one picks the 11 r-zero mode from the regulator one gets

$$\int |d^{16}\theta\theta^{11}|^2 V_1 V_2 V_3 \sim k^2 \, \hat{R}^3 + \cdots \tag{5.12}$$

which means that one must distribute two momenta on three powers of linearized Riemann tensor $\hat{R}_{mnpq} = k_{[m}\,\zeta_{n][p}\,k_{q]}$. This vanishes by the on-shell conditions. In the case where there is no contributions of r-zero mode from the regulator one get and amplitudes of the type

$$\int |d^{16}\theta|^2 V_1 V_2 V_3 \sim k^{13} \hat{R}^3 + \cdots \tag{5.13}$$

which has more powers of momenta to contract and this vanished after using the on-shell conditions. The same conclusion is reached to the contribution involving the supersymmetric partner of the graviton. This show that the massless 3-point amplitude vanish to all order in perturbation.

We hope that our considerations help to a better understanding of this intricate and interesting new field. Higher-loop and multileg computations are important for several checks in string perturbation theory and beyond, but in addition, they are needed test of the soundness of the formalism.

Acknowledgements: We would like to thank Nathan Berkovits, Michael Green, Alberto Lerda, Massimo Porrati and Stefan Theisen for useful discussions and comments on a preliminary version of this draft. We would like to thank the organizers of the KITP workshop "Fundamental Aspects of Superstring Theory" for providing a stimulating atmosphere where most of this work has been done. This research was supported in part (PV) by the ANR grant BLAN06-3-137168. This research was supported in part by the National Science Foundation under Grant No. PHY05-51164.

Appendix A. The tree-level amplitude

We consider the general form of the regulator

$$\Psi = \bar{\lambda}_\alpha \theta^\alpha f(\lambda \cdot \bar{\lambda}) \tag{A.1}$$

where f is a real function. With this choice of gauge fermion we have the following regulator

$$\widehat{\mathcal{N}} = \exp\left(-(\lambda \cdot \bar{\lambda})\, f(\lambda \cdot \bar{\lambda}) + r_\alpha M^\alpha{}_\beta \theta^\beta\right) \tag{A.2}$$

where $M^\alpha{}_\beta = \delta^\alpha{}_\beta\, f(\lambda \cdot \bar{\lambda}) + \lambda^\alpha \bar{\lambda}_\beta\, f'(\lambda \cdot \bar{\lambda})$. With this regulator we evaluate the tree-level integral

$$\langle 0|C\rangle = \int d^{16}\theta \int [d\lambda][d\bar{\lambda}][dr]\, \widehat{\mathcal{N}}\, (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \tag{A.3}$$

By performing the integration over the 11 r variables and using that $(\bar{\lambda}\cdot\theta)^2 = 0$ we get

$$\begin{aligned}\langle 0|C\rangle = \int d^{16}\theta \int [d\lambda] d\bar{\lambda}_{\alpha_1} \wedge \cdots \wedge d\bar{\lambda}_{\alpha_{11}}\, e^{-(\lambda\cdot\bar{\lambda})\, f(\lambda\cdot\bar{\lambda})}\, (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\times \\ \times\, f(\lambda\cdot\bar{\lambda})^{10}\, \theta^{\alpha_1}\cdots\theta^{\alpha_{10}}\theta^\sigma \left(\delta^{\alpha_{11}}_\sigma\, f(\lambda\cdot\bar{\lambda}) + 11\, \bar{\lambda}_\sigma\, \lambda^{\alpha_{11}} f'(\lambda\cdot\bar{\lambda})\right)\ .\end{aligned} \tag{A.4}$$

Performing the integration over the sixteen θ variables leads to

$$\begin{aligned}\langle 0|C\rangle = \int [d\lambda] d\bar\lambda_{\alpha_1} \wedge \cdots \wedge d\bar\lambda_{\alpha_{11}}\, e^{-(\lambda\cdot\bar\lambda)\, f(\lambda\cdot\bar\lambda)}\, (\gamma^m\lambda)_{r_1}(\gamma^n\lambda)_{r_2}(\gamma^p\lambda)_{r_3}(\gamma_{mnp})_{r_4r_5} \times \\ \times\, f(\lambda\cdot\bar\lambda)^{10}\, \epsilon_{16}^{\alpha_1\cdots\alpha_{10}\sigma r_1\cdots r_5}\, \big(\delta_\sigma^{\alpha_{11}}\, f(\lambda\cdot\bar\lambda) + 11\,\bar\lambda_\sigma\, \lambda^{\alpha_{11}} f'(\lambda\cdot\bar\lambda)\big)\end{aligned} \tag{A.5}$$

Using the properties of the pure spinor measure

$$\begin{aligned}&[d\lambda](\gamma^m\lambda)_{r_1}(\gamma^n\lambda)_{r_2}(\gamma^p\lambda)_{r_3}(\gamma_{mnp})_{r_4r_5} = \epsilon_{16\, r_1\cdots r_5\gamma_1\cdots\gamma_{11}} d\lambda^{\gamma_1}\cdots d\lambda^{\gamma_{11}} \\ &[d\lambda]\lambda^\alpha\lambda^\beta\lambda^\gamma\,\lambda^\delta = 4\,\epsilon_{16\, r_1\cdots r_5\gamma_1\cdots\gamma_{11}} d\lambda^{\gamma_1}\cdots d\lambda^{\gamma_{11}}\, \lambda^{(\alpha}(T^{-1})^{\beta\gamma\delta)r_1\cdots r_5}\end{aligned} \tag{A.6}$$

and the relation $\epsilon_{16r_1\cdots r_{16}}\epsilon_{16}^{s_1\cdots s_{16}} = 16!\,\delta^{s_1\cdots s_{16}}_{r_1\cdots r_{16}}$ we get that

$$\begin{aligned}\langle 0|C\rangle = 11!5! \int d\lambda^{\alpha_1}\wedge\cdots\wedge d\lambda^{\alpha_{11}} d\bar\lambda_{\alpha_1}\wedge\cdots\wedge d\bar\lambda_{\alpha_{11}}\, e^{-(\lambda\cdot\bar\lambda)\, f(\lambda\cdot\bar\lambda)}\, f(\lambda\cdot\bar\lambda)^{11} \\ + 11\,16!\,4 \int d\lambda^{\alpha_1}\wedge\cdots\wedge d\lambda^{\alpha_{11}} d\bar\lambda_{\alpha_1}\wedge\cdots\wedge d\bar\lambda_{\alpha_{11}}\, e^{-(\lambda\cdot\bar\lambda)\, f(\lambda\cdot\bar\lambda)} \times \bar\lambda_\sigma f'(\lambda\cdot\bar\lambda)\, f(\lambda\cdot\bar\lambda)^{10} \\ \times\, \lambda^{(\alpha_{11}}(T^{-1})^{\beta\gamma\delta)s_1\cdots s_5}\, T_{(\alpha\beta\gamma)r_1\cdots r_5}\, \delta^{\gamma_1\cdots\gamma_{11}}_{\alpha_1\cdots\alpha_{11}}\delta^{\alpha_1\cdots\alpha_{10}\sigma r_1\cdots r_5}_{\gamma_1\cdots\gamma_{11}s_1\cdots s_5}\end{aligned} \tag{A.7}$$

Using that $(T^{-1})^{(\alpha\beta[\gamma)r_1\cdots r_5]} = 0$ we find that $\alpha_{11} = \sigma$ in the last term, leading to

$$\langle 0|C\rangle = 11!5! \int \prod_{i=1}^{11} d\lambda^{\alpha_i} d\bar\lambda_{\alpha_i}\, e^{-(\lambda\cdot\bar\lambda)\, f(\lambda\cdot\bar\lambda)}\, f(\lambda\cdot\bar\lambda)^{10}\, (f(\lambda\cdot\bar\lambda) + f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)) \tag{A.8}$$

Setting $h(\lambda\cdot\bar\lambda) = (\lambda\cdot\bar\lambda) f(\lambda\cdot\bar\lambda)$ this gives

$$\langle 0|C\rangle = 11!5! \int \prod_{i=1}^{11} d\lambda^{\alpha_i} d\bar\lambda_{\alpha_i}\; (\lambda\cdot\bar\lambda)^{-10}\, e^{-h(\lambda\cdot\bar\lambda)}\, h(\lambda\cdot\bar\lambda)^{10}\, h'(\lambda\cdot\bar\lambda) \tag{A.9}$$

▷ We give another derivation of the same result using some Fierz identities derived in [27].

We use the following definition for the normalisations

$$\int d^{16}\theta \int [d\lambda][d\bar\lambda][dr]\, e^{-\lambda\cdot\bar\lambda\, f(\lambda\cdot\bar\lambda) - rM\theta} \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(x,\theta) = \langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(x,\theta)\rangle \tag{A.10}$$

and the Fierz identity established in [27]

$$\begin{aligned}&\int d^{16}\theta \int [d\lambda][d\bar\lambda][dr]\, e^{-\lambda\cdot\bar\lambda\, f(\lambda\cdot\bar\lambda) - rM\theta} \lambda^\alpha\lambda^\beta\lambda^\gamma \bar\lambda_\epsilon f^\epsilon_{\alpha\beta\gamma}(x,\theta) \\ &= \frac{(\lambda\cdot\bar\lambda)}{33}\, \Big(8\langle \lambda^{(\alpha}\lambda^\beta\lambda^\gamma f^{\delta)}_{\alpha\beta\gamma\delta}\rangle - \langle (\lambda\gamma^m)_\epsilon \gamma_m^{(\alpha\beta}\lambda^\gamma\lambda^{\delta)} f^\epsilon_{\alpha\beta\gamma\delta}\rangle\Big)\end{aligned} \tag{A.11}$$

The amplitude in (A.4) takes the form

$$\begin{aligned}\langle 0|C\rangle = \int d^{16}\theta \int [d\lambda][d\bar\lambda][dr]\, e^{-(\lambda\cdot\bar\lambda)\, f(\lambda\cdot\bar\lambda)}\, f(\lambda\cdot\bar\lambda)^{10}\,(r\theta)^{10}\,\times \\ \times \big(f(\lambda\cdot\bar\lambda)\,(r\cdot\theta)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \\ +11\,(\bar\lambda\cdot\theta)\,(r\cdot\lambda) f'(\lambda\cdot\bar\lambda)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\big)\end{aligned} \tag{A.12}$$

The first identity (A.10) gives

$$\langle 0|C\rangle_1 = \langle (r\cdot\theta)^{11}\, f(\lambda\cdot\bar\lambda)^{11}\,(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle \tag{A.13}$$

the second identity (A.11) on the second line with $f^{\epsilon}_{\alpha\beta\gamma\delta} = \theta^\epsilon\, r_\delta\,(\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\gamma^p\theta)_\gamma(\theta\gamma_{mnp}\theta)$ leads to

$$\begin{aligned}\langle 0|C\rangle_2 &= \frac{2}{3}\,\langle (r\cdot\theta)^{11}\, f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle \\ &- \frac{1}{3}\langle (r\cdot\theta)^{10} f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma^s\theta)\, r_\delta\gamma_s^{(\alpha\beta}\lambda^\gamma\lambda^{\delta)}\,(\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\gamma^p\theta)_\gamma(\theta\gamma_{mnp}\theta)\rangle \\ &= \frac{2}{3}\,\langle (r\cdot\theta)^{11}\, f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle \\ &- \frac{1}{6}\langle (r\cdot\theta)^{10} f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma^s\theta)\,(r\gamma_s\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle\end{aligned} \tag{A.14}$$

where we used that $(\lambda\gamma_s\theta)(\lambda\gamma^p\theta)(\theta\gamma^{mns}\theta)(\theta\gamma_{mnp}\theta) = 0$. This expression can be reduced further to

$$\begin{aligned}\langle 0|C\rangle_2 &= \frac{1}{2}\,\langle (r\cdot\theta)^{11}\, f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle \\ &- \frac{1}{6}\langle (r\cdot\theta)^{10} f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma_s\theta)\,(r\gamma^{sm}\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle\end{aligned} \tag{A.15}$$

Using the Fierz using that $3!\,16\,\theta_\alpha\theta_\beta = (\theta\gamma_{abc}\theta)(\gamma^{abc})_{\alpha\beta}$ one shows that

$$(r\gamma^{sm}\theta)(\lambda\gamma_s\theta) = 4(r\theta)(\lambda\gamma^m\theta) + (\lambda\gamma^m\theta)(r\theta) \tag{A.16}$$

And the total amplitude takes the form

$$\begin{aligned}\langle 0|C\rangle &= \langle (r\cdot\theta)^{11}\, f(\lambda\cdot\bar\lambda)^{11}(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle \\ &+ \langle (r\cdot\theta)^{11}\, f(\lambda\cdot\bar\lambda)^{10} f'(\lambda\cdot\bar\lambda)\,(\lambda\cdot\bar\lambda)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle\end{aligned} \tag{A.17}$$

which reproduces (A.9) after integration over the r and the θ variables.

References

[1] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[2] N. Berkovits, "Pure spinor formalism as an N = 2 topological string," JHEP **0510**, 089 (2005) [arXiv:hep-th/0509120].

[3] N. Berkovits and N. Nekrasov, "Multiloop superstring amplitudes from non-minimal pure spinor formalism," JHEP **0612**, 029 (2006) [arXiv:hep-th/0609012].

[4] N. Berkovits, "New higher-derivative R**4 theorems," Phys. Rev. Lett. **98**, 211601 (2007) [arXiv:hep-th/0609006].

[5] N. Berkovits, "ICTP lectures on covariant quantization of the superstring," arXiv:hep-th/0209059.

[6] M. B. Green, J. G. Russo and P. Vanhove, "Non-renormalisation conditions in type II string theory and maximal supergravity," JHEP **0702**, 099 (2007) [arXiv:hep-th/0610299];

[7] M. B. Green, J. H. Schwarz and L. Brink, "N=4 Yang-Mills And N=8 Supergravity As Limits Of String Theories," Nucl. Phys. B **198**, 474 (1982).

[8] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, "Three-Loop Superfiniteness of N=8 Supergravity," Phys. Rev. Lett. **98**, 161303 (2007) [arXiv:hep-th/0702112].
Z. Bern, L. J. Dixon and R. Roiban, "Is N = 8 supergravity ultraviolet finite?," Phys. Lett. B **644**, 265 (2007) [arXiv:hep-th/0611086].
Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of N=8 Supergravity," arXiv:0808.4112 [hep-th].
Z. Bern, J. J. M. Carrasco and H. Johansson, "Progress on Ultraviolet Finiteness of Supergravity," arXiv:0902.3765 [hep-th].

[9] M. B. Green, J. G. Russo and P. Vanhove, "Ultraviolet properties of maximal supergravity," Phys. Rev. Lett. **98**, 131602 (2007) [arXiv:hep-th/0611273].

[10] Z. Bern, J. J. Carrasco, D. Forde, H. Ita and H. Johansson, "Unexpected Cancellations in Gravity Theories," arXiv:0707.1035 [hep-th].

[11] N. E. J. Bjerrum-Bohr and P. Vanhove, "Absence of Triangles in Maximal Supergravity Amplitudes," JHEP **0810**, 006 (2008) [arXiv:0805.3682 [hep-th]].

[12] S. Badger, N. E. J. Bjerrum-Bohr and P. Vanhove, "Simplicity in the Structure of QED and Gravity Amplitudes," arXiv:0811.3405 [hep-th].

[13] N. Arkani-Hamed, F. Cachazo and J. Kaplan, "What is the Simplest Quantum Field Theory?," arXiv:0808.1446 [hep-th].

[14] Y. Aisaka and N. Berkovits, "Pure Spinor Vertex Operators in Siegel Gauge and Loop Amplitude Regularization" arXiv: arXiv:0903.3443

[15] P. A. Grassi and G. Policastro, "Super-Chern-Simons theory as superstring theory," arXiv:hep-th/0412272.

[16] N. A. Nekrasov, "Lectures on curved beta-gamma systems, pure spinors, and anomalies," arXiv:hep-th/0511008.

[17] L. Anguelova, P. A. Grassi and P. Vanhove, "Covariant one-loop amplitudes in D = 11," Nucl. Phys. B **702**, 269 (2004) [arXiv:hep-th/0408171].

[18] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, "Superstrings and WZNW models," arXiv:hep-th/0402122.

[19] I. Oda and M. Tonin, "Y-formalism and b ghost in the Non-minimal Pure Spinor Formalism of Superstrings," Nucl. Phys. B **779**, 63 (2007) [arXiv:0704.1219 [hep-th]].

[20] W. Siegel and H. d. Feng, "Gauge-covariant vertex operators," Nucl. Phys. B **683**, 168 (2004) [arXiv:hep-th/0310070].

[21] B. Zwiebach, "Closed string field theory: Quantum action and the B-V master equation," Nucl. Phys. B **390**, 33 (1993) [arXiv:hep-th/9206084].

[22] P. A. Grassi and L. Tamassia, "Vertex operators for closed superstrings," JHEP **0407**, 071 (2004) [arXiv:hep-th/0405072].

[23] J. J. Atick, J. M. Rabin and A. Sen, "An Ambiguity In Fermionic String Perturbation Theory," Nucl. Phys. B **299**, 279 (1988).
J. J. Atick, G. W. Moore and A. Sen, "Some Global Issues In String Perturbation Theory," Nucl. Phys. B **308**, 1 (1988).
J. J. Atick, G. W. Moore and A. Sen, "Catoptric Tadpoles," Nucl. Phys. B **307**, 221 (1988).

[24] E. J. Martinec, "Nonrenormalization Theorems And Fermionic String Finiteness," Phys. Lett. B **171**, 189 (1986).

[25] O. Lechtenfeld, "Factorization And Modular Invariance Of Multiloop Superstring Amplitudes In The Unitary Gauge," Nucl. Phys. B **338**, 403 (1990).
O. Lechtenfeld and A. Parkes, "On Covariant Multiloop Superstring Amplitudes," Nucl. Phys. B **332**, 39 (1990).
O. Lechtenfeld, "On The Finiteness Of The Superstring," Nucl.Phys.B **322**, 82 (1989).

[26] N. Berkovits, "Finiteness and unitarity of Lorentz covariant Green-Schwarz superstring amplitudes," Nucl. Phys. B **408**, 43 (1993) [arXiv:hep-th/9303122].

[27] C. R. Mafra, "Superstring Scattering Amplitudes with the Pure Spinor Formalism," arXiv:0902.1552 [hep-th].