A NON-DIFFERENTIABLE NOETHER'S THEOREM

by

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Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie.

Emmy Noether, Invariante Variationsprobleme, 1918

Abstract. — In the framework of the non-differentiable embedding of Lagrangian systems, defined by Cresson and Greff in [**5**], we prove a Noether's theorem based on the lifting of one-parameter groups of diffeomorphisms.

Contents

2000 ***Mathematics Subject Classification.*** — 49S05, 49J05, 26B05, 70H03.
Key words and phrases. — Noether's theorem, Symmetries, First integrals, Non-differentiable calculus of variations, Lagrangian systems.

1. Introduction

This paper is a contribution to the idea of embedding of Lagrangian systems initiated in [**3**]. A review of the subject is given in [**2**]. An embedding of an ordinary or partial differential equation is a way to give a meaning to this equation over a larger set of solutions, like stochastic processes or non-differentiable functions. As an example, Schwartz's theory of distributions can be seen as an embedding theory. In this paper, we consider an extension of a particular class of differential equations which are Euler-Lagrange equations over non-differentiable functions described in [**5**]. The Euler-Lagrange equations are second order differential equations whose solutions correspond to critical points of a Lagrangian functional, [**1**]. Lagrangian systems cover a large set of dynamical behaviors and are widely used in classical mechanics. In [**10, 9**], Nottale introduce the idea that the space-time structure at the microscopic scale becomes non-differentiable. His goal is to recover the classical equations of quantum mechanics from those of classical mechanics. Using the fact that at the macroscopic scale the space-time is differentiable and the equations of mechanics are governed by a variational principle, called the least-action principle, he formulates a scale-relativity principle. Namely, the equations of motions over the non-differentiable space-time are given by the classical equation extended to non-differentiable solutions. This extension is done by choosing a different operator of differentiation on continuous functions. In [**5**], we defined the notion of non-differentiable embedding of differential equations and proved that the solutions of an embedded Euler-Lagrange equation correspond to critical points of a non-differentiable Lagrangian functional. In particular, the classical Newton's equation of Mechanics transforms into the Schrödinger equation by a non-differentiable embedding. This is summarized by the following diagram:

$$\begin{array}{ccc}
\text{Lagrangian} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Lagrangian} \\
{\scriptstyle \text{L.A.P.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{N.D.L.A.P.}} \\
\text{Euler-Lagrange equation} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Euler-Lagrange equation}
\end{array}$$

where N.D. stands for non-differentiable, Emb. for embedding and L.A.P for least-action principle.

In this paper, we pursue our study of the non-differentiable embedding of Lagrangian systems. A classical result of Emmy Noether provides a relation between groups of symmetries of a given equation and constants of motion, i.e. first integrals. Precisely, if a Lagrangian system is invariant under a group of symmetries then it admits an explicit first integral. In the framework of the non-differentiable embedding of Lagrangian systems, we have then a natural question: Assume that the classical Lagrangian system is invariant under a group of symmetries, what can be said about the non-differentiable embedded Lagrangian system? In particular, do we have a non-differentiable notion of constants of motion? If yes, is it possible to extend the Noether's theorem? These questions can be summarized by the following diagram:

$$\begin{tikzcd}
\text{invariance of Lagrangian} \arrow[r, "\text{N.D. Emb}"] \arrow[d, "\text{Noether's thm.}"'] & \text{invariance of N.D Lagrangian} \arrow[d, "\text{N.D. Noether's thm.}"] \\
\text{First integral} \arrow[r, "\text{N.D. Emb}"] & \text{N.D. First integral}
\end{tikzcd}$$

In this paper, we prove a non-differentiable Noether's theorem. Previous attempt in this direction has been made in [**4**] using a different formalism over non-differentiable functions and not in the context of the non-differentiable embedding of Lagrangian systems. In particular, the problem of the persistence of symmetries under embedding was not discussed.

The outline of the paper is as follows: first, we recall the framework of the non-differentiable calculus of variations introduced in [**5**]. In section 3, we remind classical results about group of symmetries, first integrals, and Noether's theorem. We then introduce the notion of invariance for a non-differentiable Lagrangian functional and discuss the problem of persistence of symmetries under a non-differentiable embedding. Section 4 is devoted to the proof of the non-differentiable Noether's theorem. We conclude with application to the Navier-Stokes equation.

2. Reminder about non-differentiable calculus of variations

We recall some notations and definitions from [**5**].

2.1. Definitions. — Let $d \in \mathbb{N}$ be a fixed integer, I an open set in $\mathbb{R}$, and $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset I$, be given in the whole paper. We denote by $\mathcal{F}(I, \mathbb{R}^d)$ the set of functions $x : I \to \mathbb{R}^d$ from I to $\mathbb{R}^d$, and $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) the subset of $\mathcal{F}(I, \mathbb{R}^d)$ (respectively $\mathcal{F}(I, \mathbb{C}^d)$) which are continuous. Let $n \in \mathbb{N}$, we denote by $\mathcal{C}^n(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^n(I, \mathbb{C}^d)$) the set of functions in $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) which are differentiable up to order n.

Definition 1. — *(Hölderian functions) Let $w \in \mathcal{C}^0(I, \mathbb{R}^d)$. Let $t \in I$.*

1. *w is Hölder of Hölder exponent α, $0 < \alpha < 1$, at point t if*
$$\exists c > 0,\, \exists \eta > 0 \, s.t. \, \forall t' \in I \mid t - t' \mid \leq \eta \Rightarrow \|w(t) - w(t')\| \leq c \mid t - t' \mid^{\alpha},$$
where $\| \cdot \|$ is a norm on $\mathbb{R}^d$.
2. *w is α-Hölder and inverse Hölder with $0 < \alpha < 1$, at point t if*
$$\begin{aligned} &\exists c, C \in \mathbb{R}^{+*},\, c < C,\, \exists \eta > 0 \, s.t. \, \forall t' \in I \mid t - t' \mid \leq \eta \\ &\quad c \mid t - t' \mid^{\alpha} \leq \|w(t) - w(t')\| \leq C \mid t - t' \mid^{\alpha} \ . \end{aligned}$$

A complex valued function is α-Hölder if its real and imaginary parts are α-Hölder. We denote by $H^{\alpha}(I, \mathbb{R}^d)$ the set of continuous functions α-Hölder. For explicit examples of α-Hölder and α-inverse Hölder functions we refer to ([**11**], p.168) in particular the Knopp or Takagi function.

2.2. The quantum derivative. — Let $x \in \mathcal{C}^0(I, \mathbb{R}^d)$. For any $\epsilon > 0$, the ϵ-scale derivative of x at point t is the quantity denoted by $\frac{\Box_\epsilon x}{\Box t} : \mathcal{C}^0(I, \mathbb{R}^d) \to \mathcal{C}^0(I, \mathbb{C}^d)$, and defined by
$$\frac{\Box_\epsilon x}{\Box t} := \frac{1}{2}\Big[\big(d_\epsilon^+ x + d_\epsilon^- x\big) + i\mu\big(d_\epsilon^+ x - d_\epsilon^- x\big)\Big],$$
where $\mu \in \{1, -1, 0, i, -i\}$ and
$$d_\epsilon^\sigma x(t) := \sigma \frac{x(t + \sigma\epsilon) - x(t)}{\epsilon}, \ \sigma = \pm, \quad \forall t \in I.$$

Definition 2. — *Let $x \in \mathcal{C}^0(I,\mathbb{C}^d)$ be a continuous complex valued function. For all $\epsilon > 0$, the ϵ-scale derivative of x, denoted by $\frac{\Box_\epsilon x}{\Box t}$ is defined by*

$$\frac{\Box_\epsilon x}{\Box t} := \frac{\Box_\epsilon \mathrm{Re}(x)}{\Box t} + i\frac{\Box_\epsilon \mathrm{Im}(x)}{\Box t}, \tag{1}$$

where $\mathrm{Re}(x)$ *and* $\mathrm{Im}(x)$ *are the real and imaginary part of* x.

Let $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ be a sub-vectorial space of $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ such that for any function $f \in \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ the limit $\lim_{\epsilon\to 0} f(t,\epsilon)$ exists for any $t \in I$. We denote by E a complementary space of $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ in $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ and by π the projection onto $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ by

$$\begin{aligned} \pi : \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d) \oplus E &\rightarrow \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d) \\ f_{conv} + f_E &\mapsto f_{conv} \,. \end{aligned}$$

We can then define the operator $\langle\,.\,\rangle$ by

$$\begin{aligned} \langle\,.\,\rangle : \mathcal{C}^0(I\times]0,1],\mathbb{R}^d) &\rightarrow \mathcal{F}(I,\mathbb{R}^d) \\ f &\mapsto \langle \pi(f)\rangle : t \mapsto \lim_{\epsilon\to 0} \pi(f)(t,\epsilon)\,. \end{aligned}$$

Definition 3. — *Let us introduce the new operator $\frac{\Box}{\Box t}$ (without ϵ) on the space $\mathcal{C}^0(I,\mathbb{R}^d)$ by:*

$$\frac{\Box x}{\Box t} := \langle \pi(\frac{\Box_\epsilon x}{\Box t})\rangle. \tag{2}$$

The operator $\frac{\Box}{\Box t}$ acts on complex valued functions by $\mathbb{C}$-linearity.
For a differentiable function $x \in \mathcal{C}^1(I,\mathbb{R}^d)$, $\frac{\Box x}{\Box t} = \frac{dx}{dt}$, which is the classical derivative. More generally if $\frac{\Box^k}{\Box t^k}$ denotes $\frac{\Box^k}{\Box t^k} := \frac{\Box}{\Box t}\circ\ldots\circ\frac{\Box}{\Box t}$ and $x \in \mathcal{C}^k(I,\mathbb{R}^d)$, $k \in \mathbb{N}$, then $\frac{\Box^k x}{\Box t^k} = \frac{d^k x}{dt^k}$.
The following lemma is an analogous of the standard Leibniz (product) rule for non-differentiable functions under the action of $\frac{\Box}{\Box t}$:

***Lemma 1* ($\Box$-Leibniz rule)**. — *Let $f \in H^\alpha(I,\mathbb{R}^d)$ and $g \in H^\beta(I,\mathbb{R}^d)$, with $\alpha+\beta > 1$,*

$$\frac{\Box}{\Box t}(f\cdot g) = \frac{\Box f}{\Box t}\cdot g + f\cdot\frac{\Box g}{\Box t}\,. \tag{3}$$

We refer to **[5]** for the proof. Let us note that for $\beta = \alpha$, we must have $\alpha > \frac{1}{2}$.

2.3. Non-differentiable calculus of variations. —

Definition 4. — *An admissible Lagrangian function* L *is a function* $\mathrm{L} : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \to \mathbb{C}$ *such that* $\mathrm{L}(t, x, v)$ *is holomorphic with respect to* v*, differentiable with respect to* x *and real when* $v \in \mathbb{R}$*.*

Let us consider an admissible Lagrangian $\mathrm{L} : \mathbb{R}\times\mathbb{R}^d\times\mathbb{C}^d \to \mathbb{C}$. A Lagrangian function defines a *functional* on $\mathcal{C}^1(I, \mathbb{R}^d)$, denoted by

$$\mathcal{L} : \mathcal{C}^1(I, \mathbb{R}^d) \to \mathbb{R}, \quad x \in \mathcal{C}^1(I, \mathbb{R}^d) \longmapsto \int_a^b \mathrm{L}\big(s, x(s), \frac{dx}{dt}(s)\big)\, ds\,. \tag{4}$$

The classical *calculus of variations* analyzes the behavior of $\mathcal{L}$ under small perturbations of the initial function x. The main ingredients are a notion of differentiable functional and extremal. Extremals of the functional $\mathcal{L}$ can be characterized by an ordinary differential equation of order 2, called the Euler-Lagrange equation.

Theorem 1. — *The extremals* $x \in \mathcal{C}^1(I, \mathbb{R}^d)$*of* $\mathcal{L}$ *coincide with the solutions of the Euler-Lagrange equation denoted by (EL) and defined by*

$$\frac{d}{dt}\left[\frac{\partial \mathrm{L}}{\partial v}\left(t, x(t), \frac{dx(t)}{dt}(t)\right)\right] = \frac{\partial \mathrm{L}}{\partial x}\left(t, x(t), \frac{dx(t)}{dt}(t)\right). \tag{EL}$$

The non-differentiable embedding procedure allows us to define a natural extension of the classical Euler-Lagrange equation in the non-differentiable context.

Definition 5. — *The non-differentiable Lagrangian functional* $\mathcal{L}_\square$ *associated to* $\mathcal{L}$ *is given by*

$$\mathcal{L}_\square : \mathcal{C}^1_\square(I, \mathbb{R}^d) \to \mathbb{R}, \quad x \in \mathcal{C}^1_\square(I, \mathbb{R}^d) \longmapsto \int_a^b \mathrm{L}\big(s, x(s), \frac{\square x(s)}{\square t}\big)\, ds\,. \tag{5}$$

where $\mathcal{C}^1_\square(I, \mathbb{R})$ *is the set of continuous functions* $f \in \mathcal{C}^0(I, \mathbb{R})$ *such that* $\frac{\square f}{\square t} \in \mathcal{C}^0(I, \mathbb{C})$*.*

Let $H_0^\beta := \{h \in H^\beta(I, \mathbb{R}^d),\ h(a) = h(b) = 0\}$, and $x \in H^\alpha(I, \mathbb{R}^d)$ with $\alpha + \beta > 1$. A H_0^β-variation of x is a function of the form $x + h$, with $h \in H_0^\beta$.

For $x \in H^{\alpha}(I, \mathbb{R}^d)$ and $h \in H_0^{\beta}$, we denote by $D\mathcal{L}_{\Box}(x)(h)$ the quantity

$$\lim_{\epsilon \to 0} \frac{\mathcal{L}_{\Box}(x + \epsilon h) - \mathcal{L}_{\Box}(x)}{\epsilon}$$

if it exists and called the differential of $\mathcal{L}_{\Box}$ at point x in direction h. A H_0^{β}-extremal curve of the functional $\mathcal{L}_{\Box}$ is a curve $x \in H^{\alpha}(I, \mathbb{R}^d)$ satisfying

$$D\mathcal{L}_{\Box}(x)(h) = 0, \quad \text{for any } h \in H_0^{\beta}.$$

Theorem 2 **(Non-differentiable least-action principle)**

Let $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta > 1$. Let L be an admissible Lagrangian function of class $\mathcal{C}^2$. We assume that $x \in H^{\alpha}(I, \mathbb{R}^d)$, and $\frac{\Box x}{\Box t} \in H^{\alpha}(I, \mathbb{R}^d)$. A curve $x \in H^{\alpha}(I, \mathbb{R}^d)$ satisfying the following generalized Euler-Lagrange equation

$$\frac{\partial L}{\partial x}\big(t, x(t), \frac{\Box x(t)}{\Box t}\big) - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\big(t, x(t), \frac{\Box x(t)}{\Box t}\big)\right) = 0\,. \qquad (NDEL)$$

is an extremal curve of the functional (5) on the space of variations H_0^{β}.

We refer to [**5**] for a proof.

3. Group of symmetries and invariance of functionals

3.1. Group of symmetries. — Symmetries are defined via the action of one parameter group of diffeomorphisms.

Definition 6. — *We call $\{\phi_s\}_{s \in \mathbb{R}}$ a one parameter group of diffeomorphisms $\phi_s : \mathbb{R}^d \to \mathbb{R}^d$, of class $\mathcal{C}^1$ satisfying*

i) $\phi_0 = \mathrm{Id}$,

ii) $\phi_s \circ \phi_u = \phi_{s+u}$.

iii) *ϕ_s is of class $\mathcal{C}^1$ with respect to s.*

Classical examples of symmetries are given by translations in a given direction u

$$\phi_s : x \mapsto x + su, \quad x \in \mathbb{R}^d$$

and rotations

$$\phi_s : x \mapsto x + s, \quad x \in [0, 2\pi]^d.$$

In [**4**] we use the related notion of *infinitesimal transformations*, instead of group of diffeomorphisms. They are obtained using a Taylor expansion of $y_t(s) = \phi_s(x(t))$ in a neighborhood of 0. We obtain

$$y_t(s) = y_t(0) + s \cdot \frac{dy_t}{ds}(0) + o(s).$$

As $\phi_0 = \mathrm{Id}$, we deduce that denoting by $\xi(t,x) = \dfrac{dy_t}{ds}(0)$ an *infinitesimal transformation* is of the form

$$x(t) \mapsto x(t) + s\xi(t, x(t)) + o(s).$$

3.2. Invariance of functionals and Noether's theorem. — In this section, we recall a classical result of E. Noether, [**8, 7**], which provides a relation between symmetries and first integrals, i.e. constants of motions. The classical notion of *first integral* for a dynamical systems can be defined in various ways leading to different generalized concepts of first integrals for non-differentiable dynamical system. We consider the following one:

***Definition 7* (First integral).** — *Let $J : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a function of class $\mathcal{C}^1$, then J is said to be a first integral of the ordinary differential equation $\dot{x}(t) = f(t, x(t))$, with $f \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ if for any solution x of the ordinary equation we have*

$$\frac{d}{dt}\big(J(t, x(t))\big) = 0 \quad \textit{for any } t \in \mathbb{R}.$$

The Euler-Lagrange equation is a second order differential equation. Therefore, a first integral for the Euler-Lagrange equation is a function $J : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that for any solution x of the Euler-Lagrange equation, we have

$$\frac{d}{dt}\big(J(t, x(t), \dot{x}(t))\big) = 0 \quad \text{for any } t \in \mathbb{R}.$$

***Definition 8* (Invariance).** — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be invariant under the action of Φ if it satisfies:*

$$L\Big(t, x(t), \frac{dx}{dt}(t)\Big) = L\Big(t, \phi_s(x(t)), \frac{d}{dt}\big(\phi_s(x(t))\big)\Big), \quad \forall s \in \mathbb{R}, \forall t \in \mathbb{R}, \tag{6}$$

for any solution x of the Euler-Lagrange equation.

A Lagrangian satisfying (6) will be called *classically invariant* under $\{\phi_s\}_{s\in\mathbb{R}}$. The Noether's theorem is based on the notion of invariance of Lagrangian under a group of symmetries. Let us recall the classical Noether's theorem, [**7**].

Theorem 3 **(Noether's theorem)**. — *Let L be an admissible Lagrangian of class $\mathcal{C}^2$ invariant under $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$, a one parameter group of diffeomorphisms. Then, the function*

$$J : (t, x, v) \mapsto \frac{\partial L}{\partial v}(t, x, v) \cdot \frac{d\phi_s(x)}{ds} \mid_{s=0}$$

is a first integral of the Euler-Lagrange equation (EL).

3.3. The non-differentiable case. — The generalization of the notion of invariance of the Lagrangian to the non-differentiable case is quite natural and is deduced from the non-differentiable theory in [**5**]. This leads to the following definition.

Definition 9 **($\Box$-invariance)**. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be $\Box$-invariant under the action of Φ if*

$$(7) \qquad L(t, x(t), \frac{\Box x}{\Box t}(t)) = L(t, \phi_s(x(t)), \frac{\Box}{\Box t}(\phi_s(x(t)))), \quad \forall s \in \mathbb{R}, \quad \forall t \in I.$$

for any solution $x \in \mathcal{C}^1_\Box$ of the non-differentiable Euler-Lagrange equation (NDEL).

Remark 1. — *The regularity assumption on the family $\{\phi_s\}_{s\in\mathbb{R}}$ is related to the classical definition of invariance (6). In our case, we can weaken this assumption using for example family of homeomorphisms of class $\mathcal{C}^1_\Box$. However, as we have no examples of natural symmetries of this kind, we keep the classical definition.*

A natural question arising from the non-differentiable embedding theory of Lagrangian systems developed in [**5**] is the problem *persistence* of symmetries under embedding.

Problem 1 **(Persistence of invariance)**. — *Assuming that a Lagrangian L is classically invariant under a group of symmetries $\{\phi_s\}_{s\in\mathbb{R}}$. Do we have the $\Box$-invariance of the Lagrangian L under $\{\phi_s\}_{s\in\mathbb{R}}$?*

This problem seems difficult. However, there exits one case where we can prove the persistence of invariance.

Definition 10 **(Strong invariance)**. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be strongly invariant under the action of Φ if*

$$L(t,x,v) = L(t,\phi_s(x),\phi_s(v)), \quad \forall s \in \mathbb{R},\ \forall t \in I,\ \forall x \in \mathbb{R}^d,\ \forall v \in \mathbb{R}^d.$$

As an example we can consider the following Lagrangian L, given by:

$$L(t,x,v) = \frac{1}{2}\|v\|^2 - \frac{1}{\|x\|^2}.$$

If ϕ_s is a rotation, $\phi_s(x) := e^{is\theta}x$, then the Lagrangian L is strongly invariant.

Definition 11 **($\Box$-commutation)**. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms, such that $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$. Φ satisfies the $\Box$-commutation property, if*

$$\frac{\Box}{\Box t}(\phi_s(x)) = \phi_s\Big(\frac{\Box x}{\Box t}\Big), \quad \forall s \in \mathbb{R}. \tag{8}$$

Lemma 2 **(Sufficient condition)**. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms, $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$. If the Lagrangian L is strongly invariant and Φ satisfies the $\Box$-commutation property, then the Lagrangian L is $\Box$-invariant under the action of $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$.*

Proof. — Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ a one parameter group of diffeomorphisms. Let x be a solution of the non-differentiable Euler-Lagrange equation. Let $s \in \mathbb{R}$, applying definition 11 and condition (8), we obtain:

$$\begin{aligned} L\Big(t,\phi_s(x(t)),\frac{\Box}{\Box t}\big(\phi_s(x(t))\big)\Big) &= L\Big(t,\phi_s(x(t)),\phi_s\big(\frac{\Box x}{\Box t}(t)\big)\Big), \\ &= L\Big(t,x(t),\frac{\Box x}{\Box t}(t)\Big), \end{aligned}$$

which concludes the proof. $\Box$

***Problem 2* (Commutation)**. — *Let $\phi \in \mathcal{C}^1(\mathbb{C}^d, \mathbb{C}^d)$. Under which condition do we have the $\Box$-commutation*

$$\frac{\Box}{\Box t}\big(\phi(x)\big) = \phi\Big(\frac{\Box}{\Box t}(x)\Big)\ ?$$

Lemma 3. — *Let ϕ be a linear map, then ϕ satisfies the property of $\Box$-commutation.*

Proof. — As ϕ is linear on $\mathbb{C}^d$, there exists a matrix A such that $\phi : x \mapsto A \cdot x$. Hence, we have:

$$\frac{\Box\phi(x)}{\Box t} = \frac{\Box(A\cdot x)}{\Box t} = A \cdot \frac{\Box x}{\Box t} = \phi\big(\frac{\Box x}{\Box t}\big).$$

$\Box$

As a consequence, if L is strongly invariant under a linear group, then L is $\Box$-invariant.

We finish this section with a technical lemma which will be usefull in the proof of the non-differentiable Noether's theorem.

Lemma 4. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms $\phi_s : \mathbb{R}^d \to \mathbb{R}^d$, then we have*

$$\frac{d}{ds}\Big(\frac{\Box}{\Box t}\big(\phi_s(x(t))\big)\Big)|_{s=0} = \frac{\Box}{\Box t}\Big(\frac{d}{ds}\big(\phi_s(x(t))\big)|_{s=0}\Big). \tag{9}$$

Proof. — Using a Taylor expansion of $\phi_s(x(t))$ in $s = 0$, since $\phi_0(x(t)) = x(t)$, we obtain

$$\phi_s(x(t)) = x(t) + s\frac{d}{ds}(\phi_s(x(t)))|_{s=0} + s\, r(s, x(t)),$$

with $\lim\limits_{s\to 0} r(s,\cdot) = 0$. Then, since $\frac{\Box}{\Box t}$ is linear, we obtain

$$\frac{\Box\phi_s(x(t))}{\Box t} = \frac{\Box x(t)}{\Box t} + s\frac{\Box}{\Box t}\Big(\frac{d}{ds}\big(\phi_s(x(t))\big)|_{s=0}\Big) + s\frac{\Box r(s,x(t))}{\Box t}.$$

Taking the derivative with respect to s gives:

$$\frac{d}{ds}\Big(\frac{\Box\phi_s(x(t))}{\Box t}\Big) = \frac{\Box}{\Box t}\big(\frac{d}{ds}(\phi_s(x(t)))|_{s=0}\big) + \frac{\Box}{\Box t}\big(r(s,x(t))\big) + s\frac{d}{ds}\Big(\frac{\Box r(s,x(t))}{\Box t}\Big),$$

then, for $s = 0$, we deduce

$$\frac{d}{ds}\Big(\frac{\Box\phi_s(x(t))}{\Box t}\Big)|_{s=0} = \frac{\Box}{\Box t}\big(\frac{d}{ds}(\phi_s(x(t)))|_{s=0}\big) + \underbrace{\frac{\Box}{\Box t}\big(r(s,x(t))\big)|_{s=0}}_{=0}.$$

This concludes the proof. $\square$

4. Non-differentiable Noether's theorem

As we have a notion of $\square$-invariance of non-differentiable functionals, we can look for an analogous to Noether's theorem. This means that we need to define the corresponding notion of first integrals for $\square$-differential equations. A generalization of definition 7 to non-differential curves comes from the non-differentiable theory of [**5**], and leads to the following definition.

***Definition 12* (Generalized first integral).** — *A map $J : \mathbb{R} \times \mathbb{C}^d \to \mathbb{C}$ is a generalized first integral of an ordinary $\square$-differentiable equation*

$$\frac{\square x(t)}{\square t} = f(t, x(t))$$

with $f \in \mathcal{C}^0(\mathbb{R} \times \mathbb{C}^d, \mathbb{C})$ if for any solution x

$$\frac{\square}{\square t}\big(J(t, x(t))\big) = 0 \quad \forall t \in \mathbb{R}.$$

A non-differentiable Euler-Lagrange equation is a second order $\square$-differentiable equation, consequently an associated generalized first integral is a function $J : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \to \mathbb{C}$ such that for any solution x of (NDEL), we have

$$\frac{\square}{\square t}\Big(J\big(t, x(t), \frac{\square x(t)}{\square t}\big)\Big) = 0 \qquad \forall t \in \mathbb{R}.$$

***Theorem 4*.** — *Let L be a Lagrangian of class $\mathcal{C}^2$ $\square$-invariant under $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$, a one parameter group of diffeomorphisms, such that $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$, for any $s \in \mathbb{R}$. Then, the function*

$$J : (t, x, v) \mapsto \frac{\partial L}{\partial v}(t, x, v) \cdot \frac{d\phi_s(x)}{ds} \mid_{s=0} \tag{10}$$

is a generalized first integral of the non-differentiable Euler-Lagrange equation (NDEL) on $H^\alpha(I, \mathbb{R}^d)$ with $\frac{1}{2} < \alpha < 1$.

Proof. — Let x be a solution of the non-differentiable Euler-Lagrange equation. Let $s \in \mathbb{R}$. As the Lagrangian is $\square$-invariant under Φ,

$$L\Big(t, \phi_s(x(t)), \frac{\square}{\square t}\big(\phi_s(x(t))\big)\Big) = L\Big(t, x(t), \frac{\square}{\square t} x(t)\Big) \ , \ \forall t \in I.$$

As a consequence, we obtain for any $s \in \mathbb{R}$

$$\frac{d}{ds}\left(L\Big(t, \phi_s(x(t)), \frac{\Box}{\Box t}(\phi_s(x(t)))\Big)\right) = 0. \tag{11}$$

On the other hand, we have for any $s \in \mathbb{R}$

$$\frac{d}{ds}\Big(L(t, \phi_s(x(t)), \frac{\Box}{\Box t}(\phi_s(x(t)))\Big) = \frac{\partial L}{\partial x}(\star_s(x)) \cdot \frac{d\phi_s(x(t))}{ds} + \frac{\partial L}{\partial v}(\star_s(x)) \cdot \frac{d}{ds}\Big(\frac{\Box \phi_s(x(t))}{\Box t}\Big),$$

where

$$\star_s(x) := \Big(t, \phi_s(x(t)), \frac{\Box}{\Box t}(\phi_s(x(t)))\Big).$$

Since (9) holds, we obtain for $s = 0$

$$\frac{d}{ds}\Big(L(\star_s(x))\Big)|_{s=0} = \frac{\partial L}{\partial x}(\star_0(x)) \cdot \frac{d\phi_s(x(t))}{ds}|_{s=0} + \frac{\partial L}{\partial v}(\star_0(x)) \cdot \frac{\Box}{\Box t}\left(\frac{d\phi_s(x(t))}{ds}\right)|_{s=0}.$$

Therefore, using (11) and since x is a solution of the non-differentiable Euler-Lagrange equation leads to

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}(\star_0(x))\right) \cdot \frac{d\phi_s(x(t))}{ds}|_{s=0} + \frac{\partial L}{\partial v}(\star_0(x)) \cdot \frac{\Box}{\Box t}\left(\frac{d\phi_s(x(t))}{ds}|_{s=0}\right) = 0.$$

As $x \in H^\alpha$, $\frac{\Box x}{\Box t} \in H^\alpha$ and $\frac{d}{ds}\phi_s$, $\frac{\partial L}{\partial v}$ continuous, with $2\alpha > 1$, applying lemma 1 we obtain

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}(t, x(t), \frac{\Box x}{\Box t}) \cdot \frac{d\phi_s(x(t))}{ds}|_{s=0}\right) = 0.$$

This concludes the proof. □

5. Application

In [**5**] we define non-differentiable characteristics of a classical PDE. For the Navier-Stokes equation these non-differentiable characteristics coincide with critical points of a non-differentiable Lagrangian functional of the form

$$L(t, x, v) = \frac{1}{2}v^2 - p(x, t), \tag{12}$$

where $x \in \mathbb{R}^d$, $v \in \mathbb{C}^d$ and $t \in \mathbb{R}$ over $H^{1/2}$. We refer to [**5**] for details.

Let $d = 3$, we now study the Lagrangian (12) assuming that p is invariant with respect to the group of rotations around the vertical axis. With respect to our work on non-differentiable characteristics of the Navier-Stokes equation, this corresponds to consider the axisymmetric Navier-Stokes equations studied in [**6**]. Using the non-differentiable Noether's theorem we have the following result:

Theorem 5. — *Let L be the Lagrangian (12) where p is assumed invariant under the group of rotations $\Phi = \{\phi_\theta\}_{\theta\in\mathbb{R}}$ around the vertical axis given by*

$$\phi_\theta : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3, \\ (x,y,z) & \longmapsto & (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z). \end{array}$$

Then the function

$$J : \begin{array}{ccc} \mathbb{R}\times\mathbb{R}^3\times\mathbb{C}^3 & \longrightarrow & \mathbb{C}^3, \\ (t,(x,y,z),(v_x,v_y,v_z)) & \longmapsto & -yv_x + xv_y \end{array}$$

is a generalized first integral of the non-differentiable Euler-Lagrange equation

$$\frac{\Box}{\Box t}\left(\frac{\Box x}{\Box t}\right) = -\nabla_x p,$$

over H^α with $1/2 < \alpha < 1$.

Proof. — First, we extend Φ to $\mathbb{C}^3$ trivially. As p is invariant under the group Φ, and ϕ_θ is an isometry for each $\theta\in\mathbb{R}$, the Lagrangian L is strongly invariant under Φ. Moreover, as ϕ_θ is linear for each $\theta\in\mathbb{R}$, using lemma 3 we deduce that the group Φ satisfies $\Box$-commutation. Hence, applying lemma 2, we deduce that L is $\Box$-invariant under the action of Φ. We then apply theorem 4 to conclude. $\Box$

This result can be extended using the same argument on rotations, to the Lagrangian underlying the Schrödinger equation view as a non-differentiable Euler-Lagrange equation over $H^{1/2}$. Indeed, in this case, the function p is given by $1/\sqrt{x^2+y^2+z^2}$ defined on $\mathbb{R}^3\setminus\{0\}$ and is invariant under each groups of rotations with respect to a fixed axis.

However, due to the limitation $1/2 < \alpha$, we cannot applied our result directly to give more informations on the non-differentiable characteristic of the Navier-Stokes equations or for the Schrödinger equation. The constraint on α is mainly due to the $\Box$-Leibniz rule.

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