NONCOMMUTATIVE CATALAN NUMBERS

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Abstract. The goal of this paper is to introduce and study *noncommutative Catalan numbers* C_n which belong to the free Laurent polynomial algebra $\mathcal{L}_n$ in n generators. Our noncommutative numbers admit interesting (commutative and noncommutative) specializations, one of them related to Garsia-Haiman (q,t)-versions, another – to solving noncommutative quadratic equations. We also establish total positivity of the corresponding (noncommutative) Hankel matrices H_n and introduce accompanying *noncommutative binomial coefficients* $\left\langle\!\!\binom{n}{k}\!\!\right\rangle \in \mathcal{L}_{n+k-1}, \left\langle\!\!\binom{n}{k}\!\!\right\rangle' \in \mathcal{L}_n$.

Contents

1. Introduction

Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$ are important combinatorial objects which satisfy a number of remarkable properties such as:

- The recursion $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$ for all $n \geq 0$ (with $c_0 = c_1 = 1$).
- the determinantal identities $\det \begin{pmatrix} c_m & c_{m+1} & \dots & c_{m+n} \\ c_{m+1} & c_{m+2} & \dots & c_{m+n+1} \\ & & \dots & \\ c_{m+n} & c_{m+n+1} & \dots & c_{m+2n} \end{pmatrix} = 1$ for $n \geq 0$, $m \in \{0,1\}$.

In this paper we introduce and study *noncommutative Catalan numbers* C_n, $n \geq 1$ which are totally noncommutative Laurent polynomials in n variables and satisfy analogues of the recursion and the determinantal identities (Proposition 2.3 and equation (2.7)). It turns out that specializing these variables to appropriate powers of q, we recover Garsia-Haiman q-Catalan numbers ([5]).

It is "less well-known" that Catalan numbers satisfy a combinatorial identity involving their truncated counterparts $c_n^k = \binom{n+k}{k} - \binom{n+k}{k-1}$ (so that $c_n = c_n^n = c_n^{n-1}$):

$$c_n = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}: \\ a+b \leq n, a-b=d}} c_{n-b}^a c_{n-a}^b \tag{1.1}$$

for each $n \in \mathbb{Z}_{\geq 0}$ and each $d \in \mathbb{Z}$ with $|d| \leq n$ (e.g., the right hand side does not depend on d).

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We introduce noncommutative analogues of all truncated Catalan numbers and establish a noncommutative version of (1.1) (Theorem 2.22). It is curios that the c_n^k satisfy thee more combinatorial two of which identities involving binomial coefficients:

$$c_{n+1}^k = \sum_{j=0}^{k} c_j c_{n-j}^{k-j},\ \sum_{j=0}^{k} (-1)^j c_{n+k-j}^j \cdot \binom{n-j}{k-j} = 0,\ c_{m+n}^k = \sum_{\ell=0}^{n} c_{m+\ell}^{k-\ell} \cdot \binom{n}{\ell},$$

where $0 \le k < n$ in the first two identities and $0 \le k \le m+n$ in the third one.

We establish a noncommutative generalization of the first identity (Proposition 2.20(c)), define appropriate noncommutative versions $\left\langle\!\binom{n}{k}\!\right\rangle', \left\langle\!\binom{n}{k}\!\right\rangle$ of binomial coefficients and establish analogues of the remaining two identities with our noncommutative binomial coefficients (Corollary 2.33 and Theorem 2.34) as well as the analogue of multiplication law for both kinds of noncommutative binomial coefficients (Theorem 2.32).

In fact, these constructions and results extend our previous work on Noncommutative Laurent Phenomenon ([1, 2]) and we expect more such Phenomena to emerge in Combinatorics, Representation Theory, Topology and related fields.

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2. Notation and main results

Let F be the free group generated by x_k, $k \in \mathbb{Z}_{\ge 0}$ and F_m be the (free) subgroup of F generated by $x_0, \dots, x_m$.

Denote by $\tilde{\mathcal{P}}_n$ the of monotonic lattice paths in $[0,n]\times[0,n]$ from $(0,0)$ to (n,n). Clearly, $|\tilde{\mathcal{P}}_n| = \binom{2n}{n}$. We say that $P \in \tilde{\mathcal{P}}_n$ is *Catalan* if for each point $p = (p_1, p_2) \in P$ one has $c(p) \ge 0$, where $c(p_1,p_2) := p_1 - p_2$ is the content of p. Denote by $\mathcal{P}_n \subset \tilde{\mathcal{P}}_n$ the set of all Catalan paths in $[0,n]\times[0,n]$. Clearly, $|\mathcal{P}_n| = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, which justifies the terminology.

We say that a point $p = (p_1, p_2)$ of $P \in \tilde{\mathcal{P}}_n$ is a *southeast (resp. northwest) corner* of P if $(p_1 - 1, p_2) \in P$ and $(p_1, p_2 + 1) \in P$ (resp. $(p_1, p_2 - 1) \in P$ and $(p_1 + 1, p_2) \in P$).

To each $P \in \mathcal{P}_n$ we assign an element $M_P \in F_n$ by

$$M_P = \overrightarrow{\prod} x_{c(p)}^{\operatorname{sgn}(p)}\ , \tag{2.1}$$

where the product is over all corners $p \in P$ (taken in the natural order) and $\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } p \text{ is southeast} \\ -1 & \text{if } p \text{ is northwest} \end{cases}$.

We define the *noncommutative Catalan number* $C_n \in \mathbb{Z}F_n$ by

$$C_n = \sum_{P \in \mathcal{P}_n} M_P\ . \tag{2.2}$$

Clearly, under the counit homomorphism $\varepsilon : \mathbb{Z}F \to \mathbb{Z}$ $(x_k \mapsto 1)$ the image $\varepsilon(C_n)$ is $|\mathcal{P}_n|$, the ordinary Catalan number.

Noncommutative Catalan numbers exhibit some symmetries.

Proposition 2.1. *$\overline{C}_n = C_n$, where $\bar{\cdot}$ denotes the anti-automorphism of $\mathbb{Z}F$ such that $\overline{x}_k = x_k$ for $k \in \mathbb{Z}_{\ge 0}$.*

Example 2.2. $C_0 = x_0$, $C_1 = x_1$, $C_2 = x_2 + x_1 x_0^{-1} x_1$,

$$C_3 = x_3 + x_2 x_1^{-1} x_2 + x_2 x_0^{-1} x_1 + x_1 x_0^{-1} x_2 + x_1 x_0^{-1} x_1 x_0^{-1} x_1\ ,$$

$$C_4 = x_4 + x_3 x_2^{-1} x_3 + x_2 x_0^{-1} x_2 + x_3 x_1^{-1} x_2 + x_2 x_1^{-1} x_3 + x_3 x_0^{-1} x_1 + x_1 x_0^{-1} x_3 + x_2 x_1^{-1} x_2 x_1^{-1} x_2$$
$$+ x_1 x_0^{-1} x_2 x_0^{-1} x_1 + x_2 x_1^{-1} x_2 x_0^{-1} x_1 + x_1 x_0^{-1} x_2 x_1^{-1} x_2 + x_2 x_0^{-1} x_1 x_0^{-1} x_1 + x_1 x_0^{-1} x_1 x_0^{-1} x_2 + x_1 x_0^{-1} x_1 x_0^{-1} x_1 x_0^{-1} x_1\ .$$

It turns out that our noncommutative Catalan numbers satisfy the following generalization of the well-known classical recursion.

Proposition 2.3. *For $n \geq 0$ one has*

$$C_{n+1} = \sum_{k=0}^{n} C_k x_0^{-1} T(C_{n-k}),\ C_{n+1} = \sum_{k=0}^{n} T(C_k) x_0^{-1} C_{n-k} \tag{2.3}$$

for all $n \in \mathbb{Z}_{\geq 0}$, where $T : \mathbb{Z}F \to \mathbb{Z}F$ an endomorphism of $\mathbb{Z}F$ given by $T(x_k) = x_{k+1}$ for all $k \in \mathbb{Z}_{\geq 0}$.

For example, $C_2 = T(C_1) + C_1 x_0^{-1} T(C_0)$ and $C_3 = T(C_2) + C_1 x_0^{-1} T(C_1) + C_2 x_0^{-1} T(C_0)$.

The following is immediate.

Corollary 2.4. *The formal power series $\mathbf{C}(t) = \sum_{n=0}^{\infty} C_n t^n \in (\mathbb{Z}F)[[t]]$ satisfies:*

$$\mathbf{C}(t) = x_0 + t\mathbf{C}(t) x_0^{-1} T(\mathbf{C}(t)),\ T(\mathbf{C}(t)) x_0^{-1} \mathbf{C}(t) = \mathbf{C}(t) x_0^{-1} T(\mathbf{C}(t))\ , \tag{2.4}$$

Remark 2.5. Applying ε to (2.4), we obtain the well-known functional equation $c(t) = 1 + tc(t)^2$ for the classical generating function $c(t) = \sum_{n=0}^{\infty} \varepsilon(C_n) t^n$ of Catalan numbers.

Remark 2.6. After the first version of this paper became available, Philippe Di Francesco and Rinat Kedem pointed to us that $\mathbf{C}(t) x_0^{-1}$ is a *noncommutative Stieltjes continued fraction* which can be computed by combining methods of [4, Section 3.3.1] and [7, Section 8] as follows.

$$\mathbf{C}(t) x_0^{-1} = \lim_{k \to \infty} \mathbf{S}(x_1 x_0^{-1}, \ldots, x_k x_{k-1}^{-1}, t)\ ,$$

where $\mathbf{S}(z_1, t) = (1 - z_1 t)^{-1}$, $\mathbf{S}(z_1, \ldots, z_k, t) = \mathbf{S}(z_1, \ldots, z_{k-2}, \mathbf{S}(z_k, t) z_{k-1}, t)$ for $k \geq 2$.

Remark 2.7. In fact there is another recursion

$$C_{n+1} = C_n x_0^{-1} x_1 + \sum_{k=1}^{n} C_k x_1^{-1} T^2(C_{n-k}) = x_1 x_0^{-1} C_n + \sum_{k=0}^{n-1} T^2(C_k) x_1^{-1} C_{n-k}\ .$$

for $n \geq 1$ and a functional equation $\mathbf{C}(t) = x_0 + t(\mathbf{C}(t) x_0^{-1} x_1 - x_0 x_1^{-1} T^2(\mathbf{C}(t)) + \mathbf{C}(t) x_1^{-1} T^2(\mathbf{C}(t)))$, which we leave as an exercise to the reader.

Remark 2.8. Specializing t to the lower triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ Jacobi matrix with entries $f_{ij} = \delta_{i,j-1}$, we see that the functional equation (2.4) can be rewritten in a matrix format: $H x_0^{-1} T(H) = T(H) x_0^{-1} H = H'$, where H (resp. H') is the lower triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ Toeplitz matrix whose (i,j)-th entry is C_{i-j} (resp. C_{i-j+1}) if $i \geq j$. In particular, H^{-1} a lower triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ Toeplitz matrix whose (i,j)-th entry is $-x_0^{-1} T(C_{i-j-1}) x_0^{-1}$ for $i > j$.

It turns out that there is a remarkable specialization $\underline{C}_n \in \mathbb{Z}F_2$ of C_n. Indeed, let $\sigma : \mathbb{Z}F \to \mathbb{Z}F_2$ be a ring homomorphism given by $\sigma(x_k) = x_0^k x_1^k$, $k \in \mathbb{Z}_{\geq 0}$. Abbreviate $\underline{C}_n := \sigma(C_n)$ for $n \geq 0$.

The following result asserts, in particular, that $\underline{C}_n$ are noncommutative polynomials (rather than Laurent polynomials) and they satisfy yet another noncommutative generalization of the well-known classical recursion for Catalan numbers.

Proposition 2.9. *The elements $\underline{C}_n \in \mathbb{Z}\langle x_0, x_1 \rangle$ are determined by the following recursion: $\underline{C}_0 = 1$ and*

$$\underline{C}_{n+1} = \sum_{k=0}^{n} \underline{C}_k x_0 \underline{C}_{n-k} x_1 = \sum_{k=0}^{n} x_0 \underline{C}_k x_1 \underline{C}_{n-k}\ , \tag{2.5}$$

for $n \geq 0$. In particular, all $\underline{C}_n$ belong to the free semi-ring $\mathbb{Z}_{\geq 0}\langle x_0, x_1 \rangle \subset \mathbb{Z}_{\geq 0} F_2$.

Remark 2.10. Applying σ to the recursions from Remark 2.7 and using an argument from the proof of Proposition 2.9, we obtain another recursion for $\underline{C}_n$:

$$\underline{C}_{n+1} = \underline{C}_n x_0 x_1 + \sum_{k=1}^{n} \underline{C}_k x_1^{-1} x_0 \underline{C}_{n-k} x_1^2 = x_0 x_1 C_n + \sum_{k=0}^{n-1} x_0^2 \underline{C}_k x_1 x_0^{-1} \underline{C}_{n-k}\ .$$

Remark 2.11. One can show that the "two-variable" noncommutative Catalan numbers are invariant under the anti-involution of $\mathbb{Z}F_2$ interchanging x_0 and x_1.

In fact, we can explicitly compute each $\underline{C}_n$. Indeed, assign a monomial $\underline{M}_P \in F_2$ to each $P \in \mathcal{P}_n$ by:

$$\underline{M}_P = x_0^{j_0} x_1^{j_1} x_0^{j_2} \cdots x_1^{j_{2k}} \ ,$$

where $(j_0, j_1, \ldots, j_{2k}) \in \mathbb{Z}_{>0}^{2k+1}$ is the sequence of jumps of the path P, i.e., the r-th northwest corner is $(j_0+j_2+\cdots+j_{2r}, j_1+j_3+\cdots+j_{2r+1})$ and r-th southeast corner of P is $(j_0+j_2+\cdots+j_{2r}, j_1+j_3+\cdots+j_{2r-1})$ One can easily see that $\sigma(M_P) = \underline{M}_P$, so we obtain the following immediate corollary.

Corollary 2.12. *$\underline{C}_n = \sum\limits_{P\in\mathcal{P}_n} \underline{M}_P$ for all $n \geq 1$.*

Example 2.13. $\underline{C}_2 = x_0^2x_1^2 + x_0x_1x_0x_1$, $\underline{C}_3 = x_0^3x_1^3 + x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1 + x_0x_1x_0^2x_1^2 + x_0x_1x_0x_1x_0x_1$,

$$\underline{C}_4 = x_0^4x_1^4 + x_0^3x_1x_0x_1^3 + x_0^2x_1^2x_0^2x_1^2 + x_0^3x_1^2x_0x_1^2 + x_0^2x_1x_0^2x_1^3 + x_0^3x_1^3x_0x_1 + x_0x_1x_0^3x_1^3 + x_0^2x_1x_0x_1x_0x_1^2$$
$$+x_0x_1x_0^2x_1^2x_0x_1 + x_0^2x_1x_0x_1^2x_0x_1 + x_0x_1x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1x_0x_1 + x_0x_1x_0x_1x_0^2x_1^2 + x_0x_1x_0x_1x_0x_1x_0x_1 \ .$$

The following immediate result is a "two-variable" version of Corollary 2.4.

Corollary 2.14. *The formal power series $\underline{\mathbf{C}}(t) = \sum\limits_{n=0}^{\infty} \underline{C}_n t^n \in \mathbb{Z}\langle x_0, x_1\rangle[[t]]$ satisfies:*

$$\underline{\mathbf{C}}(t) = 1 + t\underline{\mathbf{C}}(t)x_0\underline{\mathbf{C}}(t)x_1 \ . \tag{2.6}$$

Remark 2.15. For $t = 1$, the equation (2.6) coincides with the quadratic equation on formal series $K(x_0, x_1)$ studied in [13] where a solution of this equation was presented as a "noncommutative Rogers-Ramanujan continued fraction".

Remark 2.16. In our previous work [3] on the inversion of $\sum\limits_{n\geq 0} x_0^n x_1^n$ in the ring of formal series $\mathbb{Z}\langle\langle x_0, x_1\rangle\rangle$ in noncommutative variables x_0, x_1 we encountered a quadratic equation $D = 1 - Dx_0x_1 + Dx_0Dx_1$ for some $D \in \mathbb{Z}\langle\langle x_0, x_1\rangle\rangle$ and noticed that it is very similar to (2.6). This was the starting point of the project.

Remark 2.17. In fact, there is another group homomorphism $\pi : F \to F_2$ given by $\pi(x_k) = x_0 \cdot (x_0^{-1}x_1)^k$, $k \in \mathbb{Z}_{\geq 0}$, which results in an "almost commutative" specialization of noncommutative Catalan numbers: $\pi(C_n) = \psi(x_n) \cdot \frac{1}{n+1}\binom{2n}{n}$.

For each $0 \leq k \leq n$ denote by $\mathcal{P}_n^k$, the set of all $P \in \mathcal{P}_n$ such that the rightmost southeast corner p of P satisfies $p = (n, y)$, where $y \leq k$. In particular, $\mathcal{P}_n^{n-1} = \mathcal{P}_n^n = \mathcal{P}_n$. For each $0 \leq k \leq n$ define *truncated noncommutative Catalan number* $C_n^k \in \mathbb{Z}F_n$ by

$$C_n^k := \sum_{P\in\mathcal{P}_n^k} M_P \ .$$

The following recursion on C_n^k is immediate.

Lemma 2.18. *$C_n^k = C_n^{k-1} + C_{n-1}^k x_{n-k-1}^{-1} x_{n-k}$ for all $1 \leq k \leq n$ (with the convention $C_n^\ell = 0$ if $\ell > n$).*

Example 2.19. $C_n^0 = x_n$, $C_n^{n-1} = C_n^n = C_n$ for all $n \geq 1$. Also, $C_n^1 = x_n + \sum\limits_{i=1}^{n-1} x_i x_{i-1}^{-1} x_{n-1}$,

$$C_n^2 = \sum_{1\leq i\leq j\leq n, j>1} x_i x_{i-1}^{-1} x_{j-1} x_{j-2}^{-1} x_{n-2} \ .$$

Sometimes it is convenient to express C_n^k via $y_i = x_i x_{i-1}^{-1}$, $i \in \mathbb{Z}_{\geq 0}$. Indeed, denote $\tilde{C}_n^k := C_n^k x_{n-k}^{-1}$ for $k, n \in \mathbb{Z}_{\geq 0}$, $k \leq n$.

The following result generalizes a number of basic properties of truncated Catalan numbers.

Proposition 2.20. *For all $0 \leq k \leq n$ one has:*

(a) $\tilde{C}_n^k = \sum\limits_{j_1\leq\ldots\leq j_k\leq n : j_1\geq 1,\ldots,j_k\geq k} y_{j_1} y_{j_2-1} \cdots y_{j_k-k+1}$.

(b) $\tilde{C}_n^k = \tilde{C}_{n-1}^k + \tilde{C}_n^{k-1} y_{n+1-k}$ *(with the convention $\tilde{C}_n^\ell = 0$ if $\ell > n$).*

(c) $\tilde{C}_{n+1}^k = \sum\limits_{i=0}^{k} \tilde{C}_i^i T(\tilde{C}_{n-i}^{k-i})$.

Example 2.21. $\tilde{C}_n^0 = 1$, $\tilde{C}_n^1 = y_1 + \cdots + y_n$, and $\tilde{C}_n^n = \tilde{C}_n^{n-1} y_1$ for all $n \geq 1$.

$$\tilde{C}_n^2 = \sum_{1 \leq i \leq j \leq n, j>1} y_i y_{j-1}, \ \tilde{C}_n^3 = \sum_{1 \leq i \leq j \leq k \leq n, \ j>1, \ k>2} y_i y_{j-1} y_{k-2} \ .$$

However, the following recursion is rather non-trivial (and we could not find its classical analogue in the literature).

Theorem 2.22. $C_n = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}: \\ a+b \leq n, a-b=d}} C_{n-b}^a x_{n-a-b}^{-1} \overline{C_{n-a}^b}$ *for each* $n \in \mathbb{Z}_{\geq 0}$ *and each* $d \in \mathbb{Z}$ *with* $|d| \leq n$ *(e.g., the right hand side does not depend on* d*).*

Remark 2.23. In particular, Theorem 2.22 provides another confirmation of $\bar{\cdot}$-invariance of noncommutative Catalan numbers (established in Proposition 2.1).

It turns out that the above "two-variable specialization" σ is also of interest for truncated noncommutative Catalan numbers. Indeed, in the notation as above, denote $\underline{C}_n^k := \sigma(C_n^k)$ and $\underline{\underline{C}}_n^k := \underline{C}_n^k x_1^{k-n}$.

The following is immediate.

Corollary 2.24. *In the notation of Proposition 2.9, one has*

(a) $\underline{C}_n^k = \sum_{P \in \mathcal{P}_n^k} \underline{M}_P$ *for all* $k, n \in \mathbb{Z}_{\geq 0}$, $k \leq n$.

(b) $\underline{\underline{C}}_n^k = \underline{\underline{C}}_n^{k-1} x_1 + \underline{\underline{C}}_{n-1}^k x_0$ *for all* $1 \leq k \leq n$ *(with the convention* $\underline{\underline{C}}_n^\ell = 0$ *if* $\ell > n$*). In particular, each* $\underline{\underline{C}}_n^k$ *is a noncommutative polynomial in* x_0, x_1 *of degree* $n + k$.

Example 2.25. $\underline{\underline{C}}_n^0 = x_0^n$, $\underline{\underline{C}}_n^1 = x_0^n x_1 + \sum_{i=1}^{n-i} x_0^i x_1 x_0^{n-1}$, $\underline{\underline{C}}_n^2 = \underline{\underline{C}}_n^1 x_1 + \sum_{1 \leq i \leq j \leq n-1, j>1} x_0^i x_1 x_0^{j-i} x_1 x_0^{n-j}$.

It turns out that our (truncated) noncommutative Catalan numbers $\tilde{C}_n^k$ admit another specialization into certain polynomials in $\mathbb{Z}_{\geq 0}[q]$ defined by Garsia and Haiman in [5]. Namely, let $\chi_q : \mathbb{Z}F \to \mathbb{Z}[q]$ be a ring homomorphism defined by $\chi_q(x_k) = q^{\frac{k(k-1)}{2}}$ for $k \geq 0$, i.e., $\chi_q(y_k) = q^{k-1}$ for $k \in \mathbb{Z}_{\geq 1}$.

Define polynomials $c_n^k(q,t) \in \mathbb{Z}_{\geq 0}[q,t]$, $0 \leq k \leq n$ recursively by $c_n^0(q,t) = 1$ and

$$c_n^k(q,t) = \sum_{r=1}^{k} \begin{bmatrix} r+n-k \\ r \end{bmatrix}_q t^{k-r} q^{\frac{r(r-1)}{2}} c_{k-1}^{k-r}(q,t) \ ,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient $\frac{[n]_q!}{[k]_q![n-k]_q!}$, $[n]_q! = [1]_q \cdots [n]_q$, $[k]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots q^{k-1}$.

These polynomials are closely related to polynomials $H_{n,k}(q,t)$ introduced by Garsia and Haglund ([6, Equation I.24]), namely, $c_n^k(q,t) = t^{-k} q^{-\frac{(n+1-k)(n-k)}{2}} H_{n+1,n+1-k}(q,t)$, in particular, $c_n^n(q,t) = c_n(q,t)$ is the celebrated (q,t)-Catalan number introduced in [5].

The following result shows that our (truncated) noncommutative Catalan numbers are noncommutative deformations of $(q,1)$-Catalan numbers.

Theorem 2.26. $\chi_q(\tilde{C}_n^k) = c_n^k(q,1)$ *for all* $k \leq n$*, in particular,* $\chi_q(C_n) = c_n(q,1)$ *for* $n \geq 0$.

Example 2.27. $\chi_q(\tilde{C}_n^1) = [n+1]_q$ and $\chi_q(\tilde{C}_n^k) = \chi_q(\tilde{C}_n^{k-1}) q^{n-k} + \chi_q(\tilde{C}_{n-1}^k)$ for $1 \leq k \leq n$.

Remark 2.28. It is curious that for another class of q-Catalan numbers, $q^{\frac{n(n-1)}{2}} c_n(q, q^{-1}) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$, there is no an analogue of Theorem 2.26. Also, it would be interesting to find an appropriate noncommutative deformations of (q,t)-Catalan numbers.

The following is a generalization of the well-known property of Hankel determinants of q-Catalan numbers.

Theorem 2.29. *For* $n \geq 1$, $m \in \{0, 1\}$ *the determinant of the* $(n+1) \times (n+1)$ *matrix* $(c_{i+j+m}(q,1))$, $i, j = 0, \ldots, n$, *is* $q^{\frac{n(n+1)(4n-1+6m)}{6}}$.

Define the *noncommutative binomial coefficients* $\left\langle\!\!\binom{n}{k}\!\!\right\rangle \in \mathbb{Z}F_{n+k-1}, \left\langle\!\!\binom{n}{k}\!\!\right\rangle' \in \mathbb{Z}F_n$ by

$$\left\langle\!\!\binom{n}{k}\!\!\right\rangle = \sum y_J,\ \left\langle\!\!\binom{n}{k}\!\!\right\rangle' = \sum y'_J$$

where each summation is over all subsets $J = \{j_1 < j_2 < \cdots < j_k\}$ of $[1,n]$ and we abbreviated $y_J = y_{j_k+k-1}\cdots y_{j_2+1}y_{j_1}$, $y'_J = y_{j_1+k-1}y_{j_2+k-3}\cdots y_{j_k+1-k}$ for $j \in \mathbb{Z}_{\geq 1}$.

Remark 2.30. The q-binomial coefficients can be expressed as $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum q^{j_1+\cdots+j_k-\frac{k(k+1)}{2}}$, where the summation is over all subsets $J = \{j_1 < j_2 < \cdots < j_k\}$ of $[1,n]$. Therefore, under the above specialization $\chi_q : \mathbb{Z}F \to \mathbb{Z}[q,q^{-1}]$ we have $\chi_q\left(\left\langle\!\!\binom{n}{k}\!\!\right\rangle\right) = q^{k(k-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q$, $\chi_q\left(\left\langle\!\!\binom{n}{k}\!\!\right\rangle'\right) = q^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q$ for all $k,n \in \mathbb{Z}_{\geq 0}$.

Example 2.31. $\left\langle\!\!\binom{n}{0}\!\!\right\rangle = \left\langle\!\!\binom{n}{0}\!\!\right\rangle' = 1$, $\left\langle\!\!\binom{n}{1}\!\!\right\rangle = \left\langle\!\!\binom{n}{1}\!\!\right\rangle' = \sum\limits_{i=1}^{n} y_i$, $\left\langle\!\!\binom{n}{2}\!\!\right\rangle = \sum\limits_{1\leq i<j\leq n} y_{j+1}y_i$, $\left\langle\!\!\binom{n}{n}\!\!\right\rangle = y_{2n-1}\cdots y_3y_1 = y_{[1,n]}$, $\left\langle\!\!\binom{n}{n-1}\!\!\right\rangle = \sum\limits_{i=1}^{n} y_{[1,n]\setminus\{i\}}$, $\left\langle\!\!\binom{n}{n-2}\!\!\right\rangle = \sum\limits_{1\leq i<j\leq n} y_{[1,n]\setminus\{i,j\}}$, $\left\langle\!\!\binom{n}{2}\!\!\right\rangle' = \sum\limits_{1\leq i<j\leq n} y_{i+1}y_{j-1}$, $\left\langle\!\!\binom{n}{n}\!\!\right\rangle' = y_ny_{n-1}\cdots y_1 = y'_{[1,n]}$, $\left\langle\!\!\binom{n}{n-1}\!\!\right\rangle' = \sum\limits_{i=1}^{n} y'_{[1,n]\setminus\{i\}}$, $\left\langle\!\!\binom{n}{n-2}\!\!\right\rangle' = \sum\limits_{1\leq i<j\leq n} y'_{[1,n]\setminus\{i,j\}}$.

Clearly, $\varepsilon\left(\left\langle\!\!\binom{n}{k}\!\!\right\rangle\right) = \varepsilon\left(\left\langle\!\!\binom{n}{k}\!\!\right\rangle'\right) = \binom{n}{k}$ and $\left\langle\!\!\binom{n}{k}\!\!\right\rangle = \left\langle\!\!\binom{n}{k}\!\!\right\rangle' = 0$ if $k \notin [0,n]$.

Similarly to the classical case, we have an analogue of the Pascal triangle and the multiplication law for noncommutative binomial coefficients.

Theorem 2.32. $\left\langle\!\!\binom{m+n}{k}\!\!\right\rangle = \sum\limits_{\substack{a,b\in\mathbb{Z}_{\geq 0}:\\ a+b=k}} T^{n+b}\left(\left\langle\!\!\binom{m}{a}\!\!\right\rangle\right)\left\langle\!\!\binom{n}{b}\!\!\right\rangle$, $\left\langle\!\!\binom{m+n}{k}\!\!\right\rangle' = \sum\limits_{\substack{a,b\in\mathbb{Z}_{\geq 0}:\\ a+b=k}} T^b\left(\left\langle\!\!\binom{m}{a}\!\!\right\rangle'\right)T^{m-a}\left(\left\langle\!\!\binom{n}{b}\!\!\right\rangle'\right)$ *for* $m,n,k \in \mathbb{Z}_{\geq 0}$. *In particular,* $\left\langle\!\!\binom{n+1}{k}\!\!\right\rangle = \left\langle\!\!\binom{n}{k}\!\!\right\rangle + y_{n+k}\left\langle\!\!\binom{n}{k-1}\!\!\right\rangle$, $\left\langle\!\!\binom{n+1}{k}\!\!\right\rangle' = T\left(\left\langle\!\!\binom{n}{k}\!\!\right\rangle'\right) + y_k\left(\left\langle\!\!\binom{n}{k-1}\!\!\right\rangle'\right)$ *for all* $n,k \in \mathbb{Z}_{\geq 0}$.

Actually, Theorem 2.32 together with the recursion from Proposition 2.20(b) imply the following analogue of the multiplication law for the truncated noncommutative Catalan numbers, which justified the introduction of noncommutative binomial coefficients of the "second kind."

Corollary 2.33. $\tilde{C}^k_{m+n} = \sum\limits_{\ell=0}^{n} \tilde{C}^{k-\ell}_{m+\ell}\cdot T^{m-k+\ell}\left(\left\langle\!\!\binom{n}{\ell}\!\!\right\rangle'\right)$ *for all* $m,n,k \in \mathbb{Z}_{\geq 0}$.

The following relation between truncated noncommutative Catalan numbers and the binomial coefficients of the "first kind" is rather surprising.

Theorem 2.34. $\sum\limits_{j=0}^{k}(-1)^j\tilde{C}^j_{n+k-j}\cdot\left\langle\!\!\binom{n-j}{k-j}\!\!\right\rangle = 0$ *for any* $0 < k \leq n$.

Remark 2.35. In fact, there is an accompanying identity $\sum\limits_{j=0}^{k}(-1)^j\left\langle\!\!\binom{n+k-j}{j}\!\!\right\rangle\cdot\tilde{C}^{k-j}_{n-j} = 0$ for any $0 < k \leq n$, which follows from Theorem 2.43 below. We leave this as an exercise to the readers.

This turns out to be equivalent to the following "determinantal" identities between noncommutative truncated Catalan numbers and binomial coefficients (whose classical analogues also seem to be new).

Theorem 2.36. *For all* $k,n \in \mathbb{Z}_{\geq 0}$, $k \leq n$ *one has* $\tilde{C}^k_n = \sum_J(-1)^{k+1-|J|}M_{n,J}$, $\left\langle\!\!\binom{n}{k}\!\!\right\rangle = \sum_J(-1)^{k+1-|J|}\tilde{M}_{n,J}$, *where each summation is over all subsets* $J = \{0 = j_0 < \cdots < j_\ell = k\}$ *of* $[0,k]$ *and*

$$M_{n,J} = \left\langle\!\!\binom{n+j_{\ell-1}+j_\ell-k}{j_\ell-j_{\ell-1}}\!\!\right\rangle\cdots\left\langle\!\!\binom{n+j_1+j_2-k}{j_2-j_1}\!\!\right\rangle\left\langle\!\!\binom{n+j_0+j_1-k}{j_1-j_0}\!\!\right\rangle,$$

$$\tilde{M}_{n,J} = \tilde{C}^{j_1-j_0}_{n+j_0+j_1-k} \cdot \tilde{C}^{j_2-j_1}_{n+j_1+j_2-k} \cdots \tilde{C}^{j_\ell-j_{\ell-1}}_{n+j_{\ell-1}+j_\ell-k} \ .$$

Actually, Theorems 2.26, 2.34, and 2.36 hint to some remarkable properties of Hankel matrices with noncommutative Catalan numbers as entries.

For $m, n \in \mathbb{Z}_{\geq 0}$, $n \geq 0$ define the $(n+1) \times (n+1)$ matrix $H^n_m =$ over $\mathbb{Z}F$ whose (i,j)-th entry is C_{m+i+j}, $i, j = 0, \ldots, n$ (with the convention $C_k = 0$ whenever $k < 0$).

Example 2.37. $H^1_0 = \begin{pmatrix} C_0 & C_1 \\ C_1 & C_2 \end{pmatrix}$, $H^1_1 = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_3 \end{pmatrix}$, $H^2_0 = \begin{pmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{pmatrix}$, $H^2_1 = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{pmatrix}$.

We refer to H^n_m as *noncommutative Hankel-Catalan matrix* by analogy with its classical counterpart $\varepsilon(H_{m,n}) \in Mat_{n+1,n+1}(\mathbb{Z})$.

We will finish the section by showing that each H^n_m, $m \in \{0,1\}$, $n \geq 0$ admits a Gauss factorization over $\mathbb{Z}F$ involving truncated noncommutative Catalan numbers and it inverse (which is also a matrix over $\mathbb{Z}F$) is given by an interesting combinatorial formula involving our noncommutative binomial coefficients.

For $m \in \{0,1\}$ let L_m be the lower unitriangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix whose (j,i)-th entry, $0 \leq i \leq j$, is $\tilde{C}^{i-j}_{i+j+m}$ and let U_m be the upper triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix whose (i,j)-th entry, $0 \leq i \leq j$, $\overline{C^{j-i}_{i+j+m}}$).

Theorem 2.38. *$H_m = L_m \cdot U_m$ for each $m \in \{0,1\}$.*

Remark 2.39. Most apparently, the classical version of this result, $\varepsilon(H_m) = \varepsilon(L_m) \cdot \varepsilon(U_m)$, is new.

Theorem 2.39 and [8, Theorem 4.9.7] imply the following immediate corollary.

Corollary 2.40. *C^{j-i}_{m+i+j} equals the quasideterminant* $\begin{vmatrix} C_m & C_{m+1} & \ldots & C_{m+i} \\ C_{m+1} & C_{m+2} & \ldots & C_{m+i+1} \\ & & \ldots & \\ C_{m+i-1} & C_{m+i} & \ldots & C_{m+2i-1} \\ C_{m+j} & C_{m+j+1} & \ldots & \boxed{C_{m+i+j}} \end{vmatrix}$ *for $0 \leq i \leq j$, $m \in \{0,1\}$ (see* [9, 10] *for notation). In particular,*

$$\begin{vmatrix} C_m & C_{m+1} & \ldots & C_{m+n} \\ C_{m+1} & C_{m+2} & \ldots & C_{m+n+1} \\ & & \ldots & \\ C_{m+n} & C_{m+n+1} & \ldots & \boxed{C_{m+2n}} \end{vmatrix} = x_{m+2n} \tag{2.7}$$

for all $n \in \mathbb{Z}_{\geq 0}$, $m \in \{0,1\}$.

Remark 2.41. In fact, (2.7) is noncommutative generalization of the well-known fact that $\det(\varepsilon(H^n_0)) = \det(\varepsilon(H^n_1)) = 1$ for $n \geq 0$. Moreover similarly to the classical case, noncommutative Catalan numbers are uniquely determined by equations (2.7) for $n \in \mathbb{Z}_{\geq 0}$, $m \in \{0,1\}$.

Remark 2.42. Noncommutative Hankel quasideterminants were introduced in [7] in the context of inversion of noncommutative power series. In fact, [7, Corollary 8.3] asserts that such an inverse can be expressed via continued fractions involving such quasideterminants of the coefficients of the series in question. This correlates with Remark 2.6 above.

For $m \in \{0,1\}$ let L^-_m be the lower unitriangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix whose (j,i)-th entry, $0 \leq i \leq j$, is $(-1)^{i+j}\left\langle\!\!\binom{i+j+m}{j-i}\!\!\right\rangle$ and let U^-_m be the upper triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix whose (i,j)-th entry, $0 \leq i \leq j$, is $(-1)^{i+j}\overline{\left\langle\!\!\binom{i+j+m}{j-i}\!\!\right\rangle x^{-1}_{2j+m}}$.

For any $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix M denote by $M|_n$ the principal $(n+1) \times (n+1)$-submatrix of M (e.g., $H^n_m = H_m|_n$).

Theorem 2.43. *$(U_m)^{-1} = U^-_m$ and $(L_m)^{-1} = L^-_m$, hence $(H^n_m)^{-1} = U^-_m|_n \cdot L^-_m|_n$ for $m \in \{0,1\}$, $n \geq 1$.*

Remark 2.44. Similarly to Remark 2.39 the classical version of this result, $\varepsilon(H^n_m)^{-1} = \varepsilon(L^-_m|_n) \cdot \varepsilon(U^-_m)|_n$, seems to be new.

Computation of H^{-1}_m for $m \geq 2$ is more challenging task, which we will perform elsewhere.

3. Proofs of main results

3.1. **Proof of Propositions 2.1, 2.3, 2.9, 2.20 and Theorems 2.22, 2.26.** Prove Proposition 2.1 first. Define an involution $s_n : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $s_n(x, y) = (n - y, n - x)$. Clearly, $s_n(\mathcal{P}_n) = \mathcal{P}_n$. It is easy to see that

$$\overline{M}_P = M_{s_n(P)} \tag{3.1}$$

for all $P \in \mathcal{P}_n$. Therefore, $\overline{C}_n = \sum_{P\in\mathcal{P}_n} \overline{M}_P = \sum_{P\in\mathcal{P}_n} M_{s_n(P)} = \sum_{P\in\mathcal{P}_n} M_P = C_n$ for all $n \geq 0$.

The proposition is proved. □

Prove Proposition 2.20 now. Prove (a) first. Denote by $\mathbf{J}_n^k$ the set of all sequences $\mathbf{j} = (j_1, \ldots, j_k) \in \mathbb{Z}^k$ such that $j_1 \leq \ldots \leq j_k \leq n$ and $j_1 \geq 1, \ldots, j_k \geq k$.

For each $P \in \mathcal{P}_n^k$ and $s \in [1, k]$ denote by $j_s(P)$ the minimum of x-coordinates of all points in P whose y-coordinate is s. For each $\mathbf{j} = (j_1, \ldots, j_k) \in \mathbb{Z}^k$ with $j_s \geq s$, $s \in [1, k]$ we abbreviate $y_\mathbf{j} = y_{j_1} y_{j_2-1} \ldots y_{j_k-k+1}$.

The following is immediate.

Lemma 3.1. *For all $k, n \in \mathbb{Z}_{\geq 0}$, $k \leq n$ one has:*

(a) The assignments $P \mapsto \mathbf{j}(P) := (j_1(P), \ldots, j_k(P))$ defines a bijection $\mathcal{P}_n^k \widetilde{\to} \mathbf{J}_n^k$.

(b) For each $P \in \mathcal{P}_n^k$ we have $M_P x_{n-k}^{-1} = y_{\mathbf{j}(P)}$.

Using Lemma 3.1(b), we obtain $\tilde{C}_n^k = \sum_{\mathbf{j}\in\mathbf{J}_n^k} y_\mathbf{j}$ and thus finish the proof of (a).

Prove (b). It is easy to see that $\mathbf{J}_n^k = \mathbf{J}_{n-1}^k \sqcup (\mathbf{J}_n^{k-1}, n)$. Therefore,

$$\tilde{C}_n^k = \sum_{\mathbf{j}\in\mathbf{J}_n^k} y_\mathbf{j} = \sum_{\mathbf{j}\in\mathbf{J}_n^k} y_\mathbf{j} + \sum_{\mathbf{j}\in(\mathbf{J}_n^{k-1},n)} y_\mathbf{j} = \tilde{C}_{n-1}^k + \tilde{C}_n^{k-1} y_{n+1-k} \ .$$

This proves (b).

Prove (c) now. We need the following result.

Lemma 3.2. $\mathbf{J}_{n+1}^k = \bigsqcup_{i=0}^{k} \mathbf{J}_i^i \times T^{i+1}(\mathbf{J}_{n-i}^{k-i})$ *for all $k, n \in \mathbb{Z}_{\geq 0}$, $0 \leq k \leq n$, where $T = T_r : \mathbb{Z}^r \to \mathbb{Z}^r$, $r \geq 1$ is the translation given by $x \mapsto x + \underbrace{(1, \ldots, 1)}_{r}$.*

Proof. For each $\mathbf{j} = (j_1, \ldots, j_k) \in \mathbf{J}_{n+1}^k$ denote by $i_\mathbf{j}$ the largest $i \in [1, k]$ such that $j_i = i$ and set $i(\mathbf{j}) := 0$ if such an i does not exist. This implies that, $\{\mathbf{j} \in \mathbf{J}_n^k : i_\mathbf{j} = i\} = \mathbf{J}_i^i \times T^{i+1}(\mathbf{J}_{n-i-1}^{k-i})$ for all $i \in [0, k]$ (the first factor is empty for $i = 0$).

The lemma is proved. □

Taking into account that for $\mathbf{j} = (\mathbf{j}', T^{i+1}(\mathbf{j}'')) \in \mathbf{J}_i^i \times T^{i+1}(\mathbf{J}_{n-i}^{k-i})$, we have $y_\mathbf{j} = y_{\mathbf{j}'} T(y_{\mathbf{j}''})$, we obtain:

$$\tilde{C}_{n+1}^k = \sum_{\mathbf{j}\in\mathbf{J}_{n+1}^k} y_\mathbf{j} = \sum_{i\in[0,k],\mathbf{j}'\in\mathbf{J}_i^i,\mathbf{j}''\in\mathbf{J}_{n-i}^{k-i}} y_{\mathbf{j}'} T(y_{\mathbf{j}''}) = \sum_{i=0}^{k} \tilde{C}_i^i T(\tilde{C}_{n-i}^{k-i}) \ .$$

This proves (c).

Proposition 2.20 is proved. □

Prove Theorem 2.26 now. Applying χ_q to $\tilde{C}_{n+1}^k$ given by Proposition 2.20(c) and using the fact that $\chi_q(T(y)) = q^d \chi_q(y)$ for any homogeneous noncommutative polynomial of degree d in $y_1, y_2, \ldots$, we obtain:

$$\chi_q(\tilde{C}_{n+1}^k) = \sum_{i=0}^{k} \chi_q(\tilde{C}_i^i) \chi_q(\tilde{C}_{n-i}^{k-i})$$

for all $0 \leq k \leq n$.

In view of [11, Equation (3.41)] and the fact that $F_{n,k}(q, t) = H_{n,k}(q, t) = t^{n-k} q^{\frac{k(k-1)}{2}} c_{n-1}^{n-k}(q, t)$ for all $0 \leq k < n$, we obtain same recursion $c_{n+1}^k = \sum_{i=0}^{k} c_i^i(q, 1) q^{k-i} c_{n-i}^{k-i}(q, 1)$ for all $0 \leq k \leq n$. Using this and taking into account that $\chi_q(\tilde{C}_{n+1}^n) = \chi_q(\tilde{C}_{n+1}^{n+1})$, we conclude that $\chi_q(\tilde{C}_n^k) = c_n^k(q, 1)$ for all $0 \leq k \leq n$.

The theorem is proved. □

Prove Proposition 2.3 now. Indeed, taking into account that $C_r = \tilde{C}_r^r \cdot x_0 = \tilde{C}_r^{r-1} y_1 x_0$ for all $r \geq 1$, we see that the first identity (2.3) is equivalent to $\tilde{C}_{n+1}^n = \sum_{k=0}^n \tilde{C}_k^k T(\tilde{C}_{n-k}^{n-k})$ which coincides with the assertion of Proposition 2.20(c) with $k = n$.

The second identity (2.3) follows from the first one and Proposition 2.1 by applying the anti-involution $\bar{\cdot}$.

Proposition 2.3 is proved. □

Prove Proposition 2.9 now. We say that $x \in F$ is alternating if it is of the form $x_{i_1} x_{i_2}^{-1} x_{i_3} \dots x_{i_{s-1}}^{-1} x_{i_s}$ for some $i_1, \dots, i_s \in \mathbb{Z}_{\geq 0}$ and denote by F^{alt} the set of all alternating elements in F. We also denote by $\mathbb{Z}F^{alt}$ the $\mathbb{Z}$-linear span of F^{alt} in $\mathbb{Z}F$. We need the following fact.

Lemma 3.3. *$\sigma(T(x)) = x_0 \sigma(x) x_1$ for all For $x \in \mathbb{Z}F^{alt}$.*

Proof. We first prove the assertion for all $x \in F^{alt}$. Indeed,let $x = x_{i_1} x_{i_2}^{-1} x_{i_3} \dots x_{i_{s-1}}^{-1} x_{i_s}$ for some $i_1, i_2, \dots, i_s \geq 0$. We have $\sigma(T(x)) = x_{i_1+1} x_{i_2+1}^{-1} x_{i_3+1} \dots x_{i_{s-1}+1}^{-1} x_{i_s+1}$

$$= (x_0^{i_1+1} x_1^{i_1+1})(x_0^{i_2+1} x_1^{i_2+1})^{-1} (x_0^{i_3+1} x_1^{i_3+1}) \dots (x_0^{i_{s-1}+1} x_1^{i_{s-1}+1})^{-1} (x_0^{i_s+1} x_1^{i_s+1}) =$$

$$= x_0 \cdot (x_0^{i_1} x_1^{i_1})(x_0^{i_2} x_1^{i_2})^{-1} (x_0^{i_3} x_1^{i_3}) \dots (x_0^{i_{s-1}} x_1^{i_{s-1}})^{-1} (x_0^{i_s} x_1^{i_s}) \cdot x_1 = x_0 \sigma(x) x_1 \ .$$

By linearity of σ we obtain the assertion for all $x \in \mathbb{Z}F^{alt}$.

The lemma is proved. □

Since each C_k belongs to $\mathbb{Z}F^{alt}$, Lemma 3.3 implies that $\sigma(T(C_k)) = x_0 \sigma(C_k) x_1 = x_0 \underline{C}_k x_1$ for all $k \geq 0$. Using this and applying σ to the first identity (2.3), we obtain (2.5).

Proposition 2.9 is proved. □

Prove Theorem 2.22 now. In the notation of the proof of Proposition 2.20, for all $0 \leq k \leq n$ denote by $\mathbf{J}_{n,k}$ the set of all $\mathbf{j} = (j_1, \dots, j_n) \in \mathbf{J}_n^n$ such that $j_1 \geq n - k$.

Lemma 3.4. $\overline{C}_n^k \cdot x_0^{-1} = \sum\limits_{\mathbf{j} \in \mathbf{J}_{n,k}} y_{\mathbf{j}}$ *for all* $0 \leq k \leq n$.

Proof. Indeed, in view of (3.1), we obtain using Lemma 3.1(b):

$$\overline{C_n^k} x_0^{-1} = \sum_{P \in \mathcal{P}_n^k} \overline{M}_P \cdot x_0^{-1} = \sum_{P \in \mathcal{P}_n^k} M_{s_n(P)} \cdot x_0^{-1} = \sum_{P \in \mathcal{P}_n^k} y_{\mathbf{j}(s_n(P))} = \sum_{\mathbf{j} \in \mathbf{J}_{n,k}} y_{\mathbf{j}}$$

because $\mathbf{J}_{n,k} = \mathbf{j}(s_n(\mathcal{P}_n^k))$.

The lemma is proved. □

Furthermore, after multiplying by x_0^{-1} on the right, the assertion of Theorem 2.22 is equivalent to:

$$\tilde{C}_n^n = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}: \\ a+b \leq n, a-b=d}} \tilde{C}_{n-b}^a \cdot (\overline{C_{n-a}^b} x_0^{-1}) \tag{3.2}$$

for each $n \in \mathbb{Z}_{\geq 0}$ and each $d \in \mathbb{Z}$ with $|d| \leq n$.

Prove (3.2). Indeed, fix $d \in [1-n, n-1]$. The following is immediate.

Lemma 3.5. *For each $\mathbf{j} = (j_1, \dots, j_n) \in \mathbf{J}_n^n$ there exists a unique $a = a(\mathbf{j}, d) \in [\max(0, d), n]$ such that $j_a \leq n + d - a \leq j_{a+1}$ (with the convention $j_0 = 0$, $j_{n+1} = \infty$).*

For $a \in [\max(0, d), n]$ denote by $\mathbf{J}_n^n(a, d)$ the set of all $\mathbf{j} \in \mathbf{J}_n^n$ such that $a(\mathbf{j}, d) = a$.

The following is immediate (in the notation of Lemma 3.2).

Lemma 3.6. $\mathbf{J}_n^n(a, d) = \mathbf{J}_{n+d-a}^a \times T^a(\mathbf{J}_{n-a, a-d})$.

Using these results and taking into account that for $\mathbf{j} = (\mathbf{j}', T^a(\mathbf{j}'')) \in \mathbf{J}_{n+d-a}^a \times T^a(\mathbf{J}_{a-d, n-a})$, we have $y_{\mathbf{j}} = y_{\mathbf{j}'} y_{\mathbf{j}''}$, we obtain: $\tilde{C}_{n+1}^n = \sum\limits_{\mathbf{j} \in \mathbf{J}_n^n} y_{\mathbf{j}} = \sum\limits_{\substack{a \in [\max(0,d), n], \\ \mathbf{j}' \in \mathbf{J}_{n+d-a}^a, \mathbf{j}'' \in \mathbf{J}_{n-a, a-d}}} y_{\mathbf{j}'} y_{\mathbf{j}''} = \sum_{a \in [\max(0,d), n]} \tilde{C}_{n+d-a}^a \cdot (\overline{C_{n-a}^{a-d}} x_0^{-1})$.

This proves (3.2).

Theorem 2.22 is proved.

3.2. **Proof of Theorems 2.32 and 2.34.** For any set X and $k \geq 0$ denote by $\left\{ {X \atop k} \right\}$ the set of all subsets $J \subset X$ of cardinality $|J| = k$. Clearly, $\left\{ {[1,m+n] \atop k} \right\} = \bigsqcup_{\substack{a,b\in\mathbb{Z}_{\geq 0}:\\ a+b=k}} \left\{ {[1,m] \atop a} \right\} \times T^m \left(\left\{ {[1,n] \atop b} \right\} \right)$ for all $m,n,k \in \mathbb{Z}_{\geq 0}$ in the notation of Lemma 3.2, where we view each $J \in \left\{ {[1,n] \atop k} \right\}$ naturally as an element of $\mathbb{Z}^b$.

Taking into account that for $J = (J', T^m(J'')) \in \left\{ {[1,m] \atop a} \right\} \times T^m \left(\left\{ {[1,n] \atop b} \right\} \right)$, $a+b=k$, we have $y_J = T^{m+a}(y_{J''})y_{J'}$ and $y'_J = T^b(y'_{J'})T^{m-a}(y'_{J''})$, we obtain:

$$\left(\!\! {m+n \atop k} \!\! \right) = \sum_{J \in \left\{ {[1,m+n] \atop k} \right\}} y_J = \sum_{\substack{a,b\in\mathbb{Z}_{\geq 0}: a+b=k,\\ J' \in \left\{ {[1,m] \atop k} \right\}, J'' \in \left\{ {[1,n] \atop k} \right\}}} T^{m-a}(y_{J''})y_{J'} = \sum_{\substack{a,b\in\mathbb{Z}_{\geq 0}:\\ a+b=k}} T^{m+a}\left(\left(\!\! {n \atop b} \!\! \right) \right) \left(\!\! {m \atop a} \!\! \right),$$

$$\left(\!\! {m+n \atop k} \!\! \right)' = \sum_{J \in \left\{ {[1,m+n] \atop k} \right\}} y'_J = \sum_{\substack{a,b\in\mathbb{Z}_{\geq 0}: a+b=k,\\ J' \in \left\{ {[1,m] \atop k} \right\}, J'' \in \left\{ {[1,n] \atop k} \right\}}} T^b(y'_{J'})T^{m-a}(y'_{J''}) = \sum_{\substack{a,b\in\mathbb{Z}_{\geq 0}:\\ a+b=k}} T^b\left(\left(\!\! {m \atop a} \!\! \right)' \right) T^{m-a}\left(\left(\!\! {n \atop b} \!\! \right)' \right)$$

for $m,n,k \in \mathbb{Z}_{\geq 0}$.

Theorem 2.32 is proved. □

Prove Theorem 2.34 now. For each $0 \leq j \leq k \leq n$ denote by $\mathbf{I}_{j,k;n}$ the set of all $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{Z}^k_{\geq 1}$ such that $i_j \leq n+k+1-2j$, $i_{j+1} \leq n+k-1-2j$, $i_s \leq i_{s+1}+1$ for all $s \in [1,j]$, and $i_s > i_{s+1}+1$ for all $s \in [j+1,k]$. (with the convention that if $j \in \{0,k\}$, then meaningless inequalities are omitted and $\mathbf{I}_{-1,k;n} = \mathbf{I}_{k+1,k;n} = \emptyset$).

The following is immediate.

Lemma 3.7. $\tilde{C}^j_{n+k-j} \cdot \left(\!\! {n-j \atop k-j} \!\! \right) = \sum_{\mathbf{i} \in \mathbf{I}_{j,k;n}} Y_{\mathbf{i}}$ *for all* $0 \leq j \leq k$, *where we abbreviate* $Y_{\mathbf{i}} := y_{i_1} \cdots y_{i_k}$.

For $j \in [0, k+1]$ denote $\mathbf{I}^-_{j,k;n} = \mathbf{I}_{j-1,k;n} \cap \mathbf{I}_{j,k;n}$.

By definition, $\mathbf{I}^-_{0,k} = \mathbf{I}^-_{k+1,k} = \emptyset$ and the following is immediate.

Lemma 3.8. $\mathbf{I}^-_{j,k;n}$ *is the set of all* $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbf{I}_{j,k;n}$ *such that* $i_j \leq i_{j+1}+1$ *for all* $j \in [0,k]$. *In particular,* $\mathbf{I}_{j,k;n} = \mathbf{I}^-_{j,k;n} \sqcup \mathbf{I}^-_{j+1,k;n}$ *for* $j \in [0,k]$.

Using Lemmas 3.7 and 3.8, we obtain:

$$\sum_{j=0}^{k} (-1)^j \tilde{C}^j_{n+k-j} \cdot \left(\!\! {n-j \atop k-j} \!\! \right) = \sum_{j\in[0,k], \mathbf{i} \in \mathbf{I}_{j,k;n}} (-1)^j Y_{\mathbf{i}} = \sum_{j\in[0,k], \mathbf{i} \in \mathbf{I}^-_{j,k;n}} (-1)^j Y_{\mathbf{i}} + \sum_{j\in[0,k], \mathbf{i} \in \mathbf{I}^-_{j+1,k;n}} (-1)^j Y_{\mathbf{i}} = 0$$

for all $0 < k \leq n$.

Theorem 2.34 is proved. □

3.3. **Proof of Theorems 2.38 and 2.43.** Prove Theorem 2.38 first. Indeed, the assertion is equivalent to $(H_m)_{ij} = \sum_{k=0}^{\min(i,j)} (L_m)_{ik}(U_m)_{kj}$, i.e., to $C_{m+i+j} = \sum_{k=0}^{\min(i,j)} C^{i-k}_{i+k+m} \cdot x^{-1}_{2k+m} \overline{C^{j-k}_{k+j+m}}$ for all $i,j \in \mathbb{Z}_{\geq 0}$, $m \in \{0,1\}$. This identity coincides with that from Theorem 2.22 taken with $n = m+i+j$, $a = i-k$, $b = j-k$, $d = i-j$.

Theorem 2.38 is proved.

Prove Theorem 2.43 now. It suffices to do so only for L^-_m (the argument for U^-_m is identical). Indeed, the assertion is equivalent to $\sum_{k'=i'}^{j'} (L_m)_{j'k'}(L^-_m)_{k'i'} = \delta_{i'j'}$, i.e., to $\sum_{k'=i'}^{j'} \tilde{C}^{j'-k'}_{j'+k'+m} \cdot (-1)^{i'+k'} \left(\!\! {i'+k'+m \atop k'-i'} \!\! \right) = 0$ for all $0 \leq i' < j'$. It is easy to show that this identity coincides with that from Theorem 2.34 taken with $n = i'+j'+m$, $j = j'-k'$ $k = j'-i'$.

Theorem 2.43 is proved.

3.4. **Proof of Theorems 2.29, 2.36.** Prove Theorem 2.36. We need the following well-known fact.

Lemma 3.9. *Any lower unitriangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix $A = (a_{ij})$ over an associative unital ring $\mathcal{A}$ is invertible and $(A^{-1})_{ji} = \sum_{j=i_1>i_2>\cdots>i_k=i, k\geq 1} (-1)^{k-1} a_{i_1,i_2} \cdots a_{i_{k-1},i_k}$ for all $1 \leq i \leq j \leq n$.*

Applying Lemma 3.9 with $A = L_m^-$, i.e., $a_{ji} = \tilde{C}_{i+j+m}^{i-j}$ and using Theorem 2.43 which asserts that $(A^{-1})_{ji} = (-1)^{i+j} \left\langle\!\!\left\langle {i+j+m \atop j-i} \right\rangle\!\!\right\rangle$, we obtain the first identity of the theorem. Swapping A and A^{-1}, we obtain the second one.

Theorem 2.36 is proved. □

Prove Theorem 2.29 now. Recall from [9] that for any matrix over a commutative ring, its determinant equals the product its of principal quasiminors. Let $\underline{H}_m^n = \chi_q(H_m^n) = (c_{i+j+m}(q,1))$, $i,j = 0,\ldots,n$, where $\chi_q : \mathbb{Z}F \to \mathbb{Z}[q,q^{-1}]$ is the homomorphism defined in Section 2. Since all principal submatrices of $\underline{H}_m^n$ are $\underline{H}_m^k$, $k = 0,1,\ldots,n$, these and Corollary 2.40 guarantee that

$$\det(\underline{H}_m^n) = \prod_{k=0}^{n} \chi_q(x_{m+2k}) = q^{\sum_{k=0}^{m} \frac{(m+2k)(m+2k-1)}{2}} = q^{\frac{n(n+1)(4n-1+6m)}{6}} .$$

Theorem 2.29 is proved. □

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