SMALL REPRESENTATIONS, STRING INSTANTONS, AND FOURIER MODES OF EISENSTEIN SERIES

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WITH APPENDIX "SPECIAL UNIPOTENT REPRESENTATIONS" BY DAN CIUBOTARU AND PETER E. TRAPA

Abstract. This paper concerns some novel features of maximal parabolic Eisenstein series at certain special values of their analytic parameter, s. These series arise as coefficients in the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions in the low energy expansion of the scattering amplitudes in maximally supersymmetric string theory reduced to $D = 10 - d$ dimensions on a torus, T^d $(0 \leq d \leq 7)$. For each d these amplitudes are automorphic functions on the rank $d+1$ symmetry group E_{d+1}.

Of particular significance is the orbit content of the Fourier modes of these series when expanded in three different parabolic subgroups, corresponding to certain limits of string theory. This is of interest in the classification of a variety of instantons that correspond to minimal or "next-to-minimal" BPS orbits. In the limit of decompactification from D to $D+1$ dimensions many such instantons are related to charged $\frac{1}{2}$-BPS or $\frac{1}{4}$-BPS black holes with euclidean world-lines wrapped around the large dimension. In a different limit the instantons give nonperturbative corrections to string perturbation theory, while in a third limit they describe nonperturbative contributions in eleven-dimensional supergravity.

A proof is given that these three distinct Fourier expansions have certain vanishing coefficients that are expected from string theory. In particular, the Eisenstein series for these special values of s have markedly fewer Fourier coefficients than typical maximal parabolic Eisenstein series. The corresponding mathematics involves showing that the wavefront sets of the Eisenstein series in question are supported on only a limited number of coadjoint nilpotent orbits – just the minimal and trivial orbits in the $\frac{1}{2}$-BPS case, and just the next-to-minimal, minimal and trivial orbits in the $\frac{1}{4}$-BPS case. Thus as a byproduct we demonstrate that the next-to-minimal representations occur automorphically for E_6, E_7, and E_8, and hence the first two nontrivial low energy coefficients in scattering amplitudes can be thought of as exotic θ-functions for these groups. The proof includes an appendix by Dan Ciubotaru and Peter E. Trapa which calculates wavefront sets for these and other special unipotent representations.

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Contents

1. Introduction

String theory is expected to be invariant under a very large set of discrete symmetries ("dualities"), associated with arithmetic subgroups of a variety of reductive Lie groups. For example, maximally supersymmetric string theory (type II superstring theory), compactified on a d-torus to $D = 10-d$ space-time dimensions, is strongly conjectured to be invariant under $E_{d+1}(\mathbb{Z})$, the integral points of the rank $d+1$ split real form[1] of one of the groups in the sequence E_8, E_7, E_6, $SO(5,5)$, $SL(5)$, $SL(3)\times SL(2)$, $SL(2)\times\mathbb{R}^+$, $SL(2)$ listed in table 1.

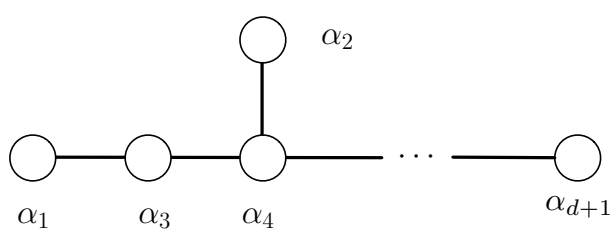

Figure 1. The Dynkin diagram for the rank $d+1$ Lie group E_{d+1}, which defines the symmetry group for $D = 10-d$.

[1]The split real forms are conventionally denoted $E_{n(n)}$, but in this paper we will truncate this to E_n since no other forms of E_n are needed.

These symmetries severely constrain the dependence of string scattering amplitudes on the symmetric space coordinates (or "moduli"), ϕ_{d+1}, which parameterise the coset E_{d+1}/K_{d+1}, where the stabiliser K_{d+1} is the maximal compact subgroup of E_{d+1}. The list of these symmetry[2] groups and stabilisers is given in table 1. These moduli are scalar fields that are interpreted as coupling constants in string theory. A general consequence of the dualities is that scattering amplitudes are functions of ϕ_{d+1} that must transform as automorphic functions under the appropriate duality group $E_{d+1}(\mathbb{Z})$. It is difficult to determine the precise restrictions these dualities impose on general amplitudes, but certain exact properties have been obtained in the case of the four-graviton interactions, where a considerable amount of information has been obtained for the first three terms in the low energy (or "derivative") expansion of the four graviton scattering amplitude in [1] (and references cited therein). These are described by terms in the effective action of the form

$$\mathcal{E}^{(D)}_{(0,0)}(\phi_{d+1})\,\mathcal{R}^4\,, \quad \mathcal{E}^{(D)}_{(1,0)}(\phi_{d+1})\,\partial^4\,\mathcal{R}^4\,, \quad \mathcal{E}^{(D)}_{(0,1)}(\phi_{d+1})\,\partial^6\,\mathcal{R}^4\,, \tag{1.1}$$

where the symbol $\mathcal{R}^4$ indicates a contraction of four powers of the Riemann tensor with a standard rank 16 tensor. The coefficient functions, $\mathcal{E}^{(D)}_{(p,q)}(\phi_{d+1})$, are automorphic functions that are the main focus of our interests (the notation is taken from [1, 2] and will be reviewed later in (2.3)). More precisely we will focus on the three terms shown in (1.1) that are protected by supersymmetry, which accounts for the relatively simple form of their coefficients.

The coefficients of the first two terms satisfy Laplace eigenvalue equations (2.6-2.7) and are subject to specific boundary conditions that are required for consistency with string perturbation theory and M-theory. The solutions to these equations are particular maximal parabolic Eisenstein series that were studied in [2] (for cases with rank ≤ 5) and [1] (for the E_6, E_7 and E_8 cases), and will be reviewed in the next section. The required boundary conditions in each limit amount to conditions on the constant terms in the expansion of these series in three limits associated with particular maximal parabolic subgroups of relevance to the string theory analysis. Such subgroups have the form $P_\alpha = L_\alpha\, U_\alpha$, where α labels a simple root, U_α is the unipotent radical and $L_\alpha = GL(1) \times M_\alpha$ is the Levi factor. The three subgroups of relevance here have Levi factors $L_{\alpha_1} = GL(1) \times SO(d,d)$, $L_{\alpha_2} = GL(1) \times SL(d+1)$, and $L_{\alpha_{d+1}} = GL(1) \times E_d$, respectively. In each case the $GL(1)$ parameter, r, can be thought of as measuring the distance to the cusp[3], as will be discussed in the next section. A key feature of the boundary conditions is that they require these constant terms to have very

[2]The continuous groups, $E_{d+1}(\mathbb{R})$, will be referred to as *symmetry groups* while the discrete arithmetic subgroups, $E_{d+1}(\mathbb{Z})$, will be referred to as *duality groups*.

[3]Each of the groups we are considering has a single cusp. The various limits correspond to different ways of approaching this cusp.

D	$E_{d+1}(\mathbb{R})$	K_{d+1}	$E_{d+1}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$	$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5)\times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

TABLE 1. The symmetry groups of maximal supergravity in $D = 10 - d \leq 10$ dimensions. The group $E_{d+1}(\mathbb{R})$ is a split real form of rank $d+1$, and K_{d+1} is its maximal compact subgroup. In string theory these groups are broken to the discrete subgroups, $E_{d+1}(\mathbb{Z})$ as indicated in the last column [3]. The split real form $E_{d+1}(\mathbb{R})$ is determined among possible covers or quotients by its maximal compact subgroup K_{d+1}, which shares the same fundamental group. The terminology $10A$ and $10B$ in the first column refers to the two possible superstring theories (types IIA and IIB) in $D = 10$ dimensions.

few components with distinct powers of the parameter r. These conditions pick out the unique solutions to the Laplace equations, which are,[4]

$$\mathcal{E}^{(10-d)}_{(0,0)} \quad = \quad 2\,\zeta(3)\,E^{E_{d+1}}_{\alpha_1;\,\frac{3}{2}}\,, \tag{1.2}$$

for the groups E_1, E_4, E_5, E_6, E_7, and E_8 [1,2] and

$$\mathcal{E}^{(10-d)}_{(1,0)} \quad = \quad \zeta(5)\,E^{E_{d+1}}_{\alpha_1;\,\frac{5}{2}}\,, \tag{1.3}$$

for the groups E_1, E_6, E_7, and E_8 [1]. Here $E^G_{\beta;s}$ is the maximal parabolic Eisenstein series for a parabolic subgroup $P_\beta \subset G$ that is specified by the node β of the Dynkin diagram (see (2.12) for a precise definition). This generalizes results for the $SL(2,\mathbb{Z})$ case (relevant to the ten-dimensional type IIB string theory). The functions $\mathcal{E}^{(10-d)}_{(0,0)}$ and $\mathcal{E}^{(10-d)}_{(1,0)}$ in the intermediate rank cases involve linear combinations of Eisenstein series [2], which will be discussed later in section 4. The third coefficient function, $\mathcal{E}^{(10-d)}_{(0,1)}$ satisfies an interesting inhomogeneous Laplace equation and is not an Eisenstein series [1,5]. Its constant terms in the three limits under consideration were

[4]In [1,2,4] the series were indexed by the Dynkin label $[1\,0\cdots 0]$ of the root α_1. In the present paper, we will index the series according the labeling of the simple root in figure 1. We have as well changed the normalisations of the Eisenstein series, since our series there was instead $\mathbf{E}^{E_{d+1}}_{[10\cdots 0];s} = 2\zeta(2s)E^{E_{d+1}}_{\alpha_1;s}$.

also analysed in the earlier references but it will not be considered in this paper, which is entirely concerned with Eisenstein series.

In other words, our previous work showed that the particular Eisenstein series in (1.2) and (1.3) have strikingly sparse constant terms as required to correctly describe the coefficients of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS interactions. But the string theory boundary conditions also determine the support of the non-zero Fourier coefficients in each of the three limits under consideration. In string theory, the non-zero Fourier modes describe instanton contributions to the amplitude. These are classified in BPS orbits obtained by acting on a representative instanton configuration with the appropriate Levi subgroup. A given instanton configuration generally depends on only a subset of the parameters of the Levi group, $L_\alpha = GL(1) \times M_\alpha$, so that a given orbit depends on the subset of the moduli that live in a coset space of the form $M_\alpha/H^{(i)}$, where $H^{(i)} \subset M_\alpha$ denotes the stabiliser of the i-th orbit. The dimension of the i-th orbit is the dimension of this coset space.

In particular, the coefficients in the $s = 3/2$ cases covered by (1.2) must be localized within the smallest possible non-trivial orbits ("minimal orbits") of the Levi actions, as required by the $\frac{1}{2}$-BPS condition. Furthermore, in the $s = 5/2$ cases covered by (1.3) the coefficients must be localized within the "next-to-minimal" (NTM) orbits (see section 2.2).

This provides motivation from string theory for the following

String motivated vanishing of Fourier modes of Eisenstein series:

(i) *The non-zero Fourier coefficients of* $E^{E_{d+1}}_{\alpha_1;\frac{3}{2}}$ *($d = 5, 6, 7$) in any of the three parabolic subgroups of relevance are localized within the smallest possible non-trivial orbits ("minimal orbits") of the action of the Levi subgroup associated with that parabolic, as required by the* $\frac{1}{2}$*-BPS condition.*

(ii) *The non-zero Fourier coefficients of* $E^{E_{d+1}}_{\alpha_1;\frac{5}{2}}$ *($d = 5, 6, 7$) are localized within "next-to-minimal" (NTM) orbits, as required by the* $\frac{1}{4}$*-BPS condition.*

While the special properties of the Fourier coefficients of the $s = 3/2$ series is implied by the results in [6], the corresponding properties for the NTM orbits at $s = 5/2$ is novel. One of the main mathematical contributions of this paper is to give a rigorous proof of these statements using techniques from representation theory, by connecting these automorphic forms to small representations of the split real groups E_{d+1}. The Fourier coefficients in the intermediate rank cases not covered by (1.2) and (1.3) satisfy analogous properties as we will determine by explicit calculation later in this paper.

2. Overview of scattering amplitudes and Eisenstein series

Since this paper covers topics of interest in both string theory and mathematics, this section will present a brief description of the background to

these topics from both points of view followed by a detailed outline of the rest of the paper.

2.1. **String theory Background.** We are concerned with exact (i.e., non-perturbative) properties of the low energy expansion of the four-graviton scattering amplitude in dimension $D = 10 - d$, which is a function of the moduli, ϕ_{d+1}, as well as of the particle momenta k_r ($r = 1, \dots, 4$) that are null Lorentz D-vectors ($k_r^2 = k_r \cdot k_r = 0$) that are conserved ($\sum_{r=1}^4 k_r = 0$). They arise in the invariant combinations (Mandelstam invariants), $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ that satisfy $s + t + u = 0$. At low orders in the low-energy expansion the amplitude can usefully be separated into analytic and nonanalytic parts

$$A_D(s,t,u) \;=\; A_D^{analytic}(s,t,u) \;+\; A_D^{nonanalytic}(s,t,u) \tag{2.1}$$

(where the dependence on ϕ_{d+1} has been suppressed). The analytic part of the amplitude has the form

$$A_D^{analytic}(s,t,u) \;=\; T_D(s,t,u)\, \ell_D^6\, \mathcal{R}^4\,, \tag{2.2}$$

where ℓ_D denotes the D-dimensional Planck length scale and the factor $\mathcal{R}^4$ represents the particular contraction of four Riemann curvature tensors, $\mathrm{tr}(\mathcal{R}^4) - (\mathrm{tr}\,\mathcal{R}^2)^2/4$, that is fixed by maximal supersymmetry in a standard fashion [7]. The scalar function T_D has the expansion (in the Einstein frame[5])

$$\begin{aligned} T_D(s,t,u) \;&=\; \mathcal{E}_{(0,-1)}\,\sigma_3^{-1} + \sum_{p,q\geq 0} \mathcal{E}^{(D)}_{(p,q)}\,\sigma_2^p\,\sigma_3^q \\ &=\; 3\,\sigma_3^{-1} + \mathcal{E}^{(D)}_{(0,0)} + \mathcal{E}^{(D)}_{(1,0)}\,\sigma_2 + \mathcal{E}^{(D)}_{(0,1)}\,\sigma_3 + \cdots\,. \end{aligned} \tag{2.3}$$

Symmetry under interchange of the four gravitons implies that the Mandelstam invariants only appear in the combinations σ_2 and σ_3 with $\sigma_n = (s^n + t^n + u^n)\,(\ell_D^2/4)^n$. Since s, t, u are quadratic in momenta the successive terms in the expansion are of order $n = 2p + 3q$ in powers of (momenta)^2. The degeneracy, $d_n = \lfloor (n+2)/2 \rfloor - \lfloor (n+2)/3 \rfloor$, of terms with power n is given by the generating function[6],

$$\frac{1}{(1-x^2)(1-x^3)} \;=\; \sum_{n=0}^{\infty} d_n\, x^n\,, \tag{2.4}$$

so $d_0 = 1$, $d_1 = 0$ and $d_n = 1$ for $2 \leq n \leq 5$.

The coefficient functions in (2.3), $\mathcal{E}^{(D)}_{(p,q)}(\phi_{d+1})$, are automorphic functions of the moduli ϕ_{d+1} appropriate to compactification on T^d. The first term on the right-hand side of (2.3) coefficient is identified with the tree-level

[5]The Einstein frame is the frame in which lengths are measured in Planck units rather than string units, and is useful for discussing dualities.

[6]This is the same as the well-known dimension formula for the space of weight $2n$ holomorphic modular forms for $SL(2,\mathbb{Z})$, which are expressed as polynomials in the (holomorphic) Eisenstein series G_4 and G_6.

contribution of classical supergravity and has a constant coefficient given by $\mathcal{E}^{(D)}_{(0,-1)}(\phi_{d+1}) = 3$. The terms of higher order in s, t, u represent stringy modifications of supergravity, which depend on the moduli in a manner consistent with duality invariance. This expansion is presented in the Einstein frame so the curvature, $\mathcal{R}$, is invariant under $E_{d+1}(\mathbb{Z})$ transformations, whereas it transforms nontrivially in the string frame since it is nonconstant in $\phi_{d+1} \in E_{d+1}(\mathbb{R})$. Apart from the first term, the power series expansion in (2.3) translates into a sum of local interactions in the effective action. The first two of these have the form

$$\ell_D^{8-D} \int d^D x \sqrt{-G^{(D)}}\, \mathcal{E}^{(D)}_{(0,0)}\, \mathcal{R}^4 \,, \quad \ell_D^{12-D} \int d^D x \sqrt{-G^{(D)}}\, \mathcal{E}^{(D)}_{(1,0)} \partial^4 \mathcal{R}^4 \,. \tag{2.5}$$

The three interactions with coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$ displayed in the second equality in (2.3) are specially simple since they are protected by supersymmetry from renormalisation beyond a given order in perturbation theory. In particular, the $\mathcal{R}^4$ interaction breaks 16 of the 32 supersymmetries of the type II theories and is thus $\frac{1}{2}$-BPS, while the $\partial^4\mathcal{R}^4$ interaction breaks 24 supersymmetries and is $\frac{1}{4}$-BPS; likewise, the $\partial^6 R^4$ interaction breaks 28 supersymmetries and is $\frac{1}{8}$-BPS. The next interaction is the $p = 2, q = 0$ term in (2.3), $\mathcal{E}^{(D)}_{(2,0)}\, \partial^8\mathcal{R}^4$. Naively this interaction breaks all supersymmetries, in which case it is expected to be much more complicated, but it would be of interest to discover if supersymmetry does constrain this interaction.[7]

It was argued in [2], based on consistency under various dualities, that the coefficients $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$ satisfy the equations

$$\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right) \mathcal{E}^{(D)}_{(0,0)} \;=\; 6\,\pi\,\delta_{D,8}\,, \tag{2.6}$$

$$\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right) \mathcal{E}^{(D)}_{(1,0)} \;=\; 40\,\zeta(2)\,\delta_{D,7}\,, \tag{2.7}$$

$$\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2}\right) \mathcal{E}^{(D)}_{(0,1)} \;=\; -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 120\zeta(3)\delta_{D,6}\,, \tag{2.8}$$

where $\Delta^{(D)}$ is the Laplace operator on the symmetric space E_{11-D}/K_{11-D}. The discrete Kronecker δ contributions on the right-hand-side of these equations arise from anomalous behaviour and can be related to the logarithmic ultraviolet divergences of loop amplitudes in maximally supersymmetric supergravity [4].

Recall that automorphic forms for $SL(2,\mathbb{Z})$ have Fourier expansions (i.e., q-expansions) in their cusp. For higher rank groups, automorphic forms have Fourier expansions coming from any one of several maximal parabolic subgroups P_{α_r}, where the simple root α_r corresponds to node r in the Dynkin

[7] A discussion of the properties of $\mathcal{E}^{(9)}_{(2,0)}$ in nine dimensions can be found in [8, section 4.1.1].

diagram for E_{d+1} in figure 1. We are particularly interested in this Fourier expansion for $r = 1, 2$, or $d+1$, because each of these expansions has a distinct string theory interpretation in terms of the contributions of instantons in the limit in which a special combination of moduli degenerate. These three limits are:

(i) *The decompactification limit* in which one circular dimension, r_d, becomes large. In this case the amplitude reduces to the $D+1$-dimensional case with $D = 10 - d$. The BPS instantons of the $D = (10-d)$-dimensional theory are classified by orbits of the Levi subgroup $GL(1) \times E_d$. Apart from one exception, these instantons can be described in terms of the wrapping of the world-lines of black hole states in the decompactified $D+1$-dimensional theory around the large circular dimension (the exception will be described later). This limit is associated with the parabolic subgroup $P_{\alpha_{d+1}}$.

(ii) *The string perturbation theory limit* of small string coupling constant, in which the string coupling constant, $\sqrt{y_D}$, is small, and string perturbation theory amplitudes are reproduced. The instantons are exponentially suppressed contributions that are classified by orbits of the Levi subgroup $GL(1) \times SO(d,d)$. This limit is associated with the parabolic subgroup P_{α_1}.

(iii) *The M-theory limit* in which the M-theory torus has large volume $\mathcal{V}_{d+1}$, and the semi-classical approximation to eleven-dimensional supergravity is valid. This involves the compactification of M-theory from 11 dimensions on the $(d+1)$-dimensional M-theory torus, where the instantons are classified by orbits of the Levi subgroup $GL(1) \times SL(d+1)$. This is associated with the parabolic subgroup P_{α_2}.

The special features of the constant terms that lead to consistency of all perturbative properties in these three limits appear to be highly nontrivial, and indicate particularly special mathematical properties of the Eisenstein series that define the coefficients of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions. The solutions to equations (2.6-2.8) satisfying requisite boundary conditions on the constant terms (zero modes) in the Fourier expansions in the limits (i), (ii), and (iii) were obtained for $7 \le D \le 10$ in [2], and for $3 \le D \le 6$ in [1]. In particular, (1.2) and (1.3) were found to be solutions for the cases with duality groups E_6, E_7 and E_8. Whereas the coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ are given in terms of Eisenstein series that satisfy Laplace eigenvalue equations on the moduli space, the coefficient $\mathcal{E}^{(D)}_{(0,1)}$, of the $\frac{1}{8}$-BPS interaction $\partial^6\mathcal{R}^4$, is an automorphic function that satisfies an inhomogeneous Laplace equation. Various properties of its constant terms in these three limits were also determined in [1, 2].

Whereas the earlier work concerned the zero Fourier modes of the coefficient functions, in this paper we are concerned with the non-zero modes in

the Fourier expansion in any of the three limits listed above. These Fourier coefficients should have the exponentially suppressed form that is characteristic of instanton contributions. In more precise terms, the angular variables involved in the Fourier expansion with respect to a maximal parabolic subgroup P_α come from the unipotent radical U_α of P_α, and are conjugate to integers that define the instanton "charge lattice". Asymptotically close to a cusp a given Fourier coefficient is expected to have an exponential factor of $\exp(-S^{(p)})$, where $S^{(p)}$ is the action for an instanton of a given charge, as will be defined in section 3.1. In the case of fractional BPS instantons the leading asymptotic behaviour in the cusp is the real part of $S^{(p)}$, and is related to the charge (B.4), which enters the phase of the mode.

In each limit the $\frac{1}{2}$-BPS orbits are minimal orbits (i.e., smallest nontrivial orbits) while the $\frac{1}{4}$-BPS orbits are "next-to-minimal" (NTM) orbits (i.e., smallest nonminimal or nontrivial orbits). The next largest are $\frac{1}{8}$-BPS orbits, which only arise for groups of sufficiently high rank; in the E_8 case there is a further $\frac{1}{8}$-BPS orbit beyond that. These come up again as "character variety orbits", a major consideration in sections 5 and 6. They are closely related to – but not to be confused with – the minimal and next-to-minimal coadjoint nilpotent orbits that are attached to the Eisenstein series that arise in the solutions for the coefficients, $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in (1.2) and (1.3), respectively.

Note on conventions. Following [1, Section 2.4], the parameter associated with the $GL(1)$ factor that parameterises the approach to any cusp will be called r and is normalised in a mathematically convenient manner. It translates into distinct physical parameters in each of the three limits described above, that correspond to parabolic subgroups defined at nodes $d+1$, 1 and 2, respectively, of the Dynkin diagram in fig. 1. These are summarised as follows:

$$\begin{aligned}
&\text{Limit (i)} \quad r^2 = r_d/\ell_{11-d}\,, \quad r_d = \text{radius of decompactifying circle}\,,\\
&\text{Limit (ii)} \quad r^{-2} = \sqrt{y_D} = \text{string coupling constant}\,,\\
&\text{Limit (iii)} \quad r^{\frac{2(1+d)}{3}} = \mathcal{V}_{d+1}/\ell_{11}^{d+1}\,, \quad \mathcal{V}_{d+1} = \text{vol. of M}-\text{theory torus}\,.
\end{aligned} \tag{2.9}$$

The D-dimensional string coupling constant is defined by $y_D = g_s^2\,\ell_s^d/V_d$, where $D = 10-d$ and g_s is either the $D=10$ IIA string coupling constant, g_A, or the IIB string coupling constant, g_B, and V_d is the volume of T^d in string units.[8] The Planck length scales in different dimensions are related

[8]We will use the symbol T^d to denote the string theory d-torus while using the symbol $\mathcal{T}^{d+1}$ for the corresponding M-theory $(d+1)$-torus expressed in eleven-dimensional units.

to each other and to the string scale, ℓ_s, by

$$\begin{aligned}
(\ell_{10}^A)^8 &= \ell_s^8\, g_A^2\,, \quad (\ell_{10}^B)^8 \;=\; \ell_s^8\, g_B^2\,, \quad \ell_{11} \;=\; g_A^{\frac{1}{3}}\,\ell_s\,, \\
(\ell_D)^{D-2} &= \ell_s^{D-2}\, y_D \;=\; (\ell_{D+1})^{D-1}\,\frac{1}{r_d}\,, \quad \text{for } D \le 8 \ (d \ge 2) \\
\ell_9^7 &= \ell_s^7\, y_9 = (\ell_{10}^A)^9\,\frac{1}{r_A} \;=\; (\ell_{10}^B)^9\,\frac{1}{r_B}\,. \qquad (2.10)
\end{aligned}$$

(note the two distinct Planck lengths in the ten-dimensional case and the distinction between $r_1 = r_A$ and $r_1 = r_B$ in the two type II theories).

2.2. **Mathematics background.** Let us begin by recalling some notions from the theory of automorphic forms that are relevant to the expansion (2.3), specifically from [1, Section 2]. Let G denote the split real Lie group E_n, $n \le 8$, defined in table 1. For convenience we fix (as we may) a Chevalley basis of the Lie algebra $\mathfrak{g}$ of G, and a choice of positive roots Φ_+ for its root system Φ. Letting $\Sigma \subset \Phi_+$ denote the positive simple roots, the Lie algebra $\mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} \;=\; \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{n}_-\,, \qquad (2.11)$$

where $\mathfrak{n}$ (respectively, $\mathfrak{n}_-$) is spanned by the Chevalley basis root vectors X_α for positive roots $\alpha \in \Phi_+$ (respectively, $\alpha \in \Phi_-$), and $\mathfrak{a}$ is spanned by their commutators $H_\alpha = [X_\alpha, X_{-\alpha}]$. Let $N \subset G$ be the exponential of $\mathfrak{n}$; it is a maximal unipotent subgroup. Likewise $A = \exp(\mathfrak{a})$ is a maximal torus, and is isomorphic to $\mathrm{rank}(G)$ copies of $\mathbb{R}^+$. The group G has an Iwasawa decomposition $G = NAK$, where $K = K_n$ is the maximal compact subgroup of G listed in table 1. There thus exists a logarithm map $H : A \to \mathfrak{a}$ which is inverse to the exponential, and which extends to all $g \in G$ via its value on the A-factor of the Iwasawa decomposition of g.

The standard maximal parabolic subgroups of G are in bijective correspondence with the positive simple roots of G. Given such a root β and a standard maximal parabolic P_β, the *maximal parabolic Eisenstein series* induced from the constant function on P_β is defined by the sum

$$E^G_{\beta;s} \;:=\; \sum_{\gamma\,\in\,(P_\beta \cap G(\mathbb{Z}))\backslash G(\mathbb{Z})} e^{2\,s\,\omega_\beta(H(\gamma g))}\,, \quad \mathrm{Re}\ s \gg 0\,, \qquad (2.12)$$

where ω_β, the fundamental weight associated to β, is defined by the condition $\langle \omega_\beta, \alpha \rangle = \delta_{\alpha,\beta}$. These series generalize the classical nonholomorphic Eisenstein series (the case of $G = SL(2)$), and more generally the Epstein Zeta functions (the case of $G = SL(n)$ and β either the first or last node of the A_{n-1} Dynkin diagram). Because of this special case, we often refer to the $\beta = \alpha_1$ series (in the numbering of figure 1) as the *Epstein* series for a particular group, even if it is not $SL(n)$. These series are the main mathematical objects of this paper.

As shorthand, we often denote a root by its "Dynkin label", that is, stringing together its coefficients when written as a linear combination of the positive simple roots Σ. Thus $\alpha_2+\alpha_3+2\alpha_4+\alpha_5$ could be denoted $0112100\cdots$ or $[0112100\cdots]$, with brackets sometimes added for clarity. Note that Eisenstein series of the type (2.12) are parameterized by a single complex variable, s, whereas the more general minimal parabolic series in (5.3) has $\mathrm{rank}(G)$ complex parameters.

The series (2.12) is initially absolutely convergent for $\mathrm{Re}\, s$ large, and has a meromorphic continuation to the entire complex plane as part of a more general analytic continuation of Eisenstein series due to Langlands. Its special value at $s = 0$ is the constant function identically equal to one. This corresponds to the trivial representation of $G(\mathbb{R})$, and clearly has no nontrivial Fourier coefficients. The main result of the following sections extends this phenomenon to other special values of s which are connected to small representations of real groups (see sections 2.2.2 and 5), and which have very few nontrivial Fourier coefficients. This will be demonstrated to be in complete agreement with a number of string theoretic predictions, in particular the one stated in section 2.2.2.

The main results of [1] were the identifications (1.2) and (1.3) of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$, respectively, in terms of special values of the Epstein series, for $3 \leq D = 10 - d \leq 5$. The more general automorphic function $\mathcal{E}^{(D)}_{(0,1)}$, which satisfies (2.8) was also analysed in [1], but will not be relevant in this paper. The case of $SO(5,5)$ was also covered in [1], but is somewhat more intricate; it will be explained separately. We will show in a precise sense that these Epstein series at the special values at $s = 0$, $3/2$, and $5/2$ correspond, respectively, to the three smallest types of representations of G (see theorem 2.13) below.

2.2.1. *Coadjoint nilpotent orbits.* Let $\mathfrak{g}$ be the Lie algebra of a matrix Lie group G, whether over $\mathbb{R}$ or $\mathbb{C}$. An element of $\mathfrak{g}$ is *nilpotent* if it is nilpotent as a matrix, i.e., some power of it is zero. The group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint action $Ad(g)X = gXg^{-1}$, and hence dually on linear functionals $\lambda : \mathfrak{g} \to \mathbb{C}$ through the coadjoint action given by $(Coad(g)\lambda)(X) = \lambda(Ad(g)X) = \lambda(gXg^{-1})$. Actually $\mathfrak{g}$ is isomorphic to its space of linear functionals via the Killing form, and so the coadjoint action is isomorphic to the adjoint action. Following common usage, we thus refer to the orbits of the adjoint action of G on $\mathfrak{g}$ as *coadjoint nilpotent orbits* (even though they are, technically speaking, adjoint orbits).

The book [9] is a standard reference for the general theory of coadjoint nilpotent orbits. When G is a real or complex semisimple Lie group there are a finite number of orbits, each of which is even dimensional. The smallest of these is the trivial orbit, $\{0\}$. On the other hand, there is always an open, dense orbit, usually refereed to as the *principal* or *regular orbit*. Another orbit which will be important for us is the *minimal* orbit, the smallest orbit

Group	Orbit Dimension	Basepoint
$SL(2)$	0	0
	2	X_1
$SL(3) \times SL(2)$	0	0
	2	an $SL(2)$ root
	4	an $SL(3)$ root
$SL(5)$	0	0
	8	X_{1111}
	12	$X_{1110} + X_{0111}$
$SO(5,5)$	0	0
	14	X_{12211}
	20	$X_{01111} + X_{11211}$
E_6	0	0
	22	X_{122321}
	32	$X_{111221} + X_{112211}$
	40	$X_{011221} + X_{111210} + X_{112211}$
E_7	0	0
	34	$X_{2234321}$
	52	$X_{1123321} + X_{1223221}$
	54	$X_{0112210} + X_{1112221} + X_{1122110}$
E_8	0	0
	58	$X_{23465432}$
	92	$X_{23354321} + X_{22454321}$
	112	$X_{22343221} + X_{12343321} + X_{12244321}$
	114	$X_{11232221} + X_{12233211}$

TABLE 2. Basepoints of the smallest coadjoint nilpotent orbits for the complexified E_n groups. The notation X_α denotes a root vector for the simple root α, which are written here in terms of the Dynkin labels described in the text. The $SL(3) \times SL(2)$ case comes from the E_3 Dynkin diagram, which is the E_8 Dynkin diagram from figure 1 after the removal of nodes 4, 5, 6, 7, and 8. It is a product of two simple Lie algebras, and has a different orbit structure than the others; its smallest orbits come from the respective factors.

aside from the trivial orbit. Since our groups G are all simply laced, it can be described as the orbit of any root vector X_α, for any root α.

Tables 2 gives a list of some orbits that are important to us, along with their basepoints.

2.2.2. *Automorphic representations.* The right translates of an automorphic function by the group G span a vector space on which G acts. For a suitable basis of square-integrable automorphic forms and most Eisenstein series, this action furnishes an irreducible representation. As we discussed in [1, Section 2], the Eisenstein series are specializations of the larger "minimal parabolic Eisenstein series" defined in (5.3). The automorphic representations connected to the latter are principal series representations, an identification which can be made by comparing the infinitesimal characters (that is, the action of all G-invariant differential operators). They are also right-K-invariant, and thus by definition their special values are spherical subrepresentations of these principal series representations.

An irreducible representation is related to coadjoint nilpotent orbits through its *wavefront set*, also known as the "associated variety" of its "annihilator ideal". It is a theorem of Borho-Brylinski [10] and Joseph [11] that this set is always the closure of a unique coadjoint nilpotent orbit. Thus a coadjoint nilpotent orbit is attached to every irreducible representation.

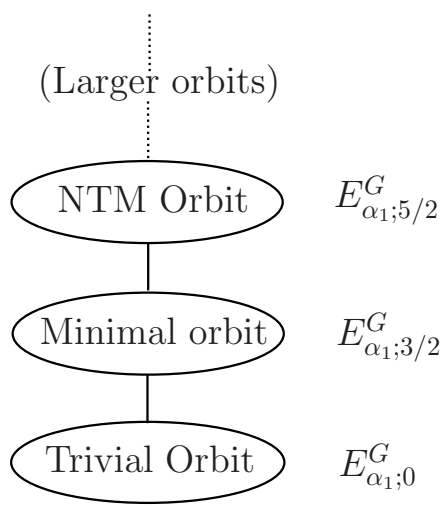

FIGURE 2. Schematic of small representations and Eisenstein special values

Part (iii) of the following theorem is the main mathematical result of this paper, in particular the cases of E_7 and E_8. Part (i) is trivial, while part (ii) is contained in results of Ginzburg-Rallis-Soudry [6], following earlier work of Kazhdan-Savin [12].

Theorem 2.13. *Let G one of the groups E_6, E_7, or E_8 from table 1. Then*

(i) *The wavefront set of the automorphic representation attached to the $s = 0$ Epstein series is the trivial orbit.*
(ii) *The wavefront set of the automorphic representation attached to the $s = 3/2$ Epstein series is the closure of the minimal orbit.*

(iii) *The wavefront set of the automorphic representation attached to the $s = 5/2$ Epstein series is the closure of the next-to-minimal (NTM) orbit.*

The closure of the minimal orbit is simply the union of the minimal orbit and the trivial orbit, while the closure of the next-to-minimal orbit is the union of itself, the minimal orbit, and the trivial orbit. Theorem 2.13 will be used in proving theorem 6.1, which is the mathematical proof of the statement concerning vanishing Fourier modes motivated by string considerations at the end of section 1.

2.3. **Outline of paper.** This paper combines information deduced from string theory with results in number theory involving properties of Eisenstein series, which we hope will be of interest to both string theorists and number theorists. In particular, each subject is used to make nontrivial statements about the other). Sections 3–4 and appendices B–E are framed in string theory language and provide information concerning the structure expected of the non-zero Fourier modes based on instanton contributions in superstring theory and supergravity. The subsequent sections provide the mathematical foundations of these observations and generalize them significantly.

Section 3 presents the classification of the expected orbits of fractional BPS instantons in the three limits (i), (ii), and (iii) considered in section 2.1, from the point of view of string theory. The BPS constraints imply that these instantons span particular small orbits generated by the action of the Levi subgroup acting on the unipotent radical associated with the parabolic subgroup appropriate to a given limit. These orbits can be thus thought of as character variety orbits, which are discussed at the beginning of section 4.

In the rest of section 4 and appendix E we will consider explicit low-rank examples (with rank $d+1 \leq 5$) of the Fourier expansions of the functions $\mathcal{E}^{(10-d)}_{(0,0)}$ and $\mathcal{E}^{(10-d)}_{(1,0)}$ in the parabolic subgroups corresponding to each limit. In the cases with $d+1 \leq 4$ ($D \geq 7$), the definition (2.12) implies that the coefficient functions are combinations of $SL(n)$ Eisenstein series that can easily be expressed in terms of elementary lattice sums. In these cases it is straightforward to use standard Poisson summation techniques to exhibit the precise form of their Fourier modes. In particular, the non-zero Fourier modes of $\mathcal{E}^{(10-d)}_{(0,0)} = 2\,\zeta(3)\,E^{E_{d+1}}_{\alpha_1;3/2}$ will be determined in the three limits under consideration for the rank $d+1 \leq 4$ cases. These modes are localized within the minimal character variety orbits that contain precisely the $\frac{1}{2}$-BPS instantons that are anticipated in section 3. We will see, in particular, that in the decompactification limit (i) the precise form for each of these coefficients matches in detail with the expression determined directly from

a quantum mechanical treatment of D-particle world-lines wrapped around a $S^1 \subset \mathrm{T}^d$.[9]

Explicit examples of the Fourier expansion of the coefficient of the $\frac{1}{4}$-BPS interaction, $\mathcal{E}^{(D)}_{(1,0)}$, will also be presented in section 4 and appendix E. This function is equal to $\zeta(5)\, E^{SL(2)}_{\alpha_1;5/2}$ for $D = 10$, but involves particular combinations of $E^{E_{d+1}}_{\alpha_1;5/2}$ and other Eisenstein series for $6 \leq D \leq 9$. In order to give a complete analysis of the contributions to $\mathcal{E}^{(7)}_{(1,0)}$, we will make use of a representation of $E^{SL(5)}_{\alpha_4;5/2}$ that expresses it as the Mellin transform of the $SO(5,5)$ Eisenstein series $E^{SO(5,5)}_{\alpha_1;3/2}$. As we are not aware of a reference for this representation in the literature, we present it in proposition 4.1. The resulting Fourier expansions contain contributions localized within the minimal ($\frac{1}{2}$-BPS) character variety orbit and the next-to-minimal ($\frac{1}{4}$-BPS) character variety orbit, comprising precisely the instantons anticipated in section 3.

The highest rank case that is amenable to classical lattice summation techniques is the $D = 6$ case (with duality group $SO(5,5,\mathbb{Z})$), where we have made use of an integral representation of the series $E^{SO(5,5)}_{\alpha_1;s}$. The coefficient $\mathcal{E}^{(6)}_{(0,0)}$ involves only this series at $s = 3/2$, and its non-zero Fourier modes are supported within the minimal ($\frac{1}{2}$-BPS) character variety orbits in any of the three limits. On the other hand the next coefficient, $\mathcal{E}^{(6)}_{(1,0)}$, involves the sum of the regularized values of $\hat{E}^{SO(5,5)}_{\alpha_1;5/2}$ and $\hat{E}^{SO(5,5)}_{\alpha_5;3}$. Although we have not computed the Fourier expansion of the second series, it is still possible to show that the non-zero Fourier coefficients of this sum are supported within the minimal and next-to-minimal (i.e., $\frac{1}{2}$- and $\frac{1}{4}$-BPS) character variety orbits in each of the three limits. This will be discussed at the end of section 4.

Sections 5, 6, and 7 are primarily concerned with the exceptional group cases, which correspond to $d \geq 5$ and $D \leq 5$. Since classical lattice summation techniques are difficult to apply in this context, we instead use results from representation theory to show a large number of the Fourier coefficients vanish. Indeed, avoiding explicit computations here is one of the main novelties of the paper. Section 5 discusses aspects connected to representation theory and contains a proof of theorem (2.13), which makes important use of appendix A by Ciubotaru and Trapa on special unipotent representations.

Section 6 then applies these results to Fourier expansions, using a detailed analysis of character variety orbits. We will see that the spectrum of instantons that are expected to vanish on the basis of string theory is precisely reproduced by the Eisenstein series in (1.2), (1.3). For the $s = 3/2$

[9]The term *D-particle* refers to any point-like BPS particle state obtained by completely wrapping the spatial directions of Dp-brane states.

case (the $\frac{1}{2}$-BPS case) we will reproduce the statements in [6, 13, 14] that only the minimal orbit and the trivial orbit contribute to the Fourier expansions of the Eisenstein series. However, we will find that this generalizes for $s = 5/2$ (the $\frac{1}{4}$-BPS case) to the statement that no orbits larger than the next-to-minimal (NTM) orbit can contribute. The analysis in [1] showed the striking fact that the particular Eisenstein series in (1.2) and (1.3) have constant terms with very few powers of r (defined in 2.9) in their expansion around any of the three limits under consideration. The analysis in this paper demonstrates analogous special features of the orbit structure of the non-zero modes. Theorem 6.1 gives a precise statement about which Fourier modes automatically vanish because of representation theoretic reasons. This set of vanishing coefficients is exactly those that are argued to vanish for string theory reasons in section 3.

It is important to point out that our methods show the vanishing of a precise set of Fourier coefficients, but typically do not show the *nonvanishing* of the remaining Fourier coefficients. However, this is accomplished in a number of low rank cases by explicit calculations in section 4.1 and appendix E, and we hope to treat some of the higher rank cases in a future paper. Section 7 discusses square-integrability of the coefficients and conditions under which $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ is square-integrable for higher rank groups.

3. Orbits of supersymmetric instantons

From the string theory point of view our main interest is in the systematics of orbits of BPS instantons that enter the Fourier expansions of the coefficients of the low order terms in the low energy expansion of the four graviton amplitude. Before describing these orbits in sections 3.3 – 3.5 we begin with a short overview of the special features of such instantons that follow from supersymmetry. A short summary of the M-theory supersymmetry algebra and BPS particle states is given in appendix B (although this barely skims the surface of a huge subject), where the structure of the eleven-dimensional superalgebra is seen to imply the presence of an extended two-brane (the $M2$-brane) and five-brane (the $M5$-brane) in eleven dimensions. Compactification on a torus also leads to Kaluza–Klein (KK) point-like states and Kaluza–Klein monopoles (KKM), one of which is interpreted in string theory as a $D6$-brane. All the particle states in lower dimensions can be obtained by wrapping the spatial directions of these objects around cycles of the torus.

3.1. **BPS instantons.** One class of BPS instantons can be described from the eleven-dimensional semi-classical M-theory point of view by wrapping euclidean world-volumes of $M2$- and $M5$- branes around compact directions so that the brane actions are finite. These branes couple to the three-form M-theory potential and its dual, and the BPS conditions constrain their charges, $Q^{(p)}$, to be proportional to their tensions, $T^{(p)}$, where $p = 2$ or 5 (as briefly reviewed in appendix B). Wrapping the world-volume of a euclidean

$M2$-brane around a 3-torus, $\mathcal{T}^3 \subset \mathcal{T}^{d+1}$, or a euclidean $M5$-brane around a 6-torus, $\mathcal{T}^6 \subset \mathcal{T}^{d+1}$, gives a $\frac{1}{2}$-BPS instanton, which has a euclidean action of the form $S^{(p)} = 2\pi\,(T^{(p)} + iQ^{(p)})$. This gives a factor in amplitude of the form $e^{-S^{(p)}}$ that has a characteristic phase determined by the charge of the brane.

In addition, the "KK instanton" is identified with the euclidean world-line of a KK charge winding around a circular dimension. The magnetic version of this is the "KKM instanton", one manifestation of which appears in string theory as a wrapped euclidean $D6$-brane. Recall that a KK monopole in eleven dimensional (super)gravity with one compactified direction labelled $x^{\#}$ has a metric of the form [15]

$$ds^2 = V^{-1}\,(dx^{\#} + \mathbf{A}\cdot\mathbf{dy})^2 + V\,\mathbf{dy}\cdot\mathbf{dy} - dt^2 + dx_6^2\,, \qquad V = 1 + \frac{R}{2|\mathbf{y}|}\,, \quad (3.1)$$

where $ds_7^2 = -dt^2 + dx_6^2$ is the seven-dimensional Minkowski metric and the other four dimensions, $x^{\#}$, $\mathbf{y} = (y_1, y_2, y_3)$, define a Taub–NUT space, and $|\mathbf{y}|^2 = \sum_{i=1}^3 y_i^2$. The coordinate $x^{\#}$ is periodic with period $2\pi R$ and the potential, $\mathbf{A}$, satisfies the equation $\nabla \times \mathbf{A} = -\nabla V = \mathbf{B}$. Poincaré duality in the ten dimensions $(t, x_6, \mathbf{y})$ relates the 1-form potential, $\mathbf{A}$, to a 7-form, i.e, $*d\mathbf{A} = dC^{(7)}$. If $x^{\#}$ is identified with the M-theory circle, $C^{(7)}$ couples to a $D6$-brane in the string theory limit. This gives an instanton when its world-volume is wrapped around a 7-torus. More generally, $x^{\#}$ can be identified with other circular dimensions of the torus $\mathcal{T}^{d+1}$, giving a further d distinct KKM's, each one of which appears as a finite action instanton when wrapped on an M-theory 8-torus, $\mathcal{T}^8$ (i.e., when $d = 7$). When describing these in the string theory parameterisation (on the string torus T^7) these will be referred to as "stringy KKM instantons". Furthermore, it is well understood how to combine wrapped branes to make $\frac{1}{2}-$, $\frac{1}{4}-$ and $\frac{1}{8}$-BPS instantons [16, 17][10] in a manner analogous to combining p-branes to make states preserving a fraction of the symmetry.

This description of instantons is directly relevant to the discussion of the semi-classical M-theory limit (case (iii)) associated with the Fourier expansion in the parabolic subgroup P_{α_2} in section 3.5. This is the large-volume limit in which eleven-dimensional supergravity is a valid approximation. Similarly, the instanton contributions in limits (i) and (ii) can be described by translating from the M-theory description to the string theory description of the wrapped branes. These wrapped string theory objects comprise: the fundamental string and the Neveu–Schwarz five-brane (NS5-brane) that couple to B_{NS}; Dp-branes that couple to the Ramond–Ramond $(p+1)$-form potentials $C^{(p+1)}$ (with $-1 \leq p \leq 9$); and KK charges and KK monopoles

[10] We are concerned with compactification on tori, but more generally the BPS condition requires branes to be wrapped on special lagrangian submanifolds (SLAGs) or on holomorphic cycles [16].

that couple to modes of the metric associated with toroidal compactification on T^d.

Knowledge of this instanton spectrum is a valuable ingredient in understanding the systematics of the Fourier modes of the Eisenstein series that enter into the definitions of the coefficients of the low order interactions in the expansion of the scattering amplitude. In particular, it connects closely with the study of the Fourier expansions of specific Eisenstein series that enter into $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ (that will be discussed later in this paper), as well as with the Fourier expansion of the more general automorphic function $\mathcal{E}^{(D)}_{(p,q)}$ (that will not be discussed in this paper).

3.2. **Fourier modes and orbits of BPS charges.** The Fourier expansion associated with any parabolic subgroup, $P_\alpha = L_\alpha U_\alpha$, of E_{d+1} is a sum over integer charges that are conjugate to the angular variables that enter in its unipotent radical U_α. These determine the phases of the modes. The Levi factor is a reductive group that has the form $L_\alpha = GL(1) \times M_\alpha$, where M_α is its semisimple component.

The conjugation action on U_α of L_α – or more specifically, its intersection with the discrete duality group $L_\alpha \cap E_{d+1}(\mathbb{Z})$ – relates these charges by Fourier duality. Thus this action carves out orbits within the charge lattice, with each given orbit only covering a subset of the total charge space. This viewpoint is expanded upon in more detail in section 4.1. In this subsection we classify these orbits in cruder form, by considering the action of the continuous group L_α on the charge lattice. Our purpose here is to isolate broad families of charges which have common features. Indeed, since we are mainly interested in the algebraic nature of the group action, we sometimes look at the less refined action of the complexification of L_α in order to avoid subtle issues about square roots.

As will be explained in section 4.1, the action of L_α on the charge lattice is related to the adjoint representation on the Lie algebra of U_α. This representation is irreducible if and only if U_α is abelian. That is the case for the unipotent radicals we consider of every symmetry group $E_{d+1}(\mathbb{R})$ of rank $d+1 < 6$. Otherwise, the Fourier expansion is only well-defined after averaging over the commutator subgroup (see (4.3)), and hence does not capture the full content of the function. We devote the rest of this section to relating these orbits to BPS instantons in the three limits we consider. In each particular case we will explain the origin of the non-abelian nature of the unipotent radicals, which have charges that do not commute with the other brane charges. A discussion of such effects within string theory can be found, for example, in [18].

We now describe the adjoint action $V_{\hat{\alpha}}$ on the unipotent radical, where $\hat{\alpha}$ labels the node immediately adjacent to α in the Dynkin diagram (fig. 1). For the three parabolic subgroups of interest to us the representations of the unipotent radical are as follows:

(i) The maximal parabolic $P_{\alpha_{d+1}}$.

In this case $\hat{\alpha} = \alpha_d$ and $L_{\alpha_{d+1}} = GL(1) \times E_d$. The following lists the representations V_{α_d} for each value of $2 \leq d \leq 7$.

E_{d+1}	$M_{\alpha_{d+1}}$	V_{α_d}
E_8	E_7	$q^i : \mathbf{56}$, $q : \mathbf{1}$
E_7	E_6	$q^i : \mathbf{27}$
E_6	$SO(5,5)$	$S_\alpha : \mathbf{16}$
$SO(5,5)$	$SL(5)$	$v_{[ij]} : \mathbf{10}$
$SL(5)$	$SL(3) \times SL(2)$	$v_{ia} : \mathbf{3} \times \mathbf{2}$
$SL(3) \times SL(2)$	$SL(2) \times \mathbb{R}^+$	$vv_a : \mathbf{2}$

The notation in the last column indicates the irreducible representations indexed by their dimensions. Both the fundamental representation and the trivial representation of E_7 occur, because the unipotent radical U_{α_8} is a Heisenberg group. The lower dimensional representations are: the fundamental representation for E_6; a spinor representation for $SO(5,5)$; the rank 2 antisymmetric tensor representation for $SL(5)$; a bivector representation for $SL(3) \times SL(2)$; and a scalar-vector representation for $SL(2) \times \mathbb{R}^+$.

(ii) The maximal parabolic P_{α_1}.

In this case $\hat{\alpha} = \alpha_3$, which is a spinor node (following the numbering of figure 1) and $L_{\alpha_1} = GL(1) \times SO(d,d)$. The representation $V_{\hat{\alpha}}$ always includes a spinor representation of $SO(d,d)$. It is irreducible except in the cases of $d = 6, 7$. The case of $SO(6,6) \subset E_7$ also includes a copy of the trivial representation, because the unipotent radical is again a Heisenberg group; the case of $SO(7,7) \subset E_8$ also includes a copy of the standard 14-dimensional "vector" representation.

(iii) The maximal parabolic P_{α_2}.

In this case $\hat{\alpha} = \alpha_4$ and $L_{\alpha_2} = GL(1) \times SL(d+1)$. The representation $V_{\hat{\alpha}}$ always includes a rank 3 antisymmetric tensor of $SL(d+1)$, v_{ijk}, of dimension $\frac{1}{3!}(d+1)d(d-1)$. It is irreducible when the rank is less than 6 (see table 3) for the dimensions in the higher rank cases.

In each case, the charges form a lattice within the first listed piece of $V_{\hat{\alpha}}$, that is, the irreducible subrepresentation coming from the "abelian part" of U_α. More precisely, these are the nontrivial representations in part (i), the spinor representations in part (ii), and the rank 3 antisymmetric tensors v_{ijk} in part (iii). This space is identical with the "character variety orbit" $\mathfrak{u}_{-1}$ introduced in section 4.1.

Before proceeding with the explicit list of orbits based on the counting of states and instantons in the next three subsections, we will recall basic properties of the space of charges. Apart from the most trivial case (with duality group $SL(2,\mathbb{Z})$), the $\frac{1}{2}$-BPS orbits only fill a subset of the whole space. For the E_{d+1} groups with $1 \leq d \leq 5$ the complementary space to the $\frac{1}{2}$-BPS space is filled out by $\frac{1}{4}$-BPS orbits. For E_7 and E_8 the full space is spanned

Group	first node		second node		last node	
$SL(3)\times SL(2)$	2	0	1	0	3	0
$SL(5)$	4	0	4	0	6	0
$SO(5,5)$	8	0	10	0	10	0
E_6	16	0	20	1	16	0
E_7	32	1	35	7	27	0
E_8	64	14	56	8 + 28	56	1

TABLE 3. Dimensions of the unipotent radical U_{α_i} for the standard maximal parabolic subgroup P_{α_i} where $i=1$, $i=2$ and $i=d$. For each node the first column gives the dimension of the character variety $\mathfrak{u}_{-1}$ (see section 4.1), and the second column gives the dimension of the derived subgroup $[U,U]$. The sum of the two is the dimension of U. The unipotent radical U is abelian when the dimension in the second column is zero; it is a Heisenberg group when this dimension equals 1 and even more non-abelian when it is >1.

by the union of $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS orbits. The Fourier coefficients of the BPS protected operators will have nonvanishing Fourier coefficients only associated to these nilpotent orbits. The classification of possible charge orbits only depends on the semi-classical nature of the associated BPS configurations, but does not provide any detailed information about strong quantum corrections. Such information should be encoded in the precise form of the instanton contributions to the Fourier modes.

The instanton spectrum will now be considered in each of these limits in turn. In each case we will list the single-particle BPS states and single instantons that form the basis of the charge orbits. These numbers are equal to the dimensions of the full space of charges spanned by the orbits. Since we will be only interested in BPS (supersymmetric) orbits we will not discuss all the possible nilpotent orbits of E_7 and E_8. A complete discussion of the orbit structure is given in section 6.1.

3.3. **BPS instantons in the decompactification limit: $P_{\alpha_{d+1}}$.**

The parabolic subgroup of relevance to the expansion of the amplitude in $D=10-d$ dimensions when the radius r_d defined in (2.9) of one circle of the torus T^d becomes large is $P_{\alpha_{d+1}}$, which has Levi factor $L_{\alpha_{d+1}}=GL(1)\times E_d$. In this limit there is a close correspondence between the spectrum of instantons in $D=10-d$ dimensions and the spectrum of black hole states in $D+1=11-d$ dimensions. This follows from the identification of the euclidean world-line of a charged black hole of mass M wrapping around a circular dimension of radius r with an instanton with action $2\pi Mr$ that gives rise to an exponential factor of $e^{-2\pi Mr}$ in the amplitude. In addition to instantons of this type, there can be instantons that do not decompactify to particle states in the higher dimension because their actions are singular

in the large-r limit. In any dimension there are also instantons with actions independent of r that are inherited from the higher dimension in a trivial manner.

The spectrum of BPS black hole states in compactified string theory has been studied extensively. We will here follow the analysis in [19, 20], which considered the spectrum of branes wrapped on T^d. This generates charged $\frac{1}{2}$- and $\frac{1}{4}$- BPS black hole states that correspond to singular solutions in supergravity since they have zero horizon size and hence zero entropy. In addition, for E_6, E_7 and E_8 there are $\frac{1}{8}$-BPS states that correspond to black holes that have non-zero entropy (as well as states with zero entropy), the prototypes being the analysis of black holes in $D = 5$ dimensions (with E_6 duality group) in [21, 22]. The discussion of the associated nilpotent orbits was given in [23]. Our main interest is to extend the analysis in order to account for BPS instantons.

We shall, for convenience, use the M-theory description starting from eleven dimensional supergravity compactified on a $(d+1)$-torus that will be denoted $\mathcal{T}^{d+1}$. The BPS particle states in any dimension are obtained by wrapping all the spatial dimensions of the various extended objects in supergravity around the torus. These include the $M2$-brane and the $M5$-brane, together with the Kaluza–Klein modes of the metric and the magnetic dual Kaluza–Klein monopoles. The BPS instantons can be listed by completely wrapping the euclidean world-volumes of these objects on these tori.

3.3.1. *Features of $P_{\alpha_{d+1}}$ orbits.*

The details of the enumeration of BPS states and instantons in the decompactification limit are reviewed in appendix C, the results of which are summarised in this subsection. These states are labelled by a set of charges that couple to components of the various tensor potentials in the theory and span a space whose dimension is given in the second column of table 4 on page 21 for each Levi group, $M_{\alpha_{d+1}}$, with $0 \leq d \leq 7$. Correspondingly, the dimension of the space of instanton charges is given in the third column. Table 5 on page 22 lists the BPS orbits for each Levi group in the range $0 \leq d \leq 7$.

Table 4 shows that, with one exception, the number of BPS instantons in dimension D equals the sum of the number of BPS particle states and the BPS instantons in dimension $D + 1$, as anticipated above. The exceptional case is the parabolic subgroup with $M_{\alpha_8} = E_7$, where the number of instantons, 120, is one greater than the number of BPS states, 56, plus instantons, 63 in $D = 4$.

The BPS orbits for each value of $d = 10 - D$ with Levi factor $L_{d+1} = GL(1) \times E_d$ are shown in table 5. The tensors v, v_a, v_{ia}, v_{ij} and the spinor S are introduced in section 3.2. I_3 and I_4 are cubic and quartic invariants of E_6 and E_7, respectively, which are defined in terms of the fundamental representation, q^i, of E_6 and E_7, as reviewed in appendices C.6 and C.7. A general feature that is valid in for each $d > 0$ is that the $\frac{1}{2}$-BPS states fill

$D = 10-d$	$M_{\alpha_{d+1}} = E_d$	# point charges (dim $U_{\alpha_{d+1}}$)	# instanton charges = # +ve roots of E_d
10A	1	1	0
10B	$SL(2)$	0	1
9	$SL(2)\times\mathbb{R}^+$	3	1
8	$SL(3)\times SL(2)$	6	4
7	$SL(5)$	10	10
6	$SO(5,5)$	16	20
5	E_6	27	36
4	E_7	56 (57)	63
3	E_8	120	120

TABLE 4. The dimensions of the spaces spanned by the BPS point-like charges and BPS instantons of maximal supergravity for the Levi subgroups in $P_{\alpha_{d+1}}$. The parenthesis for $M_{\alpha_8} = E_7$ indicates that the number of BPS states is one less than the dimension of the unipotent radical, U_{α_8}, of the parabolic subgroup P_{α_8} of E_8.

out orbits of the form

$$\mathcal{O}_{\frac{1}{2}-BPS} = \frac{E_{d+1}}{E_d \ltimes \mathbb{R}^{n_{d+1}}}\,, \qquad (n_2,\ldots,n_8) = (0,3,6,10,16,27,57)\,. \quad (3.2)$$

The integers n_{d+1} are the dimensions of the unipotent radicals, $U_{\alpha_{d+1}}$, listed in table 3 on page 19; they are also the numbers of BPS states for the symmetry groups E_{d+1} listed in table 4, apart from the case of $d = 7$ where U_{α_8} is an element of a non-abelian Heisenberg group. As mentioned earlier, U_{α_8} has dimension 57 while the E_7 point-like states (charged black holes) are labelled by only 56 charges. The missing charge arises from the fact that among the 120 instantons in $D = 3$ dimensions (see table 4) there is one that is a wrapped KKM with $x^\#$ (the fibre coordinate in (3.1)) wrapped around the direction that is identified with (euclidean) time. Since particle states in $D = 4$ dimensions are obtained by identifying the decompactified direction with time, the exceptional instanton is one for which $x^\#$ grows in the cusp and its action becomes singular. By contrast, 56 of the $D = 3$ instantons have action proportional to r_7 and are seen as point-like states in four dimensions, and the other 63 have no r_7 dependence and decompactify to instantons in four dimensions.

It is interesting to speculate about an additional line to table 5 which we did not list, namely one for $M_{\alpha_9} = E_8$ inside the affine Kac-Moody group E_9. While this latter group is infinite dimensional, one can still make sense of the orbits in terms of the finite dimensional vector space $\mathfrak{u}_-$ in (4.5). Indeed, $\mathfrak{u}_-$ here is 248-dimensional and the action of E_8 is isomorphic to the adjoint action on its Lie algebra. Thus the orbits there coincide with the coadjoint nilpotent orbits for E_8.

$M_{\alpha_{d+1}} = E_d$	BPS	BPS condition	Orbit	Dim.
$SL(2)$	$\frac{1}{2}$	-	1	0
$SL(2)\times\mathbb{R}^+$	$\frac{1}{2}$	$v\,v_a = 0$	$\frac{\mathbb{R}^+\times SL(2)}{SL(2)}$	1
	$\frac{1}{4}$	$v\,v_a \neq 0$	$\mathbb{R}^+\times\frac{SL(2)}{SO(2)}$	3
$SL(3)\times SL(2)$	$\frac{1}{2}$	$\epsilon^{ab}\,v_{i\,a}v_{j\,b} = 0$	$\frac{SL(3)\times SL(2)}{(\mathbb{R}^+\times SL(2))\ltimes\mathbb{R}^3}$	5
	$\frac{1}{4}$	$\epsilon^{ab}\,v_{i\,a}v_{j\,b} \neq 0$	$\frac{SL(3)\times SL(2)}{SL(2)\ltimes\mathbb{R}^2}$	6
$SL(5)$	$\frac{1}{2}$	$\epsilon^{ijklm}\,v_{ij}\,v_{kl} = 0$	$\frac{SL(5)}{(SL(3)\times SL(2))\ltimes\mathbb{R}^6}$	7
	$\frac{1}{4}$	$\epsilon^{ijklm}\,v_{ij}\,v_{kl} \neq 0$	$\frac{SL(5)}{O(2,3)\ltimes\mathbb{R}^4}$	10
$SO(5,5)$	$\frac{1}{2}$	$(S\Gamma^m S)=0$	$\frac{SO(5,5)}{SL(5)\ltimes\mathbb{R}^{10}}$	11
	$\frac{1}{4}$	$(S\Gamma^m S)\neq 0$	$\frac{SO(5,5)}{O(3,4)\ltimes\mathbb{R}^8}$	16
E_6	$\frac{1}{2}$	$I_3=\frac{\partial I_3}{\partial q^i}=0$, and $\frac{\partial^2 I_3}{\partial q^i\partial q^j}\neq 0$.	$\frac{E_6}{O(5,5)\ltimes\mathbb{R}^{16}}$	17
	$\frac{1}{4}$	$I_3=0$, $\frac{\partial I_3}{\partial q^i}\neq 0$	$\frac{E_6}{O(4,5)\ltimes\mathbb{R}^{16}}$	26
	$\frac{1}{8}$	$I_3\neq 0$	$\mathbb{R}^*\times\frac{E_6}{F_{4(4)}}$	27
E_7	$\frac{1}{2}$	$I_4 = \left.\frac{\partial^2 I_4}{\partial q^i\partial q^j}\right\vert_{Adj_{E_7}}=0$, and $\frac{\partial^3 I_4}{\partial q^i\partial q^j\partial q^k}\neq 0$.	$\frac{E_7}{E_6\ltimes\mathbb{R}^{27}}$	28
	$\frac{1}{4}$	$I_4=\frac{\partial I_4}{\partial q^i}=0$, and $\left.\frac{\partial^2 I_4}{\partial q^i\partial q^j}\right\vert_{Adj_{E_7}}\neq 0$.	$\frac{E_7}{(O(5,6)\ltimes\mathbb{R}^{32})\times\mathbb{R}}$	45
	$\frac{1}{8}$	$I_4=0$, $\frac{\partial I_4}{\partial q^i}\neq 0$	$\frac{E_7}{F_{4(4)}\ltimes\mathbb{R}^{26}}$	55
	$\frac{1}{8}$	$I_4>0$	$\mathbb{R}^+\times\frac{E_7}{E_{6(2)}}$	56

TABLE 5. The orbits of instantons associated with the parabolic subgroup $P_{\alpha_{d+1}}$. With one exception these are orbits of charged black hole states satisfying fractional BPS conditions that are generated by the action of the Levi subgroup, $GL(1)\times E_d$, on a representative BPS state. The notation is explained in the text.

3.4. **The string perturbation theory limit: P_{α_1}.**

In this limit BPS instantons give non-perturbative corrections to string perturbation theory. This involves an expansion in the parabolic subgroup P_{α_1}, with Levi factor $L_{\alpha_1} = GL(1)\times SO(d,d)$. This limit is analogous to the

limit considered in the previous subsection with the role of the decompactifying circle radius, r_d, replaced by the inverse string coupling in $D = 10 - d$ dimensions, which is denoted $1/\sqrt{y_D}$. In this case the orbits of BPS charges do not correspond to black hole charge orbits.

The BPS instantons that enter in this limit are easiest to analyse in terms of the wrapping of euclidean world-volumes of Dp-branes, the NS5-brane and stringy KKM instantons. The Dp-branes enter for all values of $d \geq 0$ and their contribution alone leads to an abelian unipotent radical, U_{α_1}. The NS5-branes contribute on tori of dimension $d \geq 6$ and the KKM instantons contribute for $d = 7$. Both these kinds of instantons render the unipotent radical nonabelian. In section 3.4.1 and appendix D we review the classification of Dp-brane instantons in terms of the classification of $SO(d,d)$ chiral spinor orbits, which leads to the following features:

- For $d \leq 3$ there is only one non-trivial orbit, which is $\frac{1}{2}$-BPS.
- $\frac{1}{4}$-BPS orbits arise when $d \geq 4$ and have dimensions 2^{d-1}, the same as that of the full spinor space.
- For $d = 4$ the $\frac{1}{2}$-BPS orbit is parameterised by a spinor satisfying the $SO(4,4)$ pure spinor constraint, $S \cdot S = 0$, while the full eight-component spinor space (with $S \cdot S \neq 0$) parameterises the $\frac{1}{4}$-BPS orbit.
- For $d = 5$ the $\frac{1}{2}$-BPS orbit is parameterised by a $SO(5,5)$ spinor satisfying the pure spinor constraint,[11] $S\Gamma^i S = 0$, and once again the unconstrained spinor parameterises the $\frac{1}{4}$-BPS orbit.
- For $d = 6$ the $\frac{1}{2}$-BPS orbit is defined by a $SO(6,6)$ spinor satisfying the pure spinor constraint,

$$F_2 \;:=\; \tfrac{1}{2} \sum_{i,j=1}^{12} S\Gamma^{ij} S \, dx^i \wedge dx^j \;=\; 0\,, \tag{3.3}$$

where the $\frac{1}{4}$-BPS orbit is parameterised by a spinor satisfying the weaker constraints

$$F_2 \;\neq\; 0\;, \qquad F_2 \wedge F_2 \;=\; 0\,. \tag{3.4}$$

In addition there is a $\frac{1}{8}$-BPS orbit which is identified with the space of a spinor satisfying

$$F_2 \wedge F_2 \;\neq\; 0\;, \qquad *F_2 \wedge F_2 \;=\; 0\,, \tag{3.5}$$

where $*$ is the Hodge star operator, and a second $\frac{1}{8}$-BPS orbit identified with the space spanned by an unconstrained 32-component spinor.

[11] The Dirac matrices Γ^i $(i = 1, \ldots 2d)$ form a $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ representation of the Clifford algebra $C\ell(d,d)$. We will denote the antisymmetric product of r Dirac Γ matrices by $\Gamma^{i_1 \cdots i_r} = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} (-)^\sigma \Gamma^{i_{\sigma(1)}} \cdots \Gamma^{i_{\sigma(r)}}$, where $(-)^\sigma$ is the signature of the permutation σ.

- For $d = 7$ there are nine nontrivial orbits (in addition to the trivial orbit) that were determined by Popov [24]. The $\frac{1}{2}$-BPS case is the smallest non-trivial orbit, which is the space spanned by a spinor satisfying

$$F_3 \quad := \quad \frac{1}{3!} \sum_{i,j,k=1}^{14} S\Gamma^{ijk} S \, dx^i \wedge dx^j \wedge dx^k \quad = \quad 0 \,, \tag{3.6}$$

  where S is a $SO(7,7)$ spinor and Γ^i $(i = 1, \ldots, 14)$ are corresponding Dirac matrices. However, the description of the remaining orbits in terms of covariant constraints involving F_3 analogous to those of (3.4) and (3.5) is not known to our knowledge.

We now turn to a detailed description of these orbits, which draws from the information in section 6.1.

3.4.1. *Classification of spinor orbits.* A review of the method for classifying spinor orbits of $G = Spin(d,d)$ (the subgroup of even and invertible elements of the Clifford group $C\ell(d,d)$ associated with $SO(d,d)$) can be found in [25] (based on the original work in [26] for $d \leq 6$, and [24] for $d = 7$).

The following tables will summarise some facts about these orbits, which are cosets of the form $\mathcal{O} = SO(d,d)/H$, H being the stabilizer of a point in the orbit. For each value of d we will give a representative spinor of each orbit (labelled S^0 in column 1 and defined in appendix D), together with its stabiliser (H in column 2), its dimension ($\dim(G/H)$ in column 3) and the fraction of supersymmetry it preserves – i.e., its BPS degree $N/2^{d-1}$ is determined by the number of linearly independent spinors N of the orbit representative S^0. In the following we will only list the BPS orbits appearing into the Fourier coefficients of the coefficients we are interested in. A more complete discussion is given in section 6.1.

The tables that follow have the following general properties:

- The bottom row is the trivial orbit and the top row is the dense orbit of a full spinor.
- The first non-trivial orbit is the $\frac{1}{2}$-BPS configuration with orbit parametrized by the coset[12]

$$\mathcal{O}_{\frac{1}{2}-BPS} = \frac{SO(d,d)}{SL(d) \ltimes \mathbb{R}^{\frac{d(d-1)}{2}}} \tag{3.7}$$

  of dimension $1+d(d-1)/2$. This is the orbit of a spinor satisfying the pure spinor constraint and can be obtained by acting on the ground state of the Fock space representation of the spinor with $SO(d,d)$ rotations.

[12]Although the orbits listed in this section are over $\mathbb{R}$ or $\mathbb{C}$, the structures are largely independent of the ground field. For example, this particular orbit has the same form over any field k with characteristic different from 2, but with the $\mathbb{R}$ factor replaced by $k^{\frac{d(d-1)}{2}}$.

- The second non-trivial orbit (the NTM, or $\frac{1}{4}$-BPS, orbit) arises for $d \geq 4$ and is the coset

$$\mathcal{O}_{\frac{1}{4}-BPS} \quad = \quad \frac{SO(d,d)}{(Spin(7) \times SL(d-4)) \ltimes U_{\frac{(d-4)(d+11)}{2}}} \ , \tag{3.8}$$

  where where U_s is a unipotent group of dimension s (which is non-abelian for $d \geq 6$).

In more detail, the specific orbits for each $SO(d,d)$ group are as follows:

► $SO(1,1)$ is trivial. For $SO(2,2)$ and $SO(3,3)$ the action of the spin group is transitive and there are only two orbits: the trivial one of dimension 0, and the Weyl spinor orbit. This is in accord with the discussion in the previous subsection.

$G = SO(2,2)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
1	$SL(2) \ltimes \mathbb{R}$	2	$\frac{1}{2}$
0	$Spin(4)$	0	$--$

$G = SO(3,3)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
1	$SL(3) \ltimes \mathbb{R}^3$	4	$\frac{1}{2}$
0	$Spin(6)$	0	$--$

► For $d \geq 4$ the action of the spin group is not transitive and there are several non-trivial orbits represented by constrained spinors.[13]

$G = SO(4,4)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e_{1234}$	$Spin(7)$	8	$\frac{1}{4}$
1	$SL(4) \ltimes \mathbb{R}^6$	7	$\frac{1}{2}$
0	$Spin(8)$	0	$--$

$G = SO(5,5)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e_{1234}$	$Spin(7) \ltimes \mathbb{R}^8$	16	$\frac{1}{4}$
1	$SL(5) \ltimes \mathbb{R}^{10}$	11	$\frac{1}{2}$
0	$Spin(10)$	0	$--$

[13]The symbols $e_{i_1 \cdots i_r}$ and $e^*_{i_1 \cdots i_r}$ labelling the spinor S^0 are defined in appendix D.

► The $SO(6,6)$ case involves some noncommutative unipotent subgroups U_s of dimension s. The full spinor orbit of dimension 32 is $\mathbb{R}^* \times SO(6,6)/SL(6)$.

$G = SO(6,6)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e^*_{14} + e^*_{25} + e^*_{36}$	$SL(6)$	32	0
$1 + e^*_{14} + e^*_{25}$	$Sp(6) \ltimes \mathbb{R}^{14}$	31	$\frac{1}{8}$
$1 + e^*_{14}$	$(SL(2) \times Spin(7)) \ltimes U_{17}$	25	$\frac{1}{4}$
1	$SL(6) \ltimes \mathbb{R}^{15}$	16	$\frac{1}{2}$
0	$Spin(12)$	0	$--$

► For $SO(7,7)$ the full spinor orbit of dimension 32 is $GL(1) \times SO(7,7)/(G_2 \times_{\mathbb{Z}_2} G_2)$, where G_2 is the exceptional group of rank 2 and where $H_1 \times_{\mathbb{Z}_2} H_2$ denotes the almost direct product of two groups intersecting on $\mathbb{Z}_2$. Of the total of 10 orbits obtained in [24], we only list the ones relevant for the analysis of the Fourier modes discussed in this paper.

$G = SO(7,7)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e^*_7$	$SL(6) \ltimes \mathbb{R}^{12}$	44	$\frac{1}{8}$
$1 + e^*_{147} + e^*_{257}$	$(Sp(6) \times_{\mathbb{Z}_2} \mathbb{R}) \ltimes \mathbb{R}^{26}$	43	$\frac{1}{8}$
$1 + e_{1234}$	$(SL(3) \times Spin(7)) \ltimes U_{27}$	35	$\frac{1}{4}$
1	$SL(7) \ltimes \mathbb{R}^{21}$	22	$\frac{1}{2}$
0	$Spin(14)$	0	$--$

3.4.2. *Neveu–Schwarz five-brane and stringy KKM instantons.*

The wrapped world-volume of the NS5-brane produces a new kind of instanton when $d \geq 6$, which is a source of $B_{\rm NS}$ flux. Whereas the Dp-brane instantons have actions of the form C/g_s with C independent of g_s, the wrapped NS5-brane has an action of the form C/g_s^2. This means that such NS5-instantons are suppressed by e^{-C/g_s^2}, and so, in the string perturbation theory regime they are suppressed relative to the Dp-brane instantons. The presence of the charge carried by this wrapped NS5-brane instanton leads to a non-commutativity of the unipotent radical, U_{α_1}, which lies in a Heisenberg group (this is analogous to the fact that the KKM instanton in $D = 3$ led to non-commutativity of the unipotent radical U_{α_8} in the P_{α_8} parabolic subgroup of E_8). The non-commutativity arises because the presence of a NS5-brane charge generates a non-trivial $B_{\rm NS}$ background. This affects the definition of the D-brane charges due to the dependence on $B_{\rm NS}$ of their field-strengths, $F^{(4)} := dC^{(3)} + dB_{\rm NS} \wedge C^{(1)}$ and $*F^{(4)} = dC^{(5)} + C^{(3)} \wedge dB_{\rm NS} - dC^{(3)} \wedge B_{\rm NS}$. Since there is only one euclidean NS5-brane configuration on a 6-torus (the $D = 4$ case) the non-commutative part of U_{α_1} is one-dimensional, so the unipotent radical forms a Heisenberg group.

Upon further compactification on T^7 to $D = 3$ there are 7 distinct wrapped NS5-brane world-volume instantons, one for each six-cycle. In addition, there are 8 M-theory KKM instantons that are distinguished from each other in the M-theory description by identifying the coordinate $x^\#$ with any one of the 1-cycles, as explained earlier. In string language, one of these is the wrapped euclidean $D6$-brane that has been counted as one of the 64 components of the $SO(7,7)$ spinor space and contributes to the abelian part of the unipotent radical U_{α_1}. The other 7 are KKM instantons with $x^\#$ identified with a circle in one of the 7 other directions. These are T-dual to the 7 wrapped NS5-branes. The presence of the $D6$-brane and KKM instantons leads to a higher degree of non-commutativity of the unipotent radical, due for example, to the non-linear dependence of the D6-brane field strength on $B_{\rm NS}$ through $*dC^{(1)} = dC^{(7)} + \frac{1}{2}\, B_{\rm NS} \wedge dC^{(5)} - \frac{1}{2} dB_{\rm NS} \wedge C^{(5)} - \frac{1}{3} B_{\rm NS} \wedge B_{\rm NS} \wedge dC^{(3)} + \frac{1}{3} B_{\rm NS} \wedge dB_{\rm NS} \wedge dC^{(5)}$.

We will see later that this counting coincides with that expected from a group theoretic analysis of the dimension of the abelian and non-abelian (i.e., derived subgroup) parts of the unipotent radical summarised in the columns labelled "first node" of table 3 on page 19.

3.5. **BPS instantons in the semi-classical M-theory limit: P_{α_2}.**

This is the limit in which the volume, $\mathcal{V}_{d+1}$, of the M-theory torus $\mathcal{T}^{d+1}$ becomes large and semi-classical eleven-dimensional supergravity is a good approximation. The Fourier modes of interest are those associated with the maximal parabolic subgroup P_{α_2}, which has Levi subgroup $L_{\alpha_2} = GL(1) \times SL(d{+}1)$. The constant terms in the Fourier expansion were considered in [1] and shown to match expectations based on perturbative eleven-dimensional supergravity.

The instanton charge space can be described as follows. The wrapped KK world-lines do not give instantons in this limit since their action is independent of the volume, $\mathcal{V}_{d+1}$. Wrapped euclidean $M2$-branes appear in $D \leq 8$ dimensions (corresponding to symmetry groups with rank ≥ 3), while the wrapped euclidean $M5$-brane arises for $D \leq 5$ dimensions (corresponding to symmetry groups with rank ≥ 6) and the wrapped world-volume associated with the KKM enters first in $D = 3$ dimensions (i.e., for symmetry group E_8). These instanton actions have the exponentially suppressed form $\exp(-C/\mathcal{V}_{d+1}^a)$, where C is independent of $\mathcal{V}_{d+1}$ in the limit $\mathcal{V}_{d+1} \to 0$, and $a = 3/(d+1)$ for the wrapped $M2$-brane, $a = 6/(d+1)$ for the wrapped $M5$-brane and $a = 7/(d+1)$ for the wrapped KKM.

The space spanned by the 3-form, $v_{[ijk]}$ that couples to $M2$-brane world-sheets wrapping 3-cycles inside $\mathcal{T}^{d+1}$ has dimension

$$D_{M2}^{d+1} \quad = \quad \frac{(d+1)!}{3!\,(d-2)!}\,, \tag{3.9}$$

which equals 1, 4, 10, 20, 35, and 56, respectively, for tori of dimensions 3, 4, 5, 6, 7, and 8 (corresponding to the duality groups $E_3, \ldots, E_8$). Similarly, the

space of euclidean five-branes wrapping 6-cycles inside $\mathcal{T}^{d+1}$ has dimension

$$D^{d+1}_{M5} = \frac{(d+1)!}{6!(d-5)!}\,, \tag{3.10}$$

which equals 1, 7, and 28, respectively, for $d+1 = 6$, 7, and 8 (corresponding to duality groups E_6, E_7, and E_8). Finally, a finite action KKM instanton only exists if there are 8 circular dimensions, so it only contributes for the E_8 case. As argued earlier, there are 8 distinct objects of this kind since $x^{\#}$ is distinguished from the other circular coordinates.

Again these dimensions can be compared with those listed in section 6.1 and summarised in table 3 on page 19 under the heading "second node". The wrapped euclidean $M2$-branes contribute the dimensions of abelian part of the unipotent radical for this maximal parabolic subgroup. In fact the numbers in the left-hand column of the second node heading are equal to D^{d+1}_{M2} for all $0 \leq d \leq 7$. The $M5$-brane charge space of dimension D^{d+1}_{M5}, equals the dimension of the non-commutative part (i.e., derived subgroup) of the unipotent radical for E_6 and E_7, while for E_8 there is also a contribution of 8 from the KKM instantons. In this case the non-abelian component of the unipotent radical arises from the KKM instanton dependence on the 3-form $A^{(3)}$ configurations (analogous to the way the $B_{\rm NS}$ configurations induced the non-commutativity in the previous section).

Although we have given a list of dimensions of the space spanned by the orbits, in this case we have not analysed the BPS conditions to discover how the complete space decomposes into orbits with fractional supersymmetry. However, the latter part of this paper analyses the complete orbit structure for the subgroup P_{α_2} and the list of orbits is given in table 8 on page 55. From this we can identify, for each value of d, the minimal ($\frac{1}{2}$-BPS) and NTM ($\frac{1}{4}$-BPS) orbits, as well as many others that arise when $d \geq 5$ (i.e. for E_6, E_7 and E_8).

4. Explicit examples of Fourier modes for rank ≤ 5.

4.1. **Fourier expansions for higher rank groups.** Suppose that $\phi \in C^\infty(\Gamma\backslash G)$ is an automorphic function, and that $A \subset G$ is an abelian subgroup which is isomorphic to $\mathbb{R}^m$ for some $m > 0$. If $\Gamma \cap A$ corresponds to a lattice in $\mathbb{R}^m$ under this identification, then ϕ's restriction to A, $\phi(a)$, has a Fourier expansion. The same is true for any right translate $\phi(ag)$, for g fixed. A prime example of this is A equal to the unipotent radical U of a maximal parabolic subgroup $P = LU$ of G, when U is abelian and Γ is arithmetically defined:

$$\phi(ug) \;=\; \sum_\chi \chi(u)\phi_\chi(g)\;, \quad \phi_\chi(g) \;=\; \int_{\Gamma\cap U\backslash U} \phi(ug)\,\chi(u)^{-1}\,du\,, \tag{4.1}$$

where the sum is taken over all characters χ of U which are trivial on $\Gamma \cap U$. In particular the special case $u = e$,

$$\phi(g) \;=\; \sum_{\chi} \phi_\chi(g)\,, \tag{4.2}$$

reconstructs ϕ as a sum of its Fourier coefficients ϕ_χ. When U fails to be abelian the coefficients ϕ_χ still make sense, though ϕ is no longer a sum of them. Instead, it is the integral of ϕ over the commutator subgroup[14] $[U,U]$ of U which has an expansion

$$\int_{\Gamma\cap[U,U]\backslash[U,U]} \phi(ug)\,du \;=\; \sum_{\chi} \phi_\chi(g)\,; \tag{4.3}$$

in other words, the Fourier expansion only captures a small part of ϕ's restriction to U – the part which transforms trivially under $[U,U]$.

A character on U can be viewed as a linear functional on its Lie algebra $\mathfrak{u}$, via its differential. In our case, in which U is the unipotent radical of a maximal parabolic subgroup $P = P_{\alpha_j}$ for some simple root α_j, $\mathfrak{u}$ has a graded structure

$$\mathfrak{u} \;=\; \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \cdots \tag{4.4}$$

in which $\mathfrak{u}_k$ is the span of root vectors for roots of the form $\alpha = \sum c_k\alpha_k$, with $c_j = k$. The Killing form exhibits the dual $\mathfrak{u}^*$ of $\mathfrak{u}$ as the complexification of the Lie algebra

$$\mathfrak{u}_- \;=\; \mathfrak{u}_{-1} \oplus \mathfrak{u}_{-2} \oplus \cdots\,. \tag{4.5}$$

The commutator subgroup $[U,U]$ has Lie algebra $\mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \cdots$, so the differential of a character is sensitive only to $\mathfrak{u}_1$. Again through the bilinear pairing of the Killing form, its dual space $\mathfrak{u}_1^*$ is isomorphic to the complexification $\mathfrak{u}_{-1}\otimes\mathbb{C}$ of $\mathfrak{u}_{-1}$. The exponential of any such a linear functional is a character of U, and hence $\mathfrak{u}_{-1}\otimes\mathbb{C}$ is known as the *character variety* of U.

Now let χ be a character of U which is invariant under the discrete subgroup $\Gamma\cap U$. The above correspondence guarantees the existence of a unique

$$Y \;\in\; \mathfrak{u}_{-1}\otimes\mathbb{C} \quad \text{such that} \quad \chi(e^X) \;=\; e^{B(Y,X)}\,, \tag{4.6}$$

where $B(\cdot,\cdot)$ is the Killing form. Decompose $P = LU$, where L is the Levi component. Then formula (4.1) and the automorphy of ϕ under any $\gamma \in \Gamma \cap L$ imply that

$$\begin{aligned} \phi_\chi(\gamma g) \;&=\; \int_{\Gamma\cap U\backslash U} \phi(\gamma^{-1}u\gamma g)\,\chi(u)^{-1}\,du \\ &=\; \int_{\Gamma\cap U\backslash U} \phi(ug)\,\chi(\gamma u\gamma^{-1})^{-1}\,du\,. \end{aligned} \tag{4.7}$$

[14]The commutator subgroup $[U,U]$ is the smallest normal subgroup of U which contains all elements of the form $[u_1,u_2]$, for $u_1,u_2\in U$.

Here we have changed variables $u \mapsto \gamma u \gamma^{-1}$, which preserves the measure du. In terms of (4.6)

$$\chi(\gamma e^X \gamma^{-1}) \;=\; \chi(e^{\gamma X \gamma^{-1}}) \;=\; e^{B(Y,\gamma X \gamma^{-1})} \;=\; e^{B(\gamma^{-1} Y \gamma, X)}, \qquad (4.8)$$

because of the invariance of the Killing form under the adjoint action; the character in the second line of (4.7) is hence equal to the character for the Lie algebra element $\gamma^{-1} Y \gamma \in \mathfrak{u}_{-1} \otimes \mathbb{C}$.

Consequently, the Fourier coefficients (4.1) are related for characters χ which lie in the same $\Gamma \cap L$-orbit under the adjoint action on $\mathfrak{u}_{-1} \otimes \mathbb{C}$. It should be remarked that $\mathfrak{u}_{-1}$ – like each space $\mathfrak{u}_j$ – is invariant under the adjoint action of L, and typically furnishes an irreducible representation of L. The complexification $L_{\mathbb{C}}$ of L likewise acts on $\mathfrak{u}_{-1} \otimes \mathbb{C}$ according to an irreducible representation, and carves it up into finitely many complex character variety orbits. Similarly, the adjoint action of $\Gamma \cap L$ on the set of characters of U which are trivial on $\Gamma \cap U$ refines these complex orbits into myriad further "integral" orbits. Those characters naturally form a lattice inside of $i\mathfrak{u}_{-1} \subset \mathfrak{u}_{-1} \otimes \mathbb{C}$, and this last action is that of a discrete subgroup of L on a lattice, e.g., the action of $GL(n, \mathbb{Z})$ on $\mathbb{Z}^n$ in a particular special case. These are more subtle to describe because of number-theoretic reasons; indeed, even describing $\Gamma \cap L$ for a large exceptional group is quite complicated. Recall that this is the charge lattice from section 3.

Each of these complex character variety orbits (and hence each of the $\Gamma \cap L$-orbits on the set of characters that are trivial on $\Gamma \cap U$) is thus contained in a single (complex) coadjoint nilpotent orbit. It therefore makes sense to categorize the complex character variety orbits by giving their basepoints and dimensions. This information was provided in section 3, based on the analysis of BPS states in string theory. This analysis focused on the supersymmetric orbits and did not cover all possible orbits. A systematic and detailed analysis of the remaining orbits for the maximal parabolic subgroups we study will be given in 6.1. These have long been known for the classical groups by the study of "classical rank theory"; the paper [27] contains a listing for all maximal parabolic subgroups of exceptional groups. In addition, the integral orbits are also known in many cases: Bhargava [28, Section 4] and Krutelevich [29] treat certain cases, with additional cases to appear in forthcoming work of Bhargava.

Note that the calculation (4.7) shows that each coefficient ϕ_χ – which is determined by its values on L – is automorphic under any γ that lies in both Γ and $\mathrm{Stab}_L(\chi)$, the stabilizer of χ within L. In terms of the differential, these are the elements of $\Gamma \cap L$ for which the adjoint action fixes the element $Y \in \mathfrak{u}_{-1} \otimes \mathbb{C}$ from (4.6). One can therefore use (4.7) to write the sum of $\phi_\chi(g)$, for χ ranging over one of the integral orbits, as the sum of left γ-translates of a fixed ϕ_χ, where γ now ranges over cosets of $\Gamma \cap L$ modulo the stabilizer of this fixed character. The vanishing of any Fourier coefficient ϕ_χ as a function of L is equivalent to that of all Fourier coefficients in its orbit.

The following subsections (together with details that are presented in appendix E) concern some specific, explicit examples of the Fourier modes of the coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ for the low rank duality groups with $d \leq 4$ (i.e. $D \geq 6$). In these cases standard, classical techniques can be used to obtain exact expressions, including the arithmetical divisor sums that appear. These techniques have the virtue of being relatively simple in these special low rank cases; the higher rank cases of E_6, E_7 and E_8 will be discussed in the later sections, although without precise calculations – our chief contribution is to use representation theory to show that many of them vanish.

In each particular case we will explicitly identify the character χ, which lies in the lattice of characters of U that are trivial on $\Gamma \cap U$, with a tuple of integral parameters m_i, and use the notation

$$\mathcal{F}^{(D)\alpha}_{(p,q)}(m_i) \;:=\; \Big(\mathcal{E}^{(D)}_{(p,q)}\Big)_\chi \qquad \text{and} \qquad F^{G\,\alpha}_{\beta;s}(m_i) \;:=\; \big(E^G_{\beta;s}\big)_\chi \tag{4.9}$$

to refer to the Fourier modes of $\mathcal{E}^{(D)}_{(p,q)}$ and $E^G_{\beta;s}$, respectively.

The precise details of these Fourier coefficients could, in principle, be independently checked against an explicit evaluation of instanton contributions to the graviton scattering amplitude, but in practice such detailed verification is very difficult. However, most details of the contribution of $\frac{1}{2}$-BPS instantons to these coefficients in limit (i), the decompactification limit in which $r_d \gg 1$, can be motivated directly from string theory. This is the limit in which, for these low rank cases, the instantons are identified with wrapped world-lines of small black holes of the $(D+1)$-dimensional theory. The asymptotic behaviour can be understood by studying the fluctuations around $\frac{1}{2}$-BPS D-particle configurations in a manner that generalises the arguments of [30], leading to an expression for the modes in $D = 10-d \leq 9$ dimensions of the form

$$\mathcal{F}^{(D)\alpha_{d+1}}_{(0,0)}(k) \;=\; (\tfrac{r_d}{\ell_{D+1}})^{n_D}\, \sigma_{7-D}(|k|)\, \frac{e^{-S_D(k)}}{S_D(k)^{\frac{8-D}{2}}}\, \Big(1 + O(\tfrac{\ell_{D+1}}{r_d})\Big)\,. \tag{4.10}$$

Here $S_D(k) = 2\pi|k|r_d m_{\frac{1}{2}}$ is the action for the world-line of the D-particle wound around the circle of radius r_d and $m_{\frac{1}{2}}$, which is a function of the moduli, is the mass of a "minimal" $\frac{1}{2}$-BPS point-like particle state in $D+1$ dimensions – that is, a state that is related by duality to the lightest mass single-charge D-particle. Such states can form threshold bound D-particles of mass $p\, m_{\frac{1}{2}}$. The divisor sum, $\sigma_n(k) = k^n\, \sigma_{-n}(k) = \sum_{q|k} q^n$, sums over the winding number q of the world-lines of such D-particles (where $k = p \times q$) and can be identified with a matrix model partition function. The factor of $S_D(k)^{(D-8)/2}$ comes from integration over the bosonic and fermionic zero modes and n_D is a constant that depends on the dimension D. Because of the high degree of supersymmetry preserved by the $\frac{1}{2}$-BPS configuration it turns out that this approximation is exact in several cases. We have

not completed an independent quantum calculation of the $\frac{1}{4}$-BPS instanton contributions, which are more subtle, but we hope to discuss these in a separate publication.

4.2. $D = 10B$**:** $SL(2,\mathbb{Z})$**.**

The simplest nontrivial (but very degenerate) example arises in the case of the IIB theory with $D = 10$, where the discrete duality group is $SL(2,\mathbb{Z})$.[15] In this case the $\frac{1}{2}$- and $\frac{1}{4}$-BPS interactions, $\mathcal{E}^{(10)}_{(0,0)}$ and $\mathcal{E}^{(10)}_{(1,0)}$, are given by Eisenstein series [31, 32]

$$\mathcal{E}^{(10)}_{(0,0)} = 2\zeta(3)\, E^{SL(2)}_{\frac{3}{2}}(\Omega)\,, \qquad \mathcal{E}^{(10)}_{(1,0)} = \zeta(5)\, E^{SL(2)}_{\frac{5}{2}}(\Omega)\,, \tag{4.11}$$

where $2\zeta(2s)\, E_s(\Omega)$ is a non-holomorphic Eisenstein series and $\Omega := \Omega_1 + i\Omega_2 = C^{(0)} + i/\sqrt{y_{10}}$.

It is useful to parametrize the coset $SL(2)/SO(2)$ (the upper half plane) associated with the continuous symmetry group, $SL(2,\mathbb{R})$, by the coset described by the parabolic subgroup consisting of matrices of the form

$$e_2 \;=\; \frac{1}{\Omega_2^{\frac{1}{2}}}\begin{pmatrix} 1 & \Omega_1 \\ 0 & \Omega_2 \end{pmatrix} \;=\; \begin{pmatrix} \Omega_2^{-\frac{1}{2}} & 0 \\ 0 & \Omega_2^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 & \Omega_1 \\ 0 & 1 \end{pmatrix}. \tag{4.12}$$

where the (somewhat trivial) Levi factor L is the diagonal $GL(1)$ factor and the second factor, which depends on Ω_1, is the unipotent radical, U. The $SL(2)$ Eisenstein series can be expressed as

$$2\zeta(2s)\, E^{SL(2)}_s(\Omega) := \sum_{M_2\in\mathbb{Z}^2\backslash\{0\}} (m^2_{SL(2)})^{-s} = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} \frac{\Omega_2^s}{|m+n\Omega|^{2s}}\,, \tag{4.13}$$

where the $SL(2,\mathbb{Z})$-invariant (mass)^2 is defined by

$$m^2_{SL(2)} := M_2\cdot g_2\cdot M_2^T = \frac{|m+n\Omega|^2}{\Omega_2}\,, \tag{4.14}$$

where $g_2 = e_2\cdot e_2^T$ and $M_2 = (n,m)\in\mathbb{Z}^2\backslash\{0\}$.

It is straightforward to determine the Fourier coefficients using the standard expansion of such series in terms of Bessel functions,

$$E_s(\Omega) = \sum_{n\in\mathbb{Z}} F^{SL(2)}_s(n)\, e^{2i\pi n\Omega_1}\,, \tag{4.15}$$

The zero Fourier mode is

$$F^{SL(2)}_s(0) = \Omega_2^s + \frac{\xi(2s-1)}{\xi(2s)}\,\Omega_2^{1-s}\,, \tag{4.16}$$

[15]The type IIA theory has no instantons, which means that only the 0-dimensional trivial orbit contributes.

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. The non-zero mode with phase $e^{2i\pi n\Omega_1}$ is

$$F_s^{SL(2)}(n) = \frac{2\,\Omega_2^{\frac{1}{2}}}{\xi(2s)} \frac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac{1}{2}}}\, K_{s-\frac{1}{2}}(2\pi|n|\Omega_2)\,, \tag{4.17}$$

where $\sigma_\alpha(n) = \sum_{0<d|n} d^\alpha$ is the divisor function, and the non-zero mode with frequency n is proportional to $K_{s-\frac{1}{2}}$, which is a modified Bessel function of the second kind.

In this degenerate case the only limit to consider is $\Omega_2 \to \infty$, which is the limit of string perturbation theory organized as a power series in Ω_2^{-2} corresponding to the genus expansion of a closed Riemann surface. In this limit the expansion of the coefficient functions is dominated by the two power behaved constant terms in the zero mode $F_s^{SL(2)}(0)$ in (4.16), while the non-zero modes have asymptotic behaviour at large Ω_2,

$$F_s^{SL(2)}(n) = \frac{\sigma_{2s-1}(|n|)}{\xi(2s)|n|^s}\, e^{-2\pi|n|\Omega_2}\left(1 + O(\Omega_2^{-1})\right)\,, \tag{4.18}$$

where the asymptotic expansion of the Bessel function

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + O(x^{-1})\right)\,, \tag{4.19}$$

for $x \gg 1$ has been used.

The two power behaved terms have the interpretation of terms in string perturbation theory, which is an expansion of y_{10}, the square of the string coupling constant. Furthermore, the Eisenstein series with $s = 3/2$ and with $s = 5/2$ have the correct power-behaved terms to account precisely for the known behaviour of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ terms in the low energy expansion of the four graviton amplitude in 10 dimensions. In [1] it was shown that this is in agreement with string perturbation theory extends to the higher rank cases where the pattern of constant terms is more elaborate. Furthermore, the exponential terms in the expansion in (4.18) correspond to the expected D-instantons that arise in the $D = 10$ type IIB theory. This illustrates the fact, common to all BPS instanton processes, that the exponential decay of a Fourier mode is proportional to the charge n that determines the phase of the mode. The correction term of order Ω_2^{-1} in (4.18) indicates perturbative corrections to the instanton contribution given by an expansion in powers of the string coupling constant that corresponds to the addition of boundaries in the Riemann surface.

In this case the only instantons are $\frac{1}{2}$-BPS D-instantons – there are no $\frac{1}{4}$-BPS instantons in the ten-dimensional type IIB theory. However, it is known from string theory arguments that the Eisenstein series at $s = 3/2$ is associated with the $\frac{1}{2}$-BPS $\mathcal{R}^4$ term while the series at $s = 5/2$ is associated with the $\frac{1}{4}$-BPS $\partial^4\mathcal{R}^4$ contribution. This leaves unresolved the question as to what features of these series at special values of s encode the fraction of supersymmetry that these terms preserve? This must be encoded in the

measure. Indeed in the $s = 3/2$ case it was argued in [30, 33] that the measure factor $\sigma_{-2}(|n|)$ arises from the $\frac{1}{2}$-BPS D-instanton matrix model, which was verified in [30,34]. Presumably, the $s = 5/2$ measure should arise in a similar manner.

In most of the higher-rank examples that follow there is a less subtle distinction between the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS cases since in typical cases there are $\frac{1}{4}$-BPS instanton configurations that break $\frac{3}{4}$ of the supersymmetry. As will be shown in the following, these generally enter into non-zero Fourier modes of the coefficient $\mathcal{E}^{(D)}_{(1,0)}$ for $3 \leq D < 10$ (although, as will also be seen later, only the $\frac{1}{2}$-BPS orbit contributes in the P_{α_1} parabolic with $D = 7, 8, 9$). The subtleties of the measure factor are not required in order to identify the fraction of supersymmetry preserved in such cases. However, there are no $\frac{1}{8}$-BPS configurations for $D > 5$. Therefore, for $D > 5$ the distinction between the coefficient $\mathcal{E}^{(D)}_{(0,1)}$ and the ones which preserve more supersymmetry is again not determined by the spectrum of instantons that contribute in the various limits under consideration. This indicates that the $\frac{1}{8}$-BPS nature of $\mathcal{E}^{(D)}_{(0,1)}$ must be encoded in the form of the measure factor.

4.3. $D = 9$: $SL(2,\mathbb{Z})$.

The coefficients of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions in this case are [2, 35, 36]

$$\mathcal{E}^{(9)}_{(0,0)} = 2\zeta(3)\,\nu_1^{-\frac{3}{7}}\,E^{SL(2)}_{\frac{3}{2}} + 4\zeta(2)\nu_1^{\frac{4}{7}}\,, \tag{4.20}$$

$$\mathcal{E}^{(9)}_{(1,0)} = \zeta(5)\nu_1^{-\frac{5}{7}}\,E^{SL(2)}_{\frac{5}{2}} + \frac{4\zeta(2)\zeta(3)}{15}\nu_1^{\frac{9}{7}}\,E^{SL(2)}_{\frac{3}{2}} + \frac{4\zeta(2)\zeta(3)}{15}\nu_1^{-\frac{12}{7}}\,, \tag{4.21}$$

where $\nu_1 = (\ell^B_{10}/r_B)^2 = g_A^{\frac{7}{8}}\,(r_A/\ell^A_{10})^{\frac{3}{2}}$ with r_B the radius of the compact dimension in the IIB theory and $r_A = \ell_s^2/r_B$ the radius in the IIA theory. The IIA string coupling, g_A, is related to that of the IIB theory by $g_A = g_B\,\ell_s/r_B$. Furthermore, the $D = 9$ theory can be viewed as the compactification of M-theory from 11 dimensions on a 2-torus, $\mathcal{T}^2$, with volume $\mathcal{V}_2 = \nu_1^{2/3}\,\ell_{11}^2$.

The limit $\nu_1 \to 0$ is the limit in which the $\mathbb{R}^+$ parameter of the continuous symmetry, $SL(2,\mathbb{R}) \times \mathbb{R}^+$, becomes infinite, which is the decompactification limit to the $D = 10$ IIB theory ($r_B \to \infty$) while the limit $\nu_1 \to \infty$ is the semi-classical M-theory limit in which, $\mathcal{V}_2$, the volume of $\mathcal{T}^2$ becomes infinite. Equations (4.20) and (4.21) show that there are no non-zero modes in either of these limits. Since $\Omega_2 = g_A^{-1} r_A/\ell_s$, the perturbative IIB limit, $\Omega_2 \to \infty$, is also the $D = 10$ type IIA limit, $r_A \to \infty$. This is the limit in the parabolic subgroup $GL(1) \times U$ of the $SL(2)$ factor (given in (4.12)) in which the parameter in the $GL(1)$ Levi factor in the $SL(2)$ becomes infinite. The non-zero Fourier modes of the expression for $\mathcal{E}^{(9)}_{(0,0)}$ in (4.20) that contribute in this limit are obtained by using the mode expansion of $E_{3/2}$ given in the

previous section in (4.20), giving

$$\begin{aligned}\mathcal{F}^{(9)}_{(0,0)}(k) &:= \int_{[0,1]} d\Omega_1\, \mathcal{E}^{(9)}_{(0,0)}\, e^{-2i\pi k\Omega_1} \\ &= 8\pi\Omega_2^{\frac{1}{2}}\, \nu_1^{-\frac{3}{7}}\, \frac{\sigma_2(|n|)}{|k|}\, K_1(2\pi|k|\Omega_2)\,. \end{aligned} \tag{4.22}$$

The limit $\Omega_2 \to \infty$ in the Bessel function in the second line gives the D-instanton contribution to the coefficient of the $\mathcal{R}^4$ interaction in the type IIB perturbative string theory limit, which has the form, after reinstating the power of ℓ_9 in the effective action, (2.5),

$$\frac{1}{\ell_9}\,\mathcal{F}^{(9)}_{(0,0)}(k) = \frac{r_B}{\ell_s^2}\,\sqrt{8\pi}\,\sigma_{-2}(|n|)\,\frac{e^{-2\pi|k|\Omega_2}}{(2\pi|k|\Omega_2)^{-\frac{1}{2}}}\,(1+O(\Omega_2^{-1}))\,, \tag{4.23}$$

where the factor of r_B/ℓ_s shows that this term survives the limit $r_B \to \infty$.

On the other hand, taking the large radius $r_A/\ell_{10} \to \infty$ limit in the IIA case gives

$$\frac{1}{\ell_9}\,\mathcal{F}^{(9)}_{(0,0)}(k) = \frac{1}{r_A}\,\sqrt{8\pi}\,\sigma_{-2}(|k|)\,\frac{e^{-2\pi|k|r_A m_{\frac{1}{2}}}}{(2\pi|k|r_A m_{\frac{1}{2}})^{-\frac{1}{2}}}\,(1+O(\ell_{10}/r_A))\,, \tag{4.24}$$

where $m_{\frac{1}{2}} = 1/(\ell_s g_A)$. This expression reproduces the asymptotic behaviour for the $\frac{1}{2}$-BPS contribution given in (4.10) with $D=9$, $n_D=-1$ and $S_9 = 2\pi|k|r_A m_{\frac{1}{2}}$. The exponent has the interpretation of the action of the euclidean world-line of a type IIA $D0$-brane of charge p wrapped q times around the circle of radius r_A, where $k = p\times q$ (and the sum over q is in $\sigma_{-2}(|k|)$).

A similar expansion of the two Eisenstein series in (4.21) gives the mode expansion of the coefficient $\mathcal{E}^{(9)}_{(1,0)}$ as the sum of two terms. The occurrence of both the $s=3/2$ and $s=5/2$ series demonstrates that the $\partial^4\mathcal{R}^4$ interaction contains a piece that is $\frac{1}{4}$-BPS as well as a piece that is $\frac{1}{2}$-BPS. Repeating the above analysis for the $\frac{1}{4}$-BPS part of $\mathcal{E}^{(9)}_{(1,0)}$ (the $E_{5/2}$ term in (4.21)), making use of (4.18) with $s=5/2$ gives (after multiplying by ℓ_9^3 to reproduce the $\partial^4\mathcal{R}^4$ interaction in (2.5))

$$\ell_9^3\,\mathcal{F}^{(9)}_{(1,0)}(k)\Big|_{\frac{1}{4}-BPS} \sim (\ell_{10}^A)^3\, g_A^{-\frac{1}{2}}\,\left(\frac{\ell_{10}^A}{r_A}\right)^3\,\sigma_{-4}(|k|)\,\frac{e^{-S_9(n)}}{(S_9(n))^{-\frac{3}{2}}}\,. \tag{4.25}$$

As with the $D=10$ examples, the distinction between the $s=3/2$ and $s=5/2$ Eisenstein series is not seen in the instanton orbits (both series contain the same 1-dimensional orbit) but must be encoded in the different measure factors, such as the divisor function, which takes the form $\sigma_{-4}(|k|)$ when $s=5/2$. In contrast to the $\frac{1}{2}$-BPS case we have not derived (4.25), or

the analogous expressions for $D < 9$ obtained below, by explicitly evaluating the $\frac{1}{4}$-BPS instanton contributions.

4.4. $D = 8$**:** $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$**.**

The coefficient function $\mathcal{E}^{(8)}_{(0,0)}$ is given in terms of Eisenstein series by [1,2,36,37]

$$\mathcal{E}^{(8)}_{(0,0)} := \lim_{\epsilon\to 0}\left(2\zeta(3+2\epsilon)\, E^{SL(3)}_{\alpha_1;\frac{3}{2}+\epsilon} + 4\zeta(2-2\epsilon)\, E^{SL(2)}_{1-\epsilon}(\mathcal{U})\right). \tag{4.26}$$

It was shown in [2] that the poles in ϵ of the individual series in parentheses cancel and the expression is analytic at $\epsilon = 0$. The coefficient function $\mathcal{E}^{(8)}_{(1,0)}$ is given by

$$\mathcal{E}^{(8)}_{(1,0)} = \zeta(5)\, E^{SL(3)}_{\alpha_1;\frac{5}{2}} + \frac{4\zeta(4)}{3}\, E^{SL(3)}_{\alpha_1;-\frac{1}{2}}\, E^{SL(2)}_{2}(\mathcal{U})\,. \tag{4.27}$$

We have suppressed the dependence of the $SL(3)$ series on the 5 parameters of the $SL(3)/SO(3)$ coset, but have indicated that the $SL(2)$ series depends on $\mathcal{U}$, the complex structure of the 2-torus, T^2 (see appendix E for details).

(i) **The maximal parabolic** $P_{\alpha_3} = GL(1) \times SL(2) \times \mathbb{R}^+ \times U_{\alpha_3}$

This is relevant for the decompactification limit $r_2/\ell_9 \to \infty$. The Fourier modes, which are integrals with respect to the U_{α_3} factor in (E.16), get contributions from the sum of the modes of the $SL(3)$ and $SL(2)$ Eisenstein series. The modes of $\mathcal{E}^{(D)}_{(0,0)}$ are defined by

$$\mathcal{F}^{(8)\alpha_3}_{(0,0)}(kp_1, kp_2, k') := \int_{[0,1]^3} dC^{(2)} dB_{\rm NS} d\mathcal{U}_1\, e^{-2i\pi k(p_1 C^{(2)} + p_2 B_{\rm NS}) - 2i\pi k'\mathcal{U}_1}\, \mathcal{E}^{(8)}_{(0,0)}\,, \tag{4.28}$$

where $\gcd(p_1,p_2) = 1$ and $C^{(2)}$, $B_{\rm NS}$ and $\mathcal{U}_1$ are the components of the unipotent radical in (E.16). Using the definition in (4.26) the Fourier modes of $\mathcal{E}^{(8)}_{(0,0)}$ are given by the sum of the Fourier modes of the $SL(3)$ and $SL(2)$ series defined in (E.17) and (E.19)[16]

$$\mathcal{F}^{(8)\alpha_3}_{(0,0)}(kp_1, kp_2, k) = 2\zeta(3)\, F^{SL(3)\,\alpha_2}_{\alpha_1;\frac{3}{2}}(kp_1, kp_2) + 4\zeta(2)\, F^{SL(2)}_{1}(k')\,. \tag{4.29}$$

Using the expression in (E.20) with $s = 1$ for the $SL(2)$ Fourier modes and $\mathcal{U}_2 = r_2/r_1 = r_2/r_B$ we obtain[17]

$$F^{SL(2)}_{1}(k') = 4\pi\, \sigma_{-1}(|k'|)\, e^{-2\pi\,|k'|\, r_2 \times \frac{1}{r_1}}\,. \tag{4.30}$$

The exponent can be identified with minus the action of the world-line of a $\frac{1}{2}$-BPS charge p KK state wrapped q times around a circle of radius r_2, with $p \times q = k'$. The divisor sum $\sigma_{-1}(|k'|)$ weights the different values of p with

[16]The nodes on the $SL(3)$ Fourier coefficients are labelled in the notation of the standard Dynkin diagram for $SL(3)$.

[17]Here, and in the following we will use the type IIB description, in which $r_1 = r_B$.

a factor of $1/p$. The expression (4.30) agrees with the general asymptotic formula (4.10), but it is notable that in this case there are no perturbative corrections.

The $SL(3)$ part is obtained from (E.18) with $s = 3/2$,

$$F^{SL(3)\,\alpha_2}_{\alpha_1;\frac{3}{2}}(kp_1, kp_2) = 2\pi\,\sigma_{-1}(|k|)\,e^{-2\pi|k|\frac{|p_2+p_1\Omega|}{\sqrt{\Omega}_2}\frac{1}{\sqrt{\nu}_2}}\,, \tag{4.31}$$

where $\gcd(p_1, p_2) = 1$. This expression reproduces the asymptotic behaviour (which is again exact) for the $\frac{1}{2}$-BPS contribution given in (4.10) with $D = 8$. The exponent can be written as

$$-\,2\pi|k|\frac{|p_2 + p_1\Omega|}{\sqrt{\Omega}_2}\frac{1}{\sqrt{\nu}_2} \quad = \quad -2\pi|k|r_2\, m_{p_1,p_2}\,, \tag{4.32}$$

where the $k = 1$ contribution is minus the action for the world-line of a state of mass

$$m_{p_1,p_2}\,\ell_s = |p_2 + p_1\Omega|\,\frac{r_1}{\ell_s}\,, \tag{4.33}$$

wound around the circle of radius r_2. This is the mass of a (non-threshold) bound state of p_2 fundamental strings and p_1 D-strings wound around the dimension of radius r_1. In the limit $r_2/\ell_9 \to \infty$ the Fourier coefficients with different p_1's and p_2's fill out an orbit under the action of the discrete subgroup, $SL(2,\mathbb{Z})$, of the Levi factor, which is the nine-dimensional duality group. This is made manifest by expressing m_{p_1,p_2} in nine-dimensional Planck units,

$$m_{p_1,p_2}\,\ell_9 = \frac{|p_2 + p_1\Omega|}{\sqrt{\Omega_2}}\,\nu_1^{-3/7}\,, \tag{4.34}$$

where $SL(2,\mathbb{Z})$ acts with the usual linear fractional transformation on Ω and leaves ν_1 invariant. For $k > 1$ in (4.31) describe world-line actions of threshold bound states of mass $p \times m_{p_1,p_2}$ wound q times around the circle of radius r_2 with $k = p \times q$ and the divisor sum weights the contributions with a factor of $1/|q|$.

Thus, in the decompactification limit these instantons correspond to the expected contributions from the point-like $\frac{1}{2}$-BPS black hole states in nine dimensions listed in appendix C.2. The Kaluza–Klein $\frac{1}{2}$-BPS states in (4.30) are in the singlet v and the (p,q)-string bound state in (4.31) in the doublet v_a of $SL(2)$. These contributions come from separate configurations ($v = 0$, $v_a \neq 0$) and ($v \neq 0$, $v_a = 0$) so that the condition $vv_a = 0$ is satisfied.

The Fourier modes of the coefficient $\mathcal{E}^{(8)}_{(1,0)}$ in the P_{α_3} parabolic are defined as

$$\mathcal{F}^{(8)\alpha_3}_{(1,0)}(kp_1, kp_2, k') := \int_{[0,1]^3} dC^{(2)} dB_{\rm NS} d\mathcal{U}_1\, e^{-2i\pi\,k\,(p_1 C^{(2)} + p_2 B_{\rm NS}) - 2i\pi k'\,\mathcal{U}_1}\,\mathcal{E}^{(8)}_{(1,0)}\,, \tag{4.35}$$

where we have chosen to extract the greatest common divisor k so that $\gcd(p_1, p_2) = 1$. Note that, unlike in the case of $\mathcal{E}^{(8)}_{(0,0)}$, the integral does not split into the sum of two terms even though U_{α_3} is block diagonal since $\mathcal{E}^{(8)}_{(1,0)}$

contains the product of two Eisenstein series. Substituting the expression (4.27) for $\mathcal{E}^{(8)}_{(1,0)}$ (which includes a term quadratic in Eisenstein series), it is straightforward to perform the Fourier integration with the result

$$\begin{aligned}\mathcal{F}^{(8)\alpha_3}_{(1,0)}(kp_1,kp_2,k') &= \zeta(5)\, F^{SL(3)\alpha_2}_{\alpha_1;\frac{5}{2}}(kp_1,kp_2) \\ &+ \frac{2\pi^4}{135}\, F^{SL(3)\alpha_2}_{\alpha_1;-\frac{1}{2}}(kp_1,kp_2)\, F^{SL(2)}_2(k')\end{aligned} \tag{4.36}$$

The $k=0$ or $k'=0$ terms are determined by $\frac{1}{2}$-BPS instantons arising from the winding of the nine-dimensional $\frac{1}{2}$-BPS states, listed in appendix C.2, around the decompactifying circle.

The $\frac{1}{4}$-BPS part is contained in the $k\neq 0$, $k'\neq 0$ modes of the second contribution in (4.36). Applying (E.22) with $s=-1/2$ and $s'=2$, and after extracting the greatest common divisor $\ell=\gcd(k,k')$ and setting $k=\ell q_1, k'=\ell q_2$ with $\gcd(q_1,q_2)=1$, these can be written as

$$\begin{aligned}&\frac{\pi^2}{1080}\,\frac{\Omega_2^{\frac{4}{3}}}{T_2^{\frac{1}{3}}}\,\sigma_{-3}(|\ell q_1|)\,\sigma_{-3}(|\ell q_2|)\,\frac{1+2\pi|\ell q_1||p_2+p_1\Omega|T_2}{|p_2+p_1\Omega|^3}\,\frac{1+2\pi|\ell q_2|\mathcal{U}_2}{\mathcal{U}_2} \\ &\qquad\times\exp(-2\pi|\ell q_1||p_2+p_1\Omega|T_2-2\pi|\ell q_2|\mathcal{U}_2)\,.\end{aligned} \tag{4.37}$$

Taking the limit $r_2/\ell_9\to\infty$ and recalling that $T_2=\nu_1^{-\frac{3}{7}}\,\Omega_2^{-\frac{1}{2}}\,r_2/\ell_9$ and $\mathcal{U}_2=r_2/r_1=\nu_1^{\frac{4}{7}}\,r_2/\ell_9$, the leading behaviour of this expression is

$$\frac{\zeta(4)}{3}\,\frac{\ell_9^4}{\ell_8^4}\,\sigma_3(|\ell q_1|)\,\sigma_3(|\ell q_2|)\,\frac{\exp(-2\pi\,\ell\,r_2 m_{\frac{1}{4}})}{(|\ell q_1|\,\frac{|p_2+p_1\Omega|}{\sqrt{\Omega_2}}\,\nu_1^{-\frac{3}{7}})^2\times(|\ell q_2|\,\nu_1^{\frac{4}{7}})^2}\,, \tag{4.38}$$

where the $\frac{1}{4}$-BPS mass is given by

$$m_{\frac{1}{4}}\,\ell_9=|q_1|\frac{|p_2+p_1\Omega|}{\sqrt{\Omega_2}}\,\nu_1^{-\frac{3}{7}}+|q_2|\,\nu_1^{\frac{4}{7}}\,, \tag{4.39}$$

or in string units

$$m_{\frac{1}{4}}\,\ell_s=|q_1||p_2+p_1\Omega|\frac{r_1}{\ell_s}+|q_2|\frac{\ell_s}{r_1}\,. \tag{4.40}$$

Thus, as anticipated, the instanton action is described by the world-lines of the constituents (in this case bound states of F and D strings and KK charge) of $\frac{1}{4}$-BPS bound states on a circle S^1 of radius r_2, Much as before, the divisor functions encode the combinations of winding numbers and charges carried by these world-lines although the combinatorics are here more complicated than in the $\frac{1}{2}$-BPS and deserve further study.

(ii) **The maximal parabolic** $P_{\alpha_1}=GL(1)\times SO(2,2)\times U_{\alpha_1}$

This is relevant to the string perturbation theory limit, in which the string coupling constant, y_8 gets small. The unipotent factor U_{α_1} in (E.23)

is parametrized by $(C^{(2)},\Omega_1)$. In this case the non-zero Fourier modes of $\mathcal{E}^{(8)}_{(0,0)}$ are obtained from (E.25) with $s=3/2$,

$$\begin{aligned}\mathcal{F}^{(8)\alpha_1}_{(0,0)}(kp_1,kp_2) &:= \int_{[0,1]^2} d\Omega_1 dC^{(2)} e^{-2i\pi k\,(p_1C^{(2)}+p_2\Omega_1)}\mathcal{E}^{(8)}_{(0,0)} \\ &= \frac{4\pi}{\sqrt{y_8}}\,\frac{\sigma_2(|k|)}{|k|}\,\frac{\sqrt{T_2}}{|p_2+p_1T|}K_1\left(2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2y_8}}\right).\end{aligned} \tag{4.41}$$

Its asymptotic form for $y_8\to 0$ is given by

$$\lim_{y_8\to 0}\mathcal{F}^{(8)\alpha_1}_{(0,0)}(kp_1,kp_2)\sim\frac{2\pi}{y_8}\,\sigma_2(|k|)\,\left(\frac{\sqrt{T_2y_8}}{|k|\,|p_2+p_1T|}\right)^{\frac{3}{2}}e^{-2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2y_8}}}\,, \tag{4.42}$$

where $\gcd(p_1,p_2)=1$ and the asymptotic form of the Bessel function has been used in the last line in order to extract the leading instanton contribution in the perturbative limit, $y_8\to 0$ with T_2 fixed [2] (recalling $y_8=(\Omega_2^2T_2)^{-1}$ is the square of the string coupling). In this limit these non-perturbative effects behave as $e^{-C/\sqrt{y_8}}$, as expected of D-brane instantons. The $p_1=0$ and $p_2\neq 0$ terms are D-instanton contributions and those with $p_1\neq 0$ are the wrapped D-string contributions of charge (p_1,p_2) that are related by the $SL(2,\mathbb{Z})$ action on the T modulus, which is part of the perturbative T-duality symmetry.

The Fourier modes of $\mathcal{E}^{(8)}_{(1,0)}$ are given by

$$\begin{aligned}\mathcal{F}^{(8)\alpha_1}_{(1,0)} &:= \int_{[0,1]^2} d\Omega_1 dC^{(2)}\,e^{-2i\pi k(p_1C^{(2)}+p_2\Omega_1)}\mathcal{E}^{(8)}_{(1,0)} \\ &= \frac{16\zeta(2)}{y_8^{\frac{2}{3}}}\,\frac{\sigma_4(|k|)}{|k|^2}\,\frac{T_2}{|p_2+p_1T|^2}\,K_2\left(2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2y_8}}\right) \\ &+ \frac{8\zeta(4)E_2(\mathcal{U})}{\pi\,y_8^{\frac{1}{6}}}\,\frac{\sigma_2(|k|)}{|k|}\,\frac{|p_2+p_1T|}{\sqrt{T_2}}\,K_1\left(2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2y_8}}\right),\end{aligned} \tag{4.43}$$

with $\gcd(p_1,p_2)=1$. In the limit of small string coupling, $y_8\to 0$ and recalling that $\ell_8=\ell_s\,y_8^{1/6}$, the first line on the right-hand side behaves as

$$\frac{\ell_s^4}{\ell_8^4}\,\frac{8\zeta(2)}{y_8}\,\sigma_4(|k|)\,\left(\frac{\sqrt{y_8T_2}}{|k|\,|p_2+p_1T|}\right)^{\frac{5}{2}}\exp\left(-2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2y_8}}\right), \tag{4.44}$$

which is characteristic of the $\frac{1}{2}$-BPS configuration due to a euclidean worldsheet of a (p_1,p_2) D-string wrapped k times around T^2.

The second line behaves in the small string coupling limit $y_8\to 0$ as

$$\frac{\ell_s^4}{\ell_8^4}\,4\zeta(4)y_8\,E_2(\mathcal{U})\,\sigma_{-2}(|k|)\,\left(\frac{\sqrt{y_8T_2}}{|k|\,|p_2+p_1T|}\right)^{-\frac{1}{2}}\exp\left(-2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2y_8}}\right), \tag{4.45}$$

which is suppressed relative to (4.44) by y_8^2, which is four powers of the string coupling. As in the $D = 9$ and $D = 10$ cases, the distinction between the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS cases is not seen in the argument of the Bessel function, which determines the exponential suppression at small y_8. In other words, there are no $\frac{1}{4}$-BPS instantons so the second term on the right-hand side of (4.43) has the same exponential suppression in the $y_8 \to 0$ limit as the first line. The distinction between the $\frac{1}{2}$- and $\frac{1}{4}$-BPS contributions in (4.43) again lies in the properties of the measure rather than in the spectrum of instantons.

(iii) **The maximal parabolic** $P_{\alpha_2} = GL(1) \times SL(3) \times U_{\alpha_2}$

This corresponds to the limit in which the volume of the M-theory 3-torus, $\mathcal{V}_3$, gets large. The unipotent factor U_{α_2} (E.26) depends only on $\mathcal{U}_1$ and the Fourier modes in this case only involve the modes of the $SL(2,\mathbb{Z})$ Eisenstein series,

$$\mathcal{F}^{(8)\alpha_2}_{(0,0)} := \int_{[0,1]} d\mathcal{U}_1\, e^{-2i\pi\, k\mathcal{U}_1}\, \mathcal{E}^{(8)}_{(0,0)} = 2\pi\sigma_{-1}(|k|)\, e^{-2\pi|k|\mathcal{U}_2}\,. \tag{4.46}$$

Recalling [2] that $\mathcal{U}_2 = \mathcal{V}_3/\ell_P^3$ is the volume of the M-theory 3-torus, we see that these coefficients are exponentially suppressed in $\mathcal{V}_3$, and correspond to the expected contributions from euclidean $M2$-branes wrapped k times on the 3-torus.

Furthermore, the divisor function reproduces the one derived from a direct partition function calculation in [38]. The form of this measure factor can also be seen from a simple duality argument using the fact that the wrapped $M2$-brane instanton is related to the Kaluza–Klein world-line instanton by the $SL(2,\mathbb{Z})$ part of the duality group. This duality interchanges T and $\mathcal{U}$ and, hence, the factor $\exp(-2\pi|k|/\sqrt{\Omega_2\nu_2}) = \exp(-2\pi|k|\,T_2)$ in (4.30) is related to $\exp(-2\pi|k|\,\mathcal{U}_2)$ in (4.46). This explains the fact that the measure factor, $\sigma_{-1}(|k|)$, is the same in both these equations.

4.5. $D = 7$: $SL(5,\mathbb{Z})$.

In this case the coefficient functions are given in terms of Eisenstein series by [1,2]

$$\mathcal{E}^{(7)}_{(0,0)} = 2\zeta(3)\, E^{SL(5)}_{\alpha_1;\frac{3}{2}}\,, \tag{4.47}$$

$$\mathcal{E}^{(7)}_{(1,0)} = \lim_{\epsilon\to 0}\left(\zeta(5+2\epsilon)E^{SL(5)}_{\alpha_1;\frac{5}{2}+\epsilon} + \frac{24\zeta(4-2\epsilon)\zeta(5-2\epsilon)}{\pi^2}E^{SL(5)}_{\alpha_4;\frac{5}{2}-\epsilon}\right) \tag{4.48}$$

It was shown in [2] that the pole of the individual series in the parenthesis cancel in the limit $\epsilon \to 0$ and the resulting expression is analytic at $\epsilon = 0$. The detailed properties of the Eisenstein series that appear on the right-hand side are reviewed in appendix E.2.

(i) **The maximal parabolic** $P_{\alpha_4} = GL(1) \times SL(3) \times SL(2) \times U_{\alpha_4}$

This is the decompactification limit in which $r_3/\ell_8 = r^2 \to \infty$ (where r is the $GL(1)$ parameter that parameterises the approach to the cusp). Recalling the relation between the volume of the 3-torus ν_3 and the volume of the 2-torus ν_2 [2], the limit under consideration is one in which $\nu_3 = \nu_2^{\frac{5}{6}}\,(r_3/\ell_8)^{-2} \to 0$. The unipotent radical is abelian and has the form

$$U_{\alpha_4} = \begin{pmatrix} I_2 & Q_4 \\ 0 & I_3 \end{pmatrix}, \tag{4.49}$$

where I_n is the rank n identity matrix and Q_4 is the 2×3 matrix defined in (E.37).

Specialising the Fourier modes of $E^{SL(5)}_{\alpha_1;s}$ that are given in (E.40) to the case $s = 3/2$ and using the relation between the $GL(1)$ parameter and the radius of compactification, $r^2 = r_3/\ell_8$, gives the Fourier modes of $\mathcal{E}^{(7)}_{(0,0)}$ in (4.47)

$$\begin{aligned} \mathcal{F}^{(7)\alpha_4}_{(0,0)}(k,\tilde{N}_4) &:= \int_{[0,1]^6} d^3B_{\rm NS} d^3C^{(2)}\, e^{-2i\pi k\,{\rm tr}(\tilde{N}_4\cdot Q_4)}\, \mathcal{E}^{(7)}_{(0,0)} \\ &= \left(\frac{r_3}{\ell_8}\right)^{\frac{4}{5}} 4\pi\,\sigma_0(|k|)\,K_0(2\pi|k|\,r_3\,m_{\frac{1}{2}})\,, \end{aligned} \tag{4.50}$$

where $\gcd(\tilde{N}_4) = 1$ and the support of the non vanishing Fourier coefficients is determined by the rank 1 integer-valued matrix $\tilde{N}_4$ in $M(3,2;\mathbb{Z})$ of the form $\tilde{N}_4 = m^T n$ with $n = (n_i) \in \mathbb{Z}^3$ and $m = (m_a) \in \mathbb{Z}^2$. This matrix satisfies the relation

$$\sum_{a,b=1}^{2} \epsilon_{ab} (\tilde{N}_4)_i{}^a (\tilde{N}_4)_j{}^b = 0, \qquad \forall i,j = 1,2,3 \tag{4.51}$$

with $\epsilon_{12} = \epsilon_{21} = -1$ and $\epsilon_{11} = \epsilon_{22} = 0$, which is precisely $\frac{1}{2}$-BPS condition discussed in appendix C.3. The argument of the Bessel function in (4.50) is proportional to the mass of $\frac{1}{2}$-BPS states, where

$$m^2_{\frac{1}{2}}\,\ell_8 := {\rm tr}(g_3^{-1}\tilde{N}_4 g_2 \tilde{N}_4^T) = m^2_{SL(2)} \times m^2_{SL(3)}\,, \tag{4.52}$$

where $m^2_{SL(2)}$ is given in (4.14) and $m^2_{SL(3)}$ is given in (E.8). This is the mass of a $\frac{1}{2}$-BPS bound state of fundamental strings and D-strings with Kaluza–Klein momentum. This expression is covariant under the action of the symmetry group $SL(3) \times SL(2)$ of the Levi factor. In the limit $r_3/\ell_8 \to \infty$ the expression for the Fourier modes $\mathcal{F}^{(7)\alpha_4}_{(0,0)}$ takes the form

$$\mathcal{F}^{(7)\alpha_4}_{(0,0)}(k,\tilde{N}_4) = \left(\frac{r_3}{\ell_8}\right)^{\frac{4}{5}} 2\pi\,\sigma_0(|k|)\,\frac{e^{-2\pi|k|\,r_3\,m_{\frac{1}{2}}}}{\sqrt{|k|r_3\,m_{\frac{1}{2}}}}\,(1 + O(\ell_8/r_3))\,, \tag{4.53}$$

which is in accord with the behaviour described in (4.10) with $D = 7$.

The Fourier modes of $\mathcal{E}^{(7)}_{(1,0)}$ in (4.48) in this parabolic subgroup are defined as

$$\mathcal{F}^{(7)\alpha_4}_{(1,0)}(k,N_4) := \int_{[0,1]^6} d^3B_{\rm NS} d^3C^{(2)}\, e^{-2i\pi k\,{\rm tr}(N_4\cdot Q_4)}\, \mathcal{E}^{(7)}_{(1,0)} \tag{4.54}$$

with $\gcd(N_4)=1$. The expression for these Fourier modes is obtained by adding (E.40) for the series $E^{SL(5)}_{\alpha_1;s}$ to (E.55) for the series $E^{SL(5)}_{\alpha_4;s}$ in the correct ratio and setting $s=5/2$.

The Fourier modes of the Eisenstein series $E^{SL(5)}_{\alpha_4;s}$ will be computed by noting that this series can be represented as the Mellin transform of the $E^{SO(5,5)}_{\alpha_1;\frac{3}{2}}$ series, making use of the following proposition.

We consider $H=\gamma\, g\, \gamma^T$, where $\gamma\in SL(d,\mathbb{Z})$ and g is the $SL(d)$ matrix parametrizing the coset space $SL(d)/SO(d)$. Letting H_k be the bottom right $k\times k$ minor of H the general minimal parabolic Eisenstein series associated with the minimal parabolic subgroup $P(1,\ldots,1)$,

$$E^{SL(d)}_{\beta;s_1,\ldots,s_d} = \sum_{\gamma\in SL(n,\mathbb{Z})/B(\mathbb{Z})} \prod_{k=1}^{d-1} (\det H_k)^{\frac{\lambda_{d-k+1}-\lambda_{d-k}-1}{2}}\,, \tag{4.55}$$

Here we have set $2s_k=\lambda_{d-k+1}-\lambda_{d-k}-1$ for $1\le k\le d-1$, and $\epsilon_k=1$ if $s_k\neq 0$ and $\epsilon_k=0$ if $s_k=0$ and $\beta=\sum_{i=1}^{d-1}\epsilon_i\beta_i$ where β_i are the simple roots of $SL(d)$ with the usual labelling.

Proposition 4.1. The $SL(d)$ series $E^{SL(d)}_{\beta_1;s}$ is given by the Mellin transform of the $SO(d,d)$ series $E^{SO(d,d)}_{\alpha_1;d/2-1}$ according

$$4\xi(2s)\xi(2s-1)\, E^{SL(d)}_{\beta_2;s} = 2\xi(d-2)\int_0^\infty dV\, V^{2s-1}\, E^{SO(d,d)}_{\alpha_1;\frac{d}{2}-1}(V\,g)\,, \tag{4.56}$$

where $G=V\,g$ parametrizes the coset $SO(d,d)/SO(d)\times SO(d)$ and $\det g=1$. An equivalent integral representation for the series $E^{SL(d)}_{\beta_{d-1};s}$ is obtained by the use of the functional equation.

Proof. In [2, appendix B.2] an integral representation for these $SL(d)$ Eisenstein series was given. The construction considered the integral

$$I_s(\Lambda,g) := \int_0^\Lambda dV\, V^{2s-1}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\Gamma_{(d,d)}(V\,g;\tau) \tag{4.57}$$

where $\Lambda>0$, $\Gamma_{(d,d)}(G;\tau)$ is the genus one lattice sum for the self-dual lorentzian lattice of rank d. The metric G parametrizing the coset $SO(d,d)/SO(d)\times SO(d)$ is decomposed as $G=V\,g$ with $\det g=1$ and g parametrizes the coset space $SL(d)/SO(d)$. This integral was evaluated in [2, appendix B.2] with the result

$$I_s(\Lambda,g) = 2\zeta(2s)\,\frac{\Lambda^{2s}}{2s} + \frac{\pi}{3}\frac{\Lambda^{2s-1}}{2s-1}\, E^{SL(d)}_{\beta_1;1} + 4\xi(2s)\xi(2s-1)\, E^{SL(d)}_{\beta_2;s}\,. \tag{4.58}$$

On the other hand we have the following representation of the $E^{SO(d,d)}_{\alpha_1;s}$ series [2, appendix C]

$$2\xi(2s)\, E^{SO(d,d)}_{\alpha_1;s}(G) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, E_{s+1-\frac{d}{2}}(\tau)\, \Gamma_{(d,d)}(G) \tag{4.59}$$

Since $E^{SL(2)}_0 = 1$ this implies that the series $E^{SL(d)}_{\beta_2;s}(g)$ is the Mellin transform with respect to the parameter V of the $SO(d,d)$ series $E^{SO(d,d)}_{\alpha_1;\frac{d}{2}-1}(V\, g)$ given in (4.56). The series $E^{SL(d)}_{\beta_{d-1};s}$ is obtained from $E^{SL(d)}_{\beta_2;s}$ by the functional $SL(d)$ equation

$$\xi(d-2s)\xi(d-1-2s)\, E^{SL(d)}_{\beta_{d-1};\frac{d}{2}-s} = \xi(2s)\xi(2s-1)\, E^{SL(d)}_{\beta_2;s} \tag{4.60}$$

leading to

$$\xi(2s)\xi(2s-1)\, E^{SL(d)}_{\beta_{d-1};s} = \xi(d-1) \int_0^\infty dV\, V^{2s-1-d}\, E^{SO(d,d)}_{\beta_1;\frac{d}{2}-1}(V^{-1}g)\,. \tag{4.61}$$

□

This construction, which differs from the one presented in [39], is very useful for explicitly evaluating the Fourier coefficients of the series $E^{SL(5)}_{\beta_2;s}$. By applying the proposition in the $SL(5)$ case (and noting that the relation of the standard labelling of the simple roots of $SL(5)$, β_i, to our labelling, α_i, in figure 1 implies that $E^{SL(5)}_{\alpha_3;s} = E^{SL(5)}_{\beta_2;s}$ and $E^{SL(5)}_{\alpha_4;s} = E^{SL(5)}_{\beta_4;s}$) the expression for the coefficient of the $\partial^4\mathcal{R}^4$ interaction is given by the sum of two contributions,

$$\mathcal{F}^{(7)\alpha_4}_{(1,0)}(N_4) = \mathcal{F}^{(7)\alpha_4}_{(1,0)\,I}(k,\tilde{N}_4) + \mathcal{F}^{(7)\alpha_4}_{(1,0)\,II}(N_4)\,, \tag{4.62}$$

where $\mathcal{F}^{(7)\alpha_4}_{(1,0)\,I}(k,\tilde{N}_4)$ depends on the rank 1 integer valued matrix $\tilde{N}_4$ that arises from the terms in (E.56) and contains the $\frac{1}{2}$-BPS configurations, while $\mathcal{F}^{(7)\alpha_4}_{(1,0)\,II}(N_4)$ depends on the rank 2 integer valued matrix, N_4, given in (4.64) based on the terms that arise from the contribution to the Eisenstein series, $E^{SL(5)}_{\alpha_4;s}$ in (E.57). This contains the $\frac{1}{4}$-BPS contributions.

The $\frac{1}{2}$-BPS contributions are given in (E.40) and (E.56)

$$\begin{aligned}\mathcal{F}^{(7)\alpha_4}_{(1,0)\,I}(k,\tilde{N}_4) &= 4\pi^2 \left(\frac{r_3}{\ell_8}\right)^2 \frac{\sigma_2(|k|)}{|k|}\, \frac{m_{\frac{1}{2}}}{\|n\|}\, K_1(2\pi|k|\, r_3\, m_{\frac{1}{2}}) \\ &+ \frac{2\pi}{3}\left(\frac{r_3}{\ell_8}\right)^2 \left(\sum_{u\in\mathbb{Z}^3\setminus\{0\}} \frac{\delta(u\cdot n)}{\|u\|^4}\right) \frac{1+2\pi\frac{r_3 m_{\frac{1}{2}}}{\ell_8}}{\|p\|^3}\, e^{-2\pi\, r_3\, |k| m_{\frac{1}{2}}}\end{aligned} \tag{4.63}$$

where $\tilde{N}_4 = n^T \cdot p$ with $n \in \mathbb{Z}^3$ and $p \in \mathbb{Z}^2$ with $m_{\frac{1}{2}}$ is defined in (4.52), $\|p\|^2 = p^T \cdot g_2^{-1} \cdot p$, $\|n\|^2 = n^T \cdot g_3 \cdot n$, $\|u\|^2 = u^T \cdot g_3 \cdot u$ and $\delta(x) = 1$ if $x = 0$ and 0 otherwise.

The $\frac{1}{4}$-BPS contributions are characterized by Fourier coefficients localized on the contribution from rank 2 matrix

$$N_4 = m^T p - n^T q; \qquad m = (m_i), n = (n_i) \in \mathbb{Z}^3; p = (p_a), q = (q_a) \in \mathbb{Z}^2 \,. \tag{4.64}$$

The expression derived in (E.57) reads

$$\mathcal{F}^{(7)\alpha_4}_{(1,0)\,II}(k, N_4) = \frac{3}{2\pi^3} \int_{-\infty}^{+\infty} d\tau_1 \frac{1 + 2\pi|p + q\tau_1|\sqrt{n^2}\, r_3/\ell_8}{|p + q\tau_1|^3 (q^2)^{\frac{1}{2}}}\, e^{-2\pi r_3\, |k|\, m(\tau_1)} . \tag{4.65}$$

where $\gcd(N_4) = 1$ and the mass in the exponent is given by

$$m(\tau_1)\, \ell_8 = |p + q\tau_1|\sqrt{n^2} + |m + n\tau_1|\sqrt{q^2} \,, \tag{4.66}$$

where $n^2 = n^T \cdot g_3 \cdot n$ and $q^2 = q^T \cdot g_2 \cdot q$. In the limit $r_3 \gg \ell_8$ the integral (4.65) is dominated by the minimum value of $m(\tau_1)$, which is at $\tau_1 = 0$ (using the fact that N_4 has rank 2). The result is that the dominant mass is the sum of the masses of two $\frac{1}{2}$-BPS states given in (4.52).

(ii) **The maximal parabolic** $P_{\alpha_1} = GL(1) \times SO(3,3) \times U_{\alpha_1}$

The instanton contributions to $\mathcal{E}^{(7)}_{(0,0)}$ in the perturbative string limit associated with $L_{\alpha_1} = GL(1) \times SO(3,3)$ are given by (E.44) upon setting $s = 3/2$. The relation between the $GL(1)$ parameter and the string coupling constant in 7 dimensions is $r^{-2} = y_7^{\frac{1}{2}}$ and the relation between the 7 dimension Planck length and the string length is $\ell_7 = \ell_s\, y_7^{1/5}$ [2]. In this case the unipotent radical is abelian and has the form

$$U_{\alpha_1} = \begin{pmatrix} I_4 & Q_1 \\ 0 & 1 \end{pmatrix} , \tag{4.67}$$

where Q_1 is a $SO(3,3)$ spinor defined in (E.42).

This leads to the expression for the Fourier modes

$$\begin{aligned} \mathcal{F}^{(7)\,\alpha_1}_{(0,0)}(k, N_1) \;&:=\; \int_{[0,1]^4} d^4Q_1\, e^{-2\pi i k\, N_1^T \cdot Q_1}\, \mathcal{E}^{(7)}_{(0,0)} \\ &=\; \frac{4\pi}{y_7^{\frac{7}{10}}}\, \frac{\sigma_2(|k|)}{|k|}\, \frac{K_1\left(\frac{2\pi|k|\,\|N_1\|}{\sqrt{y_7}}\right)}{\|N_1\|} \,. \end{aligned} \tag{4.68}$$

where $\|N_1\|^2 := N_1^T \cdot g_4 \cdot N_1$ with $N_1 \in \mathbb{Z}^4 \backslash \{0\}$, such that $\gcd(N_1) = 1$, and g_4 is a 4×4 matrix parametrizing the coset space $SO(3,3)/SO(3) \times SO(3)$. In the limit $y_7 \to 0$ the right hand side of (4.68) has the exponential suppression

characteristic of an instanton contribution and contributes

$$\ell_7\,\mathcal{F}^{(7)\,\alpha_1}_{(0,0)}(k,N_1)\sim \ell_s\,\frac{2\pi}{y_7}\,\sigma_2(|k|)\,\left(\frac{\sqrt{y_7}}{|k|\,\|N_1\|}\right)^{-\frac{3}{2}}\,\exp\left(-2\pi|k|\frac{\|N_1\|}{\sqrt{y_7}}\right) \tag{4.69}$$

to the effective $\mathcal{R}^4$ action with $D=7$ in (2.5).

Terms with $N_1=(1,0,0,0)$ are D-instanton contributions. Terms with $N_1\neq(1,0,0,0)$ are $\frac{1}{2}$-BPS contributions due to wrapped Euclidean bound states of fundamental and D-strings. The rank 4 integer vector N_1 is unrestricted.

The Fourier modes in this parabolic of $\mathcal{E}^{(7)}_{(1,0)}$ are given for the series $E^{SL(5)}_{\alpha_1;5/2}$ in (E.44) and in (E.64) for the series $E^{SL(5)}_{\alpha_4;5/2}$. Adding these contributions and setting $s=5/2$ gives

$$\begin{aligned}\mathcal{F}^{(7)\alpha_1}_{(1,0)}(k,N_1) &:= \int_{[0,1]^4} d^4Q_1\,e^{-2\pi i k\,N_1^T\cdot Q_1}\,\mathcal{E}^{(7)}_{(1,0)}\\ &= \frac{8\pi^2}{3\,y_7}\,\frac{\sigma_4(|k|)}{|k|^2}\,\frac{1}{\|N_1\|^2}\,K_2\left(\frac{2\pi|k|\,\|N_1\|}{\sqrt{y_7}}\right)\\ &+ \frac{4}{\sqrt{y_7}}\left(\sum_{m\in\mathbb{Z}^4\backslash\{0\}}\frac{\delta(m\cdot N_1)}{(m^2)^2}\right)\frac{\|N_1\|}{|k|}\,K_1\left(\frac{2\pi|k|\|N_1\|}{\sqrt{y_7}}\right),\end{aligned} \tag{4.70}$$

where $N_1\in\mathbb{Z}^4\backslash\{0\}$ such that $\gcd(N_1)=1$ and $m^2=m^T\cdot g_4^{-1}\cdot m$. In the limit $y_7\to 0$ these modes give instantonic contributions of the form

$$\begin{aligned}\ell_7^5\,\mathcal{F}^{(7)\alpha_1}_{(1,0)}(k,N_1)\sim\ell_s^5\,\frac{2\pi^2}{3\,y_7}\,\sigma_4(|k|)\,\left(\frac{|k|\|N_1\|}{\sqrt{y_7}}\right)^{-\frac{5}{2}}\exp\left(\frac{2\pi|k|\,\|N_1\|}{\sqrt{y_7}}\right)\\ +\ell_s^5\,2y_7\left(\sum_{m\in\mathbb{Z}^4\backslash\{0\}}\frac{\delta(m\cdot N_1)}{(m^2)^2}\right)\frac{1}{k^2}\left(\frac{|k|\|N_1\|}{\sqrt{y_7}}\right)^{\frac{1}{2}}\exp\left(\frac{2\pi|k|\|N_1\|}{\sqrt{y_7}}\right),\end{aligned} \tag{4.71}$$

to the effective $\partial^4\mathcal{R}^4$ action with $D=7$ in (2.5).

The two contributions to the Fourier modes have the same support (i.e., in both cases the charges are labelled by the matrix N_1) because there are no $\frac{1}{4}$-BPS instantons in the expansion at node α_1 (see section 3.4.1). The different BPS nature of each contribution must be encoded in the factor multiplying the Bessel functions. Once more, we see that the $\frac{1}{4}$-BPS contribution in the second line has an extra four powers of the string coupling constant y_7^2.

4.6. $D=6$: $SO(5,5,\mathbb{Z})$.

The coefficient functions in this case are given by combinations of Eisenstein series [1],

$$\mathcal{E}^{(6)}_{(0,0)}=2\zeta(3)\,E^{SO(5,5)}_{\alpha_1;\frac{3}{2}}\,, \tag{4.72}$$

and

$$\mathcal{E}^{(6)}_{(1,0)} = \lim_{\epsilon\to 0}\left(\zeta(5+2\epsilon)\,E^{SO(5,5)}_{\alpha_1;\frac{5}{2}+\epsilon} + \frac{8\zeta(6)}{45}\,E^{SO(5,5)}_{\alpha_5;3-\epsilon}\right). \tag{4.73}$$

It was shown in [1] that the pole of the individual series in the parenthesis cancel in the limit $\epsilon \to 0$ and the resulting expression is analytic at $\epsilon = 0$. Whereas the previous cases involved $SL(n)$ Eisenstein series, which could be expressed as lattice sums that were easy to manipulate, there is much less understanding of the $SO(5,5)$ series in terms of such explicit lattice sums. Various properties of $E^{SO(5,5)}_{\alpha_1;s}$ were considered in [2] (where the series was denoted $(2\zeta(2s))^{-1}\,\mathbf{E}^{SO(5,5)}_{[10000];s}$), based on the integral representation to be reviewed below. This is sufficient to discuss the Fourier modes of the coefficient $\mathcal{E}^{(6)}_{(0,0)}$, but since $\mathcal{E}^{(6)}_{(1,0)}$ also involves the series $E^{SO(5,5)}_{\alpha_5;s}$, detailed evaluation of its Fourier modes will not be performed in this paper due to space limitations. However, we are able to determine its orbit content as will be discussed later in this subsection.

The integral representation for the $SO(d,d)$ Eisenstein series $E^{SO(d,d)}_{\alpha_1;s}$ as a theta-lift of $SL(2)$ Eisenstein series was presented in [2,40,41] in the form

$$2\xi(2s)\,E^{SO(5,5)}_{\alpha_1;s} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,E_{s-\frac{3}{2}}(\tau)\,\Gamma_{(5,5)}\,, \tag{4.74}$$

where the lattice sum is

$$\Gamma_{(5,5)} = V_{(5)} \sum_{(m^i,n^i)\in\mathbb{Z}^{10}} e^{-\frac{\pi}{\tau_2}\,(m+n\tau)^T\cdot(g+B)\cdot(m+n\bar{\tau})}\,. \tag{4.75}$$

The symmetric matrix g and the antisymmetric matrix B parametrize the coset $SO(5,5)/SO(5)\times SO(5)$. The Fourier modes of this series at node α_5 and node α_1 will now be described.

(i) **The maximal parabolic** $P_{\alpha_5} = GL(1)\times SL(5)\times U_{\alpha_5}$

This parabolic subgroup has Levi factor $L_{\alpha_5} = GL(1)\times SL(5)$ (recalling from figure 1 that in our conventions α_5 is a spinor node of $E_5 = SO(5,5)$). We will here evaluate the Fourier modes using the same methods as used for computing the constant term of the series $E^{SO(d,d)}_{\alpha_1;s}$ in [2, appendix C]. The Fourier modes are defined as

$$F^{SO(5,5)\alpha_5}_{\alpha_1;s}(N_2) := \int_{[0,1]^5} dQ_2\,e^{-2i\pi\,\mathrm{tr}(N_2^T\cdot Q_2)}\,E^{SO(5,5)}_{\alpha_1;s} \tag{4.76}$$

where Q_2 is a 5×5 antisymmetric matrix parametrizing the abelian unipotent radical U_{α_5}, and N_2 is an antisymmetric 5×5 matrix with integer entries.

We find that the Fourier modes of the series $E^{SO(5,5)}_{\alpha_1;s}$ are localized on the rank 1 contributions where N_2 satisfies the constraints

$$\sum_{i,j,k,l=1}^{5} \epsilon^{ijklm}(N_2)_{ij}(N_2)_{kl} = 0, \qquad \forall 1\le m\le 5\,, \tag{4.77}$$

where ϵ^{ijklm} is the totally antisymmetric symbol with $\epsilon^{12345} = 1$. This constrains is the $\frac{1}{2}$-BPS condition discussed in appendix C.4.

This condition can be solved as

$$N_2 = m^T n - n^T m; \qquad m, n \in \mathbb{Z}^5\,, \tag{4.78}$$

with $\gcd(N_2) = 1$. Applying the method of orbits for the $SL(2)$ action on τ, the Fourier modes of $\mathcal{F}^{SO(5,5)\,\alpha_5}_{\alpha_1;s}$ take the form

$$F^{SO(5,5)\,\alpha_5}_{\alpha_1;s}(N_2) = \frac{V_{(5)}}{\xi(s)} \int_{-\infty}^{\infty} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{\tau_2^2}\, E_{s-\frac{3}{2}}(\tau)\, e^{-\pi|k|\, V_{(5)}^{\frac{2}{5}}\, \frac{(m+\tau n)^T\cdot g_5\cdot(m+\bar\tau n)}{\tau_2}}\,, \tag{4.79}$$

Setting $s = 3/2$ in this equation, using $E_0(\tau) = 1$ and $V_{(5)} = (r_4/\ell_7)^{5/2}$, (see [2, section 3.4]) gives

$$\mathcal{F}^{(6)\alpha_5}_{(0,0)}(N_2) = 4\pi \left(\frac{r_4}{\ell_7}\right)^{\frac{5}{2}} \sigma_1(|k|)\, \frac{e^{-2\pi\,|k|\,r_4\,m_{\frac{1}{2}}}}{|k| r_4\, m_{\frac{1}{2}}}\,, \tag{4.80}$$

where

$$m^2_{\frac{1}{2}}\, \ell_7^2 := \mathrm{tr}(g_5 N_2 g_5 N_2) = m^2 n^2 - (m\cdot n)^2\,. \tag{4.81}$$

with $m^2 = m^T \cdot g_5 \cdot m$, and with identical definition for n^2 and $m \cdot n$. The expression in (4.80) reproduces the asymptotic (actually exact in this case) behaviour for $\frac{1}{2}$-BPS contribution in (4.10) with $D = 6$.

The Eisenstein series $E^{SO(5,5)}_{\alpha_1;s}$ has a single pole at $s = 5/2$ with residue equal to the $s = 3/2$ series $E^{SO(5,5)}_{\alpha_1;3/2}$ discussed above. This series only receives $\frac{1}{2}$-BPS contributions. The complete coefficient $\mathcal{E}^{(6)}_{(1,0)}$, defined in (4.73), also gets a $\frac{1}{4}$-BPS contribution from $E^{SO(5,5)}_{\alpha_5;s}$, which has a pole at $s = 3$ such that the resulting combination in (4.73) is analytic as shown in [1].

(ii) **The maximal parabolic** $P_{\alpha_1} = GL(1) \times SO(4,4) \times U_{\alpha_1}$

In this parabolic subgroup the Levi factor is $L_{\alpha_1} = GL(1) \times SO(4,4)$. The elements of the unipotent radical are parametrized by the 4×2 matrix

$$Q_1 = \begin{pmatrix} Q_{1I} & Q_2^I \end{pmatrix}, \qquad \forall 1 \le I \le 4\,. \tag{4.82}$$

In the type IIA string theory description this matrix is parametrized by the 4 euclidean $D0$-brane charges, and 4 euclidean $D2$-branes wrapped on 3-cycles of T^4.

The Fourier modes of (4.74) are defined as

$$F^{SO(5,5)\alpha_1}_{\alpha_1;s}(N_1) := \int_{[0,1]^8} d^8 Q_1 e^{-2i\pi\,\mathrm{tr}(N_1^T Q_1)}\, E^{SO(5,5)}_{\alpha_1;s}\,, \tag{4.83}$$

where N_1 is the 4×2 matrix

$$N_1 := (m^I\, n_I), \qquad \forall 1 \le I \le 4\,. \tag{4.84}$$

The entries m^I corresponds to the 4 different ways of wrapping the 1-dimensional euclidean world-volume of a $D0$-brane on the 4-torus, and the

entries n_I the four ways of wrapping the three dimensional euclidean world-volume of a $D4$-brane on the 4-torus. The energy of the $D0$-brane is $E_0 = \sum_{I=1}^4 m^I R_I/(\ell_s g_s)$ and the energy of the $D4$-brane is $E_4 = V_4 \sum_{I=1}^4 n_I \ell_s/(R_I g_s)$ where $V_4 = R_1 R_2 R_3 R_4/\ell_s^4$ is the volume of the 4-torus. In order to make a contact with the orbit classification in section 3.4.1 we have introduced the vector (p_L, p_R) in the even self-dual Lorentzian lattice $\Gamma_{(4,4)}$. The 4-vector $p_L^I = m^I R_I/\ell_s \times 1/\sqrt{V_4} + n_I \ell_s/R_I \times \sqrt{V_4}$ and $p_R^I = m^I R_I/\ell_s \times 1/\sqrt{V_4} - n_I \ell_s/R_I \times \sqrt{V_4}$ for $1 \leq I \leq 4$.

The energy the ($D0$,$D2$) bound-state is given by $\sqrt{V_4}/g_s \times \sqrt{p_L^2 + p_R^2}$. Introducing $y_6 = g_s^2/V_4$ the $GL(1)$ parameter is $r = y_6^{-\frac{1}{4}}$. We remark that the lattice is even $p_L^2 - p_R^2 = 2\sum_{I=1}^4 m_I n^I \in 2\mathbb{Z}$. In terms of the modes matrix N_1 in (4.84) this is expressed as $p_L^2 - p_R^2 = \mathrm{tr}(N_1 J N_1^T)$ where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By triality the $SO(4,4)$ vector (p_L, p_R) is equivalent to a $SO(4,4)$ chiral spinor used for the orbit classification in section 3.4.1.

By extending the constant term computation in [2, Appendix C] the Fourier coefficients are given by

$$\begin{aligned} F_{\alpha_1;s}^{SO(5,5)\,\alpha_1}(N_1) &= \frac{V_{(5)}}{2\xi(2s)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_0^\infty d\tau_2\, e^{-\pi r^2 \frac{k^2}{\tau_2} - \pi\tau_2\, r^2\,(p_L^2+p_R^2)} \times \\ &\times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, E_{s-\frac{3}{2}}(\tau)\, e^{-i\pi\tau_1\,(p_L^2-p_R^2)}\,. \end{aligned} \tag{4.85}$$

It is significant that setting $s = 3/2$ and using $E_0(\tau) = 1$, the integration over τ_1 projects onto the condition $p_L^2 - p_R^2 = 0$ which is the pure spinor condition for $SO(4,4)$. Using the triality relation between vector and spinor representation of $SO(4,4)$ this condition is the $\frac{1}{2}$-BPS (pure spinor) condition $S \cdot S = 0$ discussed in section 3.4.1. It is then straightforward to evaluate the integrals in (4.85) to evaluate the Fourier modes of the coefficient function $\mathcal{E}_{(0,0)}^{(6)}$, giving

$$\mathcal{F}_{(0,0)}^{(6)\,\alpha_1}(N_1) = 4\pi V_{(4)}\, y_6^{-\frac{1}{4}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|k|}{\sqrt{2p_L^2}} K_1(2\pi|k|\, y_6^{-\frac{1}{2}} \sqrt{2p_L^2})\, \delta(p_L^2 = p_R^2)\,, \tag{4.86}$$

where the contributions are localized on the $\frac{1}{2}$-BPS pure spinor locus $p_L^2 - p_R^2 = 0$ which is the condition $\mathrm{tr}(N_1 J N_1^T) = 0$ on the mode matrix N_1. As expected, the argument of the Bessel function is proportional to $r^2 = 1/\sqrt{y_6}$, the inverse of the string coupling with $D = 6$, so its asymptotic expansion is that expected from the contribution of $\frac{1}{2}$-BPS states from wrapped D-brane on the 4-torus T^4.

When $s \neq 3/2$ the τ_1 integral in (4.85) does not impose the restriction $p_L^2 - p_R^2 = 0$ and so the solution fills a generic $SO(4,4)$ orbit and the solution

is $\frac{1}{4}$-BPS. Although the function $\mathcal{E}^{(6)}_{(1,0)}$ in (4.73) is a linear combination of the vector Eisenstein series, $E^{SO(5,5)}_{\alpha_1;5/2}$ and the spinor series, $E^{SO(5,5)}_{\alpha_5;3}$, at present we know little about the explicit structure of the latter, so we will only discuss the former here. The Fourier modes of the vector series at $s = 5/2$ are given by

$$\begin{aligned} F^{SO(5,5)\,\alpha_1}_{\alpha_1;\frac{5}{2}}(N_1) &= 8V_{(5)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_0^\infty d\tau_2\, e^{-\pi r^2 \frac{k^2}{\tau_2} - \pi\tau_2\, r^2\,(p_L^2+p_R^2)} \times \\ &\times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, \hat{E}_1(\tau)\, e^{2i\pi\tau_1\,(p_L^2-p_R^2)}\,. \end{aligned} \tag{4.87}$$

The series $E^{SO(5,5)}_{\alpha_1;s}$ has a single pole at $s = 5/2$ from the single pole of the $SL(2)$ series $E_s(\tau)$ at $s = 1$ in the integral representation in (4.85). The Fourier modes depend on the finite part $\hat{E}_1(\tau)$ defined by the expansion in (E.14). Using the fact that $\hat{E}_1(\tau) = -\pi \times \log(\tau_2|\eta(\tau)|^4)$ gives the result

$$\begin{aligned} F^{SO(5,5)\,\alpha_1}_{\alpha_1;\frac{5}{2}}(N_1) &= -16\pi V_{(4)} y_6^{-\frac{1}{4}} \sigma_{-1}(\frac{1}{2}|p_L^2 - p_R^2|) \sum_{k\in\mathbb{Z}\setminus\{0\}} |k| \times \\ &\times \frac{K_1(2\pi y_6^{-\frac{1}{2}}\,|k|\,\sqrt{p_L^2+p_R^2+|p_L^2-p_R^2|})}{\sqrt{p_L^2+p_R^2+|p_L^2-p_R^2|}}\,, \end{aligned} \tag{4.88}$$

where the mode matrix N_1 in (4.84) is unconstrained and $p_L^2 - p_R^2 = \mathrm{tr}(N_1 J N_1^T) \in 2\mathbb{Z}$ is an even integer.

In summary, the non-zero Fourier modes of $\mathcal{E}^{(6)}_{(0,0)}$ have support on the $\frac{1}{2}$-BPS orbit in limits (i), (ii) and (iii). One of the contributions to $\mathcal{E}^{(6)}_{(1,0)}$ is the regularised series $\hat{E}^{SO(5,5)}_{\alpha_1;\frac{5}{2}}$. This has non-zero Fourier modes with support on the $\frac{1}{2}$-BPS orbit in limits (i) and (iii), but on both the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS orbits in limit (ii). Although we have not computed the modes for the other contribution to $\mathcal{E}^{(6)}_{(1,0)}$ – the series $\hat{E}^{SO(5,5)}_{\alpha_5;3}$ – we do know its orbit content by use of techniques similar to those in section 6.2. The result is that the non-zero Fourier modes of this series have support on the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS orbits in limits (i) and (iii), but only on the $\frac{1}{2}$-BPS orbit in limit (ii). In other words the complete coefficient $\mathcal{E}^{(6)}_{(1,0)}$ has the expected content of both the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS in its non-zero Fourier modes in all three limits.

5. The next to minimal (NTM) representation

This section contains the proof of theorem 2.13, drawing on some results in representation theory that can be found in appendix A by Ciubotaru and Trapa. As we remarked just before its statement, cases (i) and (ii) are by now well known, and so we restrict our attention to case (iii): the $s = 5/2$ series.

To set some terminology, let $G = NAK$ be the Iwasawa decomposition of the split real Lie group G, B the minimal parabolic subgroup of G containing NA, and $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the Lie algebra of A. Without any loss of generality we may assume it is the complex span of the Chevalley basis vectors H_α, where α ranges over the positive simple roots. For any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the dual space of complex valued linear functionals on $\mathfrak{a}_{\mathbb{C}}$, define the vector space of functions on G

$$V_\lambda \;:=\; \Big\{ f : G \to \mathbb{C} \,|\, f(nag) \,=\, e^{(\lambda+\rho)(H(a))} f(g), \forall\, n \in N, a \in A, g \in G \Big\}. \tag{5.1}$$

The transformation law and Iwasawa decomposition show that all functions in V_λ are determined by their restriction to K. Then G acts on V_λ by the right translation operator

$$(\pi_\lambda(h) f)\,(g) \;:=\; f(gh)\,, \tag{5.2}$$

making (π_λ, V_λ) into a representation of G commonly called a *(nonunitary) principal series* representation. It is irreducible for λ in an open dense subset of $\mathfrak{a}^*_{\mathbb{C}}$, but reduces at special points with certain integrality properties – such as the ones of interest to us. The representation V_λ has a unique K-fixed vector up to scaling, namely any function whose restriction to K is constant. These are also known as the *spherical* vectors of the representation, and any representation which contains them is also called "spherical". When V_λ is reducible, it clearly can have at most one spherical subrepresentation.

The minimal parabolic Eisenstein series is defined as

$$E^G(\lambda, g) \;=\; \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{(\lambda+\rho)(H(\gamma g))}\,, \tag{5.3}$$

initially for λ in Godement's range $\{\lambda | \langle \lambda, \alpha \rangle > 1 \text{ for all } \alpha \in \Sigma\}$, and then by meromorphic continuation to all of $\mathfrak{a}^*_{\mathbb{C}}$. When λ has the form $\lambda = 2s\omega_\beta - \rho$, it specializes to the maximal parabolic Eisenstein series (2.12). For generic λ in the range of convergence, the right translates of $E^G(\lambda, g)$ span a subspace of functions on $G(\mathbb{Z})\backslash G(\mathbb{R})$ which furnish a representation of G that is equivalent to V_λ; the group action here is also given by the right translation operator (5.2). The spherical vectors in this representation are the scalar multiplies of $E^G(\lambda, g)$, because the function $H(g)$ – the logarithm of the Iwasawa A-component – is necessarily right invariant under K. For general λ at which $E^G(\lambda, g)$ is holomorphic, its right translates span a spherical subrepresentation of V_λ, again with the group action given by the right translation operator (5.2).

As mentioned above, the principal series V_λ reduces for special values of λ. This reducibility reflects special behavior of the Eisenstein series $E^G(\lambda, g)$. This is most apparent at the point $\lambda = -\rho$, where the transformation law (5.1) indicates that the constant functions on K extend to constants on G, and hence that the trivial representation is a subrepresentation of $V_{-\rho}$.

	$G = E_6$	$G = E_7$	$G = E_8$
$s = 0$			
λ_{dom}	[1,1,1,1,1,1]	[1,1,1,1,1,1,1]	[1,1,1,1,1,1,1,1]
s_{GRS}			
z_{KS}			
$s = 3/2$			
λ_{dom}	[1,1,1,0,1,1]	[1,1,1,0,1,1,1]	[1,1,1,0,1,1,1,1]
s_{GRS}	1/4	5/18	19/58
z_{KS}	7/22	11/34	19/58
$s = 5/2$			
λ_{dom}	[0,1,1,0,1,1]	[1,1,1,0,1,0,1]	[1,1,1,0,1,0,1,1]
s_{GRS}	-1/2	1/18	11/58
z_{KS}	none	33/34	11/58

TABLE 6. The values of λ for the three values of s and three groups in theorem 2.13. Weights $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ are listed here in terms of their inner products with the positive simple roots as $[\langle\lambda, \alpha_1\rangle, \langle\lambda, \alpha_2\rangle, \ldots]$. For comparison with [6, 12], we have listed the parameters s_{GRS} (the quantity s on [6, p.71]) and z_{KS} (the quantify $z(G)$ from [6, p.86]) for $s = 3/2$, as well as their corresponding generalizations for $s = 5/2$. These parameters coincide for the group E_8. The parameter z_{KS} is not relevant to the $s = 5/2$ case for E_6 because the relevant Weyl orbits do not intersect (cf. [1, Section 3.1]).

Likewise, the specialization of the minimal parabolic Eisenstein series at $\lambda = -\rho$ is the constant function identically equal to 1, a compatible fact.

The proof of theorem 2.13 rests upon special properties of the spherical subrepresentation of V_λ at the values of λ relevant to the $s = 5/2$ Epstein series. We recall that for this maximal parabolic Eisenstein series, λ has the form $\lambda = 2s\omega_{\alpha_1} - \rho$; it is characterized by having inner product $2s-1$ with α_1, and inner product -1 with each α_j, $j \geq 2$. Write λ_{dom} for a dominant weight in the Weyl orbit of λ, i.e., one whose inner product with all positive roots is nonnegative. Table 6 on page 52 gives dominant weights for the groups in Theorem 2.13 as well as its three values of $s \in \{0, 3/2, 5/2\}$, although of course only the last value is of immediate relevance in this section.

The case of $G = E_6$ is slightly easier than the others because of a low-dimensional coincidence, which in fact is mostly independent of the actual value of s in that the same statement holds for generic s. Namely, the representation V_λ we consider is part of a family of degenerate principal series representations, induced from the trivial representation on the reductive $SO(5,5)$ factor of the Levi component $GL(1) \times SO(5,5)$ of the maximal parabolic subgroup P_{α_1}. These representations are indexed by the one dimensional family $\lambda = 2s\omega_{\alpha_1} - \rho$, $s \in \mathbb{C}$, which is related to the $GL(1)$ factor.

Though they may reduce at particular points, their Gelfand-Kirillov dimension[18] is equal to the dimension of the unipotent radical of that parabolic, 16; likewise, any subrepresentation of it cannot have larger dimension. Since the dimension of the wavefront set of a representation is twice the Gelfand-Kirillov dimension, it is bounded by 32. For E_6, the orbits in Figure 2 have dimensions 0, 22, and 32; all other orbits have larger Gelfand-Kirillov dimension. Hence the orbit attached to the $s = 5/2$ Eisenstein series for E_6 is either the trivial orbit, the minimal orbit, or the next-to-minimal orbit. It cannot be the trivial orbit, because only the trivial representation is attached to it. Likewise, Kazhdan-Savin [12] proved a uniqueness statement for the minimal orbit, that (up to Weyl equivalence) only the $s = 3/2$ series is related to the minimal representation. We thus conclude it is attached to the next-to-minimal orbit.

To explain the $s = 5/2$ cases for E_7 and E_8 we need to rely on some recent results from representation theory, and some notions from there concerning *unipotent* and *special unipotent* representations. A striking feature from the table is that $\langle \lambda_{\mathrm{dom}}, \alpha_j \rangle$ has all 1's except for a single zero for the $s = 3/2$ case, and two zeroes for the $s = 5/2$ case. This phenomenon, which came up here because of physical arguments, also arose in work on special unipotent representations. These λ_{dom} take the same value on simple roots as a particular element H of the Cartan subalgebra of $\mathfrak{g}$. In our three examples there is a unique coadjoint nilpotent orbit containing a nilpotent element X such that there is a homomorphism from $\mathfrak{sl}_2$ to $\mathfrak{g}$ carrying $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to X and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to H. In terms of Figure 3 these three related "dual" orbits are the top three listed, though in the reverse order. Appendix A describes a related construction for more general types of orbits beyond the ones considered in this paper.

As part of the more general result given in appendix A, corollary A.6 then asserts that the spherical subquotient of each of the three principal series $V_{\lambda_{\mathrm{dom}}}$ has wavefront set equal to the closure of the dually related orbit listed in figure 3. This proves theorem 2.13 for E_7 and E_8.

6. Fourier coefficients and their vanishing

6.1. **Dimensions of orbits in the character variety.** In sections 3.3-3.5 we listed a number of explicit features of the orbits of instantons for the parabolic subgroups P_{α_1}, P_{α_2}, and $P_{\alpha_{d+1}}$ (in the numbering of figure 1). These are the character variety orbits discussed at the beginning of section 4.1. In this section we give more details, in particular basepoints and dimensions for each of the finite number of orbits under the complexification $L_{\mathbb{C}}$ of the Levi factor of the parabolic. As shorthand, we will refer to these as the "complex

[18]The Gelfand-Kirillov dimension is a numerical index of how "large" a representation is; it is half the dimension of the associated coadjoint nilpotent orbit (i.e., the orbit whose closure is the wavefront set of the representation). For example, finite dimensional representations have Gelfand-Kirillov dimension equal to zero.

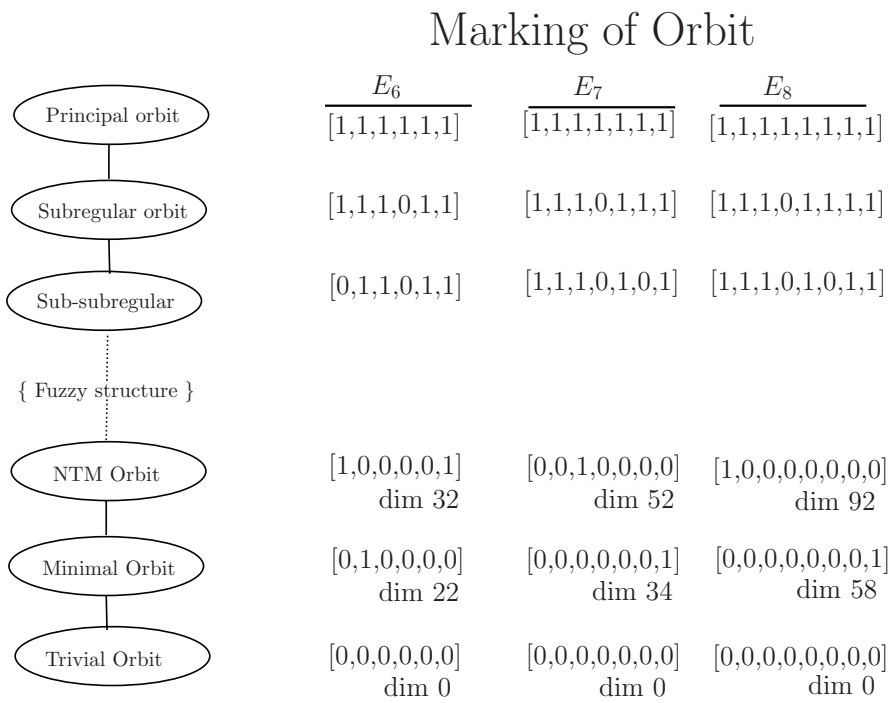

FIGURE 3. The largest and smallest orbits, with markings.

orbits of the Levi". We shall also use the notation Y_α to refer to the root vector $X_{-\alpha}$, in order to keep the listing of basepoints more readable.

This information is quoted from the paper [27], which lists the corresponding information for any maximal parabolic subgroup of an exceptional group. We also describe the group action of the Levi in some of the cases, the rest being described in [27]. Recall that the dimensions of the character varieties were given earlier in table 3 on page 19. In the following subsections, we give more details for the groups $E_5 = SO(5,5)$, E_6, E_7, and E_8. For ease of reference, tables 7 on page 55, 8 on page 55, and 9 on page 56 give the orbit dimensions for the parabolic subgroups P_{α_1}, P_{α_2}, and $P_{\alpha_{d+1}}$ of each of these groups, respectively.

6.1.1. $SO(5,5)$. Recall that we label our $E_5 = D_5$ Dynkin diagram according to the numbering in Figure 1. This does not match the customary numbering of the D_5 Dynkin diagram, but has the advantage of allowing for a uniform discussion of all of our cases of interest.

Node 1 is the so-called "vector" node, because P_{α_1} has Levi component isomorphic to $GL(1) \times SO(4,4)$, which acts on the 8-dimensional, abelian unipotent radical by the usual 8-dimensional representation of $SO(4,4)$. This action breaks into 3 complex orbits: the trivial orbit; a 7-dimensional

Group	dimensions									
$SL(2)$	0	1	-	-	-	-	-	-	-	-
$SL(3) \times SL(2)$	0	2	-	-	-	-	-	-	-	-
$SL(5)$	0	4	-	-	-	-	-	-	-	-
$SO(5,5)$	0	7	8	-	-	-	-	-	-	-
E_6	0	11	16	-	-	-	-	-	-	-
E_7	0	16	25	31	32	-	-	-	-	-
E_8	0	22	35	43	44	50	54	59	63	64

TABLE 7. Dimensions of character variety orbits for the Levi component of the parabolic formed by deleting the first node of $E_4 = SL(5)$, $E_5 = SO(5,5)$, E_6, E_7, and E_8. A dash, $-$, signifies that there is no orbit. The character variety orbits in this parabolic subgroup are the $SO(d,d)$ spinor orbits listed in section 3.4.1.

Group	dimensions									
$SL(2)$	0	-	-	-	-	-	-	-	-	-
$SL(3) \times SL(2)$	0	1	-	-	-	-	-	-	-	-
$SL(5)$	0	4	-	-	-	-	-	-	-	-
$SO(5,5)$	0	7	10	-	-	-	-	-	-	-
E_6	0	10	15	19	20	-	-	-	-	-
E_7	0	13	20	21	25	26	28	31	34	35
E_8	0	16	25	28	31	32	35	38	40	$\cdots$

TABLE 8. Dimensions of character variety orbits of the Levi component for the parabolic formed by deleting the second node of $E_4 = SL(5)$, $E_5 = SO(5,5)$, E_6, E_7, and E_8. A dash, $-$, signifies that there is no orbit. Not all E_8 orbits are listed (there are 23 total).

orbit with basepoint Y_{α_1}; and the open, dense 8-dimensional orbit with basepoint $Y_{11110} + Y_{11101}$ (see table 7).

Nodes 2 and 5 are the "spinor nodes", and have identical orbit structure (up to relabeling the nodes). Here the Levi component of P_{α_2} or P_{α_5} is now isomorphic to $GL(1) \times SL(5)$, and acts on the 10-dimensional abelian unipotent radical by the second fundamental representation, also known as the exterior square representation. In other words, the action of the $SL(5)$ piece is equivalent to that on antisymmetric 2-tensors $x \wedge y = -y \wedge x$, where x and y are 5-dimensional vectors. This action also has 3 complex orbits (part of a general description for abelian unipotent radicals of maximal parabolic subgroups given in [42]): the trivial orbit; a 7-dimensional orbit with basepoint Y_{α_2} in the case of node 2, and Y_{α_5} in the case of node 5; and the open, dense 10-dimensional orbit with basepoint $Y_{01211} + Y_{11111}$ (see

Group	dimensions				
$SL(2)$	0	-	-	-	-
$SL(3) \times SL(2)$	0	1	3	-	-
$SL(5)$	0	5	6	-	-
$SO(5,5)$	0	7	10	-	-
E_6	0	11	16	-	-
E_7	0	17	26	27	-
E_8	0	28	45	55	56

TABLE 9. Dimensions of character variety orbits of the Levi component for the parabolic formed by deleting the last node of $E_4 = SL(5)$, $E_5 = SO(5,5)$, E_6, E_7, and E_8. A dash, $-$, signifies that there is no orbit. The character variety orbits in this parabolic subgroup were also listed in table 5 based on enumeration of instanton orbits.

table 8, or table 9). This last basepoint is in the open dense orbit for either P_{α_2} or P_{α_5}.

6.1.2. E_6. Node 1 and 6 are related by an automorphism of Dynkin diagram, and have identical orbit structure (up to relabeling the nodes). Here the Levi component is isomorphic to $GL(1) \times SO(5,5)$, which acts on the 16-dimensional, abelian unipotent radical by the spin representation of $SO(5,5)$. There are three complex orbits: the trivial orbit; an 11-dimensional orbit with basepoint Y_{α_1} in the case of node 1, and Y_{α_6} in the case of node 6; and the open, dense 16-dimension orbit with basepoint $Y_{111221} + Y_{112211}$ for either nodes 1 or 6 (see table 7 or table 9).

Node 2 is the first case we encounter with a non-abelian unipotent radical. It is instead a 21-dimensional Heisenberg group, and its character variety has 5 complex orbits (another general fact for Heisenberg unipotent radicals of maximal parabolic subgroups [43]): the trivial orbit; a 10-dimensional orbit with basepoint α_2; a 15-dimensional orbit with basepoint $Y_{111221}+Y_{112211}$; a 19-dimensional orbit with basepoint $Y_{011221}+Y_{111211}+Y_{112210}$; and the open, dense 20-dimensional orbit with basepoint $Y_{010111} + Y_{112210}$ (see table 8).

6.1.3. E_7. This is the first group for which the three nodes have mathematically different structures. Node 1 has a 33-dimensional unipotent radical which is a Heisenberg group, and Levi component isomorphic to $GL(1) \times SO(6,6)$. The action on the 32-dimensional character variety again has 5 complex orbits: the trivial orbit; a 16-dimensional orbit with basepoint Y_{α_1}; a 25-dimensional orbit with basepoint $Y_{1123321} + Y_{1223221}$; a 31-dimensional orbit with basepoint $Y_{1122221}+Y_{1123211}+Y_{1223210}$; and the open, dense 32-dimensional orbit with basepoint $Y_{1011111} + Y_{1223210}$ (see Table 7 on page 55).

Node 2 has a 42-dimensional unipotent radical, and a 35-dimensional character variety. The Levi component $GL(1) \times SL(7)$ acts with 10 complex

orbits: the trivial orbit; a 13-dimensional orbit with basepoint Y_{α_2}; a 20-dimensional orbit with basepoint $Y_{1122221} + Y_{1123211}$; a 21-dimensional orbit with basepoint $Y_{0112221} + Y_{1112211} + Y_{1122111}$; a 25-dimensional orbit with basepoint $Y_{1112221}+Y_{1122211}+Y_{1123210}$; a 26-dimensional orbit with basepoint $Y_{1111111}+Y_{1123210}$; a 28-dimensional orbit with basepoint $Y_{0112221}+Y_{1112211}+Y_{1122111}+Y_{1123210}$; a 31-dimensional orbit with basepoint $Y_{0112221}+Y_{1111111}+Y_{1123210}$; a 34-dimensional orbit with basepoint $Y_{0112211}+Y_{1112111}+Y_{1112210}+Y_{1122110}$; and the open, dense 35-dimensional orbit with basepoint $Y_{0112111}+Y_{0112210}+Y_{1111111}+Y_{1112110}+Y_{1122100}$ (see table 8 on page 55).

Node 7 has a 27-dimensional abelian unipotent radical, and Levi component isomorphic to $GL(1) \times E_{6,6}$. The latter acts with 4 complex orbits: the trivial orbit, a 17-dimensional orbit with basepoint Y_{α_7}, a 26-dimensional orbit with basepoint $Y_{1123321} + Y_{1223221}$, and the open, dense 27-dimensional orbit with basepoint $Y_{0112221} + Y_{1112211} + Y_{1122111}$ (see Table 9 on page 56).

6.1.4. E_8. This is the biggest of our groups, and the unipotent radicals of its maximal parabolics are never abelian.

Node 1 has a 78-dimensional unipotent radical, and a 64-dimensional character variety. The Levi component is isomorphic to $GL(1) \times SO(6,6)$ and acts according to the spin representation of $SO(6,6)$, with 10 complex orbits: the trivial orbit; a 22-dimensional orbit with basepoint Y_{α_1}; a 35-dimensional orbit with basepoint $Y_{12244321} + Y_{12343321}$; a 43-dimensional orbit with basepoint $Y_{12233321} + Y_{12243221} + Y_{12343211}$; a 44-dimensional orbit with basepoint $Y_{11122221} + Y_{12343211}$; a 50-dimensional orbit with basepoint $Y_{11233321} + Y_{12233221} + Y_{12243211} + Y_{12343210}$; a 54-dimensional orbit with basepoint $Y_{11222221} + Y_{12243211} + Y_{12343210}$; a 59-dimensional orbit with basepoint $Y_{11122221} + Y_{11233211} + Y_{12232211} + Y_{12343210}$; a 63-dimensional orbit with basepoint $Y_{11222221} + Y_{11232211} + Y_{11233210} + Y_{12232111} + Y_{12232210}$; and the open, dense 64-dimensional orbit with basepoint $Y_{11122111} + Y_{11221111} + Y_{11233210} + Y_{12232210}$ (see table 7 on page 55).

Node 2 has a 92-dimensional unipotent radical, and a 56-dimensional character variety. The Levi component is isomorphic to $GL(1) \times SL(8)$ and acts according to the third fundamental representation of $SL(8)$, also known as the exterior cube representation. It acts with 23 complex orbits, the four smallest of which are: the trivial orbit; a 16-dimensional orbit with basepoint Y_{α_2}; a 25-dimensional orbit with basepoint $Y_{1123221} + Y_{11233211}$; and a 28-dimensional orbit with basepoint $Y_{11122221} + Y_{11222211} + Y_{112321111}$ (see table 8 on page 55).

Node 8 has a 57 dimensional unipotent radical which is a Heisenberg group. The Levi factor is isomorphic to $GL(1) \times E_7$ and acts with 5 complex orbits on the 56-dimensional character variety: the trivial orbit; a 28-dimensional orbit with basepoint Y_{α_8}; a 45-dimensional orbit with basepoint $Y_{22454321} + Y_{23354321}$; a 55-dimensional orbit with basepoint $Y_{12244321} + Y_{12343321} + Y_{22343221}$; and the open, dense 56-dimensional orbit with basepoint $Y_{01122221} + Y_{22343211}$ (see table 9 on page 56).

6.2. **Applications of Matumoto's theorem.** In Section 2.2.2 we mentioned that representations of real groups have an invariant attached to them, the wavefront set, that in a sense measures how big the representation is. Theorem A.5 indeed computes this wavefront set in many cases, including ours. There is a theorem due to Matumoto [44] that asserts, in a precise sense, that small representations cannot have large Fourier coefficients. Namely, he proves that if an element $Y \in \mathfrak{u}_{-1}$ associated to the character χ from (4.6) does not lie in the wavefront set, then the Fourier coefficient ϕ_χ from (4.1) must vanish identically.

For example, the trivial representation has wavefront set $\{0\}$, and likewise the constant function does not have any nontrivial Fourier coefficients. In [27] a detailed analysis is given of the different character variety orbits for each parabolic subgroup of an exceptional group, and which coadjoint nilpotent orbits they are contained in. It is then a simple matter to apply Matumoto's theorem and determine a set of Fourier coefficients which automatically vanishes because their containing coadjoint nilpotent orbits lie outside the wavefront set. In particular, it is shown in [27] that the closure of the minimal coadjoint nilpotent orbit contains the two smallest character variety orbits in each of the examples of P_{α_1}, P_{α_2}, and $P_{\alpha_{d+1}}$ for the groups E_{d+1}, $5 \leq d \leq 7$ (this was known to experts, at least in special cases – see for example [6]). Likewise, it is also verified there that the closure of the next-to-minimal coadjoint nilpotent orbit contains the three smallest character variety orbits in each of these nine examples.

Combining this with the characterization in Theorem 2.13 of the wavefront sets for the Epstein series at $s = 0$, $3/2$, and $5/2$, we get the following statement about the vanishing of Fourier coefficients. This gives a rigorous proof of the vanishing statements on page 5.

Theorem 6.1. *Let $5 \leq d \leq 7$ and $G = E_{d+1}$ as defined in table 1 on page 3. Then:*

(i) *All Fourier coefficients of the $s = 0$ Epstein series vanish in any of the parabolics P_{α_1}, P_{α_2}, or $P_{\alpha_{d+1}}$, with the exception of the constant terms (which were calculated in [1]).*

(ii) *All Fourier coefficients of the $s = 3/2$ Epstein series $E^G_{\alpha_1;3/2}$ vanish in any of the parabolics P_{α_1}, P_{α_2}, or $P_{\alpha_{d+1}}$, with the exceptions of the constant term and the smallest dimensional character variety orbit. This orbit has: dimension 11 for E_6 and either P_{α_1} or P_{α_6}, and dimension 10 for P_{α_2}; dimensions 16, 13, and 17 for E_7 and P_{α_1}, P_{α_2}, and P_{α_7}, respectively; and dimensions 22, 16, and 28 for E_8 and P_{α_1}, P_{α_2}, and P_{α_8}, respectively.*

(iii) *All Fourier coefficients of the $s = 5/2$ Epstein series $E^G_{\alpha_1;5/2}$ vanish in any of the parabolics P_{α_1}, P_{α_2}, or $P_{\alpha_{d+1}}$, with the exceptions of the constant term and the next two smallest dimensional character variety orbits. This additional character variety orbit is: the 16, 15,*

and 16-dimensional orbits for E_6 and P_{α_1}, P_{α_2}, and P_{α_6}, respectively; the 25, 20, and 26-dimensional orbits for E_7 and P_{α_1}, P_{α_2}, and P_{α_7}, respectively; and the 35, 25, and 45-dimensional orbits for E_8 and P_{α_1}, P_{α_2}, and P_{α_8}, respectively.

7. Square integrability of special values of Eisenstein series

In this section we remark that some of the coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ from the expansion (2.3) provide examples of square-integrable automorphic forms on higher rank groups. In particular, we will prove this is the case for $\mathcal{E}^{(D)}_{(1,0)}$ on E_7 and E_8. In light of (1.3), this proves the associated automorphic representation is unitary, since it can be realized in the Hilbert space $L^2(E_{d+1}(\mathbb{Z})\backslash E_{d+1}(\mathbb{R}))$. This unitary can also be demonstrated by purely representation theoretic methods. At present this is more of a curiosity, since we are not aware of any particular importance for our applications. The analysis in the proof also determines the exact asymptotics of these coefficients in various limits, generalizing those studied in [1].

Theorem 7.1. *Let G denote the group E_{d+1} defined in table 1 on page 3.*

(i) *The Epstein series $E^G_{\alpha_1;0}$ is constant, and hence always square-integrable.*

(ii) *The Epstein series $E^G_{\alpha_1;3/2}$ and hence $\mathcal{E}^{(10-d)}_{(0,0)}$ is square-integrable if $4 \leq d \leq 7$.*

(iii) *The Epstein series $E^G_{\alpha_1;5/2}$ and hence $\mathcal{E}^{(10-d)}_{(1,0)}$ is square-integrable if $6 \leq d \leq 7$.*

Case (i) is obvious since the quotient $E_{d+1}(\mathbb{Z})\backslash E_{d+1}(\mathbb{R})$ has finite volume, while case (ii) was proven earlier by [6]. We have included them here in the statement for convenience and comparison. It should be stressed, though, that $E^G_{\alpha_1;s}$ is certainly not square integrability for *general* s. The same method treats the lower rank groups as well, though since the statements are not needed here we refer to papers [6] and [45] for D_5.

Proof. Recall that the series $E^G_{\alpha_1;s}$ is a specialization of the *minimal parabolic* Eisenstein series $E^G(\lambda, g)$ from (5.3) at $\lambda = 2s\omega_1 - \rho$. This is explained in our context in [1, Section 2], where Langlands' constant term formula is also given in Theorem 2.18. The latter shows that the constant term of $E^G(\lambda, g)$ along any maximal parabolic subgroup P is a sum of other minimal parabolic Eisenstein series on its Levi component. By induction, this is also true if P is not maximal. In particular, since these Eisenstein series on smaller groups are orthogonal to all cusp forms on those groups, the constant terms are therefore orthogonal to all cusp forms on the Levi components – a meaningful statement only, of course, when the parabolic P is not the Borel subgroup B (so that the Levi is nontrivial). This means $E^G(\lambda, g)$ has "zero cuspidal component along any such P" in the sense of [46, Section 3], or equivalently that it is "concentrated" on the Borel subgroup B.

The constant term along B is explicitly given in terms of a sum over the Weyl group:

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^G(\lambda, ng)\, dn \quad = \quad \sum_{w\,\in\,\Omega} e^{(w\lambda+\rho)(H(g))} M(w,\lambda)\,, \tag{7.2}$$

where $M(w,\lambda)$ is given by the explicit product over roots whose sign is flipped by w,

$$M(w,\lambda) \quad = \quad \prod_{\substack{\alpha\,>\,0\\ w\alpha\,<\,0}} c(\langle\lambda,\alpha\rangle)\,, \tag{7.3}$$

with

$$c(s) \quad := \quad \frac{\xi(s)}{\xi(s+1)} \qquad \text{and} \qquad \xi(s) \quad := \quad \pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \tag{7.4}$$

(see, for example, [1, (2.16)-(2.21)]). This formula is valid for generic λ, and develops logarithmic terms at special points via meromorphic continuation. Moreover, certain coefficients $M(w,\lambda)$ may vanish, for example when $\langle\lambda,\alpha\rangle = -1$ and the respective factor in (7.4) has a zero owing to the pole of $\xi(s)$ at $s=0$ (unless it is canceled by a pole from another factor). Because $E^G(\lambda,g)$ is "concentrated on B", Langlands' criteria in [46, Section 5] asserts that it is square-integrable if and only if the surviving exponents $w\lambda$ have negative inner product with each fundamental weight:

$$\langle w\lambda, \omega_\alpha\rangle \quad < \quad 0 \quad \text{for each } \alpha \,>\, 0\,. \tag{7.5}$$

The rest of the proof involves an explicit calculation to check that for each $w\in\Omega$, either $M(w,\lambda)$ vanishes or (7.5) holds. Actually, despite the enormous size of the Weyl groups involved, $M(w,\lambda)$ vanishes for all but very few w (because of the special nature of λ).

Though the individual terms in (7.2) are frequently singular at the values of λ in question, the overall sum can be calculated explicitly by taking limits. We now present the result of this calculation. To make the condition (7.5) more transparent, we take $g=a$ to be an element of the maximal torus A (as we of course may, given that $H(g)$ depends only on the A-component of g's Iwasawa decomposition). We then furthermore parameterize a by real numbers $r_1, r_2, \ldots$ via the condition that the simple roots on a take the values

$$a^{\alpha_1} \;=\; e^{r_1}\,,\; a^{\alpha_2} \;=\; e^{r_2}\,,\; \ldots. \tag{7.6}$$

For example, for $G=E_6$ the limiting value of (7.2) as λ approaches $3\omega_1-\rho$ can be calculated explicitly as $e^{2r_1+3r_2+4r_3+6r_4+4r_5+2r_6}$ times

$$\frac{3\zeta(3)\left(e^{2r_1+r_3}+e^{r_5+2r_6}\right)+\pi^2(e^{r_2}+e^{r_3}+e^{r_5})+6\pi(r_4+\gamma-\log(4\pi))}{3\zeta(3)}\,. \tag{7.7}$$

This expression is dominated by $e^{\rho(H(g))} = e^{8r_1+11r_2+15r_3+21r_4+15r_5+8r_6}$ for $r_i>0$, that is, (7.5) holds and hence $E^G_{\alpha_1;3/2}$ is square-integrable – verifying a fact proven in [6].

We now turn to the two new cases, those of the $s = 5/2$ series for E_7 and E_8. We recall the computational method of [1, Section 2.4] to find the minimal parabolic constant terms, namely to precompute the set

$$\mathcal{S} \;:=\; \{\, w \in \Omega \;\mid\; w\alpha_i > 0 \quad \text{for all } i \neq 1 \,\}\,. \tag{7.8}$$

For $w \notin \mathcal{S}$, $M(w,\lambda)$ will include the factor $c(\langle\lambda,\alpha_i\rangle) = c(\langle 2s\omega_1 - \rho, \alpha_i\rangle) = c(-\langle\rho,\alpha_i\rangle) = c(-1) = 0$ for some $i > 1$. At the same time, at least for $\mathrm{Re}\, s < \frac{1}{2}$, all inner products $\langle\lambda,\alpha\rangle$ will be negative, and hence none of the other factors in (7.3) can have a pole (after all, $c(s)$ is holomorphic for $\mathrm{Re}\, s < 0$). Thus the term for w in (7.2) vanishes identically in s by analytic continuation, and the sum in (7.2) reduces to one over $w \in \mathcal{S}$.

For E_7 there are only 126 elements in $\mathcal{S}$ out of the 2,903,040 elements of the full Weyl group Ω. It can be calculated that all but three w of these 126 satisfy Langlands' condition (7.5), and the three that do not have the following expressions for $M(w,\lambda)$ for $s = 5/2 + \varepsilon$:

$$\begin{aligned}
\text{Exception 1}:\;& c(2(\epsilon-5))c(2\epsilon)^2c(2\epsilon-9)c(2\epsilon-8)^2c(2\epsilon-7)^2c(2\epsilon-6)^3 \times \\
&\times\; c(2\epsilon-5)^3c(2\epsilon-4)^3c(2\epsilon-3)^3c(2\epsilon-2)^3c(2\epsilon-1)^3 \times \\
&\times\; c(2\epsilon+1)^2c(2\epsilon+2)c(2\epsilon+3)c(2\epsilon+4)c(4\epsilon-7)\,, \\
\text{Exception 2}:\;& c(2\epsilon)^2c(2\epsilon-9)c(2\epsilon-8)^2c(2\epsilon-7)^2c(2\epsilon-6)^3c(2\epsilon-5)^3 \times \\
&\times\; c(2\epsilon-4)^3c(2\epsilon-3)^3c(2\epsilon-2)^3c(2\epsilon-1)^3c(2\epsilon+1)^2 \times \\
&\times\; c(2\epsilon+2)c(2\epsilon+3)c(2\epsilon+4)c(4\epsilon-7)\,, \\
\text{Exception 3}:\;& c(2(\epsilon-5))c(2\epsilon)^2c(2\epsilon-11)c(2\epsilon-9)c(2\epsilon-8)^2c(2\epsilon-7)^2 \times \\
&\times\; c(2\epsilon-6)^3c(2\epsilon-5)^3c(2\epsilon-4)^3c(2\epsilon-3)^3c(2\epsilon-2)^3 \times \\
&\times\; c(2\epsilon-1)^3c(2\epsilon+1)^2c(2\epsilon+2)c(2\epsilon+3)c(2\epsilon+4)c(4\epsilon-7)\,.
\end{aligned} \tag{7.9}$$

Each of these terms is in fact zero by dint of the triple zero counterbalancing the double pole at $\epsilon = 0$. (Incidentally, the overall series $E^G_{\alpha_1;5/2}$ was shown to be non-zero in [1] for both $G = E_7$ and $G = E_8$).

For E_8 there are 2160 elements in $\mathcal{S}$ out of the 696,729,600 elements of the full Weyl group Ω. Likewise, all but 258 of these 2160 w satisfy (7.5). Again, all 258 of these terms vanish at $s = 5/2$ because their products have a triple zero (coming from three $c(s)$ factors evaluated at near $s = -1$) that balance two poles (coming from two $c(s)$ factors evaluated near $s = 1$).

□

8. Discussion and future problems

In this paper we have studied the Fourier modes of the Eisenstein series that define the coefficients of the first two nontrivial interactions in the low energy expansion of the four-graviton amplitude in maximally supersymmetric string theory compactified on T^d, and verified they have certain expected features. In particular, we have shown that their non-zero Fourier

coefficients contain the expected minimal and next-to-minimal ($\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS) instanton orbits for any of the symmetry groups, E_{d+1} ($0 \leq d \leq 7$). This extends the analysis of these functions in [1], where the constant terms of these functions were shown to reproduce all the expected features of string perturbation theory and semi-classical M-theory. Furthermore, in low rank cases we were able to present the explicit Fourier coefficients of these functions and show that they have the form expected of BPS-instanton contributions. Indeed, the form of the $\frac{1}{2}$-BPS contributions match those deduced from string theory calculations as summarised by (4.10).

For high rank cases this involved a detailed analysis of the automorphic representations connected to these coefficients. Namely, we explained that they are automorphic realizations of the smallest two types of nontrivial representations of their ambient Lie groups, and why this property automatically implies the vanishing of a slew of Fourier coefficients – precisely the Fourier coefficients that the BPS condition ought to force to vanish. We furthermore showed the most interesting cases – those of the next-to-minimal representation for E_7 and E_8 – occur in $L^2(E_{d+1}(\mathbb{Z})\backslash E_{d+1}(\mathbb{R}))$.

This raises some obviously interesting questions, both from the string theory perspective and from the mathematical perspective.

An immediately interesting mathematical direction would be the explicit computation of the non-zero Fourier modes of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ for the high rank cases with groups E_6, E_7 and E_8, in particular to get finer information using the work of Bhargava and Krutelevich on the integral structure of the character variety orbits. In a different direction, as mentioned in section 3.3.1 it would be of interest to extend the considerations of this paper to affine E_9 and behind that to hyperbolic extensions.

Another question that is natural to ask in the context of string theory is to what extent does our analysis generalise to higher order interactions in the low energy expansion, which preserve a smaller fraction of supersymmetry? Could there be a rôle for Eisenstein series with other special values of the index s in the description of such terms? However, the evidence is that such higher order terms involve automorphic functions that are not Eisenstein series. For example, $\mathcal{E}^{(D)}_{(0,1)}$ (the coefficient of the $\frac{1}{8}$-BPS $\partial^6\,\mathcal{R}^4$ interaction) is expected to satisfy a particular inhomogeneous Laplace eigenvalue equation [5]. Although its constant term has, to a large extent, been analysed for the relevant values of D [1], it would be most interesting to analyse the non-zero Fourier modes of $\mathcal{E}^{(D)}_{(0,1)}$, which should describe the couplings of $\frac{1}{8}$-BPS instantons in the four-graviton amplitude for low enough dimensions, D. This should reveal a rich structure. For example, the instantons that contribute in the limit of decompactification from D to $D+1$ include the $\frac{1}{8}$-BPS black holes of $D+1$ dimensions, which can have non-zero horizon size and exponential degeneracy. It is not apparent at first sight whether this degeneracy should be encoded in the solutions of the inhomogeneous

equation satisfied by $\mathcal{E}^{(D)}_{(0,1)}$. Indeed, we have seen in the $\frac{1}{4}$-BPS cases that the Fourier expansion of the coefficient function $\mathcal{E}^{(D)}_{(1,0)}$ in the decompactification limit does not determine the Hagedorn-like degeneracy of $\frac{1}{4}$-BPS small black holes in $D+1$ dimensions. Rather, the divisor functions weight particular combinations of charges and windings of the wrapped world-lines of such objects.

These issues involve mathematical challenges. For example, the study of inhomogeneous Laplace equations for the group $SL(2,\mathbb{R})$ heavily relies on explicit formulas for automorphic Green functions, which do not generalize in an obvious manner to higher rank groups because they involve automorphic Laplace eigenfunction forms which do not have moderate growth in the cusps (at present the existence of such functions is itself an open problem).

Another issue is to what extent this analysis can be extended to discuss the automorphic properties of yet higher order terms in the expansion of the four-graviton amplitude. Further afield are issues concerning the extension of these ideas to multi-particle amplitudes, to amplitudes that transform as modular forms of non-zero weight, and extensions to processes with less supersymmetry.

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Appendix A. Special unipotent representations, by Dan Ciubotaru and Peter E. Trapa

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The representations considered in Theorem 2.13 are examples of a wider class of representations which have attracted intense attention in the mathematical literature. The purpose of this appendix is to recall certain results (from a purely local point of view) which are especially relevant for the discussion of Section 5.

To begin, let G denote a real reductive group arising as the real points of a connected complex algebraic group $G_{\mathbb{C}}$. In [Ar1] and [Ar2], Arthur set forth a conjectural description of irreducible (unitary) representations contributing to the automorphic spectrum of G. In many cases, these conjectures could be reduced to a fundamental set of representations attached to (integral) "special unipotent" parameters. In the real case, Arthur's conjectures — and, in particular, the definition of the corresponding special unipotent representations — are made precise and refined in the work of Barbasch-Vogan [BV1] and, more completely, in the work of Adams-Barbasch-Vogan [ABV]. The perspective of these references is entirely local. (Of course an extensive literature approaching Arthur's conjectures by global methods exists and, for classical groups, is summarized in [Ar3].) As we now explain, the representations appearing in Theorem 2.13 are indeed special unipotent in the sense of Adams-Barbasch-Vogan.

Write $\mathfrak{g}_{\mathbb{C}}$ for the Lie algebra of $G_{\mathbb{C}}$ and fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ arising as the Lie algebra of a maximal torus in $G_{\mathbb{C}}$. Write Ω for the Weyl group of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. The classification of connected reductive algebraic groups naturally leads from $G_{\mathbb{C}}$ to the Langlands dual $G_{\mathbb{C}}^{\vee}$, a connected reductive complex algebraic group, e.g. [Sp]. Let $\mathfrak{g}_{\mathbb{C}}^{\vee}$ denote the Lie algebra of $G_{\mathbb{C}}$. The construction of $G_{\mathbb{C}}^{\vee}$ includes the definition of a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}^{\vee}$ which canonically identifies with the linear dual of $\mathfrak{h}_{\mathbb{C}}$,

$$\mathfrak{h}_{\mathbb{C}}^{\vee} \simeq (\mathfrak{h}_{\mathbb{C}})^{*}. \tag{A.1}$$

Let $\mathcal{N}$ denote the cone of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$, and likewise let $\mathcal{N}^{\vee}$ denote the cone of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}^{\vee}$. Write $G_{\mathbb{C}}\backslash\mathcal{N}$ and $G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee}$ for the corresponding sets of adjoint orbits. These sets are partially ordered by the inclusion of closures. Spaltenstein defined an order-reversing map

$$d \,:\, G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee} \longrightarrow G_{\mathbb{C}}\backslash\mathcal{N}$$

with many remarkable properties which were refined in [BV1, Appendix]; see Theorem A.4 below.

Example A.1. Suppose the Dynkin diagram corresponding to $\mathfrak{g}_{\mathbb{C}}$ is simply laced (as is the case for the groups E_{d+1} from figure 1 and table 1). Then $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}_{\mathbb{C}}^{\vee}$ and $G_{\mathbb{C}}^{\vee}$ and $G_{\mathbb{C}}$ are isogenous. Thus $G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee}$ can be identified with $G_{\mathbb{C}}\backslash\mathcal{N}$ and d can be viewed as an order reversing map from the latter set to itself. With this in mind, consider Figure 3. The map d interchanges the top three orbits with the bottom three orbits (in an order reversing way, of course). In particular d applied to the sub-subregular orbit is the next to minimal orbit. The complete calculation of d is given in [Ca].

Fix an element $\mathcal{O}^{\vee}$ of $G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee}$. According to the Jacobson-Morozov Theorem, there exists a Lie algebra homomorphism

$$\phi \,:\, \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{g}_{\mathbb{C}}^{\vee}$$

such that the image under of ϕ of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ lies in $\mathcal{O}^\vee$ and

$$\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}_{\mathbb{C}}^\vee \simeq \mathfrak{h}_{\mathbb{C}}^*, \tag{A.2}$$

with the last isomorphism as in (A.1).

The element in (A.2) depends on the choice of ϕ. Its Weyl group orbit is well-defined however (independent of how ϕ is chosen). So define

$$\lambda(\mathcal{O}^\vee) := (1/2)\, \phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}_{\mathbb{C}}^*/\Omega. \tag{A.3}$$

According to the Harish-Chandra isomorphism, $\lambda(\mathcal{O}^\vee)$ specifies a maximal ideal $Z(\mathcal{O}^\vee)$ in the center of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. Recall that an irreducible admissible representation of G is said to have infinitesimal character $\lambda(\mathcal{O}^\vee)$ if its Harish-Chandra module is annihilated by $Z(\mathcal{O}^\vee)$.

A result of Dixmier implies that there is a unique primitive ideal $I(\mathcal{O}^\vee)$ in $U(\mathfrak{g}_{\mathbb{C}})$ which is maximal among all primitive ideals containing $Z(\mathcal{O}^\vee)$. (A primitive ideal in $U(\mathfrak{g}_{\mathbb{C}})$ is, by definition, a two-sided ideal which arises as the annihilator of a simple $U(\mathfrak{g}_{\mathbb{C}})$ module.) Given any two-sided ideal I in $U(\mathfrak{g}_{\mathbb{C}})$, we can consider the associated graded ideal $gr(I)$ with respect to the canonical grading on $U(\mathfrak{g}_{\mathbb{C}})$. According to the Poincaré-Birkoff-Witt Theorem, $gr(I)$ is an ideal in $gr(U(\mathfrak{g}_{\mathbb{C}})) \simeq S(\mathfrak{g}_{\mathbb{C}})$, the symmetric algebra of $\mathfrak{g}_{\mathbb{C}}$, and hence cuts out a subvariety (the so-called associated variety, $\mathrm{AV}(I)$, of I) of $\mathfrak{g}_{\mathbb{C}}^*$.

It will be convenient to identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}_{\mathbb{C}}^*$ (by means of the choice of an invariant form) and view $\mathrm{AV}(I)$ as a subvariety of $\mathfrak{g}_{\mathbb{C}}$. (The choice of form is well-defined up to scalar; since $\mathrm{AV}(I)$ is a cone, $\mathrm{AV}(I)$ becomes a well-defined subvariety of $\mathfrak{g}_{\mathbb{C}}$.) A theorem of Joseph [11] and Borho-Brylinski [BoBr1] (cf. the short proof in [V2]) implies that if I is primitive, $\mathrm{AV}(I)$ is indeed the closure of a single nilpotent orbit of $G_{\mathbb{C}}$.

Theorem A.4 ([BV1, Corollary A.3])**.** *In the setting of the previous paragraph,*

$$\mathrm{AV}(I(\mathcal{O}^\vee)) = \overline{d(\mathcal{O}^\vee)}.$$

Example A.2. Suppose $G_{\mathbb{C}}$ is simply laced and make identifications as in Example A.1. Suppose $\mathcal{O}^\vee$ is respectively the regular, subregular, or sub-subregular, orbit in Figure 3. Then $\mathrm{AV}(I(\mathcal{O}^\vee))$ is the closure respectively of the zero, minimal, or next-to-minimal orbit.

Definition A.3 (Barbasch-Vogan [BV1])**.** Fix an orbit $\mathcal{O}^\vee$ as above. Suppose further that $\mathcal{O}^\vee$ is even or, equivalently, that $\lambda(\mathcal{O}^\vee)$ is integral. An irreducible admissible representation of G is said to be *(integral) special unipotent attached to* $\mathcal{O}^\vee$ if the annihilator of its Harish-Chandra module is $I(\mathcal{O}^\vee)$.

Note that since $I(\mathcal{O}^\vee)$ is a maximal primitive ideal, special unipotent representations are, in a precise sense, as small as possible.

Theorem A.5. *Suppose G is split and π is an irreducible spherical representation with infinitesimal character $\lambda(\mathcal{O}^\vee)$ (with notation as in* (A.3)*). Suppose further that $\mathcal{O}^\vee$ is even. Then π is special unipotent in the sense of Definition A.3.*

Sketch. Chapter 27 in [ABV] defines special unipotent Arthur packets. Roughly speaking, such a packet is parametrized by a rational form of an orbit $\mathcal{O}^\vee$ in $G^\vee_{\mathbb{C}}\backslash\mathcal{N}^\vee$ ([ABV, Theorem 27.10]). In the case that $\mathcal{O}^\vee$ is even, these packets are known to consist of representations appearing in Definition A.3 ([ABV, Corollary 27.13]). As a consequence of [ABV, Definition 22.6] (see also the discussion after [ABV, Definition 1.33]), such a packet also contains a (generally nontempered) L-packet. In the case at hand, the special unipotent Arthur packet parametrize by $\mathcal{O}^\vee$ contains the L-packet consisting of the spherical representation with infinitesimal character $\lambda(\mathcal{O}^\vee)$. This completes the sketch. □

Corollary A.6. *The spherical subrepresentations of the principal series representations $V_{\lambda_{dom}}$ from section 5 are integral special unipotent attached to $\mathcal{O}^\vee$ (Definition A.3) where $\mathcal{O}^\vee$ is, respectively, the regular, subregular, and sub-subregular nilpotent orbit (all of which are even). According to Corollary A.4 and Example A.2, the wavefront sets of these representations are, respectively, the zero, minimal, and next to minimal orbits.*

Finally, we remark that since the special unipotent representation of Definition A.3 are predicted by Arthur to appear in spaces of automorphic forms, they should be unitary.

Conjecture A.7. *Suppose π is integral special unipotent in the sense of Definition A.3. Then π is unitary.*

The representations appearing in Theorem A.5 are known to be unitary if $G_{\mathbb{C}}$ is classical or of Type G2. This was proved by purely local methods in [V1], [V2], and [B]. For a summary of results obtained by global methods, see [Ar3].

For completeness, we discuss the analogs of these results in the p-adic case. Let F be a p-adic field, with ring of integers $\mathfrak{O}$, and finite residue field F_q. The group G is now the F-points of a connected algebraic group $G_{\overline{F}}$ defined over $\overline{F}$. We assume for simplicity that G is split and of adjoint type. Let K be the $\mathfrak{O}$-points of $G_{\overline{F}}$, a maximal compact open subgroup of G. Let I be the inverse image in K under the natural projection $K \to G_{\overline{F}}(F_q)$ of a Borel subgroup over F_q. The compact open subgroup I is called an Iwahori subgroup.

The Iwahori-Hecke algebra $\mathcal{H}(G, I)$ is the convolution algebra (with respect to a fixed Haar measure on G) of compactly supported, locally constant, I-biinvariant complex functions on G. It is a Hilbert algebra, in the sense of Dixmier, with respect to the trace function $f \mapsto f(1)$, and the $*$-operation $f^*(g) = \overline{f(g^{-1})}$, $f \in \mathcal{H}(G, I)$. Thus, there is a theory of unitary remodules of $\mathcal{H}(G, I)$ and an abstract Plancherel formula.

If (π, V) is a complex smooth G-representation, such that $V^I \neq 0$, the algebra $\mathcal{H}(G, I)$ acts on V^I via

$$\pi(f)v = \int_G f(x)\pi(x)v \, dx, \quad v \in V^I, \ f \in \mathcal{H}(G, I).$$

Theorem A.8 ([Bo])**.** *The functor $V \to V^I$ is an equivalence of categories between the category of smooth admissible G-representations and finite dimensional $\mathcal{H}(G, I)$-modules*

Borel conjectured that this functor induces a bijective correspondence of unitary representations. This conjecture was proved by Barbasch-Moy [BM1] (subject to a certain technical assumption which was later removed).

Theorem A.9 ([BM1])**.** *An irreducible smooth G-representation (π, V) is unitary if and only if V^I is a unitary $\mathcal{H}(G, I)$-module.*

The algebra $\mathcal{H}(G, I)$ contains the finite Hecke algebra $\mathcal{H}(K, I)$ of functions whose support is in K. Under the functor η, K-spherical representations of G correspond to spherical $\mathcal{H}(G, I)$-modules, i.e., modules whose restriction to $\mathcal{H}(K, I)$ contains the trivial representation of $\mathcal{H}(K, I)$.

The classification of simple $\mathcal{H}(G, I)$-modules is given by Kazhdan-Lusztig [KL].

Theorem A.10 ([KL])**.** *The simple $\mathcal{H}(G, I)$-modules are parameterized by $G^\vee_{\mathbb{C}}$-conjugacy classes of triples $(s^\vee, e^\vee, \psi^\vee)$, where:*

(i) *$s^\vee \in G^\vee_{\mathbb{C}}$ is semisimple;*
(ii) *$e^\vee \in \mathcal{N}^\vee$ such that $Ad(s)e = qe$;*
(iii) *$\psi^\vee$ is an irreducible representation of Springer type of the group of components of the mutual centralizer $Z_{G^\vee_{\mathbb{C}}}(s^\vee, e^\vee)$ of $s^\vee$ and $e^\vee$ in $G^\vee_{\mathbb{C}}$.*

Let $\pi(s^\vee, e^\vee, \psi^\vee)$ denote the simple $\mathcal{H}(G, I)$-module parametrized by $[(s^\vee, e^\vee, \psi^\vee)]$.

Example A.4. In the Kazhdan-Lusztig parametrization, the simple spherical $\mathcal{H}(G, I)$-modules correspond to the classes of triples $[(s^\vee, 0, 1)]$. Here $s^\vee$ is the Satake parameter of the corresponding irreducible spherical G-representation. On the other hand, let $\mathcal{O}^\vee$ be a fixed $G^\vee_{\mathbb{C}}$-orbit in $\mathcal{N}^\vee$, and set $s^\vee_{\mathcal{O}^\vee} = q^{\lambda_0(\mathcal{O}^\vee)}$ where $\lambda_0(\mathcal{O}^\vee)$ is any choice of representative of the element in (A.3). If $e^\vee_0$ belongs to the unique open dense orbit of $Z_{G^\vee_{\mathbb{C}}}(s^\vee)$ on $\mathfrak{g}^\vee_q = \{x \in \mathfrak{g}^\vee_q : Ad(s^\vee)x = qx\}$ (in particular $e^\vee_0 \in \mathcal{O}^\vee$), then the simple $\mathcal{H}(G, I)$-module (and the corresponding irreducible G-representation) parametrized by $[(s^\vee_{\mathcal{O}^\vee}, e^\vee_0, \psi^\vee)]$ is tempered.

The Iwahori-Hecke algebra has an algebra involution τ, called the Iwahori-Matsumoto involution, defined on the generators as in [IM]. It induces an involution on the set of simple $\mathcal{H}(G, I)$-modules, which is easily seen to map unitary modules to unitary modules. The effect of τ on the set of Kazhdan-Lusztig parameters is given by a Fourier transform of perverse sheaves [EM],

and therefore it is hard to compute effectively in general, except in type A [MW]. (For a general algorithm, see [L].) However, it is easy to see that if $\pi(s^\vee_{\mathcal{O}^\vee}, 0, 1)$ is a simple spherical $\mathcal{H}(G, I)$-module, then

$$\tau(\pi(s^\vee_{\mathcal{O}^\vee}, 0, 1)) = \pi(s^\vee_{\mathcal{O}^\vee}, e^\vee_0, 1), \tag{A.11}$$

where the notation is as in Example A.4. In particular, $\pi(s^\vee_{\mathcal{O}^\vee}, 0, 1)$ is unitary. Together with Theorem A.9, this gives the following corollary (cf. Conjecture A.7).

Corollary A.12. *If π is an irreducible spherical G-representation with Satake parameter $s^\vee_{\mathcal{O}^\vee} \in G^\vee_{\mathbb{C}}$, then π is unitary.*

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Appendix B. Supersymmetry and instantons

The constraints of maximal supersymmetry are efficiently described by starting with the superalgebra generated by the 32-component Majorana spinor supercharge, $Q_\alpha = \int J^0_\alpha d^{10}x$, where J^I_α is the supercurrent (with spinor index $\alpha, \beta = 1, \ldots, 32$ and vector index $I = 0, 1, \ldots, 10$). This satisfies the anti-commutation relations,

$$\{Q_\alpha \, , Q_\beta\} = P_{I_1} \left(\Gamma^0 \Gamma^{I_1}\right)_{\alpha\beta} + Z_{\alpha\beta} \tag{B.1}$$

where the central charge is

$$Z_{\alpha\beta} = Z_{I_1 I_2} \, \left(\Gamma^0 \Gamma^{I_1 I_2}\right)_{\alpha\beta} + Z_{I_1 \cdots I_5} \, \left(\Gamma^0 \Gamma^{I_1 \cdots I_5}\right)_{\alpha\beta} \, , \tag{B.2}$$

where $\Gamma^I_{\alpha\beta}$ are $SO(1,10)$ Dirac matrices[19] and P_I is the eleven-dimensional translation operator.

B.1. **BPS particle states.** Positivity of the anticommutator in (B.1) leads to the Bogomol'nyi bound that restricts the masses of states to be larger than or equal to the central charge. States saturating the bound are BPS states that form supermultiplets, the lengths of which depend on the fraction of supersymmetry broken by their presence. The shortest multiplets are $\frac{1}{2}$-BPS, with longer multiplets for smaller fractions.

The presence of the 2-form component of the central charge indicates that the theory contains a membrane-like state (the $M2$-brane) carrying a conserved charge $Q^{(2)}$, while the 5-form component indicates the presence of a 5-brane state (the $M5$-brane) carrying a charge $Q^{(5)}$. The 2-form and 5–form in (B.1) are given by integration of the spatial directions of the $M2$ and $M5$ branes over 2-cycles $A_{I_1 I_2}$ or 5-cycles $A_{I_1 \cdots I_5}$,

$$Z_{I_1 I_2} = Q^{(2)} \int_{A_{I_1 I_2}} d^2 X \, , \quad Z_{I_1 \cdots I_5} = Q^{(5)} \int_{A_{I_1 \cdots I_5}} d^5 X \, . \tag{B.3}$$

The $M2$ and $M5$-branes are $\frac{1}{2}$-BPS states that preserve 16 of the 32 components of supersymmetry. The 2-form charge couples to a 3-form potential ($C^{(3)}_{I_1 I_2 I_3}$), with field strength $H^{(4)} = dC^{(3)}$. This is analogous to the manner in which the Maxwell 1-form potential couples to a point-like electric charge (a 0-brane), and $H^{(4)}$ is the analogue of the Maxwell field. Poincare duality gives a 7-form field strength defined by $*H^{(4)} := H^{(7)}$, which is solved locally as $H^{(7)} = dC^{(6)} + C^{(3)} \wedge dC^{(3)}$ and defines the six-form potential,

[19] $\Gamma^{I_1 \cdots I_r}_{\alpha\beta}$ is the antisymmetrized product of r Gamma matrices normalised so that $\Gamma^{1 \cdots r} = \Gamma^1 \cdots \Gamma^r$.

$C^{(6)}$, that couples to the five-brane. In other words, the $M5$-brane couples to the magnetic charge that is dual to the electric charge carried by the $M2$-brane. The BPS condition implies that the charge on the brane is equal to its tension, $T^{(r)}$,

$$Q^{(r)} = T^{(r)} \,. \tag{B.4}$$

The integrals in (B.3) are well-defined when all the spatial directions of the branes are wound around the compact cycles of the M-theory torus, $\mathcal{T}^{d+1}$, in which case the state is point-like from the point of view of the $D = 10-d$ non-compact dimensions (so there are finite-mass point-like states due to wrapped $M2$-branes when $d \geq 1$ as well as wrapped $M5$-branes when $d \geq 4$).[20] Other kinds of $\frac{1}{2}$-BPS states also arise in the toroidal background, such as point-like Kaluza–Klein (KK) charges, which are modes of the metric that contribute for any $d \geq 0$. The magnetic dual of a KK state is a KKM, which is described by a Taub-NUT geometry in four spatial dimensions, leaving six more spatial dimensions that are interpreted as the directions on a six-brane. This has a finite mass when wrapped around $\mathcal{T}^6$, so it can arise when $d \geq 5$.

The complete spectrum of BPS states in an arbitrary toroidal compactification of type IIA or IIB string theory can be deduced by considering the toroidal compactification of the M-theory algebra (B.1) with appropriate rescalings of the moduli [47]. Combining completely wrapped branes in various combinations leads to point-like $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS states that are of importance in discussing the spectrum of black holes in string theory [21,22]. This spectrum is of significance in classifying the orbits of instantons that decompactify to black hole states in one higher dimension associated with the parabolic subgroup $P_{\alpha_{d+1}}$ (where we will follow the discussion in [19,20]).

Appendix C. Orbits of BPS instantons in the decompactification limit

In this limit a finite action instanton in $D = 10-d$ dimensions corresponds to an embedded euclidean world-volume that can be one of three types: (a) It has an action that does not depend on r_d as $r_d \to \infty$ and so is also an instanton of the $(D+1)$-dimensional theory – this contributes only to the constant term in this parabolic and does not appear in non-zero Fourier modes; (b) It is a euclidean world-line of a $(D+1)$-dimensional point-like BPS black hole with mass M_{BH}, which gives a term suppressed by a factor of $e^{-2\pi\, r_d\, M_{BH}}$ in the amplitude in the limit $r_d/\ell_{D+1} \to \infty$; (c) It has an action that grows faster than r_d/ℓ_{D+1} so it does not decompactify to give either a particle state or an instanton in $D+1$ dimensions.

[20] There is a huge literature of far more elaborate windings of such branes around supersymmetric cycles in curved manifolds, in which case a fraction of the supersymmetry may or may not be preserved.

In order to illustrate this pattern the following list summarises the spectrum of particle states and instantons in each dimension in the range $3 \leq D \leq 10$ (i.e. $0 \leq d \leq 7$).[21]

C.1. $D = 10$.
In this case there are no $\frac{1}{4}$-BPS states. In the type IIA theory the $\frac{1}{2}$-BPS particle states consist of threshold bound states of $D0$-branes. There are no instantons in the IIA theory and there is no symmetry group.

The type IIB theory has no BPS particle states but has the $\frac{1}{2}$-BPS D-instanton, multiples of which can contribute to the amplitude. The duality group of the IIB theory is $SL(2,\mathbb{Z})$ and there is only one orbit,

$$\mathcal{O}_{\mathbf{2}} = \frac{SL(2,\mathbb{R})}{\mathbb{R}}\,. \tag{C.1}$$

The bold face subscript, in this example and in the following, gives the dimensions of the coset, $\dim(G/H) = \dim(G) - \dim(H)$.

C.2. $D = 9$.
This is obtained by considering M-theory on a 2-torus $\mathcal{T}^2$, where the discrete duality group $SL(2,\mathbb{Z})$ is identified with the group of large diffeomorphisms of $\mathcal{T}^2$.

- The BPS particle states consist of the $M2$-brane wrapping $\mathcal{T}^2$, and two Kaluza–Klein modes arising from the two cycles of the torus, giving a total of 3 BPS states. Their charges are parametrized by a scalar v and a $SL(2)$ vector v_a. The charges of the $\frac{1}{2}$-BPS states are given by the condition [19] $v\,v_a = 0$ and the $\frac{1}{4}$-BPS states by $v\,v_a \neq 0$.
- There is a single BPS instanton that can be identified with the wrapping of the euclidean world-line of a Kaluza–Klein state formed on one cycle around the second cycle of the torus – in this sense a euclidean Kaluza–Klein state wraps a 2-cycle on a torus.

C.3. $D = 8$.
M-theory on a 3-torus $\mathcal{T}^3$ (duality group $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$).

- There are 3 BPS states from the $M2$-brane wrapping 2-cycles, and 3 BPS states from the Kaluza–Klein states associated with 1-cycles, giving a total of 6 BPS states.
- There is 1 BPS instanton from the world-volume of the $M2$-brane wrapping the whole of $\mathcal{T}^3$, and 3 BPS instantons from the Kaluza–Klein states wrapping 2-cycles, giving a total of 4 BPS instantons.

The 6 BPS states are parametrized by $v_{i\,a}$ transforming in the $\mathbf{3} \times \mathbf{2}$ of $SL(3) \times SL(2)$. The $\frac{1}{2}$-BPS states are given by the condition [19] $\epsilon^{ab}\, v_{i\,a} v_{j\,b} =$

[21] In addition to the massive black hole states listed there is, in each case, the standard charge-zero $\frac{1}{2}$-BPS massless supergraviton.

0 and the $\frac{1}{4}$-BPS states by $\epsilon^{ab}\, v_{i\,a} v_{j\,b} \neq 0$. This determines two BPS orbits given by [23]

$$\tfrac{1}{2} - BPS \qquad : \qquad \mathcal{O}_{\mathbf{5}} = \frac{SL(3,\mathbb{R}) \times SL(2,\mathbb{R})}{(\mathbb{R}^+ \times SL(2,\mathbb{R})) \ltimes \mathbb{R}^3}\,, \tag{C.2}$$

$$\tfrac{1}{4} - BPS \qquad : \qquad \mathcal{O}_{\mathbf{6}} = \frac{SL(3,\mathbb{R}) \times SL(2,\mathbb{R})}{SL(2,\mathbb{R}) \ltimes \mathbb{R}^2}\,. \tag{C.3}$$

C.4. $D = 7$**.**
M-theory on a 4-torus $\mathcal{T}^4$ (duality group $SL(5,\mathbb{Z})$).

- There are 6 BPS states from the $M2$-brane wrapping 2-cycles, and 4 BPS states from the Kaluza–Klein states associated with 1-cycles, giving a total of 10 BPS states.
- There are 4 BPS instantons from the $M2$-brane wrapping 3-cycles and 6 BPS instantons from the Kaluza–Klein states wrapping 2-cycles, giving a total of 10 BPS instantons.

The 10 BPS states are parametrized by the rank-2 antisymmetric representation $v_{[ij]}$ $(i,j = 1,\ldots,5)$ in the $\mathbf{10}$ of $SL(5)$. The $\frac{1}{2}$-BPS states are given by the condition [19] $\epsilon^{ijklm}\, v_{ij}\, v_{kl} = 0$ and the $\frac{1}{4}$-BPS by $\epsilon^{ijklm}\, v_{ij}\, v_{kl} \neq 0$. This determines two BPS orbits given by [23]

$$\tfrac{1}{2} - BPS \qquad : \qquad \mathcal{O}_{\mathbf{7}} = \frac{SL(5,\mathbb{R})}{(SL(3,\mathbb{R}) \times SL(2,\mathbb{R})) \ltimes \mathbb{R}^6}\,, \tag{C.4}$$

$$\tfrac{1}{4} - BPS \qquad : \qquad \mathcal{O}_{\mathbf{10}} = \frac{SL(5,\mathbb{R})}{O(2,3) \ltimes \mathbb{R}^4}\,. \tag{C.5}$$

C.5. $D = 6$**.**
M-theory on a 5-torus $\mathcal{T}^5$ (duality group $SO(5,5,\mathbb{Z})$).

- There are 10 BPS states from the $M2$-brane wrapping 2-cycles and 5 BPS states from the Kaluza–Klein states associated with 1-cycles. There is an additional BPS state due to the $M5$-brane wrapping the whole of $\mathcal{T}^5$, giving a total of 16 BPS states.
- There are 10 BPS instantons from the $M2$-brane world-volume wrapping 3-cycles and 10 BPS instantons from euclidean Kaluza–Klein states wrapping 2-cycles, giving a total of 20 BPS instantons.

The 16 BPS states are in a chiral spinor representation S^α $(\alpha = 1,\ldots,16)$ of $SO(5,5)$. Such a spinor satisfies the identity $(S\Gamma^m S)(S\Gamma^m S) = 0$, where Γ^m $(m = 1,\ldots,10)$ are Dirac matrices with suppressed spinor indices. The configurations are $\frac{1}{2}$-BPS if S satisfies the pure spinor condition, $S\Gamma^m S = 0$ [19]. A standard way to analyse this condition is to decompose S into $U(5)$ representations, $\mathbf{16} = \mathbf{1}_5 \oplus \bar{\mathbf{5}}_{-3} \oplus -\mathbf{10}_1$ (where the subscripts denote the $U(1)$ charges), so it has components

$$S = (s, v_a, v^{ab}), \qquad a,b = 1,\ldots,5\,. \tag{C.6}$$

The pure spinor ($\frac{1}{2}$-BPS) condition, $S\Gamma^m S = 0$ is $v_a = \frac{s^{-1}}{5!}\epsilon_{abcde}\, v^{bc}v^{de}$, which implies that the **5** is not independent of the other $U(5)$ representations, so the space of such spinors has dimension **11**. The $\frac{1}{4}$-BPS solution is the unconstrained spinor space (excluding $S\Gamma^m S = 0$) and has dimension **16**. There are two BPS orbits given by [23]

$$\tfrac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{11}} = \frac{SO(5,5,\mathbb{R})}{SL(5,\mathbb{R}) \ltimes \mathbb{R}^{10}}\,, \tag{C.7}$$

$$\tfrac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{16}} = \frac{SO(5,5,\mathbb{R})}{O(3,4) \ltimes \mathbb{R}^{8}}\,. \tag{C.8}$$

C.6. $D = 5$**.**

For M-theory on a 6-torus $\mathcal{T}^6$ (duality group $E_6(\mathbb{Z})$):

- There are 15 BPS states from the $M2$-brane wrapping 2-cycles, 6 BPS Kaluza–Klein states associated with 1-cycles, and 6 BPS states from the $M5$-brane wrapping 5-cycles, giving a total of 27 BPS states.
- There are 20 BPS instantons from the world-volume of the $M2$-brane wrapping 3-cycles, 15 BPS instantons from Kaluza–Klein states wrapping 2-cycles, and 1 BPS instanton from the world-volume of the $M5$-brane wrapping the whole of $\mathcal{T}^6$, giving a total of 36 BPS instantons.

The 27 BPS states are in the fundamental representation, q^i ($i = 1, \ldots, 27$), of E_6 and lead to $\frac{1}{2}$-, $\frac{1}{4}$- or $\frac{1}{8}$-BPS configurations depending on the following conditions on the E_6 cubic invariant $I_3 = \sum_{1\le i,j,k\le 27}(I_3)_{ijk}q^iq^jq^k$ [19]

$$\tfrac{1}{8} - BPS: \quad I_3 \neq 0\,, \tag{C.9}$$

$$\tfrac{1}{4} - BPS: \quad I_3 = 0, \qquad \frac{\partial I_3}{\partial q^i} \neq 0\,, \tag{C.10}$$

$$\tfrac{1}{2} - BPS: \quad I_3 = 0, \qquad \frac{\partial I_3}{\partial q^i} = 0,\ \frac{\partial^2 I_3}{\partial q^i \partial q^j} \neq 0\,. \tag{C.11}$$

Clearly the first of these conditions (the $\frac{1}{8}$-BPS condition) is of dimension **27**. The other conditions may be analysed by decomposing the **27** of E_6 into $SO(5,5)\times U(1)$ irreducible representations, $\mathbf{27} = \mathbf{1}_4 \oplus \mathbf{10}_{-2} \oplus \mathbf{16}_1$. This means that q^i decomposes as

$$q^i = (s, v_m, S^\alpha)\,, \tag{C.12}$$

where s is a scalar, v_m is a $SO(5,5)$ vector of dimension **10** and S^α is a spinor of dimension **16** (and the $U(1)$ charges have been suppressed). The cubic invariant I_3 decomposes as [19], $I_3 = \mathbf{10}_{-2} \otimes \mathbf{10}_{-2} \otimes \mathbf{1}_4 \oplus \mathbf{16}_1 \otimes \mathbf{16}_1 \otimes \mathbf{10}_{-2}$, which implies that

$$I_3 = s\, v\cdot v + (S\Gamma S)\cdot v\,, \tag{C.13}$$

where $v \cdot v$ is the $SO(5,5)$ (norm)^2 of the vector v, and $(S\Gamma S)\cdot v$ is the $SO(5,5)$ scalar product between the vector $S\Gamma^m S$ and v^m.

The $\frac{1}{4}$-BPS solution reduces to the condition

$$s\, v\cdot v + (S\Gamma S)\cdot v = 0\,, \tag{C.14}$$

with non-vanishing derivative with respect to s, v_m and S_a. Therefore the solution is given by the **26** dimensional space

$$(q^i)_{\frac{1}{4}-BPS} = (-(v\cdot v)^{-1}\,(S\Gamma S)\cdot v, v_m, S^\alpha)\,. \tag{C.15}$$

The $\frac{1}{2}$-BPS condition implies the following conditions

$$\begin{aligned} v\cdot v &= 0\,, && \text{(C.16)}\\ (S\Gamma^m S) + s\, v^m &= 0\,, && \text{(C.17)}\\ (S\Gamma^m)_a\, v^m &= 0\,, && \text{(C.18)} \end{aligned}$$

which are solved by $v^m = S\Gamma^m S$ (using the relation $(S\Gamma^m S)(S\Gamma^m S) = 0$). The $\frac{1}{2}$-BPS solution is therefore given by the **17**-dimensional solution

$$(q^i)_{\frac{1}{2}-BPS} = (s, S\Gamma^m S, S_a)\,. \tag{C.19}$$

To summarise, the BPS orbits in $D = 5$ are given by [23]

$$\frac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{17}} = \frac{E_6}{SO(5,5)\ltimes \mathbb{R}^{16}}\,, \tag{C.20}$$

$$\frac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{26}} = \frac{E_6}{O(4,5)\ltimes \mathbb{R}^{16}}\,, \tag{C.21}$$

$$\frac{1}{8} - BPS \quad : \quad \mathcal{O}_{\mathbf{27}} = \mathbb{R}^* \times \frac{E_6}{F_{4(4)}}\,. \tag{C.22}$$

The charges in the $\frac{1}{4}$-BPS orbit can be generated by applying $E_6(\mathbb{Z})$ transformations to a 2-charge state corresponding to a null vector in the 27 dimensional BPS state space. The charges in the $\frac{1}{8}$-BPS orbit can be generated from a 3-charge state corresponding to space-like or time-like vectors with $I_3 \neq 0$ in the 27 dimensional BPS state space (note that, unlike [20] we have included the scale factor $\mathbb{R}^*$ in the definition of the orbit which is of dimension **27**).

C.7. $D = 4$.
M-theory on a 7-torus $\mathcal{T}^7$ (duality group $E_7(\mathbb{Z})$).

- There are 21 BPS states from the $M2$-brane wrapping 2-cycles, 7 BPS states from the Kaluza–Klein states wrapping 1-cycles, 7 BPS states from the KKM's wrapping 6-cycles, and 21 BPS states from the $M5$-brane wrapping 5-cycles. This gives a total of 56 BPS states
- There are 35 BPS instantons from the $M2$-brane wrapping 3-cycles, 21 BPS instantons from the Kaluza–Klein states wrapping 2-cycles, and 7 BPS instantons from the $M5$-brane wrapping 6-cycles. This gives a total of 63 BPS instantons.

The 56 BPS states are in the fundamental representation, q^i $(i = 1, \ldots, 56)$, of E_7. The $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS configurations are classified by the following conditions on the quartic symmetric polynomial invariant I_4 [19, 48]

$$\tfrac{1}{8} - BPS: \quad I_4 \quad > 0\,, \tag{C.23}$$

$$\tfrac{1}{8} - BPS: \quad I_4 \quad = 0, \quad \frac{\partial I_4}{\partial q^i} \neq 0\,, \tag{C.24}$$

$$\tfrac{1}{4} - BPS: \quad I_4 \quad = 0, \quad \frac{\partial I_4}{\partial q^i} = 0, \quad \left.\frac{\partial^2 I_4}{\partial q^i \partial q^j}\right|_{Adj_{E_7}} \neq 0\,, \tag{C.25}$$

$$\tfrac{1}{2} - BPS: \quad I_4 \quad = 0, \quad \left.\frac{\partial^2 I_4}{\partial q^i \partial q^j}\right|_{Adj_{E_7}} = 0, \quad \frac{\partial^3 I_4}{\partial q^i \partial q^j \partial q^k} \neq 0\,. \tag{C.26}$$

The following is a summary of the BPS orbits [19, 20, 23]

$$\tfrac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{28}} = \frac{E_7}{E_6 \ltimes \mathbb{R}^{27}}\,, \tag{C.27}$$

$$\tfrac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{45}} = \frac{E_7}{(O(5,6) \ltimes \mathbb{R}^{32}) \times \mathbb{R}}\,, \tag{C.28}$$

$$\tfrac{1}{8} - BPS \quad : \quad \mathcal{O}_{\mathbf{55}} = \frac{E_7}{F_{4(4)} \ltimes \mathbb{R}^{26}}\,, \tag{C.29}$$

$$\tfrac{1}{8} - BPS \quad : \quad \mathcal{O}_{\mathbf{56}} = \mathbb{R}^* \times \frac{E_7}{E_{6(2)}}\,. \tag{C.30}$$

The $\frac{1}{2}$-BPS orbit can be obtained by acting on a single charge, the $\frac{1}{4}$-BPS orbit can be obtained by acting on a 2-charge system, the first $\frac{1}{8}$-BPS (with dimension 55) has zero entropy and can be obtained by acting on a 3-charge system the last orbit of dimension **56** is the $\frac{1}{8}$-BPS orbit with $I_4 > 0$ that has entropy $S = \pi\sqrt{I_4}/G_N$ can be obtained by acting on a 4-charge system in the **56** representation of E_7 as detailed in [23]. We have included the overall scale factor in the definition of the orbit. Another orbit of dimension **56** is $\mathbb{R}^- \times E_7/E_6$ that has $I_4 < 0$ and does not correspond to a BPS solution at all [19, 20]. All these charge orbits can be understood in terms of the superpositions of branes at angles and constructed from combinations of $(D0, D2, D4, D6)$ [49].

C.8. $D = 3$.
M-theory on an 8-torus $\mathcal{T}^8$ (duality group $E_8(\mathbb{Z})$).

- There are 28 BPS states from the $M2$-brane wrapping 2-cycles, 8 BPS states from the Kaluza–Klein states wrapping 1-cycles, 28 BPS states from the KKMs wrapping 6-cycles, and 56 BPS states from the $M5$-brane wrapping 5-cycles. This gives a total of 120 BPS states.
- There are 56 BPS instantons from the $M2$-brane wrapping 3-cycles, 28 BPS instantons from the Kaluza–Klein states wrapping 2-cycles, 8 BPS instantons from the KKM wrapping 7-cycles, and 28 BPS

instantons from $M5$-branes wrapping 6-cycles. This gives a total of 120 BPS instantons.[22]

Appendix D. Euclidean Dp-brane instantons.

We here sketch the background to the analysis of the euclidean Dp-brane instanton configurations that contribute in the perturbative limit of string theory discussed in section 3.4, based on an analysis of supersymmetry conditions on the embeddings of world-sheets on the string theory torus T^d. Contributions from wrapped NS5-brane world-sheets also arise for $d = 6, 7$ and KK monopoles for $d = 7$.

Wrapping a euclidean Dp-brane world-volume of either ten-dimensional type II string theory on a $(p+1)$-cycle leads to an instanton in the transverse $\mathbb{R}^{1,8-p}$ space-time. This $\frac{1}{2}$-BPS condition preserves a linear combination of the supersymmetries that act on the left-moving and right-moving modes of a closed superstring. This leads to the following constraint on the supersymmetry parameters,

$$\tilde{\varepsilon} = \prod_{i=1}^{p+1} \Gamma^i \varepsilon \tag{D.1}$$

where ε and $\tilde{\varepsilon}$ are chiral sixteen-component $SO(1,9)$ spinors parameterizing the left- and right-moving super symmetries and Γ^i are the usual $SO(1,9)$ Gamma matrices that satisfy the Clifford algebra $\{\Gamma^i, \Gamma^j\} = -2\eta^{ij}$, where η is the Minkowski metric with signature $(- + \cdots +)$.

When compactifying on a d-torus space-time becomes $\mathbb{R}^{1,9-d} \times \mathrm{T}^d$ and a $SO(1,9)$ spinor decomposes into a sum of bispinors, $\varepsilon = \hat{\varepsilon} \otimes \eta$, where $\hat{\varepsilon}$ is a $SO(1, 9-d)$ spinor and η is a $SO(d)$ spinor. The condition (D.1) becomes a condition relating η and $\tilde{\eta}$. T-duality transforms the Γ matrices in (D.1) by the action of the spin group $SO(d,d)$, $R^{-1} \prod_i \Gamma^i R$. This, in general, transforms a wrapped Dp-brane into a Dq-brane so that the supersymmetry conditions

$$\tilde{\eta} = \prod_{i=1}^{q+1} \Gamma^i \eta = \prod_{i=1}^{p+1} \Gamma^i \eta\,, \tag{D.2}$$

are satisfied. As remarked in [50], this this means the two spinors $\prod_{i=1}^{q+1} \Gamma^i \varepsilon$ and $\prod_{i=1}^{p+1} \Gamma^i \varepsilon$ must be in the same $SO(d,d)$ orbit.

A euclidean Dp-brane can be wrapped over cycles of a d-torus of dimension $0 \leq p+1 \leq d$ with $p = 0 \mod 2$ for type IIA superstring theory and $p = -1 \mod 2$ for type IIB. These instanton configurations fill out a chiral spinor representation, S_A, of dimension $\sum_{p \equiv s \pmod 2} \binom{d}{p+1} = 2^{d-1}$ with $s = 0$ or 1 of the T-duality group $SO(d,d)$. The BPS condition on Dp-branes wrapping a torus in (D.2) can be interpreted as a condition on

[22] One of the KKM instantons wraps the euclidean time dimension and gives a vanishing contribution upon decompactification to $D = 4$ dimensions, as discussed following (3.2).

the spinor S_A. The various brane configurations are then classified by orbits of S_A under the action of the T-duality group $SO(d,d)$ (actually the Spin group). In this manner the spinor parametrizes the commuting set of instanton charges in the perturbative regime.

For $d=6$ or $d=7$ there are also contributions from NS5-branes wrapping six-cycles. Such NS5-brane configurations give contributions to the instanton charges that do not commute with those of the wrapped Dp-branes. In other words, the Dp-brane charges in the spinor representation parametrize the $\mathfrak{u}_{-1}$ component part of the unipotent radical U (the abelian part) for the standard parabolic subgroup P_{α_1} of E_{d+1} and the NS5-brane charge are in the derived subgroup $[U,U]$ component part of the unipotent radical for the standard parabolic subgroup P_{α_1} of E_{d+1} in table 3 on page 19. For $d=6$ this provides one extra charge configuration since there is a unique six-cycle. For $d=7$ there are 7 distinct six-cycles so there are 7 NS5-brane charges. In addition there are 7 stringy KKM instantons. Recall that these arise from Kaluza–Klein monopoles in ten-dimensional string theory in which the fibre direction $x^{\#}$ is identified with a circle in T^7 (whereas the $D6$-brane is seen in M-theory as a KKM formed by identifying $x^{\#}$ with the M-theory circle).

Although it is very complicated to describe how all possible compactifications of euclidean Dp-branes fit into different spinor orbits, the following discussion will indicate the procedure. For this purpose it is convenient to start in ten dimensions by defining chiral spinors of the complexified group, $SO(10;\mathbb{C})$ (complexification does not affect the BPS classification), by means of the raising and lowering operators,

$$b_{k+1} = \frac{1}{2}(\Gamma^{2k+1} - i\Gamma^{2k})\,, \qquad b^{\dagger}_{k+1} = -\frac{1}{2}(\Gamma^{2k+1} + i\Gamma^{2k})\,, \quad 0 \le k \le 4\,, \tag{D.3}$$

so that $b^k = (b^{\dagger}_k)$ and $\{b_k, b^l\} = \delta^l_k$, and $\{b_k, b_l\} = \{b^k, b^l\} = 0$. A ground state $|----\rangle$ is defined so that $b_k|----\rangle = 0$, for $1 \le k \le 5$. Acting with b^1 gives the state $b^1|-----\rangle = |+----\rangle$, with analogous states created by any linear combination of the b^r's, giving a total of 2^5 states with $+$ or $-$ labelling each of the 5 positions. These states are graded according to whether there an even or odd number of $+$ signs. There are therefore two chiral spinor representations of $SO(10;\mathbb{C})$ of dimension **16**. Upon compactification on T^d the spinor η in (D.2) is represented as a state of the Fock space built by acting with b^i on the ground state $|-^5\rangle$. It is convenient to introduce the notation $e_{i_1\cdots i_r} := b^{i_1}\cdots b^{i_r}|-^{d/2}\rangle$ and $e^*_{i_1\cdots i_r} := b_{i_1}\cdots b_{i_r}|+^{d/2}\rangle$, which was used in section 3.4.1..

Spinors that are related by an $SO(d,d)$ transformation $\exp(\sum_{i,j} x_{ij}\gamma^{ij})$ are associated with D-brane configurations that are equivalent under T-duality. Each orbit listed in section 3.4.1, is characterized by a representative S^0. Therefore a $SO(d,d)$ pure spinor is equivalent to the ground state of the Fock space that we can denote by 1, corresponding to a pure spinor defining

a D-brane wrapping a supersymmetric cycle. The notation $e_{i_1\cdots i_r}$ and corresponds to a D-brane configuration wrapping the directions $\{i_1,\cdots,i_r\}$ in T^d and $e^*_{i_1\cdots i_r}$ a D-brane configuration wrapping the complementary directions to $\{i_1,\cdots,i_r\}$ in T^d.

Upon compactifying on a torus of dimension $d\leq 3$, all possible brane world-volumes are parallel, up to identification under $SO(d,d;\mathbb{Z})$, and the condition (D.1) ensures in this case that all instanton configurations are $\frac{1}{2}$-BPS. These are $p=0$ and $p=2$ wrappings in type IIA, and $p=-1$ and $p=1$ in type IIB.

The theory compactified on a 4-torus T^4 in type IIA (for instance), includes instantons due to wrapping $D0$-brane world-lines on any of the four 1-cycles and $D2$-brane world-volumes on any of the four 3-cycles. These configurations in general fill out an eight-dimension chiral spinor representation of $SO(4,4)$, $S_A=\sum_{i=a}^4 v_a b^a+\sum_{a,b,c=1}^4 v_{abc}b^{abc}/3!$. This parametrization makes explicit the action of $SL(4)$ on v_a or $u^a=\epsilon^{abcd}v_{abc}$ (or $SU(4)$ in the complexified case).

With a single $D0$-brane or a single $D2$-brane world-volume wrapped on T^4 the condition (D.1) is always satisfied, and the configuration is $\frac{1}{2}$-BPS. However, wrapping both a $D0$-brane world-line and a $D2$-brane world-volume results in further breaking of supersymmetry unless v_a and u^a satisfy condition (D.2). It is easily seen that this condition is satisfied for all $\eta=|\pm\pm\rangle$ if $v\cdot u=0$. But if $u\cdot v\neq 0$ only $\eta=|+\pm\rangle$ satisfy the solution which is $\frac{1}{4}$-BPS. These two conditions are invariant under the action of the T-duality group $SO(4,4)$ acting on a spinor S_A. The $\frac{1}{2}$-BPS condition corresponds to imposing the pure spinor constraint $S\cdot S=0$ while the $\frac{1}{4}$-BPS corresponds to the complementary condition, $S\cdot S\neq 0$, which defines the configuration with the $D0$-brane world-line orthogonal to the $D2$-brane world-volume.

Extensions of these arguments lead to a classification of all BPS configurations of euclidean Dp-brane world-volumes that are completely wrapped on a torus. The orbits of such configurations are obtained by imposing generalisations of the pure spinor constraint on the $SO(d,d)$ spinor that parametrizes the orbits. Orbits which preserve a smaller fractions of supersymmetry are larger and are associated with spinors satisfying weaker constraints. The resulting orbits are described in section 3.4.1.

Appendix E. Details of modes in rank 3 and rank 4 cases

This appendix presents details of the modes of Eisenstein series that enter in the expressions for the coefficients, $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in dimensions $D=8$ and 7, with symmetry groups $SL(3)\times SL(2)$ and $SL(5)$, that are used in sections 4.4 and 4.5 in the text. This summarises and extends the string theory results in [2] (see [40,41,51–53] for related investigations). The $D=6$ case, with symmetry group $SO(5,5)$, is discussed in section 4.6.

E.1. $\mathcal{E}^{(8)}_{(0,0)}$ and $\mathcal{E}^{(8)}_{(1,0)}$: $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$.

The coefficients are functions of the $SL(2)/SO(2)$ symmetric space which depends on $\mathcal{U}=\mathcal{U}_1+i\mathcal{U}_2$, the complex structure of the 2-torus, T^2, while the $SL(3)/SO(3)$ space depends on 5 parameters. We will parametrise the $SL(2)/SO(2)$ coset by (4.12) (with Ω replaced by $\mathcal{U}$) while the $SL(3)/SO(3)$ coset will be parameterised by the string fluxes as

$$e_3=\begin{pmatrix}1 & B_{\rm NS} & C^{(2)}+\Omega_1 B_{\rm NS}\\ 0 & 1 & \Omega_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\nu_2^{-\frac{1}{3}} & 0 & 0\\ 0 & \nu_2^{\frac{1}{6}}\sqrt{\Omega_2} & 0\\ 0 & 0 & \frac{\nu_2^{\frac{1}{6}}}{\sqrt{\Omega_2}}\end{pmatrix}, \tag{E.1}$$

where $\nu_2^{-\frac{1}{2}}=r_1r_2/\ell_{10}^2=\sqrt{\Omega_2}T_2$ is the volume of the 2-torus in 10 dimensional Planck units and $T_2=r_1r_2/\ell_s^2$ is the volume in string units. The five parameters of the coset are packaged into $(\Omega,T,C^{(2)})$, where $\Omega=\Omega_1+i\Omega_2$ and $T=T_1+iT_2$ (where $T_1=B_{\rm NS}$). We shall also make use of the combination $y_8^{-1}=\Omega_2^2T_2$, which is the square of the inverse string coupling. The complex parameters T is interpreted as the Kähler structure of T^2.

The coefficient functions $\mathcal{E}^{(8)}_{(0,0)}$ and $\mathcal{E}^{(8)}_{(1,0)}$ are solutions of (2.6) and (2.7) with $D=8$ [2,36],

$$\Delta^{(8)}\,\mathcal{E}^{(8)}_{(0,0)} = 6\pi \tag{E.2}$$

$$(\Delta^{(8)}-\frac{10}{3})\,\mathcal{E}^{(8)}_{(1,0)} = 0\,, \tag{E.3}$$

where the $SL(3)\times SL(2)$ Laplace operator is defined in terms of the parameters introduced above by

$$\Delta^{(8)}:=\Delta^{SL(3)}+2\Delta_U^{SL(2)}\,, \tag{E.4}$$

with

$$\Delta^{SL(3)} = \Delta_\Omega+\frac{|\partial_{B_{\rm NS}}-\Omega\partial_{C^{(2)}}|^2}{\nu_2\Omega_2}+3\partial_{\nu_2}(\nu_2^2\partial_{\nu_2}) \tag{E.5}$$

$$\Delta_{\mathcal{U}}^{SL(2)} = \mathcal{U}_2^2\,(\partial^2_{\mathcal{U}_1}+\partial^2_{\mathcal{U}_2})\,. \tag{E.6}$$

The fact that the eigenvalue in (E.2) vanishes, together with the presence of the 6π on the right-hand side is related to the presence of a 1-loop ultraviolet divergence in eight-dimensional maximally supersymmetric supergravity [4].

The solutions to these equations are given in terms of $SL(2)$ and $SL(3)$ Eisenstein series. The $SL(2)$ series is given by (4.13) while the $SL(3)$ Eisenstein (Epstein) series is given by

$$2\zeta(2s)\,E^{SL(3)}_{\alpha_1;s}(g_3)=\sum_{M_3\in\mathbb{Z}^3\setminus\{0\}}(m^2_{SL(3)})^{-s}\,, \tag{E.7}$$

where, setting $M_3 = (m_1\, m_2\, m_3) \in \mathbb{Z}^3$, the mass squared is given by

$$\begin{aligned} m^2_{SL(3)} &:= M_3 \cdot g_3 \cdot M_3^T \\ &= \frac{\nu_2^{\frac{1}{3}}}{\Omega_2} \left(|m_1 + m_2\Omega + \mathcal{B} m_3|^2 + (m_3 \Omega_2 T_2)^2 \right) \end{aligned} \tag{E.8}$$

with

$$g_3 := e_3 \cdot e_3^T = \nu_2^{\frac{1}{3}} \begin{pmatrix} \nu_2^{-1} + (g_2)_{ab} B^a B^b & (g_2)_{ab} B^b \\ (g_2)_{ab} B^a & (g_2)_{ab} \end{pmatrix}, \tag{E.9}$$

where

$$g_2 := \frac{1}{\Omega_2} \begin{pmatrix} |\Omega|^2 & \Omega_1 \\ \Omega_1 & 1 \end{pmatrix}; \qquad B := \begin{pmatrix} B_{\rm NS} \\ C^{(2)} \end{pmatrix}, \qquad \mathcal{B} := C^{(2)} + \Omega B_{\rm NS}\,. \tag{E.10}$$

The Eisenstein series $E^{SL(3)}_{\alpha_1;s}$ is related to $E^{SL(3)}_{\alpha_2;s}$ by the following functional relation,

$$\xi(2s)\; E^{SL(3)}_{\alpha_1;s}(g_3) = \xi(3-2s)\, E^{SL(3)}_{\alpha_2;\frac{3}{2}-s}(g_3^{-1})\,. \tag{E.11}$$

The solutions of (E.2) and (E.3) with appropriate boundary conditions are combinations of these Eisenstein series, [2, 35–37]

$$\mathcal{E}^{(8)}_{(0,0)} = \lim_{\epsilon\to 0} \left(2\zeta(3) E^{SL(3)}_{\alpha_1;\frac{3}{2}+\epsilon} + 4\zeta(2-2\epsilon) E^{SL(2)}_{1-\epsilon}(\mathcal{U}) \right) \tag{E.12}$$

$$\mathcal{E}^{(8)}_{(1,0)} = \zeta(5)\, E^{SL(3)}_{\alpha_1;\frac{5}{2}} + \frac{2\pi^4}{135} E^{SL(3)}_{\alpha_1;-\frac{1}{2}}\; E^{SL(2)}_2(\mathcal{U})\,. \tag{E.13}$$

The expression for $\mathcal{E}^{(8)}_{(0,0)}$ is the sum of two series that each have poles in the $\epsilon \to 0$ limit. However, these poles cancel between the two terms [2], leaving the hatted series that are defined by subtracting the pole terms, using

$$2\zeta(2+2\epsilon)\, E^{SL(2)}_{1+\epsilon}(\mathcal{U}) = \frac{\pi}{\epsilon} + \hat{E}^{SL(2)}_1(\mathcal{U}) + 2\pi(\gamma_E - \log(2)) + O(\epsilon)\,, \tag{E.14}$$

and

$$2\zeta(3+2\epsilon)\, E^{SL(3)}_{\alpha_1;\frac{3}{2}+\epsilon} = \frac{2\pi}{\epsilon} + 4\pi(\gamma_E - 1) + \hat{E}^{SL(3)}_{\alpha_1;\frac{3}{2}} + O(\epsilon)\,. \tag{E.15}$$

The Fourier modes of the coefficient functions can now be considered in each of the three parabolic subgroups of interest, after putting the $SL(3,\mathbb{Z})$ part together with the $SL(2,\mathbb{Z})$ part. The unipotent radicals in these three cases are given by:

i) The unipotent radical U_{α_3} in the maximal parabolic $P_{\alpha_3} = GL(1) \times SL(2) \times \mathbb{R}^+ \times U_{\alpha_3}$ associated with the decompactification limit is parametrized by $(C^{(2)}, B_{\rm NS})$ and takes the block diagonal form,

$$U_{\alpha_3} = \begin{pmatrix} \begin{pmatrix} 1 & B_{\rm NS} & C^{(2)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & \mathcal{U}_1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}. \tag{E.16}$$

In this maximal parabolic subgroup the Fourier coefficients of the $SL(3)$ Eisenstein series in (E.7) are defined by [23]

$$F^{SL(3)\alpha_2}_{\alpha_1;s}(kp_1,kp_2) := \int_{[0,1]^2} dB_{\rm NS} dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)}+p_2 B_{\rm NS})}\, E^{SL(3)}_{\alpha_1;s}\,, \quad \text{(E.17)}$$

with $\gcd(p_1,p_2)=1$. Extending the constant term computation in [2] the non vanishing Fourier coefficients are

$$F^{SL(3)\,\alpha_2}_{\alpha_1;s}(kp_1,kp_2) = \frac{1}{\xi(2s)}\,\Omega_2^{1-\frac{2s}{3}}\,T_2^{1-\frac{s}{3}}\,\frac{\sigma_{2s-2}(k)}{|k|^{s-1}}\,\frac{K_{s-1}(2\pi|k|\,|p_2+p_1\Omega|T_2)}{|p_2+p_1\Omega|^{1-s}}\,. \quad \text{(E.18)}$$

The Fourier modes of the $SL(2)$ series are defined as

$$F^{SL(2)}_s(k') := \int_{[0,1]} d\mathcal{U}_1\, e^{-2i\pi k'\mathcal{U}_1}\, E^{SL(2)}_s(\mathcal{U})\,, \quad \text{(E.19)}$$

with $\gcd(p_1,p_2)=1$. The non vanishing Fourier coefficients are

$$F^{SL(2)}_s(k') = \frac{2\sqrt{\mathcal{U}_2}}{\xi(2s)}\frac{\sigma_{2s-1}(|k'|)}{|k'|^{s-\frac12}}\,K_{s-\frac12}(2\pi|k'|\mathcal{U}_2)\,. \quad \text{(E.20)}$$

Finally, the Fourier modes of the product of the $SL(3)$ and the $SL(2)$ series is given by

$$\begin{aligned} F^{SL(3)SL(2)\alpha_3}_{\alpha_1;s,s'}(kp_1,kp_2,k') \;&:=\; \int_{[0,1]^2} dB_{\rm NS} dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)}+p_2 B_{\rm NS})}\, E^{SL(3)}_{\alpha_1;s} \\ &\quad\times \int_{[0,1]} d\mathcal{U}_1\, e^{-2i\pi k'\mathcal{U}_1}\, E^{SL(2)}_{s'}(\mathcal{U})\,, \end{aligned} \quad \text{(E.21)}$$

with $\gcd(p_1,p_2)=1$. The non vanishing Fourier coefficients are

$$F^{SL(3)SL(2)\alpha_3}_{\alpha_1;s,s'}(kp_1,kp_2,k') = \mathcal{F}^{SL(3)\alpha_2}_{\alpha_1;s}(kp_1,kp_2)\mathcal{F}^{SL(2)\alpha_3}_{s'}(k')\,. \quad \text{(E.22)}$$

ii) The unipotent radical U_{α_1} in the maximal parabolic $P_{\alpha_1} = GL(1)\times SO(2,2)\times U_{\alpha_1}$ associated with the string perturbation regime is parametrized by $(\Omega_1, C^{(2)})$ and takes the form,

$$U_{\alpha_1} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & C^{(2)} \\ 0 & 1 & \Omega_1 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}. \quad \text{(E.23)}$$

In this maximal parabolic only the $SL(3)$ series have non-vanishing Fourier coefficients, which are defined by

$$F^{SL(3)\alpha_1}_{\alpha_1;s}(kp_1,kp_2) := \int_{[0,1]^2} d\Omega_1 dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)}+p_2\Omega_1)}\, E^{SL(3)}_{\alpha_1;s} \quad \text{(E.24)}$$

[23] The labelling of the simple roots α_1 and α_2 on these Fourier coefficients uses the conventional labelling of the $SL(3)$ Dynkin diagram.

with $\gcd(p_1,p_2)=1$. Extending the constant term calculation in [2] leads to

$$F^{SL(3)\,\alpha_1}_{\alpha_1;s}(kp_1,kp_2)=\frac{1}{\xi(2s)}\,T_2^{\frac{2s}{3}}\,\Omega_2^{\frac{1}{2}+\frac{s}{3}}\,\frac{\sigma_{2s-1}(k)}{|k|^{s-\frac{1}{2}}}\,\frac{K_{s-\frac{1}{2}}(2\pi|k|\,|p_1T+p_2|\Omega_2)}{|p_1T+p_2|^{s-\frac{1}{2}}}\,. \tag{E.25}$$

iii) The unipotent radical U_{α_2} in parabolic $P_{\alpha_2}=GL(1)\times SL(3)\times U_{\alpha_2}$ associated with the semi-classical M-theory limit is parametrized by $\mathcal{U}_1$ and takes the form

$$U_{\alpha_2}=\begin{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & 0\\ 0 & \begin{pmatrix}1&\mathcal{U}_1\\0&1\end{pmatrix}\end{pmatrix}, \tag{E.26}$$

In this maximal parabolic subgroup only the $SL(2)$ series has non-vanishing Fourier coefficients defined as

$$F^{SL(2)}_s(k'):=\int_{[0,1]}d\mathcal{U}_1\,e^{-2i\pi k'\mathcal{U}_1}\,E^{SL(2)}_s(\mathcal{U})\,, \tag{E.27}$$

which equals

$$F^{SL(2)}_s(k')=\frac{2\sqrt{\mathcal{U}_2}}{\xi(2s)}\frac{\sigma_{2s-1}(|k'|)}{|k'|^{s-\frac{1}{2}}}\,K_{s-\frac{1}{2}}(2\pi|k'|\Omega_2)\,. \tag{E.28}$$

The evaluation of the non-zero Fourier coefficients of $\mathcal{E}^{(8)}_{(0,0)}$ and $\mathcal{E}^{(8)}_{(1,0)}$ is straightforwardly obtained by using the above expressions and is discussed in section 4.4.

E.2. $\mathcal{E}^{(7)}_{(0,0)}$ and $\mathcal{E}^{(7)}_{(1,0)}$: $SL(5,\mathbb{Z})$.

In $D=7$ dimensions the coefficient functions are automorphic under the action of the duality group $SL(5,\mathbb{Z})$ and are functions on the 14-dimensional coset space $SL(5)/SO(5)$, which is parametrized, using the notation that arises from string theory, by

$$e_5=\begin{pmatrix} & B^1_{\rm NS} & C^{(2)\,1}+\Omega_1B^1_{\rm NS}\\ N_3 & B^2_{\rm NS} & C^{(2)\,2}+\Omega_1B^2_{\rm NS}\\ & B^3_{\rm NS} & C^{(2)\,3}+\Omega_1B^3_{\rm NS}\\ 0 & 1 & \Omega_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} & & & 0 & 0\\ & \frac{\Omega_2^{\frac{1}{3}}}{\nu_3^{\frac{1}{5}}}\,D_3 & & 0 & 0\\ & & & 0 & 0\\ 0 & 0 & 0 & \nu_3^{\frac{1}{5}}\sqrt{\Omega_2} & 0\\ 0 & 0 & 0 & 0 & \frac{\nu_3^{\frac{1}{5}}}{\sqrt{\Omega_2}}\end{pmatrix}, \tag{E.29}$$

where Ω_2 is the inverse string coupling constant, Ω_1 is the type IIB RR pseudoscalar, and $B^i_{\rm NS}$ and $C^{(2)\,i}$ $(i=1,\dots,3)$ the NS and RR charges. The quantity N_3 is a rank 3 upper triangular matrix and D_3 a rank 3

diagonal matrix. These are defined so that $\tilde{e}_3 = N_3 D_3$ parametrizes the coset $SL(3)/SO(3)$. We will make use of the following combinations,

$$\nu_3^{-1} = \left(\frac{r_1 r_2 r_3}{\ell_{10}^3}\right)^2 = \Omega_2^{\frac{3}{2}} \left(\frac{r_1 r_2 r_3}{\ell_s^3}\right)^2 , \qquad y_7^{-1} = \Omega_2^2 \, \frac{r_1 r_2 r_3}{\ell_s^3} , \tag{E.30}$$

where r_1, r_2 and r_3 are the radii of T^3 and y_7 is the 7-dimensional string coupling. Note that ν_3 is invariant under the action of $SL(2) \times SL(3)$.

The coset space $SL(5)/SO(5)$ is parametrized by the metric $g_5 = e_5 \cdot e_5^T$

$$g_5 = \nu_3^{\frac{2}{5}} \begin{pmatrix} \nu_3^{-\frac{2}{3}} \, (g_3)_{ij} + (g_2)_{ab} B_i^a B_j^b & (g_2)_{ab} B_j^b \\ (g_2)_{ab} B_j^a & (g_2)_{ab} \end{pmatrix} . \tag{E.31}$$

This parametrisation is adapted to the maximal parabolic subgroup P_{α_4}, which has Levi subgroup $L_{\alpha_4} = GL(1) \times SL(3) \times SL(2)$ where g_3 parametrizes the $SL(3)/SO(3)$ coset and g_2 the $SL(2)/SO(2)$ coset

$$g_2 = \frac{1}{\Omega_2} \begin{pmatrix} |\Omega|^2 & \Omega_1 \\ \Omega_1 & 1 \end{pmatrix} ; \qquad B = \begin{pmatrix} B_{\rm NS}^1 & B_{\rm NS}^2 & B_{\rm NS}^3 \\ C^{(2)\,1} & C^{(2)\,2} & C^{(2)\,3} \end{pmatrix} . \tag{E.32}$$

The $SL(5)$ mass squared is given by the quadratic form

$$\begin{aligned} m_{SL(5)}^2 \;&:=\; M_5 \cdot g_5 \cdot M_5^T \qquad\qquad (\text{E.33}) \\ &=\; \nu_3^{\frac{2}{5}} \, \frac{|m_1 + m_2 \Omega + n^T \cdot (C^{(2)} + \Omega B_{\rm NS})|^2}{\Omega_2} + \frac{n^T \cdot g_3 \cdot n}{\nu_3^{\frac{4}{15}}} , \end{aligned}$$

where $M_5 = (n_1, n_2, n_3, m_2, m_1) \in \mathbb{Z}^5 \backslash \{0\}$ and we have set $n := (n_1, n_2, n_3)$ and defined $B_{\rm NS}$ and $C^{(2)}$ as the first and second rows of the matrix B. This expression will later be useful for describing the $SL(5)$ Eisenstein series.

The $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS coefficients, $\mathcal{E}^{(7)}_{(0,0)}$ and $\mathcal{E}^{(7)}_{(1,0)}$, that solve (2.6) and (2.7) together with the appropriate boundary conditions are given[24] in [2] by linear combinations of the $E^{SL(5)}_{\alpha_1;s}$ and $E^{SL(5)}_{\alpha_4;s}$ Eisenstein series

$$\mathcal{E}^{(7)}_{(0,0)} \;=\; 2\zeta(3) \, E^{SL(5)}_{\alpha_1;\frac{3}{2}} , \tag{E.34}$$

$$\mathcal{E}^{(7)}_{(1,0)} \;=\lim_{\epsilon \to 0} \left(\zeta(5+2\epsilon) \, E^{SL(5)}_{\alpha_1;\frac{5}{2}+\epsilon} + \frac{24\zeta(4-2\epsilon)\zeta(5-2\epsilon)}{\pi^2} \, E^{SL(5)}_{\alpha_4;\frac{5}{2}-\epsilon} \right) \tag{E.35}$$

The definition and Fourier expansions of the Eisenstein series in this expression will now be reviewed.

• **Fourier modes for the series $E^{SL(5)}_{\alpha_1;s}$**

The $E^{SL(5)}_{\alpha_1;s}$ series may be written using (E.33) in the form

$$2\zeta(2s) \, E^{SL(5)}_{\alpha_1;s} = \sum_{M_5 \in \mathbb{Z}^5 \backslash \{0\}} (M_5 \cdot g_5 \cdot M_5)^{-s} . \tag{E.36}$$

[24] In [2] these series were defined as $\mathbf{E}^{SL(5)}_{[1000];s} = 2\zeta(2s) E^{SL(5)}_{\alpha_1;s}$ and $\mathbf{E}^{SL(5)}_{[0010];s} = 4\zeta(2s)\zeta(2s-1) E^{SL(5)}_{\alpha_4;s}$

The constant terms with respect to the (2,3) parabolic subgroup P_{α_4}, the (4,1) parabolic subgroup P_{α_1}, and the $(1,4)$ parabolic subgroup P_{α_2}, were evaluated in [2].

(i) **The parabolic** $P_{\alpha_4} = GL(1) \times SL(2) \times SL(3) \times U_{\alpha_4}$**.**

The unipotent radical for this parabolic subgroup is abelian and is given by

$$U_{\alpha_4} = \begin{pmatrix} I_2 & Q_4 \\ 0 & I_3 \end{pmatrix} , \qquad Q_4 = \begin{pmatrix} g_{13} & B^1_{\rm NS} & C^{(2)\,1} + \Omega_1 B^1_{\rm NS} \\ g_{23} & B^2_{\rm NS} & C^{(2)\,2} + \Omega_1 B^2_{\rm NS} \end{pmatrix} . \tag{E.37}$$

Poisson resummation on two integers, keeping the off-diagonal terms in the parametrisation of [2, section B.5.2] results in the following Fourier expansion of $E^{SL(5)}_{\alpha_1;s}$ with respect to P_{α_4} to get

$$F^{SL(5)\alpha_4}_{\alpha_1;s}(N_4) := \int_{[0,1]^6} d^6 Q_4 \, e^{-2i\pi \, {\rm tr}(N_4^T \cdot Q_4)} \, E^{SL(5)}_{\alpha_1;s} \, , \tag{E.38}$$

where $N_4 \in M(2,3;\mathbb{Z})$.

For all values of s the Fourier modes are only non-zero when N_4 has rank 1 and is given by given by $N_4 = k\,\tilde{N}_4$ with $\gcd(\tilde{N}_4) = 1$

$$\tilde{N}_4 = m^T n = \begin{pmatrix} m_1 n_1 & m_2 n_1 \\ m_1 n_2 & m_2 n_2 \\ m_1 n_3 & m_2 n_3 \end{pmatrix} , \quad n = (n_i) \in \mathbb{Z}^3, m = (m_a) \in \mathbb{Z}^2 , \tag{E.39}$$

and takes the form

$$F^{SL(5)\,\alpha_4}_{\alpha_1;s}(k, \tilde{N}_4) = \frac{2\pi^s}{\Gamma(s)} \, r^{3-\frac{2s}{5}} \, \frac{\sigma_{2s-3}(|k|)}{|k|^{s-\frac{3}{2}}} \, (\tfrac{\|\tilde{N}_4\|}{\|m\|})^{s-\frac{3}{2}} \, K_{s-\frac{3}{2}} \left(2\pi |k| \, r^2 \, \|\tilde{N}_4\| \right) , \tag{E.40}$$

where we have defined

$$\|\tilde{N}_4\|^2 := {\rm tr}(g_3^{-1} \, \tilde{N}_4 \, g_2 \, \tilde{N}_4^T); \quad \|m\|^2 := m^T \cdot g_2 \cdot m \, . \tag{E.41}$$

The matrix $\tilde{N}_4$ is transformed by the action of $SL(3,\mathbb{Z})$ on the left by the action of $SL(2,\mathbb{Z})$ on the right. This matrix has rank 1 and therefore satisfies the $\frac{1}{2}$-BPS conditions $\epsilon_{ab}(N_4)_i{}^a(N_4)_j{}^b = 0$ of section C.3.

In other words, for any value of s, the Fourier modes fill out $\frac{1}{2}$-BPS orbits – one for each value of k.

(ii) **The parabolic** $P_{\alpha_1} = GL(1) \times SO(3,3) \times U_{\alpha_1}$**.**

The unipotent radical for this parabolic is abelian and is given, in our parameterisation by

$$U_{\alpha_1} = \begin{pmatrix} I_4 & Q_1 \\ 0 & 1 \end{pmatrix} , \qquad Q_1 = \begin{pmatrix} C^{(2)\,1} + \Omega_1 B^1_{\rm NS} \\ C^{(2)\,2} + \Omega_1 B^2_{\rm NS} \\ C^{(2)\,3} + \Omega_1 B^3_{\rm NS} \\ \Omega_1 \end{pmatrix} , \tag{E.42}$$

where I_4 is the 4×4 unit matrix and Q_1 is a $SO(3,3)$ spinor (a vector of $SL(4)$).

The Fourier modes are defined by

$$F^{SL(5)\alpha_1}_{\alpha_1;s}(k,N_1) := \int_{[0,1]^4} d^4Q_1\, e^{-2i\pi\, kN_1^T\cdot Q_1}\, E^{SL(5)}_{\alpha_1;s}\,, \tag{E.43}$$

where $N_1 \in \mathbb{Z}^4$ is such that $\gcd(N_1) = 1$. These Fourier modes are evaluated by a straightforward extension of the expansion given in [2, section B.5.1], which kept only the constant terms (for which it is sufficient to set $Q_1 = 0$) and used the fact that $SO(3,3) \cong SL(4)$. The result is

$$F^{SL(5)\,\alpha_1}_{\alpha_1;s}(k,N_1) = \frac{2\pi^s}{\Gamma(s)}\, r^{\frac{5+6s}{5}}\, \frac{\sigma_{2s-1}(|k|)}{|k|^{s-\frac{1}{2}}}\, \frac{K_{s-\frac{1}{2}}\left(2\pi|k|\, r^2\, \|N_1\|\right)}{\|N_1\|^{s-\frac{1}{2}}}\,, \tag{E.44}$$

where $\|N_1\|^2 := N_1^T \cdot g_4 \cdot N_1$ and $\gcd(N_1) = 1$.

(iii) **The parabolic** $P_{\alpha_2} = GL(1) \times SL(4) \times U_{\alpha_2}$

The unipotent radical is abelian and given by

$$U_{\alpha_2} = \begin{pmatrix} 1 & Q_2 \\ 0 & I_4 \end{pmatrix}\,, \qquad Q_2 = \left(C_{123}\; C_{124}\; C_{234}\; C_{134}\right)\,, \tag{E.45}$$

where Q_2 is again a $SL(4)$ (row) vector. The notation indicates that it is parametrized by the 3-form flux of the $M2$-brane world-volume wrapped on the M-theory 4-torus, $\mathcal{T}^4$. This translates into the NS components of flux, $B_{\mathrm{NS}\,12}, B_{\mathrm{NS}\,23}, B_{\mathrm{NS}\,13}$, and the RR $D2$-brane flux, $C^{(3)}_{123}$. In type IIB language these components become the NS flux $B_{\mathrm{NS}\,12}$, the RR D-string flux $C^{(2)}_{12}$ and the Kaluza–Klein momenta from the components of the metric $g_{i\,3}$ with $i = 1, 2$.

The Fourier coefficients in this parabolic are defined by for $k \in \mathbb{Z}$ and $N_4 \in \mathbb{Z}^4$ with $\gcd(N_4) = 1$

$$F^{SL(5)\alpha_2}_{\alpha_1;s}(k,N_4) := \int_{[0,1]^4} d^4Q_2\, e^{-2i\pi\, k\, N_4^T\cdot Q_2}\, E^{SL(5)}_{\alpha_1;s}\,. \tag{E.46}$$

These coefficient can again be evaluated by an extension of the computation of [2, section B.5.1] keeping the off-diagonal terms, which gives

$$F^{SL(5)\,\alpha_2}_{\alpha_1;s}(k,N_4) = \frac{2\pi^s}{\Gamma(s)}\, r^{4-\frac{2s}{5}}\, \frac{\sigma_{2s-4}(|k|)}{|k|^{s-2}}\, \|N_4\|^{s-2}\, K_{s-2}\left(2\pi|k|\, r^2\, \|N_4\|\right)\,, \tag{E.47}$$

where $\|N_4\|^2 := N_4^T \cdot g_4 \cdot N_4$ with $\gcd(N_4) = 1$.

• **Fourier modes for the series $\mathbf{E}^{SL(5)}_{\alpha_4;s}$**

The expression for $\mathcal{E}^{(7)}_{(1,0)}$ involves the Eisenstein series $E^{SL(5)}_{\alpha_4;s}$ in (E.35), which is not related to the series $E^{SL(5)}_{\alpha_1;s}$ by the functional equations. The Fourier modes will be evaluated using the Mellin transform representation given in (4.56) from the proposition 4.1.

(i) **The parabolic** $P_{\alpha_4} = GL(1) \times SL(2) \times SL(3) \times U_{\alpha_4}$

It is convenient to start from the expression for $\Gamma_{(5,5)}$ after Poisson resummation on two of the ten integers in (4.75),

$$\Gamma_{(5,5)} = \frac{r^{\frac{24}{5}}}{V^3} \sum_{\substack{(m_i,n_i)\in\mathbb{Z}^6\setminus\{0\}\\(p^a,q^b)\in\mathbb{Z}^4\setminus\{0\}}} e^{-\pi\, r^{\frac{8}{5}}\, \frac{(m+n\tau)\cdot g_3^{-1}\cdot(m+n\bar\tau)}{V\tau_2} - \pi\, V\, r^{\frac{12}{5}}\, \frac{(p+q\tau)\cdot g_2\cdot(p+q\bar\tau)}{\tau_2} - 2i\pi\, \mathrm{tr}(N_4^T\cdot Q_4)}\,, \tag{E.48}$$

where the $SL(2)$ metric g_2 is an element of the coset $SL(2)/SO(2)$ and the $SL(3)$ metric, g_3, is an element of the coset $SL(3)/SO(3)$. The integer-valued matrix $N_4 \in M(3,2;\mathbb{Z})$ can be written in the form

$$N_4 \;:=\; M_3 \cdot J \cdot P_2\,, \tag{E.49}$$

where

$$J \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{E.50}$$

and

$$M_3 := \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \\ m_3 & n_3 \end{pmatrix} \in M(3,2;\mathbb{Z})\,, \quad P_2 := \begin{pmatrix} p^1 & q^1 \\ p^2 & q^2 \end{pmatrix} \in M(2,2;\mathbb{Z})\,. \tag{E.51}$$

Since $\gamma^T \cdot J \cdot \gamma = J$ for all $\gamma \in SL(2,\mathbb{Z})$, the matrix N_4 is invariant under the action of since $M_3 \to M_3 \cdot \gamma$ and $P_2 \to \gamma^T \cdot P_2$.

The integral (4.57) can be analysed by use of the method of orbits in [2, appendix B.2] applied to the left action on P_2. This gives the sum of three types of contributions arising from the singular orbit with $P_2 = 0$, the degenerate orbit with $\det P_2 = 0$, $P_2 \neq 0$, which can be reduced to terms with $q_1 = q_2 = 0$, and the non degenerate orbit with $\det P_2 \neq 0$, which can be written as a sum over matrices of the form

$$P_2 = \begin{pmatrix} k & 0 \\ j & p \end{pmatrix}\,, \qquad 0 \le j < k, p \neq 0\,. \tag{E.52}$$

The result is

$$\begin{aligned}
\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \Gamma_{(5,5)} &= \frac{r^{\frac{24}{5}}}{V^3} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \Gamma_{(3,3)} \qquad\qquad (E.53)\\
&+ \int_{-\frac12}^{\frac12} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{\substack{p\in\mathbb{Z}^2\setminus\{0\}\\(m,n)\in\mathbb{Z}^6}} e^{-\pi V\, r^{\frac{12}{5}}\, \frac{p\cdot g_2\cdot p}{\tau_2}}\; e^{-\pi\, r^{\frac{8}{5}}\, \frac{(m+n\tau)\cdot g_3^{-1}\cdot(m+n\bar\tau)}{V\tau_2} + 2i\pi\, \mathrm{tr}(\tilde N_4^T Q_4)}\\
&+ \; 2\int_{-\infty}^{+\infty} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{\substack{0\le j<k\\p\neq 0}} \sum_{(m,n)\in\mathbb{Z}^6} e^{-\pi V\, r^{\frac{12}{5}}\, \frac{(p+q\tau)\cdot g_2\cdot(p+q\bar\tau)}{\tau_2}} \times\\
&\times \; e^{-\pi\, r^{\frac{8}{5}}\, \frac{(m+n\tau)\cdot g_3^{-1}\cdot(m+n\bar\tau)}{V\tau_2} + 2i\pi\, \mathrm{tr}(N_4^T Q_4)}\,,
\end{aligned}$$

where the matrix $\tilde{N}_4 = -n^T \cdot p$ has rank 1 and $N_4 = m^T \cdot q - n^T \cdot p$ has rank 2.

The Fourier modes of $E^{SL(5)}_{\alpha_4;s}$ in the P_{α_4} parabolic are given by

$$F^{SL(5)\alpha_4}_{\alpha_4;s}(N_4) := \int_{[0,1]^6} d^6Q\, e^{-2i\pi\,\mathrm{tr}(N_4^T Q_4)}\, E^{SL(5)}_{\alpha_4;s}\,, \tag{E.54}$$

with $N_4 \in M(3,2;\mathbb{Z})$ and Q_4 is defined in (E.37). This can be written as the sum of two types of contributions, one in which N_4 is of rank 1 and one in which N_4 is of rank 2

$$F^{SL(5)\alpha_4}_{\alpha_4;s}(N_4) = \mathcal{F}^{SL(5)\alpha_4}_{\alpha_4;s\,I}(\tilde{N}_4) + \mathcal{F}^{SL(5)\alpha_4}_{\alpha_4;s\,II}(N_4)\,. \tag{E.55}$$

Substituting (E.53) in the representation in (4.56), the rank 1 contribution is given by

$$F^{SL(5)\,\alpha_4}_{\alpha_4;s\,I}(\tilde{N}_4) = \frac{r^{\frac{6s+10}{5}}}{4\xi(2s)\zeta(2s-1)} \sum_{m\in\mathbb{Z}^3\setminus\{0\}} \frac{\delta(m\cdot n)}{(m^2)^{s-\frac{1}{2}}\,(p^2)^{s-1}}\, \frac{K_{s-1}(2\pi r^2\,\|\tilde{N}_4\|)}{\|\tilde{N}_4\|^{1-s}}\,, \tag{E.56}$$

where $\tilde{N}_4 = n^T \cdot p$ with $n \in \mathbb{Z}^3$ and $p \in \mathbb{Z}^2$ with $\|\tilde{N}_4\|^2 = \mathrm{tr}(g_3^{-1}\tilde{N}_4 g_2^{-1}\tilde{N}_4^T)$, and $m^2 = m^T \cdot g_3^{-1} \cdot m$ and $p^2 = p^T \cdot g_2 \cdot p$.

The contribution when $N_4 = m^T \cdot q - n^T \cdot p$ has rank 2 arises from the non-degenerate orbit and has the form

$$\begin{aligned} F^{SL(5)\,\alpha_4}_{\alpha_4;s\,II}(N_4) &= 4r_3^{\frac{30-4s}{5}} \int_{-\infty}^{+\infty} d\tau_1\, \frac{(n^2)^{\frac{s-1}{2}}\,((m+n\tau_1)^2)^{\frac{s-2}{2}}}{((p+q\tau_1)^2)^{\frac{s-1}{2}}\,(q^2)^{\frac{s-2}{2}}} \\ &\times\; K_{s-2}(2\pi\, r^2\sqrt{(m+n\tau_1)^2\, q^2})K_{s-1}(2\pi\, r^2\sqrt{(p+q\tau_1)^2\, n^2})\,. \end{aligned} \tag{E.57}$$

(ii) **The parabolic** $P_{\alpha_1} = GL(1) \times SO(3,3) \times U_{\alpha_1}$

The unipotent radical is parametrized by Q_1 and after Poisson resummation the lattice sum becomes

$$\Gamma_{(5,5)}|_{P(1,4)} = (V^{-1}r^{\frac{4}{5}})^4 \sum_{\substack{(p,q)\in\mathbb{Z}\times\mathbb{Z}\\ (m,n)\in\mathbb{Z}^4\times\mathbb{Z}^4}} e^{-\pi V r^{\frac{16}{5}}\frac{|p+q\tau|^2}{\tau_2} - \pi r^{\frac{4}{5}}\frac{(m+n\tau)\cdot g_4^{-1}\cdot(m+n\bar{\tau})}{V\tau_2} - 2i\pi\,\tilde{N}_1^T\cdot Q_1} \tag{E.58}$$

where g_4 parametrizes the coset $SL(4)/SO(4)$ and $\tilde{N}_1$ is the rank 4 vector defined by

$$\tilde{N}_1 := M_3 \cdot J \cdot P_1\,, \tag{E.59}$$

where J is the matrix defined in (E.50) and

$$M_3 := \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \\ m_3 & n_3 \\ m_4 & n_4 \end{pmatrix} \in M(4,2;\mathbb{Z})\,, \quad P_1 := \begin{pmatrix} p \\ q \end{pmatrix} \in M(2,1;\mathbb{Z})\,. \tag{E.60}$$

The vector $\tilde{N}_1$ is invariant under the action of $\gamma \in SL(2,\mathbb{Z})$ on the integers $P \to \gamma^T \cdot P$ and $M \to M \cdot \gamma$ because $\gamma \cdot J \cdot \gamma^T = J$ for all $\gamma \in SL(2,\mathbb{Z})$.

The integral (4.57) can be evaluated when the lattice sum is given by (E.58) by unfolding from the left $SL(2)$ action on P_1 to give

$$\begin{aligned} I_s(\Lambda, g_5)|_{P(1,4)} &= r^{\frac{8s}{5}} \int_0^\Lambda dV\, V^{2s-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \Gamma_{(4,4)}(V\, g_4) \\ + \quad & r^{\frac{8}{5}} \int_0^\Lambda dV\, V^{2s-5} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \sum_{\substack{m\in\mathbb{Z}\setminus\{0\}\\ (n,m)\in\mathbb{Z}^8}} e^{-\pi \frac{r^{\frac{16}{5}}\, V\, p^2}{\tau_2} - 2i\pi p\, n^T\cdot Q_1} \\ \times \quad & e^{-\pi r^{\frac{4}{5}} \frac{(m+n\tau)\cdot g_4^{-1}\cdot(m+n\bar\tau)}{V\tau_2}} . \end{aligned} \tag{E.61}$$

Poisson resumming on m in the last term leads to

$$\begin{aligned} I_s(\Lambda, g_5)|_{P(1,4)} &= r^{\frac{8s}{5}}\, I_s(\Lambda, g_4) \\ + \quad & r^{\frac{8}{5}} \int_0^\Lambda dV\, V^{2s-3} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^{+\infty} d\tau_2 \sum_{\substack{p\in\mathbb{Z}\setminus\{0\}\\ (\hat m,n)\in\mathbb{Z}^8}} e^{-\pi V \frac{p^2\, r^{\frac{16}{5}}}{\tau_2} - \pi r^{\frac{4}{5}}\, \tau_2\, \frac{n\cdot g_4^{-1}\cdot n}{V}} \\ \times \quad & e^{-\pi\tau_2 V\, r^{-\frac{4}{5}}\, \hat m\cdot g_4\cdot \hat m + 2i\pi \hat m\cdot n\, \tau_1 - 2i\pi p n\cdot Q_1} . \end{aligned} \tag{E.62}$$

The integral over τ_1 projects on the sector $\hat m\cdot n = 0$, for which the winding and the Kaluza Klein numbers are orthogonal.

The finite part when $\Lambda\to\infty$ gives the Fourier coefficients

$$\begin{aligned} F^{SL(5)\,\alpha_1}_{\alpha_4;s}(k, N_1) &:= \int_{[0,1]^4} d^4Q_1 e^{2i\pi k\, N_1^T\cdot Q_1}\, E^{SL(5)}_{\alpha_4;s} \qquad \text{(E.63)} \\ = \quad & \frac{r^{3-\frac{2s}{5}}}{\xi(2s)\zeta(2s-1)} \left(\sum_{m\in\mathbb{Z}^4\setminus\{0\}} \frac{\delta(N_1\cdot m)}{\|m\|^{2s-1}} \right) \frac{\|N_1\|^{s-\frac{3}{2}}}{|k|^{s-\frac{3}{2}}}\, K_{s-\frac{3}{2}}(2\pi r^2\, |k|\|N_1\|) , \end{aligned}$$

with $N_1\in\mathbb{Z}^4$ with $\gcd(N_1)=1$ and $\|N_1\|^2 := N^T\cdot g_4^{-1}\cdot N_1$ and $\|m\|^2 := m^T\cdot g_4\cdot m$.

(iii) **The parabolic** $P_{\alpha_2} = GL(1)\times SL(4)\times U_{\alpha_2}$

Applying the same techniques for the P_{α_2} parabolic the Fourier coefficients

$$F^{SL(5)\,\alpha_2}_{\alpha_4;s}(k, N_2) := \int_{[0,1]^4} d^4Q_2\, e^{2i\pi\, k\, N_2^T\cdot Q_2} E^{SL(5)\,\alpha_2}_{\alpha_4;s} \tag{E.64}$$

are given by the rank 1 contribution $\tilde N_2 = p\, N_2$ with $k\in\mathbb{Z}$ and $N_2\in\mathbb{Z}^4$ with $\gcd(N_2)=1$

$$\begin{aligned} F^{SL(5),\alpha_2}_{\alpha_4;s}(k\,\tilde N_2) \quad &= \quad r^{2+\frac{2s}{5}} \frac{\Gamma(2-s)\pi^{s-2}}{4\zeta(2s)\zeta(2s-1)\Gamma(2s-1)} \times \qquad \text{(E.65)} \\ & \times \quad \left(\sum_{m\in\mathbb{Z}^4\setminus\{0\}} \frac{\delta(m\cdot N_2)}{\|m\|^{4-2s}} \right) \frac{|k|^{s-1}}{\|N_2\|^{s-1}}\, K_{s-1}(2\pi r^2\, |k|\|N_2\|) , \end{aligned}$$

and where $\|N_2\|^2 := N_2^T\cdot g_4^{-1}\cdot N_2$ and $\|m\|^2 := m^T\cdot g_4\cdot m$.

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