DISTANCE AND TUBE ZETA FUNCTIONS OF FRACTALS AND ARBITRARY COMPACT SETS

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Abstract. Recently, the first author has extended the definition of the zeta function associated with fractal strings to arbitrary bounded subsets A of the N-dimensional Euclidean space $\mathbb{R}^N$, for any integer $N \geq 1$. It is defined by $\zeta_A(s) = \int_{A_\delta} d(x,A)^{s-N} \mathrm{d}x$ for all $s \in \mathbb{C}$ with Re s sufficiently large, and we call it the *distance zeta function* of A. Here, $d(x,A)$ denotes the Euclidean distance from x to A and A_δ is the δ-neighborhood of A, where δ is a fixed positive real number. We prove that the abscissa of absolute convergence of ζ_A is equal to $\overline{\dim}_B A$, the upper box (or Minkowski) dimension of A. Particular attention is payed to the principal complex dimensions of A, defined as the set of poles of ζ_A located on the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B A\}$, provided ζ_A possesses a meromorphic extension to a neighborhood of the critical line. We also introduce a new, closely related zeta function, $\tilde{\zeta}_A(s) = \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t$, called the *tube zeta function* of A. Assuming that A is Minkowski measurable, we show that, under some mild conditions, the residue of $\tilde{\zeta}_A$ computed at $D = \dim_B A$ (the box dimension of A), is equal to the Minkowski content of A. More generally, without assuming that A is Minkowski measurable, we show that the residue is squeezed between the lower and upper Minkowski contents of A. We also introduce *transcendentally quasiperiodic sets*, and construct a class of such sets, using generalized Cantor sets, along with Baker's theorem from the theory of transcendental numbers.

Contents

2010 *Mathematics Subject Classification.* Primary: 11M41, 28A12, 28A75, 28A80, 28B15, 42B20, 44A05. Secondary: 40A10, 42B35, 44A10.

Key words and phrases. zeta function, distance zeta function, tube zeta function, fractal set, fractal string, box dimension, principal complex dimensions, Minkowski content, Minkowski measurable set, residue, Dirichlet integral, transcendentally quasiperiodic set.

The research of Michel L. Lapidus was partially supported by the National Science Foundation under grants DMS-0707524 and DMS-1107750, as well as by the Institut des Hautes Études Scientifiques (IHÉS) where the first author was a visiting professor in the Spring of 2012 while part of this research was completed. Goran Radunović and Darko Žubrinić express their gratitude to the Croatian Science Foundation for its support.

1. Introduction

In this article, we provide a far-reaching extension of the theory of zeta functions for fractal strings, to arbitrary fractal sets in Euclidean spaces of any dimension. Fractal strings have been introduced by the first author (M. L. Lapidus) in the early 1990s. The related theory of zeta functions of fractal strings, developed in the course of the last more than two decades of active research, can be seen in an extensive monograph of the first author with van Frankenhuijsen [Lap-vFr3].

The new zeta function ζ_A, associated with any fractal set A in $\mathbb{R}^N$, has been introduced in 2009 by the first author, and its definition can be found in Equation (2.1) below. We refer to it as the *distance zeta function* of A. Here, by a fractal set, we mean any bounded set A of the Euclidean space $\mathbb{R}^N$, with $N \geq 1$. The reason is that, in this paper, the fundamental role is played by a certain notion of fractal dimension, more specifically, by the upper box dimension of a bounded set (also called the upper Minkowski dimension, Bouligand dimension, or limit capacity, etc.). This new class of zeta functions enables us to obtain a nontrivial extension of the theory of *complex dimensions of fractal strings*, studied by the authors of [Lap-vFr1–3], to arbitrary bounded fractal sets in Euclidean spaces of any dimension.

A systematic study of the zeta functions associated with fractal strings and fractal sprays was motivated and undertaken, in particular, in the 1990s in papers of the first author, [Lap1–3], as well as in joint papers of the first author with C. Pomerance [LapPo1–3] and with H. Maier [LapMa1–2]. In a series of papers, as well as in three monographs with M. van Frankenhuijsen [Lap-vFr1–3], and in the book [Lap4], it has grown into a well-established theory of fractal complex dimensions, and is still an active area of research, with applications to a variety of subjects, including spectral theory, harmonic analysis, number theory, dynamical systems, probability theory and mathematical physics. We also draw the attention of the reader to [DubSep], [ElLapMacRo], [Es1–2], [EsLi1–2], [Fal2], [HamLap], [HeLap], [HerLap1–2], [LalLap1–2], [Lap5–6], [LapLéRo], [LapLu], [LapLu-vFr1–2], [LapPe1–2], [LapPeWi1–2], [LapRaŽu1–7], [LapRo], [LapRoŽu], [LéMen], [MorSep], [MorSepVi1–2], [Ol1–2], [RatWi], [Tep1–2], along with the relevant references therein.

The contemporary state of the theory of fractal strings, fractal sprays and their complex dimensions, as well as its trends and developments, is described in the research monograph [Lap-vFr3] and, in particular, in [Lap-vFr3, Chapter 13].

Other, very different approaches to a higher-dimensional theory of some special classes of fractal sets, namely, fractal sprays and self-similar tilings, were developed by the first author and E. Pearse in [LapPe2–3], as well as by the first author, E. Pearse and S. Winter in [LapPeWi1–2] via fractal tube formulas and the associated scaling and tubular zeta functions. (See also [Pe] and [PeWi].) An earlier approach, based directly on tube formulas but not using any kind of zeta function was proposed in [LapPe1]. An exposition of these approaches can be found, respectively, in [Lap-vFr3, Sections 13.1 and 12.2.1].

The definitions of zeta functions introduced in [LapPe2–3] and [LapPeWi1–2] differ considerably from those studied in this article. Therefore, it would be of interest to see when they give rise to the same complex dimensions. We note that, according to an example provided in [LapPe2], complex dimensions of self-similar sets depend on the choice of the iterated function system generating them. It would be of great interest to compute the tube zeta function, introduced in this article in Definition 3.6 below, in the case of the Koch snowflake curve, and to compare the resulting formula with the one obtained in [LapPe1]. A similar question can be raised for fractal sprays and self-similar tilings, for the tube formulas obtained in [LapPe2–3] and [LapPeWi1–2]. We point out that using the fractal zeta functions introduced in this paper, it is possible to generalize the fractal tube formulas and a Minkowski measurability criterion obtained for fractal strings in [Lap-vFr3] to arbitrary compact sets in Euclidean spaces; see [LapRaŽu4–5].

1.1. **Contents.** The rest of this paper is organized as follows:

In Section 2, the distance zeta function ζ_A of a bounded set $A \subset \mathbb{R}^N$ is introduced in Definition 2.1. Then, the main result of Section 2 is obtained in Theorem 2.5, in which it is shown (among others) that the abscissa of (absolute) convergence of the distance zeta function ζ_A of any bounded subset A of $\mathbb{R}^N$ is equal to $\overline{\dim}_B A$, i.e., to the upper box dimension (or the upper Minkowski dimension) of A. (All of the subsets denoted by A appearing in this paper are implicitly assumed to be nonempty.) As a useful technical tool in the study of fractal zeta functions, we introduce the notion of 'equivalence' between tamed Dirichlet-type integrals (see Definition 2.22). We also define the set of 'principal complex dimensions' of A, denoted by $\dim_{PC} A$ (see Definition 2.18), as a refinement of the notion of the upper box dimension of A. Moreover, in the one-dimensional case (i.e., in the case of a bounded fractal string $\mathcal{L}$), we show that ζ_A, the distance zeta function of A (the boundary of the string $\mathcal{L}$), and $\zeta_{\mathcal{L}}$, the geometric zeta function of $\mathcal{L}$, contain essentially the same information. In particular, ζ_A and $\zeta_{\mathcal{L}}$ are equivalent in the above sense, and hence, have the same principal complex dimensions (see Sections 2.3 and 2.4); they also have the same (visible) complex dimensions (with the same multiplicities) in every domain of $\mathbb{C} \setminus \{0\}$ to which one (and hence, both) of them can be meromorphically continued. Finally, we show that the distance zeta function has a nice scaling property; see Proposition 2.24.

In Section 3, we introduce the so-called 'tube zeta function' $\tilde{\zeta}_A$ of the set A (which is closely related to the distance zeta function ζ_A; see Theorem 3.1 and the associated functional equation (3.14)), and study its properties; see, in particular,

Definition 3.6 in Section 3.2. Under suitable natural conditions, we show that the residue of the tube zeta function $\tilde{\zeta}_A$, computed at $D = \dim_B A$ (assuming that the box dimensions exists), always lies between the lower and upper (D-dimensional) Minkowski contents of A; see Theorem 3.7. In particular, if A is Minkowski measurable, then the residue of $\tilde{\zeta}_A$ at D coincides with the Minkowski content of A. Similar results are obtained for the distance zeta function ζ_A of the fractal set A; see Theorem 3.3. In fact, we also show that ζ_A and $\tilde{\zeta}_A$, the distance and tube zeta functions of A, contain essentially the same information. These results are illustrated by means of several examples, including a class of generalized Cantor sets (Examples 3.4, 3.9 and 3.17), a-strings (Example 3.10), as well as 'fractal grills' introduced in Section 3.4; see Theorem 3.15.

In Section 4, we introduce a class of 'n-quasiperiodic sets' (Definition 4.10). The main result is stated in Theorem 4.14, which can be considered as a fractal set-theoretic interpretation of Baker's theorem from transcendental number theory and in which we construct a family of transcendentally n-quasiperiodic sets, for any integer $n \geq 2$. An important role in the construction of quasiperiodic sets is played by the class of generalized Cantor sets $C^{(m,a)}$ depending on two paramters, introduced in Definition 4.1. Moreover, in Section 4.4, we close the main part of this paper by connecting the present work to future extensions (notably, the construction of transcendentally ∞-quasiperiodic sets), the notion of hyperfractal (and even, maximally hyperfractal) set, and more broadly, the notion of fractality within the context of this new general theory of complex dimensions. In short, much as in [Lap-vFr1–3], we say that a bounded subset $A \subset \mathbb{R}^N$ is *fractal* if its associated zeta function (i.e., the distance or the tube zeta function, ζ_A or $\tilde{\zeta}_A$, of A or when $N = 1$, the geometric zeta function $\zeta_{\mathcal{L}}$, where $\mathcal{L}$ is the fractal string associated with A) has at least one nonreal complex dimension (with positive real part) or else has a natural boundary beyond which it cannot be meromorphically continued (i.e., A is "hyperfractal"). Observe that, unlike in the one-dimensional theory of complex dimensions developed in [Lap-vFr1–3], we now have at our disposal precise definitions of fractal zeta functions of arbitrary bounded subsets of $\mathbb{R}^N$ and hence, of the complex dimensions of those sets (i.e., of the poles of these fractal zeta functions); see Definition 2.19 and the beginning of Section 3.2. The complex dimensions of a variety of classic and less well-known fractals will be computed in subsequent papers [LapRaŽu2–5] and in the forthcoming monograph [LapRaŽu1]. (See also the survey articles [LapRaŽu6,7].)

The aim of Appendix A is to introduce the class of 'extended Dirichlet-type integrals' (or functions), i.e., of EDTIs, which contains all of the fractal zeta functions studied in the present paper (and in [LapRaŽu1–7]); see Definition A.1. We study some of the key properties of EDTIs and introduce two closely related (but distinct) notions of equivalence; see Definitions A.2 and A.6.

1.2. **Notation.** Throughout this paper, we shall use the following notation. By $|E| = |E|_N$, we denote the N-dimensional Lebesgue measure of a measurable subset E of $\mathbb{R}^N$. Given $r \geq 0$, the *lower and upper r-dimensional Minkowski contents* $\mathcal{M}^{*r}(A)$ and $\mathcal{M}^r_*(A)$ of a bounded subset A of $\mathbb{R}^N$ are defined by

$$\mathcal{M}^r_*(A) = \liminf_{t\to 0^+} \frac{|A_t|}{t^{N-r}}, \quad \mathcal{M}^{*r}(A) = \limsup_{t\to 0^+} \frac{|A_t|}{t^{N-r}}. \tag{1.1}$$

Here, $A_t := \{x \in \mathbb{R}^N : d(x, A) < t\}$ denotes the *t-neighborhood* (or *tubular neighborhood of radius* t) of A, and $d(x, A)$ is the Euclidean distance from x to A. The function $t \mapsto |A_t|$, defined for t positive and close to 0, is called the *tube function* associated with A. From our point of view, one of the basic tasks of fractal analysis is to understand the nature of the tube functions for various fractal sets. The notion of Minkowski content has been introduced by H. Federer in [Fed]; the above definition coincides with Federer's definition up to a (positive) multiplicative constant depending only on N and r, the value of which is not important for the purposes of this article.

The *upper box dimension* of A is defined by

$$\overline{\dim}_B A = \inf\{r \geq 0 : \mathcal{M}^{*r}(A) = 0\}; \tag{1.2}$$

it is easy to see that we also have

$$\overline{\dim}_B A = \sup\{r \geq 0 : \mathcal{M}^{*r}(A) = +\infty\}. \tag{1.3}$$

The lower box dimension of A, denoted by $\underline{\dim}_B A$, is defined analogously, with $\mathcal{M}^r_*(A)$ instead of $\mathcal{M}^{*r}(A)$ on the right-hand side of (1.2) and (1.3). Clearly, since A is bounded, we always have $0 \leq \underline{\dim}_B A \leq \overline{\dim}_B A \leq N$. If both $\overline{\dim}_B A$ and $\underline{\dim}_B A$ coincide, their common value is denoted by $\dim_B A$ and is called the *box dimension* of A (or *Minkowski–Bouligand dimension*, or else, *Minkowski dimension*). Various properties of the box dimension can be found, e.g., in [Fal1], [Lap1–3], [Mat], [LapPo2], [Tri] and [Lap-vFr3]. A discussion of equivalent forms of the definition of the Minkowski dimension can be found in [LapRoŽu].

If there exists a nonnegative real number D such that

$$0 < \mathcal{M}^D_*(A) \leq \mathcal{M}^{*D}(A) < \infty,$$

we say that A is *Minkowski nondegenerate*. If A is nondegenerate, it then follows that $\dim_B A$ exists and is equal to D. If $\mathcal{M}^D_*(A) = \mathcal{M}^{*D}(A)$, their common value is denoted by $\mathcal{M}^D(A)$ and called the *Minkowski content* of A. If, in addition,

$$\mathcal{M}^D(A) \in (0, +\infty),$$

then A is said to be *Minkowski measurable*. The notion of Minkowski measurability has been introduced by L. L. Stachó [Sta], inspired by [Fed]. The notion of Minkowski nondegeneracy has been introduced in [Žu4] (and was studied earlier in [LapPo2] for $N = 1$). Throughout, we will assume implicitly that the bounded set $A \subset \mathbb{R}^N$ is nonempty; see also [LapPo3] when $N \geq 3$.

We note that since $|A_t| = |(\overline{A})_t|$ for every $t > 0$, the values of $\mathcal{M}^r_*(A)$, $\mathcal{M}^{*r}(A)$, $\underline{\dim}_B A$, $\overline{\dim}_B A$ (as well as of $\mathcal{M}^D(A)$ and $\dim_B A$, when they exist) do not change when we replace the bounded set $A \subset \mathbb{R}^N$ by its closure $\overline{A}$ in $\mathbb{R}^N$. Therefore, throughout this paper, we might as well assume a priori that A is an arbitrary (nonempty) compact subset of $\mathbb{R}^N$. Observe that, as is well known, this is in sharp contrast with the Hausdorff dimension (and associated Hausdorff measure $\mathcal{H}_H$); see, e.g., [Fal1]. For example, if $A = \{1/j : j \in \mathbb{N}\}$, then (since A is countable), $\dim_H A = 0$ and $\mathcal{H}_H(A) = 0$, while $D := \dim_B A = 1/2$ and $\mathcal{M}^D(A) = 2\sqrt{2}$; see [Lap1, Example 5.1].

Moreover, for notational simplicity and because this is the case of most fractal sets of interest in the applications of the theory, we also implicitly assume that the closure of A has N-dimensional Lebesgue measure zero: $|\overline{A}|_N = 0$. We refer to [LapRaŽu1] for a brief discussion of the case when $|\overline{A}| > 0$ (and hence, $\overline{\dim}_B A =$

$\underline{\dim}_B A = N$), which also applies to the broader notion of relative fractal drum (RFD) that is the object of [LapRaŽu3–5] and of [LapRaŽu1, Chapters 4 and 5].

Finally, given an extended real number $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, we denote by $\{\operatorname{Re} s > \alpha\}$ the open right half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ (which coincides with $\mathbb{C}$ or $\emptyset$ if $\alpha = -\infty$ or $+\infty$, respectively). Furthermore, if $\alpha \in \mathbb{R}$, we denote by $\{\operatorname{Re} s = \alpha\}$ the vertical line $\{s \in \mathbb{C} : \operatorname{Re} s = \alpha\}$. Also, we let $\mathtt{i} := \sqrt{-1}$.

2. Distance and tube zeta functions of fractal sets

2.1. **Definition of the distance zeta functions of fractal sets.** In this section, we study some basic properties of the distance zeta function $\zeta_A = \zeta_A(s)$ associated with an arbitrary bounded subset A of $\mathbb{R}^N$, introduced by the first author in 2009. Throughout this paper, we assume that $|\overline{A}|_N = 0$.

Definition 2.1. Let δ be any given positive number. The *distance zeta function* ζ_A of a bounded subset A of $\mathbb{R}^N$ is defined by

$$\zeta_A(s) := \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x. \tag{2.1}$$

Here, the integral is taken in the sense of Lebesgue (hence, the complex-valued function $d(\,\cdot\,, A)^{s-N}$ is absolutely integrable on A_δ) and we assume that $s \in \mathbb{C}$ is such that $\operatorname{Re} s$ is sufficiently large.

As we shall see in Theorem 2.5, the Lebesgue integral in (2.1) is well defined if $\operatorname{Re} s$ is larger than $\overline{\dim}_B A$, the upper box dimension of A; furthermore, $\overline{\dim}_B A = D(\zeta_A)$, the *abscissa of (absolute) convergence* of ζ_A. Moreover, under the additional hypotheses of Theorem 2.5(c), $\overline{\dim}_B A$ also coincides with $D_{\mathrm{hol}}(\zeta_A)$, the *abscissa of holomorphic continuation* of ζ_A. Here, by definition,

$$D(\zeta_A) := \inf\left\{\alpha \in \mathbb{R} : \int_{A_\delta} d(x, A)^{\alpha-N} \mathrm{d}x < \infty\right\} \tag{2.2}$$

while

$$D_{\mathrm{hol}}(\zeta_A) := \inf\left\{\alpha \in \mathbb{R} : \zeta_A \text{ is holomorphic on } \{\operatorname{Re} s > \alpha\}\right\}. \tag{2.3}$$

Hence, the *half-plane of (absolute) convergence* of ζ_A, $\Pi(\zeta_A) := \{\operatorname{Re} s > D(\zeta_A)\}$ (resp., the *half-plane of holomorphic continuation* of ζ_A, $\mathcal{H}(\zeta_A) := \{\operatorname{Re} s > D_{\mathrm{hol}}(\zeta_A)\}$) is the largest open half-plane of the form $\{\operatorname{Re} s > \alpha\}$, for some $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, on which the Lebesgue integral $\int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x$ is convergent or, equivalently, absolutely convergent (resp., to which ζ_A can be holomorphically continued). It will follow from our results that $D(\zeta_A) \in [0, N]$ while $D_{\mathrm{hol}}(\zeta_A) \in [-\infty, D(\zeta_A)]$, and that both $D(\zeta_A)$ and $D_{\mathrm{hol}}(\zeta_A)$ are independent of the choice of $\delta > 0$; see Proposition 2.23 along with Definition 2.22.

Again, the same comment can be made about $D(\tilde{\zeta}_A)$ and $D_{\mathrm{hol}}(\tilde{\zeta}_A)$, given exactly as in (2.2) and (2.3), respectively, except for ζ_A replaced by $\tilde{\zeta}_A$ (the tube zeta function of A, see Definition 3.6). Actually, if $\overline{\dim}_B A < N$, then $D(\zeta_A) = D(\tilde{\zeta}_A)$ and $D_{\mathrm{hol}}(\zeta_A) = D_{\mathrm{hol}}(\tilde{\zeta}_A)$; see Corollary 3.2.

Given any meromorphic function f, the *abscissa of holomorphic continuation* of f, denoted by $D_{\mathrm{hol}}(f)$, can be defined in exactly the same way as $D_{\mathrm{hol}}(\zeta_A)$, except with ζ_A replaced by f in the counterpart of (2.3). The same comment is not true for $D(f)$, which may not make sense unless f is given by a Dirichlet-type integral (DTI); see Section 2.4 and Appendix A below.

As will be shown in Proposition 2.23, the dependence of ζ_A on the choice of δ is inessential, since the difference of two distance zeta functions corresponding to the same set A and different values of δ can be identified with an entire function. Note that without loss of generality (in fact, simply by replacing A by its closure), we could assume that A is an arbitrary (nonempty) compact subset of $\mathbb{R}^N$. Similar comments could be made about the tube zeta functions introduced in Definition 3.6 below.

We shall see in Theorem 2.5 below that ζ_A is holomorphic in the half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$, and that the bound $\overline{\dim}_B A$ is the best possible for the Lebesgue (i.e., absolute) convergence of ζ_A. Also, we shall extend the definition of the zeta function so that the value of δ will become unimportant. Furthermore, we will supplement the original definition of the complex dimensions of fractal strings introduced by the first author and M. van Frankenhuijsen in [Lap-vFr1–3]. Here, we mostly deal with the *principal complex dimensions* in the higher-dimensional case; see Definition 2.18. The situation with general complex dimensions is already quite nontrivial in the one-dimensional case; see [Lap-vFr1–3].

2.2. **Analyticity of the distance zeta functions.** The main result of this section is stated in Theorem 2.5. It shows that the zeta function ζ_A is analytic (i.e., holomorphic) in the half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$, and that (under the mild hypotheses of part (c) of Theorem 2.5) the lower bound is optimal. In other words, the *abscissa of absolute convergence* $D(\zeta_A)$ of the Dirichlet-type integral defined by the right-hand side of (2.1) is always equal to the upper box dimension of A and under the additional hypotheses of Theorem 2.5(c), it also coincides with the abscissa of holomorphic continuation $D_{\mathrm{hol}}(\zeta_A)$.

In order to prove Theorem 2.5, we shall need a result due to Harvey and Polking (see [HarvPo, p. 42]), that we formulate in a different, but equivalent way:

$$\text{If} \quad \gamma \in (-\infty, N - \overline{\dim}_B A), \quad \text{then} \quad \int_{A_\delta} d(x,A)^{-\gamma} \mathrm{d}x < \infty, \tag{2.4}$$

where δ is an arbitrary positive number. This result and its various extensions is discussed in [Žu3, Lemma 1 and Theorem 2], [Žu4, Sections 3 and 4] and [Žu5, Theorem 4.1]. For the sake of completeness, we provide an extension of (2.4), that we shall need later on. We omit the proofs of the following two lemmas. They can be obtained by using, e.g., the identity $\int_{\mathbb{R}^N} f(x)^\alpha \mathrm{d}x = \alpha \int_0^\infty t^{\alpha-1} |\{f > t\}| \, \mathrm{d}t$, where $f : \mathbb{R}^N \to [0, +\infty]$ is a Lebesgue measurable function and $\alpha \in (0, +\infty)$ (see [Fol, p. 198]) and by using the definition of the upper box dimension $\overline{\dim}_B A$ given in (1.2) and (1.3) above.

Lemma 2.2. *Let A be a bounded subset of $\mathbb{R}^N$, $\delta > 0$ and $\gamma \in (-\infty, N - \overline{\dim}_B A)$. Then*

$$\int_{A_\delta} d(x,A)^{-\gamma} \, \mathrm{d}x = \delta^{-\gamma} |A_\delta| + \gamma \int_0^\delta t^{-\gamma-1} |A_t| \, \mathrm{d}t. \tag{2.5}$$

Furthermore, both of the integrals appearing in (2.5) *are finite; hence, they are convergent Lebesgue integrals.*

Lemma 2.3. *Let A be a bounded subset of $\mathbb{R}^N$, $\delta > 0$ and $\gamma > N - \overline{\dim}_B A$. Then $\int_{A_\delta} d(x,A)^{-\gamma} \mathrm{d}x = +\infty$.*

Remark 2.4. If $\gamma := N - \overline{\dim}_B A$, then the conclusion of Lemma 2.3 does not hold, in general. Indeed, a class of counterexamples is provided in [Žu4, Theorem 4.3].

In the formulation of the following theorem, we shall need the *abscissa of holomorphic continuation* of ζ_A, denoted by $D_{\mathrm{hol}}(\zeta_A)$ and defined so that $\{\operatorname{Re} s > D_{\mathrm{hol}}(\zeta_A)\}$ be the largest open right half-plane on which ζ_A is holomorphic; see (2.3) above and the discussion following it.

On the other hand, recall that the *abscissa of Lebesgue* (*i.e., absolute*) *convergence of* ζ_A, given by (2.2) is denoted by $D(\zeta_A)$ and defined so that $\{\operatorname{Re} s > D(\zeta_A)\}$ be the largest open right half-plane on which ζ_A is Lebesgue (i.e., absolutely) convergent. For a more general setting, see Definition 2.12 below and the ensuing discussion (from which it will follow that both $\Pi(\zeta_A)$ and $\Pi(\tilde{\zeta}_A)$ are well defined and have the claimed maximality property). We shall also usually say more briefly that $D(\zeta_A)$ is the *abscissa of convergence* of ζ_A, meaning the abscissa of Lebesgue (i.e., absolute) convergence of ζ_A; see (2.2) and the comment following it.

Theorem 2.5. *Let A be an arbitrary bounded subset of $\mathbb{R}^N$ and let $\delta > 0$. Then*:

(*a*) *The zeta function ζ_A defined by* (2.1) *is holomorphic in the half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$, and for all complex numbers s in that region, we have*

$$\zeta_A'(s) = \int_{A_\delta} d(x, A)^{s-N} \log d(x, A)\, \mathrm{d}x. \tag{2.6}$$

(*b*) *We have*

$$\overline{\dim}_B A = D(\zeta_A), \tag{2.7}$$

where $D(\zeta_A)$ is the abscissa of Lebesgue (*i.e., absolute*) *convergence of ζ_A. Furthermore, in light of part* (*a*), *we always have $D_{\mathrm{hol}}(\zeta_A) \leq D(\zeta_A)$.*

(*c*) *If the box* (*or Minkowski*) *dimension $D := \dim_B A$ exists, $D < N$, and $\mathcal{M}_*^D(A) > 0$, then $\zeta_A(s) \to +\infty$ as $s \to D^+$, $s \in \mathbb{R}$. In particular, in this case, we also have that*

$$\dim_B A = D(\zeta_A) = D_{\mathrm{hol}}(\zeta_A). \tag{2.8}$$

Proof. (*a*) Denoting the right-hand side of (2.6) by $I(s)$, and choosing any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B A$, it suffices to show that

$$\begin{aligned} R(h) &:= \frac{\zeta_A(s+h) - \zeta_A(s)}{h} - I(s) \\ &= \int_{A_\delta} \left(\frac{d(x, A)^h - 1}{h} - \log d(x, A) \right) d(x, A)^{s-N} \mathrm{d}x \end{aligned} \tag{2.9}$$

converges to zero as $h \to 0$ in $\mathbb{C}$, with $h \neq 0$.

Let $d := d(x, A) \in (0, \delta)$. Defining

$$f(h) := \frac{d^h - 1}{h} - \log d = \frac{1}{h}(\mathrm{e}^{(\log d)h} - 1) - \log d, \tag{2.10}$$

and using the MacLaurin series $\mathrm{e}^z = \sum_{j \geq 0} \frac{z^j}{j!}$, we obtain that

$$f(h) = h(\log d)^2 \sum_{k=0}^{\infty} \frac{1}{(k+2)(k+1)} \cdot \frac{(\log d)^k h^k}{k!}. \tag{2.11}$$

Furthermore, assuming without loss of generality that $0 < \delta \leq 1$, and hence $\log d \leq 0$, we have

$$\begin{aligned} |f(h)| &\leq \frac{1}{2}|h|\,(\log d)^2 \sum_{k=0}^{\infty} \frac{(|\log d|\,|h|)^k}{k!} \\ &= \frac{1}{2}|h|\,(\log d)^2 \mathrm{e}^{-(\log d)|h|} = \frac{1}{2}|h|\,(\log d)^2 d^{-|h|}. \end{aligned}$$

Therefore,

$$|R(h)| \leq \frac{1}{2}|h| \int_{A_\delta} |\log d(x,A)|^2 d(x,A)^{\operatorname{Re} s - N - |h|} \mathrm{d}x. \tag{2.12}$$

Let $\varepsilon > 0$ be a sufficiently small number, to be specified below. Taking $h \in \mathbb{C}$ such that $|h| < \varepsilon$, since $\delta \leq 1$ and hence $d(x,A) \leq 1$ for all $x \in A_\delta$, we have

$$|R(h)| \leq \frac{1}{2}|h| \int_{A_\delta} |\log d(x,A)|^2 d(x,A)^{\varepsilon} d(x,A)^{\operatorname{Re} s - N - 2\varepsilon} \mathrm{d}x.$$

Since there exists a positive constant $C = C(\delta, \varepsilon)$ such that $|\log d|^2 d^{\varepsilon} \leq C$ for all $d \in (0, \delta)$, we see that

$$|R(h)| \leq \frac{1}{2}C|h| \int_{A_\delta} d(x,A)^{\operatorname{Re} s - N - 2\varepsilon} \mathrm{d}x. \tag{2.13}$$

Letting $\gamma := 2\varepsilon + N - \operatorname{Re} s$, we see that the integrability condition $\gamma < N - \overline{\dim}_B A$ stated in (2.4) is equivalent to $\operatorname{Re} s > \overline{\dim}_B A + 2\varepsilon$. Observe that this latter inequality holds for all positive ε small enough, due to the assumption $\operatorname{Re} s > \overline{\dim}_B A$. Hence, $R(h) \to 0$ as $h \to 0$ in $\mathbb{C}$, with $h \neq 0$. This proves part (a).

(b) Lemma 2.3 implies that for any real number $\alpha < D = \overline{\dim}_B A$, we have $\int_{A_\delta} d(x,A)^{\alpha - N}\,\mathrm{d}x = +\infty$. On the other hand, in light of estimate (2.4), we know that $\zeta_A(\alpha) = \int_{A_\delta} d(x,A)^{\alpha - N}\,\mathrm{d}x < \infty$ for any $\alpha > D$. We therefore deduce from the definition (2.2) of $D(\zeta_A)$ that $D(\zeta_A) = \overline{\dim}_B A$. This completes the proof of part (b).

(c) Condition $\mathcal{M}_*^D(A) > 0$ implies that for any fixed $\delta > 0$ there exists $C > 0$ such that for all $t \in (0, \delta)$, we have $|A_t| \geq C t^{N-D}$. Using (2.4) and Lemma 2.2, we see that for any $\gamma \in (0, N - D)$,

$$\begin{aligned} \infty &> I(\gamma) := \int_{A_\delta} d(x,A)^{-\gamma} \mathrm{d}x = \delta^{-\gamma}|A_\delta| + \gamma \int_0^\delta t^{-\gamma-1}|A_t|\,dt \\ &\geq \gamma C \int_0^\delta t^{N-D-\gamma-1} \mathrm{d}t = \gamma C \frac{\delta^{N-D-\gamma}}{N-D-\gamma}. \end{aligned}$$

Therefore, if $\gamma \to N - D$ from the left, then $I(\gamma) \to +\infty$. Equivalently, if $s \in \mathbb{R}$ is such that $s \to D^+$, then $\zeta_A(s) \to +\infty$. Hence, ζ_A has a singularity at $s = D$. Since, in light of part (a), we know that ζ_A is holomorphic for $\operatorname{Re} s > D$, we deduce that $\{\operatorname{Re} s > D\}$ is the maximal right half-plane to which ζ_A can be holomorphically continued; i.e., $\mathcal{H}(\zeta_A) = \{\operatorname{Re} s > D\}$ and so $D_{\mathrm{hol}}(\zeta_A) = D$. Since, in light of part (b) (and because $\dim_B A$ exists, according to the assumptions of part (c)), $D := \dim_B A = D(\zeta_A)$, we conclude that (2.8) holds and hence, the proof of part (c) is complete. This concludes the proof of the theorem. □

Remark 2.6. An alternative proof of part (a) of Theorem 2.5 can be given by using a well-known theorem concerning the holomorphicity of functions defined by integrals on A_δ depending holomorphically on a parameter. In applying this theorem (see [LapRaŽu1] and the text of Definition 2.12 below) one needs to use the (obvious) fact according to which the function $x \mapsto d(x, A)$ is bounded from above (by δ); in other words, ζ_A (as defined by (2.1)) is a tamed DTI (in the sense of Definition 2.12 below).

Next, we comment on some of the hypotheses and conclusions of Theorem 2.5.

Remark 2.7. (i) The condition $\mathcal{M}_*^D(A) > 0$ in the hypotheses of Theorem 2.5(c) cannot be omitted. Indeed, for $N = 1$, there exists a class of subsets $A \subset [0, 1]$ such that $D = \dim_B A$ exists and $\mathcal{M}_*^D(A) = 0$, while $\zeta_A(D) = \int_{A_\delta} d(x, A)^{D-N} \mathrm{d}x < \infty$; see [Žu4, Theorem 4.3].

This class of bounded subsets of $\mathbb{R}$ can be easily extended to $\mathbb{R}^N$ for any $N \geq 2$ by letting $B := A \times [0, 1]^{N-1} \subset [0, 1]^N$.

(ii) The inequality $D_{\mathrm{hol}}(\zeta_A) \leq D(\zeta_A)$ is sharp. Indeed, there exist compact subsets of $\mathbb{R}^N$ such that $D_{\mathrm{hol}}(\zeta_A) = D(\zeta_A)$. For example, $A = C \times [0, 1]^{N-1}$, where C is the ternary Cantor set or, more generally, $C = \partial\Omega$ is the boundary of any (nontrivial) fractal string $\Omega \subset \mathbb{R}$. (In that case, we have $D_{\mathrm{hol}}(\zeta_A) = D(\zeta_A) = \overline{\dim}_B A = N - 1 + \dim_B C$.) This follows from Theorem 2.10 in Section 2.3 below and the comment following it.

(iii) The assumptions of part (c) of Theorem 2.5 are satisfied by most fractals of interest to us. (One notable exception is the boundary A of the Mandelbrot set (viewed as as a subset of $\mathbb{R}^2 \simeq \mathbb{C}$), for which $\dim_H A = 2$ (and consequently, $\dim_B A = 2$), according to Shishikura's well-known theorem [Shi].) We note that, on the other hand, there exists a bounded subset of $\mathbb{R}^N$ not satisfying the hypotheses of part (c) of Theorem 2.5 and such that $D_{\mathrm{hol}}(\zeta_A) < D(\zeta_A)$. Indeed, an easy computation shows that, for example, for $N = 1$ and $A = [0, 1]$, we have that $D_{\mathrm{hol}}(\zeta_A) = 0$ and $D(\zeta_A) = 1$. At present, however, we do not know whether there exist nontrivial subsets A of $\mathbb{R}$ (or, more generally, of $\mathbb{R}^N$) for which $D_{\mathrm{hol}}(\zeta_A) < D(\zeta_A)$.

2.3. **Zeta functions of fractal strings and of associated fractal sets.** In Example 2.9 below, we show that Definition 2.1 provides a natural extension of the zeta function associated with a (bounded) *fractal string* $\mathcal{L} = (\ell_j)_{j\geq 1}$, where $(\ell_j)_{j\geq 1}$ is a nonincreasing sequence of positive numbers such that $\sum_{j=1}^{\infty} \ell_j < \infty$:

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s, \tag{2.14}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. Note that the sequence $(\ell_j)_{j\geq 1}$ of positive numbers is assumed to be infinite.

The study of zeta functions of fractal strings arose naturally in the early 1990s in joint work of the first author (Michel Lapidus) with Carl Pomerance [LapPo1–3] and with Helmut Maier [LapMa1–2] (see also, e.g., [Lap1–3] and [HeLap]) while investigating direct and inverse spectral problems associated with the vibrations of a fractal string. Such a zeta function, $\zeta_{\mathcal{L}}$, called the *geometric zeta function* of $\mathcal{L}$, has since then been studied in a number of references, including several monographs; see [Lap-vFr1–3]. (See also the broader list of references given in the introduction.)

Recall that, geometrically, a *fractal string* is a bounded open set $\Omega \subseteq \mathbb{R}$. It can be uniquely written as a disjoint union of open intervals I_j ($\Omega = \cup_{j=1}^{\infty} I_j$) with lengths ℓ_j (i.e., $\ell_j = |I_j|$ for all $j \geq 1$). Without loss of generality, one may assume that $(\ell_j)_{j\geq 1}$ is written in nonincreasing order and that $\ell_j \to 0$ as $j \to \infty$: $\ell_1 \geq \ell_2 \geq \cdots$. In order to avoid trivial special cases, we will assume implicitly throughout this paper that $\mathcal{L}$ is nontrivial; i.e., that $\mathcal{L}$ consists of an infinite sequence of lengths (or 'scales') and hence, that Ω does not consist of a finite union of bounded open intervals. If $\mathcal{L}$ is trivial, then we must replace $D_{\rm hol}(\zeta_{\mathcal{L}})$ by $\max\{D_{\rm hol}(\zeta_{\mathcal{L}}), 0\}$ in (2.15) of Theorem 2.8 (since then, $D_{\rm hol}(\zeta_{\mathcal{L}}) = -\infty$ and $D(\zeta_{\mathcal{L}}) = \delta_{\partial\Omega} \geq 0$). From the point of view of fractal string theory, one may identify a fractal string with the sequence $\mathcal{L}$ of its lengths (or *scales*): $\mathcal{L} = (\ell_j)_{j\geq 1}$. The bounded open set Ω is then called a *geometric realization of* $\mathcal{L}$. Note that $|\Omega| = \sum_{j=1}^{\infty} \ell_j < \infty$, where $|\Omega| = |\Omega|_1$ denotes the 1-dimensional Lebesgue measure (or length) of Ω.

We now recall a basic property of $\zeta_{\mathcal{L}}$, first observed in [Lap2], using a key result of Besicovich and Taylor [BesTay]. (For a direct proof, see [Lap-vFr3, Theorem 1.10 or Theorem 13.111]; see also [LapLu-vFr2].)

Theorem 2.8. *If $\mathcal{L}$ is a nontrivial fractal string (i.e., $\mathcal{L} = (\ell_j)_{j\geq 1}$ is an infinite sequence), then the abscissa of convergence $D(\zeta_{\mathcal{L}})$ of $\zeta_{\mathcal{L}}$ coincides with the (inner) Minkowski dimension $\delta_{\partial\Omega}$ of $\partial\mathcal{L} = \partial\Omega$:*

$$D(\zeta_{\mathcal{L}}) = D_{\rm hol}(\zeta_{\mathcal{L}}) = \delta_{\partial\Omega}. \tag{2.15}$$

Recall that, by definition,

$$D(\zeta_{\mathcal{L}}) := \inf\Big\{\alpha \in \mathbb{R} \ : \ \sum_{j=1}^{\infty} \ell_j^{\alpha} < \infty\Big\}, \tag{2.16}$$

while $\delta_{\partial\Omega}$ is then defined in terms of the volume (i.e., length) of the inner epsilon (or tubular) neighborhoods of $\partial\Omega$, namely, $(\partial\Omega)_{\varepsilon} \cap \Omega = \{x \in \Omega \ : \ d(x, \partial\Omega) < \varepsilon\}$; see [Lap-vFr3, Chapter 1]. In [Lap-vFr3], the abscissa of convergence $D(\zeta_{\mathcal{L}})$ of $\zeta_{\mathcal{L}}$ is denoted by $\sigma_{\mathcal{L}}$.

In order to establish the equality $D(\zeta_{\mathcal{L}}) = D_{\rm hol}(\zeta_{\mathcal{L}})$ from Theorem 2.8, one first notes that $\zeta_{\mathcal{L}}$ is holomorphic for $\operatorname{Re} s > D(\zeta_{\mathcal{L}})$ and that $\{\operatorname{Re} s > D(\zeta_{\mathcal{L}})\}$ is the largest open right half-plane having this property; i.e., $D(\zeta_{\mathcal{L}}) = D_{\rm hol}(\zeta_{\mathcal{L}})$. The latter property follows from the fact that (because $\zeta_{\mathcal{L}}(s)$ is initially given in (2.14) by a Dirichlet series with positive coefficients) $\zeta_{\mathcal{L}}(s) \to +\infty$ as $s \to D^+$, $s \in \mathbb{R}$, where $D := D(\zeta_{\mathcal{L}}) = \delta_{\partial\Omega}$; see, e.g., [Ser, Section VI.2.3]. The proof of the equality $D(\zeta_{\mathcal{L}}) = \delta_{\partial\Omega}$ requires significantly more work; see the aforementioned references.

Note that, more precisely, $\overline{\dim}_B A_{\mathcal{L}} = \delta_{\partial\Omega}$ is equal to $\overline{\dim}_B(\partial\Omega, \Omega)$, the Minkowski dimension of $\partial\Omega$ relative to Ω (also called the inner Minkowski dimension of $\partial\Omega$, or, equivalently, of $\mathcal{L}$) which is defined (as in [Lap1–3] and [Lap-vFr3, Chapter 1]) in terms of the volume (i.e., length) of the inner tubular neighborhoods of Ω. More specifically, $\delta_{\partial\Omega}$ is given by (1.2) or (1.3), except for $|A_t|$ replaced by $|A_t \cap \Omega|_1$, with $A := \partial\Omega$, in the counterpart of the second equality of (1.1).

In fractal string theory, one is particularly interested in the meromorphic continuation of $\zeta_{\mathcal{L}}$ to a suitable region (when it exists), along with its poles, which are called the *complex dimensions* of $\mathcal{L}$. In particular, in the theory of complex dimensions developed in [Lap-vFr1–3], are obtained explicit formulas applicable to various counting functions associated with the geometry and the spectra of fractal strings,

as well as to $V(\varepsilon)$, now defined as the volume of the inner tubular neighborhood of $\partial\Omega$ (i.e., of $\mathcal{L}$). These explicit formulas are expressed in terms of the complex dimensions (i.e., the poles of $\zeta_{\mathcal{L}}$) and the associated residues. Furthermore, they enable one to obtain a very precise understanding of the oscillations underlying the geometry and spectra of fractal strings (as well as of more general fractal-like objects); see [Lap-vFr3], especially Chapters 5–8.

From the perspective of the theory developed in the present work, a convenient choice for the set $A_{\mathcal{L}}$ corresponding to the fractal string $\mathcal{L} = (\ell_j)_{j\geq 1}$ is

$$A_{\mathcal{L}} := \{a_k : k \geq 1\}, \quad \text{where} \quad a_k := \textstyle\sum_{j\geq k} \ell_j \quad \text{for each} \quad k \geq 1. \tag{2.17}$$

As follows easily from Theorem 2.8 and the definition of $A_{\mathcal{L}}$, the function $\zeta_{\mathcal{L}}$ in (2.14) is holomorphic for $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A_{\mathcal{L}}$. Moreover, this bound is optimal. In other words, $\overline{\dim}_B A$ coincides with the abscissa of convergence of $\mathcal{L}$. Furthermore, $\zeta_{\mathcal{L}}(s) \to +\infty$ as $s \in \mathbb{R}$ converges to $\overline{\dim}_B A_{\mathcal{L}}$ from the right (see [Lap-vFr3, p. 15]); compare with Theorem 2.5 above. In light of Theorem 2.8, Theorem 2.5(b) and Equation (2.15), we then have the following equalities:

$$\overline{\dim}_B A_{\mathcal{L}} = D(\zeta_{A_{\mathcal{L}}}) = D(\zeta_{\mathcal{L}}) = D_{\mathrm{hol}}(\zeta_{\mathcal{L}}) = \delta_{\partial\Omega}. \tag{2.18}$$

The following example shows that the study of the geometric zeta function $\zeta_{\mathcal{L}}$ of any (bounded) fractal string $\mathcal{L}$ can be reduced to the study of the distance zeta function $\zeta_{A_{\mathcal{L}}}$ of the associated bounded set $A_{\mathcal{L}}$ on the real line. (See also Remark 2.11 below.)

Example 2.9. Let $(I_k)_{k\geq 1}$ be a sequence of bounded intervals, $I_k = (a_{k+1}, a_k)$, $k \geq 1$, where the a_k's are defined by (2.17), and let s be a complex variable. Using (2.1), we see that the distance zeta function of $A = A_{\mathcal{L}}$ is given by

$$\zeta_A(s) = 2\int_0^{\delta} x^{s-1}\mathrm{d}x + \sum_{k=1}^{\infty}\int_{I_k} d(x, \partial I_k)^{s-1}\mathrm{d}x = 2s^{-1}\delta^s + \sum_{k=1}^{\infty} J_k(s), \tag{2.19}$$

where the first term in this last expression corresponds to the boundary points of the interval $(0, a_1)$. Assuming that $\delta \geq \ell_1/2$, we have that for all $k \geq 1$,

$$J_k(s) = s^{-1}2^{1-s}\ell_k^s. \tag{2.20}$$

We also assume that $s \in \mathbb{C}$ is such that $\operatorname{Re} s > D(\zeta_{\mathcal{L}})$, so that the series $\sum_{k=1}^{\infty} J_k(s)$ appearing in (2.19) is convergent. In light of (2.14)–(2.17) and (2.19), we then obtain the following relation:

$$\zeta_A(s) = s^{-1}2^{1-s}\zeta_{\mathcal{L}}(s) + 2s^{-1}\delta^s. \tag{2.21}$$

The case when $0 < \delta < \ell_1/2$ yields an analogous relation:

$$\zeta_A(s) = u(s)\zeta_{\mathcal{L}}(s) + v(s), \tag{2.22}$$

where again $u(s) := s^{-1}2^{1-s}$, with a simple pole at $s = 0$. Note that here, $u(s)$ and $v(s) = v(s, \delta)$ are holomorphic functions in the right half-plane $\{\operatorname{Re} s > 0\}$. Hence, by the principle of analytic continuation and since $\zeta_{\mathcal{L}}$ is holomorphic for $\operatorname{Re} s > \overline{\dim}_B A$, the same relation still holds for the meromorphic extensions of ζ_A and of $\zeta_{\mathcal{L}}$ (when they exist, see Theorem 2.10) within the right half-plane $\{\operatorname{Re} s > 0\}$.

The following result is in accordance with Theorem 2.8.

Theorem 2.10. *Let $\mathcal{L} = (\ell_j)_{j\geq 1}$ be a (nontrivial) fractal string such that $\sum_{j\geq 1} \ell_j < \infty$, and let $A_{\mathcal{L}} = \left\{a_k = \sum_{j\geq k} \ell_j : k \geq 1\right\}$. Then*

$$D(\zeta_{A_{\mathcal{L}}}) = D(\zeta_{\mathcal{L}}) = D_{\mathrm{hol}}(\zeta_{\mathcal{L}}) = \overline{\dim}_B A_{\mathcal{L}}. \tag{2.23}$$

Furthermore, given $c \geq 0$, the sets of poles of the meromorphic extensions of $\zeta_{A_{\mathcal{L}}}$ and $\zeta_{\mathcal{L}}$ (if one, and therefore both, of the extensions exist) to the open right half-plane $\{\operatorname{Re} s > c\}$ coincide. Moreover, the poles of $\zeta_{A_{\mathcal{L}}}$ and $\zeta_{\mathcal{L}}$ (in such a half-plane) have the same multiplicities.

More generally, given any subdomain U of $\mathbb{C} \setminus \{0\}$, $\zeta_{A_{\mathcal{L}}}$ has a meromorphic continuation to U if and only $\zeta_{\mathcal{L}}$ does, and in that case, $\zeta_{A_{\mathcal{L}}}$ and $\zeta_{\mathcal{L}}$ have the same visible poles in U and with the same multiplicities.

Proof. The first claim follows from Theorem 2.8 combined with parts (a) and (b) of Theorem 2.5. The second and the third claims are an immediate consequence of the identity (2.22) in Example 2.9. □

Remark 2.11. An entirely similar proof shows that, in Example 2.9 and Theorem 2.10, we can replace $A_{\mathcal{L}}$ with $A := \partial\Omega$, where the bounded open set $\Omega \subset \mathbb{R}$ is any geometric realization of the (nontrivial) fractal string $\mathcal{L}$, provided $\overline{\dim}_B A := \delta_{\partial\Omega}$, as defined in the comments following (2.16). Hence, with the notation used in (2.16), we also have the following counterpart of (2.18) in this more general situation:

$$D(\zeta_{\mathcal{L}}) = D_{\mathrm{hol}}(\zeta_{\mathcal{L}}) = D(\zeta_{\partial\Omega}) = \delta_{\partial\Omega} := \overline{\dim}_B(\partial\Omega, \Omega). \tag{2.24}$$

2.4. **Equivalent zeta functions.** In this section, we shall introduce an equivalence relation $\sim$ on the set of zeta functions (see Definition 2.22). Let us illustrate its purpose in the case of the distance zeta function ζ_A of a given decreasing infinite sequence $A = (a_k)_{k\geq 1}$, converging to zero in $\mathbb{R}$. As we saw in Example 2.9, it makes sense to identify it with its simpler form $\zeta_{\mathcal{L}}$, where $\mathcal{L} = (\ell_j)_{j\geq 1}$ is the associated bounded fractal string, defined by $\ell_j = a_j - a_{j+1}$. This is done by removing the inessential functions $u(s)$ and $v(s)$ appearing in Equation (2.22) above. Therefore, $\zeta_A \sim \zeta_{\mathcal{L}}$.

Throughout this section (and Appendix A in which this topic is further developed), we will assume that E is a locally compact, Hausdorff topological (and metrizable) space and that μ is a *local* (roughly speaking, locally bounded) positive or complex measure (in the sense of [DolFr], [JohLap], [JohLapNi] or [Lap-vFr3, Chapter 4]). In short, a *local measure* is a $[0, +\infty]$-valued or $\mathbb{C}$-valued set-function on $\mathcal{B} := \mathcal{B}(E)$ (the Borel σ-algebra of E), whose restriction to $\mathcal{B}(K)$, where K is an arbitrary compact subset of E, is a bounded positive measure or is a complex (and hence, bounded) measure, respectively. The *total variation measure* of μ (see, e.g., [Coh] or [Ru]) is denoted by $|\mu|$; it is a (local) positive measure and, if μ is itself positive, then $|\mu| = \mu$. We refer to [Coh, Fol, Ru] for the theory of standard positive or complex measures.

We assume that the μ-measurable function $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ appearing in Definition 2.12 just below is *tamed*, in the following sense: there exists a positive constant $C = C(\varphi)$ such that

$$|\mu|(\{\varphi > C\}) = 0; \tag{2.25}$$

i.e., φ is essentially bounded from above with respect to $|\mu|$. We then say that f, defined by (2.26) below, is a *tamed* DTI.

Definition 2.12. Given a *tamed Dirichlet-type integral* (tamed DTI, in short) function $f = f(s)$ of the form

$$f(s) := \int_E \varphi(x)^s \, \mathrm{d}\mu(x), \tag{2.26}$$

where μ is a suitable (positive or complex) local (i.e., locally bounded) measure on a given (measurable) space E [i.e., $\mu : \mathcal{B} \to [0, +\infty]$ or $\mu : \mathcal{B} \to \mathbb{C}$], and $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is a μ-measurable function such that $\varphi \geq 0$ μ-a.e. on E, we define the *abscissa of convergence* $D(f) \in \mathbb{R} \cup \{\pm\infty\}$ by

$$\begin{aligned} D(f) &:= \inf \left\{ \alpha \in \mathbb{R} : \int_E \varphi(x)^\alpha \mathrm{d}|\mu|(x) < \infty \right\} \\ &= \inf \left\{ \alpha \in \mathbb{R} : \varphi(x)^s \text{ is Lebesgue integrable for } \operatorname{Re} s > \alpha \right\}. \end{aligned} \tag{2.27}$$

It follows that the *half-plane of* (*absolute*) *convergence of* f, namely, $\Pi(f) := \{\operatorname{Re} s > D(f)\}$, is the *maximal* open right half-plane (of the form $\{\operatorname{Re} s > \alpha\}$, for some $\alpha \in \mathbb{R} \cup \{\pm\infty\}$) on which the function $x \mapsto \varphi(x)^s$ is absolutely (i.e., Lebesgue) integrable. (Note that $D(f)$ is well defined for any tamed Dirichlet-type integral f.)

In (2.27), by definition, $\inf \emptyset := +\infty$ and $\inf \mathbb{R} = -\infty$. Using a classic theorem about the holomorphicity of integrals depending analytically on a parameter, one can show that f is holomorphic on $\{\operatorname{Re} s > D(f)\}$. Hence, it follows that $D_{\mathrm{hol}}(f) \leq D(f)$. Here, $D_{\mathrm{hol}}(f) \in \mathbb{R} \cup \{\pm\infty\}$, the *abscissa of holomorphic continuation of* f, is defined exactly as $D_{\mathrm{hol}}(\zeta_A)$ in (2.3), except for ζ_A replaced by f.

In (2.27), the integral is taken with respect to $|\mu|$, the total variation measure of μ; recall that if μ is positive, then $|\mu| = \mu$. Note that we may clearly replace $\varphi(x)^s$ by $\varphi(x)^{\operatorname{Re} s}$ in the second equality of (2.27), since for a measurable function, Lebesgue integrability is equivalent to absolute integrability.

Remark 2.13. There are many examples for which $D_{\mathrm{hol}}(f) = D(f)$ (see, e.g., Equation (2.23) in Theorem 2.10) and other examples for which $D_{\mathrm{hol}}(f) < D(f)$ (this is so for Dirichlet L-functions with a nontrivial primitive character, in which case $D_{\mathrm{hol}}(f) = -\infty$ but $D(f) = 1$; see, e.g., [Ser, Section VI.3]).

Remark 2.14. All of the fractal zeta functions encountered in this work, namely, the distance and tube zeta functions (see Section 2.1 above and Section 3.2 below), their counterparts for relative fractal drums (see [LapRaŽu3] along with [LapRaŽu1–2]), the geometric zeta function of (possibly generalized) fractal strings ([Lap-vFr3, Chapters 1 and 4]), as well as the spectral zeta functions of (relative) fractal drums (see [Lap2–3], along with [LapRaŽu1,6]) are tamed DTIs, i.e., they are Dirichlet-type integrals (in the sense of (2.26), and for a suitable choice of set E, function φ and measure μ) satisfy condition (2.25). This justifies, in particular, the use of the expression "abscissa of (absolute) convergence" and "half-plane of (absolute) convergence" for all of these fractal zeta functions, including the tube and distance zeta functions which are key objects in the present paper.

For example, for the distance zeta function ζ_A (as in Definition 2.1 above), we can choose $E := A_\delta$ (or else, $E := A_\delta \setminus \overline{A}$), $\varphi(x) := d(x, A)$ for $x \in E$ and $\mu(\mathrm{d}x) := d(x, A)^{-N} \mathrm{d}x$, while for the tube zeta function (as in Definition 3.6 below), we can choose $E := (0, \delta)$, $\varphi(t) := t$ for $t \in E$ and $\mu(\mathrm{d}x) := t^{-N-1}|A_t|\mathrm{d}t = t^{-N}|A_t|\,(\mathrm{d}t/t)$. In both cases, it is easy to check that the tameness condition (2.25) is satisfied, with $C := \delta$.

In closing, we note that the class of tamed Dirichlet-type integrals also contains all arithmetic zeta functions (that is, all zeta functions occurring in number theory); see, e.g., [ParSh1–2, Ser, Tit], [Lap-vFr3, Appendix A] and [Lap4], including Appendices B, C and E, as well as the relevant references therein.

Recall from part (b) of Theorem 2.5 that we have the following result, which is very useful for the computation of the upper box dimension of fractal sets (for applications, we refer the interested reader to [LapRaŽu1–3]).

Corollary 2.15. *Let A be any bounded subset of $\mathbb{R}^N$. Then*

$$\overline{\dim}_B A = D(\zeta_A). \tag{2.28}$$

Hence, we have $0 \leq D(\zeta_A) \leq N$.

Following [Lap-vFr3, Sections 1.2.1 and 5.1], assume that the set A has the property that ζ_A can be extended to a meromorphic function defined on $G \subseteq \mathbb{C}$, where G is an open and connected neighborhood of the *window* W defined by

$$W := \{s \in \mathbb{C} : \operatorname{Re} s \geq S(\operatorname{Im} s)\}.$$

Here, the function $S : \mathbb{R} \to (-\infty, D(\zeta_A)]$, called the *screen*, is assumed to be Lipschitz continuous. Note that the closed set W contains the *critical line* (of convergence) $\{\operatorname{Re} s = D(\zeta_A)\}$. In other words, we assume that A is such that its distance zeta function can be extended meromorphically to an open domain G containing the closed right half-plane $\{\operatorname{Re} s \geq D(\zeta_A)\}$. (Following the usual conventions, we still denote by ζ_A the meromorphic continuation of ζ_A to G, which is necessarily unique due to the principle of analytic continuation. Furthermore, as in [Lap-vFr3], we assume that the screen does not contain any poles of ζ_A.) A set A satisfying this property and for which ζ_A is 'languid' (in the sense of [Lap-vFr3, Definition 5.2], that is, grows at most polynomially along the screen and a suitable sequence of horizontal lines avoiding the poles of ζ_A) is said to be *admissible*. (There exist nonadmissible fractal sets; see [Lap-vFr3, Example 5.32] and [LapRaŽu1].) In the present article, we will need to consider *the set of poles of ζ_A located on the critical line* $\{\operatorname{Re} s = D(\zeta_A)\}$, where $D(\zeta_A)$ is assumed to be a real number (see Definition 2.18):

$$\mathcal{P}_c(\zeta_A) = \{\omega \in W : \omega \text{ is a pole of } \zeta_A \text{ and } \operatorname{Re}\omega = D(\zeta_A)\}. \tag{2.29}$$

It is a subset of *the set of all poles of ζ_A* in W, that we denote by $\mathcal{P}(\zeta_A)$ or $\mathcal{P}(\zeta_A, W)$ (see Definition 2.19).

Remark 2.16. We assume in the definition of $\mathcal{P}_c(\zeta_A)$ that $D(\zeta_A) \in \mathbb{R}$, which is the case for example if A is bounded, according to Corollary 2.15. Note that clearly (and in contrast to $\mathcal{P}(\zeta_A) = \mathcal{P}(\zeta_A, W)$, to be introduced in Definition 2.19), $\mathcal{P}_c(\zeta_A)$ is independent of the choice of the window W.

Remark 2.17. We stress that because, in this paper, we will not use or extend the pointwise and distributional explicit formulas obtained in [Lap-vFr1–3] (and for the validity of which the above polynomial growth conditions are essential, see [Lap-vFr3, Chapters 5 and 8]), we do not need to include these polynomial growth conditions in the above definition of admissibility. (Fractal tube formulas in the context of the present higher-dimensional theory of fractal dimensions, are obtained in [LapRaŽu4–5]; see also [LapRaŽu1, Chapter 5].) Therefore, throughout this article, an admissible set A is one for which a meromorphic continuation of ζ_A exists

in a connected open neighborhood of the given window W (but without requiring any growth conditions on ζ_A). The problem of constructing meromorphic extensions of fractal zeta functions is studied in [LapRaŽu2], under suitable hypotheses on the geometry of A; see also [LapRaŽu1, Sections 2.3, 4.3 and 4.5].

In [LapRaŽu2] (as well as in [LapRaŽu3], the counterpart for relative fractal drums of the present paper and of its continuation in [LapRaŽu2]), we will provide many examples of computations of the set of complex dimensions $\mathcal{P}(\zeta_A)$ (in Definition 2.19 below) and of principal complex dimensions $\mathcal{P}_c(\zeta_A)$ (in Equation (2.29) above and in Definition 2.18 just below) for a variety of fractals (and relative fractal drums).

The following definition is a slight modification of the notion of complex dimension for fractal strings introduced by the first author and Machiel van Frankenhuijsen in [Lap-vFr1], which depends not only on the string, but also on the window W; see [Lap-vFr3, Section 1.2.1].

Definition 2.18. Let A be an admissible subset of $\mathbb{R}^N$ such that $D(\zeta_A) \in \mathbb{R}$. The *set of principal complex dimensions* of A, denoted by $\dim_{PC} A$, is defined as the set of poles of ζ_A which are located on the critical line $\{\operatorname{Re} s = D(\zeta_A)\}$:

$$\dim_{PC} A := \mathcal{P}_c(\zeta_A), \tag{2.30}$$

where $\mathcal{P}_c(\zeta_A)$ is given by (2.29).

As we see, in Definition 2.18, if $A \subset \mathbb{R}^N$ is bounded, the singularities of ζ_A we are interested in are located on the vertical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$.

Following and extending the definition of complex dimensions of fractal strings (and other fractals) provided in [Lap-vFr1–3], we also introduce the following natural higher-dimensional generalization in our context.

Definition 2.19. Let A be an admissible subset of $\mathbb{R}^N$. Then, the *set of visible complex dimensions* of A *with respect to a given window* W (often called in short, the *set of complex dimensions of* A *relative to* W, or simply the *set of* (*visible*) *complex dimensions* of A if no ambiguity may arise or if $W = \mathbb{C}$), is defined as the set of all the poles of ζ_A which are located in the window W:

$$\mathcal{P}(\zeta_A) = \{\omega \in W : \omega \text{ is a pole of } \zeta_A\}. \tag{2.31}$$

Instead of $\mathcal{P}(\zeta_A)$ we can also write $\mathcal{P}(\zeta_A, W)$, in order to stress that the set depends on W as well.

Next, we would like to extend the class of zeta functions to which a slight modification of Definition 2.18 and Definition 2.19 can be applied. Given a meromorphic function f on a domain $G \subseteq \mathbb{C}$ containing the vertical line $\{\operatorname{Re} s = D(f)\}$ (as in Remark 2.16 above, we assume here that $D(f) \in \mathbb{R}$), and which (for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large) is given by a convergent Dirichlet-type integral of the form (2.26) and satisfying condition (2.25), so that $D(f) < \infty$ is well defined by (2.27)), we define the set $\mathcal{P}_c(f)$ in much the same way as in (2.29):

$$\mathcal{P}_c(f) = \{\omega \in G : \omega \text{ is a pole of } f \text{ and } \operatorname{Re}\omega = D(f)\}. \tag{2.32}$$

It is a subset of the set $\mathcal{P}(f)$ of all the poles of f belonging to G. In other words,

$$\mathcal{P}(f) = \{\omega \in G : \omega \text{ is a pole of } f\}. \tag{2.33}$$

Remark 2.20. If $f = \zeta_A$, where A is an admissible set for a given window W, then (with $G := \mathring{W}$, the interior of the window) $\mathcal{P}_c(f) = \mathcal{P}_c(\zeta_A)$, the set of principal complex dimensions of A, while $\mathcal{P}(f, \mathring{W}) = \mathcal{P}(f) = \mathcal{P}(\zeta_A) = \mathcal{P}(\zeta_A, W)$, the set of (visible) complex dimensions of A (relative to W). This follows from the fact that since A is admissible, ζ_A does not have any poles along the screen S; see the discussion following Corollary 2.15.

Remark 2.21. Observe that $\mathcal{P}_c(f)$ is independent of the choice of the domain G containing the vertical line $\{\operatorname{Re} s = D(f)\}$. Moreover, since as was noted earlier, the function f is holomorphic for $\operatorname{Re} s > D(f)$, there are no poles of f located in the open half-plane $\{\operatorname{Re} s > D(f)\}$; this is why we could equivalently require that the domain $G \subseteq \mathbb{C}$ contains the closed half-plane $\{\operatorname{Re} s \geq D(f)\}$ in order to define $\mathcal{P}_c(f)$ and $\mathcal{P}(f)$.

Finally, we note that since $\mathcal{P}(f)$ is the set of poles of a meromorphic function, it is a discrete subset of $\mathbb{C}$; in particular, it is at most countable. Since $\mathcal{P}_c(f) \subseteq \mathcal{P}(f)$, the same is true for $\mathcal{P}_c(f)$. (An entirely analogous comment can be made about $\mathcal{P}_c(\zeta_A)$ and $\mathcal{P}(\zeta_A)$ in Definition 2.18 and Definition 2.19, respectively.)

We next define the equivalence of a given distance zeta function f to a suitable meromorphic function g (of a preferably simpler form), a notion which will be useful to us in the sequel. Note that the relation $\sim$ introduced in Definition 2.22 is clearly an equivalence relation on the set of all tamed DTIs.

Definition 2.22. Let f and g be tamed Dirichlet-type integrals, as in Definition 2.12, both admitting a (necessarily unique) meromorphic extension to an open connected subset U of $\mathbb{C}$ which contains the closed right half-plane $\{\operatorname{Re} s \geq D(f)\}$. (As follows from the complete definition, this closed half-plane is actually the closure of the common half-plane of convergence of f and g, given by $\Pi := \Pi(f) = \Pi(g)$.) Then, the function f is said to be *equivalent* to g, and we write $f \sim g$, if $D(f) = D(g)$ (and this common value is a real number) and furthermore, the sets of poles of f and g, located on the common critical line $\{\operatorname{Re} s = D(f)\}$, coincide. Here, the multiplicities of the poles should be taken into account. In other words, we view the set of principal poles $\mathcal{P}_c(f)$ of f as a multiset. More succinctly,

$$f \sim g \quad \overset{\text{def.}}{\iff} \quad D(f) = D(g)\ (\in \mathbb{R}) \quad \text{and} \quad \mathcal{P}_c(f) = \mathcal{P}_c(g). \tag{2.34}$$

If a tamed Dirichlet-type integral f is given (for example, a distance zeta function ζ_A corresponding to a given fractal set A), the aim is to find an equivalent meromorphic function g, defined by a simpler expression. Satisfactory results can already be obtained with functions g of the form $g(s) = u(s)f(s) + v(s)$, for a suitable choice of the functions u and v, as we have seen in Example 2.9.

We refer to Definition A.2 in Appendix A to this paper for an extension of Definition 2.22 to the broader class of Dirichlet-type integrals (extended DTIs, for short), as introduced in Definition A.1.

We also refer to Definition A.6 (and the comments surrounding it) at the end of Appendix A for a closely related, but somewhat different (and perhaps more practical) definition, allowing the meromorphic function g not to be a DTI (or more generally, an EDTI of type I, in the terminology of Appendix A). These new definitions (Definitions A.2 and A.6) can be applied to (essentially) all the examples of interest in this paper and in [LapRaŽu1–7]. Towards the end of Appendix A, the interested reader can find a large class of functions g giving the "leading behavior"

of fractal zeta functions f. They will arise in practice in the theory developed in this paper and in [LapRaŽu1–7]. (See Theorem A.3 in Appendix A, along with its consequences.)

In the following proposition, we consider the dependence of the distance zeta function ζ_A on $\delta > 0$. For this reason, we denote ζ_A by $\zeta_A(\,\cdot\,, A_\delta)$.

Proposition 2.23. *Let A be a bounded subset of $\mathbb{R}^N$. Then, for any two positive real numbers δ_1 and δ_2, we have $\zeta_A(\,\cdot\,, A_{\delta_1}) \sim \zeta_A(\,\cdot\,, A_{\delta_2})$.*

Proof. We assume without loss of generality that $\delta_1 < \delta_2$, since for $\delta_1 = \delta_2$ there is nothing to prove. For $\operatorname{Re} s > \overline{\dim}_B A$, the difference of the functions $\zeta_A(s, A_{\delta_2})$ and $\zeta_A(s, A_{\delta_1})$ is equal to

$$\int_{A_{\delta_2} \setminus A_{\delta_1}} d(x, A)^{s-N} \mathrm{d}x. \tag{2.35}$$

Note that $\delta_1 \le d(x, A) < \delta_2$ for every $x \in A_{\delta_2} \setminus A_{\delta_1}$. Hence, the integral given by (2.35) is an entire function. □

The following result deals with the scaling property of the distance zeta function. Here, we write $\zeta_A(s, A_\delta) := \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x$ for $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A$.

Proposition 2.24 (Scaling property of distance zeta functions)**.** *For any bounded subset A of $\mathbb{R}^N$, $\delta > 0$ and $\lambda > 0$, we have $D(\zeta_{\lambda A}(\,\cdot\,, \lambda(A_\delta))) = D(\zeta_A(\,\cdot\,, A_\delta)) = \overline{\dim}_B A$ and*

$$\zeta_{\lambda A}(s, \lambda(A_\delta)) = \lambda^s \zeta_A(s, A_\delta), \tag{2.36}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A$. Furthermore, if $\omega \in \mathbb{C}$ is a simple pole of the meromorphic extension of $\zeta_A(s, A_\delta)$ to some open connected neighborhood of the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$ (we use the same notation for the meromophically extended function), then

$$\operatorname{res}(\zeta_{\lambda A}(\,\cdot\,, \lambda(A_\delta)), \omega) = \lambda^\omega \operatorname{res}(\zeta_A, \omega). \tag{2.37}$$

Proof. Equation (2.36) follows easily by noting that $\lambda(A_\delta) = (\lambda A)_{\lambda\delta}$; we leave the details to the interested reader. To prove Equation (2.37), note that $\zeta_{\lambda A}(s, A_\delta) - \zeta_{\lambda A}(s, \lambda(A_\delta))$ is an entire function; see Proposition 2.23. Therefore, also using (2.36), we obtain that

$$\begin{aligned} \operatorname{res}(\zeta_{\lambda A}(\,\cdot\,, A_\delta), \omega) &= \operatorname{res}(\zeta_{\lambda A}(\,\cdot\,, \lambda(A_\delta)), \omega) = \lim_{s \to \omega} (s - \omega) \zeta_{\lambda A}(s, \lambda A) \\ &= \lim_{s \to \omega} (s - \omega) \lambda^s \zeta_A(s, A) = \lambda^\omega \operatorname{res}(\zeta_A, \omega), \end{aligned}$$

which concludes the proof of the proposition. □

This scaling result is useful, in particular, in the study of fractal sprays and self-similar sets in Euclidean spaces; see [LapRaŽu3,5], as well as [LapRaŽu1].

3. Residues of zeta functions and Minkowski contents

In this section, we show that the residue of any suitable meromorphic extension of the distance zeta function ζ_A of a fractal set A in $\mathbb{R}^N$ is closely related to the Minkowski content of the set; see Theorems 3.3 and 3.7. Therefore, the distance zeta functions, as well as the tube zeta functions that we introduce below (see Definition 3.6), can be considered as a useful tool in the study of the geometric properties of fractals.

3.1. **Distance zeta functions of fractal sets and their residues.** Here we use the notation $\zeta_A(s, A_\delta)$ for the distance zeta function instead of $\zeta_A(s)$, in order to stress the dependence of the zeta function on δ. We start with an identity, which will motivate us to introduce a new class of zeta functions, described by (3.6).

Theorem 3.1. *Let A be a bounded subset of $\mathbb{R}^N$, and let δ be a fixed positive number. Then, for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B A$, the following identity holds*:

$$\int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x = \delta^{s-N}|A_\delta| + (N-s)\int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t. \tag{3.1}$$

Furthermore, the function $\tilde{\zeta}_A(s) := \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t$ is absolutely convergent (and hence, holomorphic) on $\{\operatorname{Re} s > \overline{\dim}_B A\}$. The function $\tilde{\zeta}_A$, which we have just introduced, is called the tube zeta function of A (see Definition 3.6) and will be studied in Section 3.2.

Proof. Equality (3.1) holds for all real numbers $s \in (\overline{D}, +\infty)$, where $\overline{D} := \overline{\dim}_B A$. Indeed, it follows immedately from Lemma 2.2, if we take $\gamma := N - s$ (note that then $\gamma < N - \overline{D}$).

Let us denote the left-hand side of (3.1) by $f(s)$, and the right-hand side by $g(s)$. Since $f(s) = g(s)$ on the subset $(\overline{D}, +\infty) \subset \mathbb{C}$, to prove the theorem, it suffices to show that $f(s)$ and $g(s)$ are both holomorphic in the region $\{\operatorname{Re} s > \overline{D}\}$. Indeed, the fact that (3.1) then holds for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{D}$ follows from the principle of analytic continuation; see, e.g., [Con, Corollary 3.8]. The holomorphicity of $f(s)$ in that region is precisely the content of Theorem 2.5(a).

In order to prove the holomorphicity of $g(s)$ on $\{\operatorname{Re} s > \overline{D}\}$, it suffices to show that $\tilde{\zeta}_A(s)$ is absolutely convergent on $\{\operatorname{Re} s > \overline{\dim}_B A\}$. Note that $\tilde{\zeta}_A(s)$ is the Dirichlet-type integral, $\tilde{\zeta}_A(s) = \int_E \varphi(t)^s \mathrm{d}\mu(x)$, where $E := (0, \delta)$, $\varphi(t) := t$, $d\mu(x) := t^{-N-1}|A_t|\,dt$, and the latter measure is positive. Therefore, it suffices to show that for any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{D}$, the Dirichlet-type integral $\tilde{\zeta}_A(s)$ is well defined. To see this, let $\varepsilon > 0$ be small enough, so that $\operatorname{Re} s > \overline{D} + \varepsilon$. Since $\mathcal{M}^{*(\overline{D}+\varepsilon)}(A) = 0$, there exists $C_\delta > 0$ such that $|A_t| \le C_\delta t^{N-\overline{D}-\varepsilon}$ for all $t \in (0, \delta]$. Then

$$\begin{aligned} |\tilde{\zeta}_A(s)| &\le \int_0^\delta t^{\operatorname{Re} s - N - 1}|A_t|\,\mathrm{d}t \\ &\le C_\delta \int_0^\delta t^{\operatorname{Re} s - \overline{D} - \varepsilon - 1}\mathrm{d}t = C_\delta \frac{\delta^{\operatorname{Re} s - \overline{D} - \varepsilon}}{\operatorname{Re} s - \overline{D} - \varepsilon} < \infty, \end{aligned}$$

which concludes the proof of the theorem. □

Corollary 3.2. *If $\overline{\dim}_B A < N$, then*

$$D(\zeta_A) = D(\tilde{\zeta}_A) \quad \textit{and} \quad D_{\mathrm{hol}}(\zeta_A) = D_{\mathrm{hol}}(\tilde{\zeta}_A). \tag{3.2}$$

Proof. This follows at once from Equation (3.1) of Theorem 3.1 and from the definition of $D(f)$ and $D_{\mathrm{hol}}(f)$, for $f = \zeta_A$ or $f = \tilde{\zeta}_A$. □

The following theorem is, in particular, a higher-dimensional generalization of [Lap-vFr3, Theorem 1.17] and yields more information than the latter result, when $N = 1$. (The problem of constructing meromorphic extensions of fractal zeta functions is studied in [LapRaŽu2] and in the monograph [LapRaŽu1].)

Theorem 3.3. *Assume that the bounded set $A \subset \mathbb{R}^N$ is Minkowski nondegenerate (that is, $0 < \mathcal{M}_*^D(A) \leq \mathcal{M}^{*D}(A) < \infty$, and, in particular, $\dim_B A = D$), and $D < N$. If $\zeta_A(\cdot, A_\delta)$ can be extended meromorphically to a neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta_A(\cdot, A_\delta)$, and the value of the residue of $\zeta_A(\cdot, A_\delta)$ at D, $\operatorname{res}(\zeta_A(\cdot, A_\delta), D)$, does not depend on $\delta > 0$. Furthermore,*

$$(N-D)\mathcal{M}_*^D(A) \leq \operatorname{res}(\zeta_A(\cdot, A_\delta), D) \leq (N-D)\mathcal{M}^{*D}(A), \tag{3.3}$$

and in particular, if A is Minkowski measurable, then

$$\operatorname{res}(\zeta_A(\cdot, A_\delta), D) = (N-D)\mathcal{M}^D(A). \tag{3.4}$$

Proof. Since $\mathcal{M}_*^D(A) > 0$, using Theorem 2.5(c) we conclude that $s = D$ is a pole of $\zeta_A = \zeta_A(\cdot, A_\delta)$. Therefore, it suffices to show that the order of the pole at $s = D$ is not larger than 1. Let us take any fixed $\delta > 0$, and let

$$C_\delta = \sup_{t \in (0,\delta]} \frac{|A_t|}{t^{N-D}}. \tag{3.5}$$

Note that $C_\delta < \infty$ because $\mathcal{M}^{*D}(A) < \infty$. Then, in light of (3.1), for all $s \in \mathbb{R}$ with $D < s < N$, we have

$$\begin{aligned} \zeta_A(s, A_\delta) &= \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x = \delta^{s-N}|A_\delta| + (N-s)\int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t \\ &\leq C_\delta \delta^{s-D} + C_\delta(N-s)\frac{\delta^{s-D}}{s-D} = C_\delta(N-D)\delta^{s-D}\frac{1}{s-D}. \end{aligned} \tag{3.6}$$

Therefore, $0 < \zeta_A(s, A_\delta) \leq C_1(s-D)^{-1}$ for all $s \in (D, N)$. This shows that $s = D$ is a pole of $\zeta_A(s, A_\delta)$ which is at most of order 1, and the first claim is established. Namely, D is a simple pole of $\zeta_A(s, A_\delta)$.

The fact that the residue of $\zeta_A(s, A_\delta)$ at $s = D$ is independent of the value of $\delta > 0$ follows immediately from Proposition 2.23. In order to prove the second inequality in (3.3), is suffices to multiply (3.6) by $s - D$, with s real, and take the limit as $s \to D^+$ along the real axis:

$$\operatorname{res}(\zeta_A(\cdot, A_\delta), D) \leq (N-D)\lim_{s \to D^+} C_\delta \delta^{s-D} = (N-D)C_\delta. \tag{3.7}$$

Since the residue of $\zeta_A(s, A_\delta)$ at D does not depend on δ, (3.3) follows from (3.7) by recalling the definition of C_δ given in (3.5) and passing to the limit as $\delta \to 0^+$ (note that the function $\delta \mapsto C_\delta$ is nondecreasing and that $C_\delta \to \mathcal{M}^{*D}(A)$ as $\delta \to 0^+$) on the right-hand side of (3.7). The first inequality in (3.3) is proved analogously by replacing the supremum by an infimum in the definition of C_δ given in (3.5). □

Example 3.4 (Residues of the zeta function of the generalized Cantor set)**.** Let $A = C^{(a)}$ be the generalized Cantor set defined by the parameter $a \in (0, 1/2)$. Recall that $C^{(a)}$ is obtained by deleting the middle interval of length $1 - 2a$ from the interval $[0, 1]$, and then continuing in the usual way, scaling by the factor a at each step (for $a = 1/3$, we obtain the middle third Cantor set, which is studied in detail in [Lap-vFr1–3] from the point of view of geometric zeta functions and the associated complex dimensions). (An even more general class of Cantor sets, depending on two auxilliary paramteres, will be introduced in Definition 4.1.) By a direct computation, or using [Žu3, Equation (15) with $\gamma := N - s$], we obtain the

corresponding zeta function:

$$\zeta_A(s, A_\delta) := \frac{2^{1-s}(1-2a)^s}{s(1-2a^s)} + 2\delta^s s^{-1}. \tag{3.8}$$

Its residue computed at $D = D(a) := \dim_B A = \log_{1/a} 2$ is given by

$$\operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), D) = \frac{2}{\log 2}\left(\frac{1}{2} - a\right)^D. \tag{3.9}$$

On the other hand, the values of the lower and upper D-dimensional Minkowski contents are respectively equal to (see [Žu2, Equations (3.12) and (3.13) for $m = 2$]):

$$\mathcal{M}_*^D(A) = \frac{1}{D}\left(\frac{2D}{1-D}\right)^{1-D}, \quad \mathcal{M}^{*D}(A) = 2(1-a)\left(\frac{1}{2} - a\right)^{D-1}, \tag{3.10}$$

and thus $\mathcal{M}_*^D(A) < \mathcal{M}^{*D}(A)$ (see also Remark 3.5 below). It follows that $C^{(a)}$ is not Minkowski measurable (for a much more general result, see [Lap-vFr3, Theorem 2.16]). (We note that in the case of the classical Cantor set, where $a = 1/3$ and $D = \log_3 2$, the values in (3.10), and hence, the Minkowski nonmeasurability of $C^{(1/3)}$, have been first obtained in [LapPo2, Theorem 2.4].) Therefore, for any generalized Cantor set $A = C^{(a)}$, with $a \in (0, 1/2)$, we have that

$$(1-D)\mathcal{M}_*^D(A) < \operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), D) < (1-D)\mathcal{M}^{*D}(A). \tag{3.11}$$

This is in agreement with (3.3) in Theorem 3.3. In particular, since the functions $(0, 1/2) \ni a \mapsto \mathcal{M}_*^D(A)$ and $a \mapsto \mathcal{M}^{*D}(A)$ are bounded, and $D = \log_{1/a} 2 \to 1^-$ as $a \to 1/2^-$, we have that for any positive δ,

$$\lim_{a \to 1/2^-} \operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), D) = 0.$$

The residues of $\zeta_A(s, A_\delta)$ at the poles $s_k := D + k\mathbf{p}\mathbb{i}$, $k \in \mathbb{Z}$, on the critical line $\{\operatorname{Re} s = D\}$, expressed in terms of the residue at D and the 'oscillatory period' (see [Lap-vFr3]) $\mathbf{p} := 2\pi/\log(1/a)$, are the following:

$$\operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), s_k) = \frac{D 2^{-k\mathbf{p}\mathbb{i}}(1-2a)^{k\mathbf{p}\mathbb{i}}}{s_k a^{k\mathbf{p}\mathbb{i}}} \operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), D), \quad k \in \mathbb{Z}. \tag{3.12}$$

Remark 3.5. As we have already noted, the two inequalities in (3.11) are in agreement with (3.3) in Theorem 3.3. In [LapRaŽu2] (see also [LapRaŽu1]), we shall prove that the strict inequalities in (3.3) are not just a coincidence: indeed, they hold for a large class of Minkowski nonmeasurable sets in Euclidean spaces. An analogous remark applies to the inequalities (3.16) in Theorem 3.7 below, dealing with tube zeta functions. Also, it is shown in [Lap-vFr3, Section 8.4, Theorem 8.23] that all lattice self-similar strings (and hence, under mild assumptions, all lattice self-similar sets in $\mathbb{R}$ satisfying the open set condition) are not Minkowski measurable. In [Lap3], the same result is conjectured to hold in higher dimensions, and we suspect that the methods developed in this paper (as well as in [LapRaŽu1–7] and, especially, in [LapRaŽu4,5]) combined with those of [Lap-vFr3, Section 8.4] should eventually enable one to prove it.

3.2. **Tube zeta functions of fractal sets and their residues.** Going back to Theorem 3.1, we see that it is natural to introduce a new fractal zeta function of bounded subsets A of $\mathbb{R}^N$.

Definition 3.6. Let δ be a fixed positive number, and let A be a bounded subset of $\mathbb{R}^N$. Then, the *tube zeta function* of A, denoted by $\tilde{\zeta}_A$, is defined by

$$\tilde{\zeta}_A(s) = \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t, \tag{3.13}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. As we know from Theorem 3.1, the tube zeta function is (absolutely) convergent (and hence, holomorphic) on the open right half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$.

We call $\tilde{\zeta}_A$ the tube zeta function of A since its definition involves the tube function $(0,\delta) \ni t \mapsto |A_t|$. Relation (3.1) can be written as follows (with $\zeta_A(s) = \zeta_A(s, A_\delta)$, as before, and $\tilde{\zeta}_A(s) = \tilde{\zeta}_A(s, A_\delta)$, for emphasis):

$$\zeta_A(s, A_\delta) = \delta^{s-N}|A_\delta| + (N-s)\tilde{\zeta}_A(s, A_\delta), \tag{3.14}$$

for any $\delta > 0$ and for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B A$.

From the functional equation (3.14) relating ζ_A and $\tilde{\zeta}_A$, it would seem that $\tilde{\zeta}_A$ has a singularity at $s = N$. However, from the second part of Theorem 3.1 we see that for $\overline{\dim}_B A < N$, the value $s = N$ is regular (i.e., holomorphic) for $\tilde{\zeta}_A$. It then follows from (3.14) that the two fractal zeta functions ζ_A and $\tilde{\zeta}_A$ contain essentially the same information.

In particular, still assuming that $\overline{\dim}_B A < N$, $\tilde{\zeta}_A$ has a meromorphic continuation to a given domain $U \subseteq \mathbb{C}$ if and only if ζ_A does, and in that case (according to the principle of analytic continuation), the unique meromorphic continuations to U of ζ_A and $\tilde{\zeta}_A$ are still related by the functional equation (3.14). Also in that case, the residues (or, more generally, the principal parts) of ζ_A and $\tilde{\zeta}_A$ of a given simple (resp., multiple) pole of $s = \omega \in U$ are related in a very simple manner; see, e.g., Equation (3.15) below in the case of the simple pole $s = \overline{\dim}_B A$. Furthermore, $\mathcal{P}(\zeta_A) = \mathcal{P}(\tilde{\zeta}_A)$ and (assuming that U contains the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$), $\mathcal{P}_c(\zeta_A) = \mathcal{P}_c(\tilde{\zeta}_A)$.

Moreover, we have that $D(\tilde{\zeta}_A) = D(\zeta_A)$, $D_{\mathrm{hol}}(\tilde{\zeta}_A) = D_{\mathrm{hol}}(\zeta_A)$ and $D_{\mathrm{mer}}(\tilde{\zeta}_A) = D_{\mathrm{mer}}(\zeta_A)$. (Here, $D_{\mathrm{mer}}(f)$, the *abscissa of meromorphic continuation* of a given meromorphic function f, is defined exactly as $D_{\mathrm{hol}}(f)$ in Equation (2.3) and the surrounding text, except for "holomorphic" replaced by "meromorphic"; and similarly for the half-plane of meromorphic continuation of f.) Also, we have $\Pi(\tilde{\zeta}_A) = \Pi(\zeta_A)$ and $\mathcal{H}(\tilde{\zeta}_A) = \mathcal{H}(\zeta_A)$; similarly, the half-planes of meromorphic continuation of $\tilde{\zeta}_A$ and ζ_A coincide.

Still in light of (3.14), it follows from Theorem 3.1 that $\tilde{\zeta}_A$ is holomorphic on $\{\operatorname{Re} s > \overline{\dim}_B A\}$ and that (provided $\overline{\dim}_B A < N$), the lower bound $\overline{\dim}_B A$ is optimal from the point of the convergence of the Lebesgue integral defining ζ_A in (3.13); i.e., $D(\tilde{\zeta}_A)\,(= D(\zeta_A)) = \overline{\dim}_B A$. More generally, the exact analog of Theorem 2.5 holds for $\tilde{\zeta}_A$ (instead of ζ_A), except for the fact that in the counterpart of part (c) of Theorem 2.5 we no longer need to assume that $D < N$ (where $D := \dim_B A$).

Assuming that there exists a meromorphic extension of $\zeta_A(s, A_\delta)$ to an open connected neighborhood of $\overline{D} := \overline{\dim}_B A$, and $\overline{D}$ is a simple pole, $\overline{D} < N$, then it

easily follows from (3.14) that

$$\operatorname{res}(\tilde{\zeta}_A, \overline{D}) = \frac{1}{N-\overline{D}} \operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), \overline{D}). \tag{3.15}$$

Indeed,

$$\begin{aligned} \operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), \overline{D}) &= \lim_{s\to\overline{D}} (s-\overline{D})[\delta^{s-N}|A_\delta| + (N-s)\tilde{\zeta}_A(s)] \\ &= (N-\overline{D}) \lim_{s\to\overline{D}} (s-\overline{D})\tilde{\zeta}_A(s) \\ &= (N-\overline{D}) \operatorname{res}(\tilde{\zeta}_A, \overline{D}). \end{aligned}$$

Hence, the following result, in the case of $D < N$, is an immediate consequence of Theorem 3.3 and relation (3.1) (or, equivalently, (3.14)), while in the case when $D = N$, it can be shown directly.

Theorem 3.7. *Assume that A is a bounded subset of $\mathbb{R}^N$ such that $D := \dim_B A$ exists, $0 < \mathcal{M}_*^D(A) \le \mathcal{M}^{*D}(A) < \infty$, and there exists a meromorphic extension of $\tilde{\zeta}_A$ to an open neighborhood of D. Then D is a simple pole, and for any positive δ, the value of $\operatorname{res}(\tilde{\zeta}_A, D)$ is independent of δ. Furthermore, we have*

$$\mathcal{M}_*^D(A) \le \operatorname{res}(\tilde{\zeta}_A, D) \le \mathcal{M}^{*D}(A), \tag{3.16}$$

and, in particular, if A is Minkowski measurable, then

$$\operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}^D(A). \tag{3.17}$$

In the following example, we compute the complex dimensions of the unit $(N-1)$-dimensional sphere in $\mathbb{R}^N$, using the tube zeta function of the sphere.

Example 3.8. Let $A := \partial B_1(0)$ be the unit $(N-1)$-dimensional sphere in $\mathbb{R}^N$ centered at the origin. We would like to compute its complex dimensions. To this end, we first compute the corresponding tube zeta function $\tilde{\zeta}_A$. Let us fix any $\delta \in (0,1)$. Since $|A_t| = \omega_N(1+t)^N - \omega_N(1-t)^N$, where $t \in (0,1)$ and ω_N is the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$, we have that for any fixed $\delta \in (0,1)$,

$$\begin{aligned} \tilde{\zeta}_A(s) &= \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t = \omega_N \int_0^\delta t^{s-N-1}((1+t)^N - (1-t)^N)\,\mathrm{d}t \\ &= \omega_N \int_0^\delta t^{s-N-1} \left(\sum_{k=0}^N \binom{N}{k} \big(1-(-1)^k\big) t^k \right) \mathrm{d}t \\ &= \omega_N \sum_{k=1}^N \big(1-(-1)^k\big) \binom{N}{k} \frac{\delta^{s-N+k}}{s-(N-k)}, \end{aligned}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > N-1$. The last expression can be meromorphically extended to the whole complex plane, and we still denote it by $\tilde{\zeta}_A(s)$. Therefore, we have

$$\tilde{\zeta}_A(s) = \omega_N \sum_{k=0}^N \big(1-(-1)^k\big) \binom{N}{k} \frac{\delta^{s-N+k}}{s-(N-k)}, \tag{3.18}$$

for all $s \in \mathbb{C}$, where the constants c_k are defined as above. It follows that

$$\begin{aligned} \dim_B A = D(\tilde{\zeta}_A) = D(\zeta_A) = N-1, \\ \mathcal{P}_c(\tilde{\zeta}_A) = \mathcal{P}_c(\zeta_A) = \{N-1\}, \end{aligned} \tag{3.19}$$

as expected. (Note that $\dim_B A = N-1 < N$, so that $\mathcal{P}_c(\tilde{\zeta}_A) = \mathcal{P}_c(\zeta_A)$ and $\mathcal{P}(\tilde{\zeta}_A) = \mathcal{P}(\zeta_A)$.) Moreover, still in light of (3.18), the set of complex dimensions of A is given by

$$\begin{aligned}\mathcal{P}(\tilde{\zeta}_A) = \mathcal{P}(\zeta_A) &= \Big\{N-(2j+1) : j = 0,1,2,\dots,\Big\lfloor\frac{N-1}{2}\Big\rfloor\Big\} \\ &= \Big\{N-1, N-3, \dots, N-\Big(2\Big\lfloor\frac{N-1}{2}\Big\rfloor+1\Big)\Big\}.\end{aligned} \tag{3.20}$$

For odd N, the last number in this set is equal to 0, while for even N, it is equal to 1. Furthermore, the residue of the tube zeta function $\tilde{\zeta}_A$ at any of its poles $N-k \in \mathcal{P}(\tilde{\zeta}_A)$ is given by $\operatorname{res}(\tilde{\zeta}_A, N-k) = 2\omega_N\binom{N}{k}$; that is,

$$\operatorname{res}(\tilde{\zeta}_A, m) = 2\omega_N\binom{N}{m}, \quad \text{for all} \quad m \in \mathcal{P}(\tilde{\zeta}_A). \tag{3.21}$$

Note that in the case where $m = D := N-1$, we obtain

$$\operatorname{res}(\tilde{\zeta}_A, D) = 2N\omega_N = \mathcal{M}^D(A), \tag{3.22}$$

where the last equality is easily obtained from the definition of the Minkowski content, as follows:

$$\mathcal{M}^D(A) = \lim_{t\to 0^+}\frac{|A_t|}{t^{N-D}} = \lim_{t\to 0^+}\frac{\omega_N(1+t)^N - \omega_N(1-t)^N}{t} = 2N\omega_N.$$

In other words, A is Minkowski measurable and

$$\mathcal{M}^D(A) = 2\,\mathcal{H}^D(A), \tag{3.23}$$

where $\mathcal{H}^D$ denotes the D-dimensional Hausdorff measure. (Equation (3.23) is a special case of a much more general result proved by Federer in [Fed, Theorem 3.2.39].) Equation (3.22) is in agreement with Equation (3.17) in Theorem 3.7.

3.3. **Residues of tube zeta functions of generalized Cantor sets and a-strings.** We provide here two simple examples illustrating some of the main results of this section.

Example 3.9 (Generalized Cantor sets, Example 3.4 continued)**.** As an illustration of inequality (3.16), we consider generalized Cantors sets, $A = C^{(a)}$, $a \in (0, 1/2)$. We obtain

$$\mathcal{M}_*^D(A) < \operatorname{res}(\tilde{\zeta}_A(\,\cdot\,, A_\delta), D) < \mathcal{M}^{*D}(A), \tag{3.24}$$

where the values of the lower and upper Minkowski contents, $\mathcal{M}_*^D(A)$ and $\mathcal{M}^{*D}(A)$, are given by (3.10) and $D = D(a) = \log_{1/a} 2$. It is worth observing that $C^{(a)}$ becomes almost like a Minkowski measurable set for a close to $1/2$, since both $\mathcal{M}^{*D}(A)$ and $\mathcal{M}_*^D(A)$ tend to the common limit 1 as $a \to 1/2^-$. Intuitively, this is to be expected, since in the limit where $a \to (1/2)^-$, $C^{(a)}$ tends (with respect to the Hausdorff metric) to the unit interval $[0,1]$, for which $D = 1$ and $\mathcal{M}^1([0,1]) = 1$. Therefore, both $\mathcal{M}_*^D(C^{(a)})$ and $\mathcal{M}^{*D}(C^{(a)})$ tend to $1 = \mathcal{M}^1([0,1])$ as $a \to (1/2)^-$. (Using the definition of upper and lower Minkowski content, it is easy to prove that for any bounded subset A of $\mathbb{R}^N$ we have that $\mathcal{M}^N(A)$ exists and, moreover, $\mathcal{M}^N(A) = |\overline{A}|_N$. This shows that Federer's theorem stated in [Fed, Theorem 3.2.39] holds without any rectifiability assumption on A, provided we let $m = N$ in the statement of that theorem.)

On the other hand, in the limit where $a \to 0^+$, $C^{(a)}$ remains Minkowski non-measurable since

$$\lim_{a\to 0^+} \mathcal{M}^{*D}(A) = 4, \quad \lim_{a\to 0^+} \mathcal{M}_*^D(A) = 2. \tag{3.25}$$

Example 3.10 (a-strings)**.** Given $a > 0$, the associated *a-string* is defined by $\mathcal{L} = (\ell_j)_{j\geq 1}$, where $\ell_j = j^{-a} - (j+1)^{-a}$. Let $A = A_{\mathcal{L}} = \{j^{-a} : j \in \mathbb{N}\}$ be the associated set; see Example 2.9 and the discussion preceding it. This set is Minkowski measurable,

$$\mathcal{M}^D(A) = \frac{2^{1-D}}{D(1-D)} a^D, \quad D = D(a) = \frac{1}{1+a}. \tag{3.26}$$

This fractal string, introduced by the first author, has been studied from various points of view in [Lap1, Example 5.1], [Lap2–3], [HeLap], [LapPo3] and [Lap-vFr3, Section 6.5.1]. Due to (3.4) and (3.17), we know that

$$\operatorname{res}(\zeta_A(\,\cdot\,, A_\delta), D) = (1-D)\mathcal{M}^D(A), \quad \operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}^D(A). \tag{3.27}$$

3.4. **Distance and tube zeta functions of fractal grills.** It is of interest to understand the dependence of the distance and tube zeta functions with respect to the Cartesian products of sets. In this section, we restrict our attention to Cartesian products of the form $A \times [0,1]^k \subset \mathbb{R}^{N+k}$, which we call *fractal grills*. Here, A is a bounded subset of $\mathbb{R}^N$ and k is any positive integer.

Since the set A can be naturally identified with $A \times \{0\} \subset \mathbb{R}^{N+1}$, it will be convenient to introduce the following notation for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large:

$$\zeta_A^{[N]}(s) := \int_{A_\delta} d(x,A)^{s-N}\,\mathrm{d}x, \quad \tilde{\zeta}_A^{[N]}(s) := \int_0^\delta t^{s-N-1}|A_t|_N \mathrm{d}t, \tag{3.28}$$

where the index $[N]$ indicates that we view A as a subset of $\mathbb{R}^N$ and $|A_t|_N$ is the N-dimensional Lebesgue measure of the t-neighborhood of A in $\mathbb{R}^N$. Hence, $\tilde{\zeta}_A^{[N+1]}(s) = \int_0^\delta t^{s-N-2}|A_t|_{N+1}\mathrm{d}t$. Note that, by writing $|A_t|_{N+1}$, we interpret A_t as the t-neighborhood of $A \times \{0\}$ in $\mathbb{R}^{N+1}$. Furthermore, observe that, in (3.28), $\zeta_A^{[N]}$ and $\tilde{\zeta}_A^{[N]}$, are, respectively, the usual distance and tube zeta functions of A (viewed as a bounded subset of $\mathbb{R}^N$) whereas, for example, $\tilde{\zeta}_A^{[N+1]}$ is the tube zeta function of A, but now viewed instead as a subset of $\mathbb{R}^{N+1}$. Moreover, in (3.30) and (3.31) of Lemma 3.14 just below, $\zeta_{A\times[0,1]}^{[N+1]}$ and $\tilde{\zeta}_{A\times[0,1]}^{[N+1]}$ stand, respectively, for the usual distance and tube zeta functions of $A \times [0,1]$ (naturally viewed as a subset of $\mathbb{R}^{N+1}$).

In the sequel, if Σ is a given set of complex numbers and $m \in \mathbb{C}$ a fixed complex number, we let $\Sigma + m := \{s + m : s \in \Sigma\}$. We shall also need the following definition.

Definition 3.11. Assume that $f(s)$ and $g(s)$ are two tamed Dirichlet-type integrals (DTIs, in short) which are (absolutely) convergent on an open right half-plane $\{\operatorname{Re} s > \alpha\}$, for some $\alpha \in \mathbb{R}$. Let their difference $h(s) := f(s) - g(s)$ be a tamed DTI such that $D(h) < D(g)$. (Or, equivalently, that there exists a real number β, with $\beta < D(g)$, such that the integral defining h is absolutely convergent (and hence, holomorphic) on $\{\operatorname{Re} s > \beta\}$.) Then we say that f and g are *weakly equivalent* and write $f \simeq g$.

Remark 3.12. It can be checked that if f and g are tamed DTIs, then $f-g$ (or, more generally, any linear combination of f and g) is a tamed DTI (as is required in Definition 3.11 just above) provided both the DTIs f and g are based on the same underlying pair (E,φ) in the notation of Defintion 2.12. Therefore, $D(h)$ and $\Pi(h)$ are well defined in that case. This situation arises, for example, for the tube zeta function discussed in the present section. We then have $E:=(0,\delta)$, $\varphi(t):=t$ for all $t\in E$.

Note that in Definition 3.11, we do not assume that g possesses a meromorphic continuation to a neighborhood of any point on its critical line $\{\operatorname{Re} s=D(g)\}$. Case (c) of Lemma 3.13 below provides a simple and useful condition for the implication $f\simeq g \implies f\sim g$ to hold, where the equivalence $\sim$ is described in Definition 2.22 above.

Lemma 3.13. *Assume that f and g are two tamed Dirichlet-type integrals such that $f\simeq g$. Then, the following properties hold*:

(a) *We have $D(f)=D(g)$.*

(b) *The relation $\simeq$ is reflexive and symmetric.*

(c) *If there exists a connected open set $U\subseteq\{\operatorname{Re} s>D(h)\}$ containing the critical line $\{\operatorname{Re} s=D(g)\}$ and such that g can be meromorphically continued to U, then f has the same property and $\mathcal{P}_c(f)=\mathcal{P}_c(g)$. In particular, $f\sim g$ in the sense of Definition 2.22.*

Proof. (a) Since, by Definition 3.11, $f(s)=g(s)+h(s)$ and $D(h)<D(g)$, we conclude that $D(f)\le D(g)$. If we had $D(f)<D(g)$, then we would have

$$\max\{D(f),D(h)\}<D(g). \tag{3.29}$$

On the other hand, the function (i.e., the DTI) $g(s)=f(s)-h(s)$ is absolutely convergent on $\{\operatorname{Re} s>\max\{D(f),D(h)\}\}$, which is impossible due to (3.29). This contradiction proves that $D(f)=D(g)$.

Property (b) follows at once from (a) and Definition 3.11. Finally, property (c) follows easily from the relation $f(s)=g(s)+h(s)$. □

Lemma 3.14. *Let A be a bounded subset of $\mathbb{R}^N$. Then*

$$\zeta_{A\times[0,1]}^{[N+1]}(s)=\zeta_A^{[N]}(s-1)+\zeta_A^{[N+1]}(s) \tag{3.30}$$

and

$$\tilde{\zeta}_{A\times[0,1]}^{[N+1]}(s)=\tilde{\zeta}_A^{[N]}(s-1)+\tilde{\zeta}_A^{[N+1]}(s) \tag{3.31}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s>\overline{\dim}_B A+1$. In particular, if A is such that ζ_A or (equivalently, provided $\overline{\dim}_B A<N$) $\tilde{\zeta}_A$ admits a (necessarily unique) meromorphic continuation to a connected open neighborhood of the critical line of Lebesgue (absolute) convergence $\{\operatorname{Re} s=D(\zeta_A)\}$ (recall from Theorem 2.5 that $D(\zeta_A)=\overline{\dim}_B A$), then

$$\zeta_{A\times[0,1]}^{[N+1]}(s)\simeq\zeta_A^{[N]}(s-1)\quad\text{and}\quad\tilde{\zeta}_{A\times[0,1]}^{[N+1]}(s)\simeq\tilde{\zeta}_A^{[N]}(s-1). \tag{3.32}$$

Hence, if ζ_A can be meromorphically continued to a connected, open set U containing the critical line $\{\operatorname{Re} s=D(\zeta_A)\}$, then $\mathcal{P}_c(\zeta_{A\times[0,1]}^{[N+1]})=\mathcal{P}_c(\zeta_A^{[N]})+1$; that is,

$$\dim_{PC}(A\times[0,1])=\dim_{PC}A+1. \tag{3.33}$$

In particular, if $\overline{\dim}_B A < N$*, then*

$$\begin{aligned} D(\zeta^{[N+1]}_{A\times[0,1]}) &= D(\zeta^{[N]}_A) + 1 = D(\tilde{\zeta}^{[N]}_A) + 1 = D(\tilde{\zeta}^{[N+1]}_{A\times[0,1]}) \\ &= \overline{\dim}_B(A\times[0,1]) = \overline{\dim}_B A + 1. \end{aligned} \tag{3.34}$$

Proof. Let us first prove Equation (3.31). It is easy to see (cf. [Res, Remark 1]) that:

$$|(A\times[0,1])_t|_{N+1} = |A_t|_N \cdot 1 + |A_t|_{N+1}. \tag{3.35}$$

Substituting into the second equality of (3.28), we conclude that

$$\begin{aligned} \tilde{\zeta}^{[N+1]}_{A\times[0,1]}(s) &= \int_0^\delta t^{s-N-2}(|A_t|_N + |A_t|_{N+1})\,\mathrm{d}t \\ &= \int_0^\delta t^{(s-1)-N-1}|A_t|_N \mathrm{d}t + \int_0^\delta t^{s-(N+1)-1}|A_t|_{N+1}\mathrm{d}t \\ &= \tilde{\zeta}^{[N]}_A(s-1) + \tilde{\zeta}^{[N+1]}_A(s) \end{aligned} \tag{3.36}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A + 1$. (Here, we also use the fact that $\overline{\dim}_B A$ is the same in the case of $A\subset\mathbb{R}^{N+1}$, as in the case of $A\subset\mathbb{R}^N$; that is, the upper box dimension of a set, as well as the lower box dimension, does not depend on N; see [Res, Proposition 1].)

Let us next establish Equation (3.30). To this end, we use (3.14), which we write in the following form:

$$\tilde{\zeta}^{[N]}_A(s) = \frac{\zeta^{[N]}_A(s) - \delta^{s-N}|A_\delta|_N}{N-s}, \tag{3.37}$$

for $s\in\mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A$ and $s\neq N$. Making use of Equation (3.36), we deduce that

$$\begin{aligned} \frac{\zeta^{[N+1]}_{A\times[0,1]}(s) - \delta^{s-N-1}|(A\times[0,1])_\delta|_{N+1}}{(N+1)-s} &= \frac{\zeta^{[N]}_A(s-1) - \delta^{(s-1)-N}|A_\delta|_N}{N-(s-1)} \\ &\quad + \frac{\zeta^{[N+1]}_A(s) - \delta^{s-(N+1)}|A_\delta|_{N+1}}{(N+1)-s}, \end{aligned} \tag{3.38}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A$ and $s\neq N+1$. Since, in light of (3.35), we have $|(A\times[0,1])_\delta|_{N+1} = |A_\delta|_N + |A_\delta|_{N+1}$, after a short computation we conclude from (3.38) that

$$\zeta^{[N+1]}_{A\times[0,1]}(s) = \zeta^{[N]}_A(s-1) + \zeta^{[N+1]}_A(s), \tag{3.39}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A + 1$, where we have also used the principle of analytic continuation. Note that, according to Theorem 2.5, both $\zeta^{[N]}_A(s-1)$ and $\zeta^{[N+1]}_{A\times[0,1]}(s)$ are holomorphic on $\{\operatorname{Re} s > \overline{\dim}_B A + 1\}$ (recall that $\overline{\dim}_B(A\times[0,1]) = \overline{\dim}_B A + 1$, see [Fal1]), while, according to the same theorem, the function $\zeta^{[N+1]}_{A\times[0,1]}(s) - \zeta^{[N]}_A(s-1) = \zeta^{[N+1]}_A(s)$ is holomorphic on $\{\operatorname{Re} s > \overline{\dim}_B A\}$. Therefore, since $D(\zeta^{[N+1]}_A) = \overline{\dim}_B A < \overline{\dim}_B A + 1 = D(\zeta^{[N]}_A(\,\cdot\,-1))$, it follows from Definition 3.11 that $\zeta^{[N+1]}_{A\times[0,1]}(s) \simeq \zeta^{[N]}_A(s-1)$.

The remaining part of Lemma 3.14 can be deduced from part (c) of Lemma 3.13 by noting that since $\zeta_A(s)$ can be meromorphically continued to the set U,

then $\zeta_A(s-1)$ can be meromorphically continued to the set $U+1$. Hence, by Lemma 3.13(c), we have $\zeta^{[N+1]}_{A\times[0,1]}(s) \sim \zeta^{[N]}_A(s-1)$ in the sense of Definition 2.22, and therefore,

$$\mathcal{P}_c\big(\zeta^{[N+1]}_{A\times[0,1]}\big) = \mathcal{P}_c\big(\zeta^{[N]}_A(\,\cdot\,-1)\big) = \mathcal{P}_c\big(\zeta^{[N]}_A\big) + 1,$$

or, equivalently, $\dim_{PC}(A\times[0,1]) = \dim_{PC} A + 1$. This completes the proof of the lemma. □

Theorem 3.15. *Let A be a bounded subset of $\mathbb{R}^N$ and let m be a positive integer. Then the following properties hold:*

(a) *The distance and tube zeta functions of $A\times[0,1]^m \subset \mathbb{R}^{N+m}$ are given, respectively, by*

$$\zeta^{[N+m]}_{A\times[0,1]^m}(s) = \sum_{k=0}^{m} \binom{m}{k} \zeta^{[N+k]}_A(s-m+k) \tag{3.40}$$

and

$$\tilde{\zeta}^{[N+m]}_{A\times[0,1]^m}(s) = \sum_{k=0}^{m} \binom{m}{k} \tilde{\zeta}^{[N+k]}_A(s-m+k), \tag{3.41}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B A + m$.

(b) *If the distance zeta function ζ_A (or equivalently, the tube zeta function $\tilde{\zeta}_A$) can be meromophically extended to a connected open set containing the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$, then*

$$\zeta^{[N+m]}_{A\times[0,1]^m}(s) \sim \zeta^{[N]}_A(s-m), \quad \tilde{\zeta}^{[N+m]}_{A\times[0,1]^m}(s) \sim \tilde{\zeta}^{[N]}_A(s-m) \tag{3.42}$$

and $\mathcal{P}_c(\zeta_{A\times[0,1]^m}) = \mathcal{P}_c(\zeta_A) + m$; that is,

$$\dim_{PC}(A\times[0,1]^m) = \dim_{PC} A + m. \tag{3.43}$$

In particular, if $\overline{\dim}_B A < N$, then

$$\begin{aligned} D(\zeta^{[N+m]}_{A\times[0,1]^m}) &= D(\zeta^{[N]}_A) + m = D(\tilde{\zeta}^{[N]}_A) + m = D(\tilde{\zeta}^{[N+m]}_{A\times[0,1]^m}) \\ &= \overline{\dim}_B(A\times[0,1]^m) = \overline{\dim}_B A + m. \end{aligned} \tag{3.44}$$

Proof. (a) Let us first prove Equation (3.40). We do so by using mathematical induction on m. The case where $m=1$ has already been established in Lemma 3.14.

Now, let us assume that the claim holds for some fixed positive integer $m\geq 1$. From (3.30) we see that

$$\zeta^{[N+m+1]}_{A\times[0,1]^{m+1}}(s) = \zeta^{[N+m]}_{A\times[0,1]^m}(s-1) + \zeta^{[(N+1)+m]}_{A\times[0,1]^m}(s).$$

Therefore,

$$\begin{aligned}\zeta^{[N+m+1]}_{A\times[0,1]^{m+1}}(s) &= \sum_{k=0}^{m}\binom{m}{k}\zeta_A^{[N+k]}(s-1-m+k)+\sum_{k=0}^{m}\binom{m}{k}\tilde{\zeta}_A^{[N+1+k]}(s-m+k)\\ &= \zeta_A^{[N]}(s-m-1)+\sum_{k=0}^{m-1}\binom{m}{k+1}\zeta_A^{[N+k+1]}(s-m+k)\\ &\quad+\sum_{k=0}^{m-1}\binom{m}{k}\zeta_A^{[N+1+k]}(s-m+k)+\zeta_A^{[N+1+m]}(s)\\ &= \sum_{k=0}^{m+1}\binom{m+1}{k}\zeta_A^{[N+k]}(s-(m+1)+k),\end{aligned}$$

where in the last equality we have used the fact that $\binom{m}{k}+\binom{m}{k+1}=\binom{m+1}{k+1}$. This completes the proof of Equation (3.40).

Equation (3.41) can be proved by mathematical induction in much the same way as in the case of the distance zeta function. This completes the proof of part (a) of the theorem.

(b) To prove that $\zeta^{[N+m]}_{A\times[0,1]^m}(s)\sim\zeta_A^{[N]}(s-m)$, it suffices to note that, by Equation (3.40), the function

$$\zeta^{[N+m]}_{A\times[0,1]^m}(s)-\zeta_A^{[N]}(s-m)=\sum_{k=1}^{m}\binom{m}{k}\zeta_A^{[N+k]}(s-m+k)=:h(s) \tag{3.45}$$

has for abscissa of convergence $D(h)=\overline{\dim}_B A+(m-1)\}<\overline{\dim}_B A+m=D(\zeta_A^{[N]}(\,\cdot\,-m))$, so that $\zeta^{[N+m]}_{A\times[0,1]^m}(s)\simeq\zeta_A^{[N]}(s-m)$. Using part (c) of Lemma 3.13, we deduce that $\zeta^{[N+m]}_{A\times[0,1]^m}(s)\sim\zeta_A^{[N]}(s-m)$ in the sense of Definition 2.22, which proves the first relation in (3.42) The second relation in (3.42) can be proved along the same lines. This completes the proof of claim (b), as well as of the entire theorem. □

Remark 3.16. The relations appearing in (3.42) can be written in a less precise form as follows:

$$\zeta_{A\times[0,1]^m}(s)\sim\zeta_A(s-m)\quad\text{and}\quad\tilde{\zeta}_{A\times[0,1]^m}(s)\sim\tilde{\zeta}_A(s-m). \tag{3.46}$$

We propose to call these two properties the *shift properties* of the distance and tube zeta functions, respectively.

Example 3.17. Assume that $C^{(m,a)}$ is the generalized Cantor set introduced below in Definition 4.1 below and d is a positive integer. Then, using (3.42) and (4.6) below, we obtain that

$$\zeta_{C^{(m,a)}\times[0,1]^d}(s)\sim\frac{1}{1-ma^{s-d}}.$$

Furthermore, from (3.43) we conclude that

$$\dim_{PC}(C^{(m,a)}\times[0,1]^d)=(\log_{1/a}m+d)+\frac{2\pi}{\log m}\,\mathbb{i}\mathbb{Z}. \tag{3.47}$$

Moreover, by noticing that $\zeta_{C^{(m,a)}\times[0,1]^d}$ can be meromorphically extended to the whole complex plane, we conclude from Equation (3.40) above and from the first

part of Equation (4.7) below that the set of all complex dimensions of $C^{(m,a)}\times[0,1]^d$ is well defined in $\mathbb{C}$ and given by

$$\mathcal{P}(\zeta_{C^{(m,a)}\times[0,1]^d}) = \{0,1,\dots,d\} \cup \bigcup_{k=0}^{d} \Big((\log_{1/a} m + k) + \frac{2\pi}{\log m}\,\mathbb{i}\mathbb{Z}\Big). \tag{3.48}$$

The sets of the form $C^{(m,a)} \times [0,1]^d$ (with $m = 2$, $a = 1/3$, $d = 1$) appear, for example, in the study of the Smale horseshoe map; see, e.g., [Sma]. They also appear in the study of the singularities of Sobolev functions and of weak solutions of elliptic equations; see, e.g., [Žu1] and [HorŽu], where they are called the 'Cantor grills'.

Example 3.18. Similarly as in Example 3.17, sets of the form $\partial\Omega \times [0,1]^{N-1}$, where $\Omega = \Omega_a$ is a geometric realization of a fractal string (for example, the so-called a-string, $\Omega = \cup_{j=1}^{\infty}((j+1)^{-a}, j^{-a}))$, where $a > 0$ and for which $\partial\Omega = \{j^{-a} : j \geq 1\}\cup\{0\}$ satisfies $\overline{\dim}_B\partial\Omega = 1/(a+1)$, are used in the study of fractal drums to extend certain results from one to higher dimensions $N \geq 2$; see [Lap1, Examples 5.1 and 5.1']. The open set $\Omega \times (0,1)^{N-1}$, whose boundary is

$$(\partial\Omega \times [0,1]^{N-1}) \cup \big([0,1] \times \partial((0,1)^{N-1})\big), \tag{3.49}$$

and where $\partial\big([0,1]^{N-1}\big)$ is taken in the space $\mathbb{R}^{N-1}$, is called a 'fractal comb' in [Lap1–3]. (See also [LapRaŽu6].) The subset $\partial\big((0,1)^{N-1}\big)$ of $\mathbb{R}^{N-1}$ is an $(N-2)$-dimensional Lipschitz surface (which for $N = 2$ degenerates to a pair of points), so that the box dimension of $[0,1] \times \partial((0,1)^{N-1})$ is equal to $N-1$. Therefore, by the property of 'finite stability' of the upper box dimension (see [Fal1]), we have $\overline{\dim}_B(\Omega\times(0,1)^{N-1}) = \max\{\overline{\dim}_B(\partial\Omega\times[0,1]^{N-1}), N-1\} = \overline{\dim}_B(\partial\Omega\times[0,1]^{N-1}) = \overline{\dim}_B\partial\Omega + N - 1$.

Since, according to [Lap-vFr3, Theorem 6.21],

$$\mathcal{P}(\zeta_{\partial(\Omega_a)}) = \{\rho, -\rho, -2\rho, -3\rho, \dots\}, \tag{3.50}$$

where $\rho := 1/(a+1)$, we deduce from Theorem 3.15 that

$$\begin{aligned}\mathcal{P}(\zeta_{\partial(\Omega_a\times(0,1)^{N-1})}) &= \mathcal{P}(\zeta_{\partial(\Omega_a)\times[0,1]^{N-1}})\\ &= \{N-1+\rho, N-1-\rho, N-1-2\rho, N-1-3\rho, \dots\},\end{aligned} \tag{3.51}$$

still with $\rho = 1/(a+1)$. Furthermore, all of these complex dimensions are simple.

Remark 3.19. More precisely, it could be that beside ρ, which is always a (simple) pole of $\zeta_{\partial\Omega}$, some of the numbers $-n\rho$ ($n \geq 1$) appearing in (3.50) are not the poles of $\zeta_{\partial\Omega}$ (because the coresponding residue of $\zeta_{\partial\Omega}$ happens to vanish, for some arithmetic reason connected with the value of a). And, hence, similarly, in (3.51).

Note that if, in Example 3.18 just above, $\Omega = \Omega_{CS}$ is the Cantor string (i.e., the complement of the classic ternary Cantor set in $[0,1]$), then according to [Lap-vFr3, Subsection 1.2.2, Equation (1.30)] and Equation (3.51), we have

$$\dim_{PC}\partial(\Omega \times (0,1)^{N-1}) = \big((N-1) + \log_3 2\big) + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}, \tag{3.52}$$

which is a special case of (3.47).

4. Transcendentally n-quasiperiodic sets and their distance zeta functions

The goal of this section is to describe a construction of some of the simplest classes of quasiperiodic sets, a notion which we introduce in Definition 4.10 below. The main result is obtained in Theorem 4.14. The construction will be carried out by using a class of generalized Cantor sets depending on two auxiliary parameters. We note that, as will be briefly discussed in Section 4.4 below, this construction and its natural generalizations will play a key role in future developments of the present higher-dimensional theory of complex dimensions of fractals; see [LapRaŽu1–7].

4.1. **Generalized Cantor sets defined by two parameters.** Let us introduce a class of generalized Cantor sets $C^{(m,a)}$, depending on two parameters. As a special case, we obtain the Cantor sets of the form $C^{(a)} := C^{(2,a)}$ discussed in Example 3.4. The classical ternary Cantor set $C^{(1/3)}$ corresponds to the case where $m = 2$ and $a = 1/3$.

Definition 4.1. The generalized Cantor sets $C^{(m,a)}$ are determined by an integer $m \geq 2$ and a positive real number a such that $ma < 1$. In the first step of the analog of Cantor's construction, we start with m equidistant, closed intervals in $[0, 1]$ of length a, with $m - 1$ holes, each of length $(1 - ma)/(m - 1)$. In the second step, we continue by scaling by the factor a each of the m intervals of length a; and so on, ad infinitum. The (two-parameter) *generalized Cantor set* $C^{(m,a)}$ is defined as the intersection of the decreasing sequence of compact sets constructed in this way.

It can be shown that the generalized Cantor sets $C^{(m,a)}$ have the following properties, which extend the ones established for the sets $C^{(a)}$. Apart from the proof of (4.5), which is easily obtained, the proof of the proposition is similar to that for the standard Cantor set (see [Lap-vFr3, Equation (1.11)]), and therefore, we omit it.

Proposition 4.2. *If $C^{(m,a)} \subseteq \mathbb{R}$ is the generalized Cantor set introduced in Definition 4.1, then*

$$D := \dim_B C^{(m,a)} = D(\zeta_A) = \log_{1/a} m. \tag{4.1}$$

Furthermore, the tube formula associated with $C^{(m,a)}$ is given by

$$|C_t^{(m,a)}| = t^{1-D} G\Big(\log \frac{1}{t}\Big) \tag{4.2}$$

for all $t \in (0, \frac{1-ma}{2(m-1)})$, where $G = G(\tau)$ is the following nonconstant, positive and bounded periodic function, with minimal period equal to $T = \log(1/a)$, and defined by

$$G(\tau) = c^{D-1}(ma)^{g\left(\frac{\tau - c}{T}\right)} + 2\, c^D m^{g\left(\frac{\tau - c}{T}\right)}. \tag{4.3}$$

Here, $c = \frac{1-ma}{2(m-1)}$, and $g : \mathbb{R} \to \mathbb{R}$ is the 1-periodic function defined by $g(x) = 1 - x$ for $x \in (0, 1]$.

Moreover, the lower and upper Minkowski contents of $C^{(m,a)}$ are respectively given by

$$\begin{aligned} \mathcal{M}_*^D(C^{(m,a)}) &= \min G = \frac{1}{D}\left(\frac{2D}{1-D}\right)^{1-D}, \\ \mathcal{M}^{*D}(C^{(m,a)}) &= \max G = \left(\frac{1-ma}{2(m-1)}\right)^{D-1}\frac{m(1-a)}{m-1}. \end{aligned} \tag{4.4}$$

Therefore, $C^{(m,a)}$ is Minkowski nondegenerate but is not Minkowski measurable.

Finally, if we assume that $\delta \geq \frac{1-ma}{2(m-1)}$, then, the distance zeta function of $A := C^{(m,a)}$ is given by

$$\zeta_A(s) := \int_{-\delta}^{1+\delta} d(x,A)^{s-2}\mathrm{d}x = \left(\frac{1-ma}{2(m-1)}\right)^{s-1}\frac{1-ma}{s(1-ma^s)} + \frac{2\delta^s}{s}. \tag{4.5}$$

As a result, $\zeta_A(s)$ admits a meromorphic continuation to all of $\mathbb{C}$, given by the last expression in (4.5)*. In particular,*

$$\zeta_A(s) \sim \frac{1}{1-ma^s}, \tag{4.6}$$

and the set of poles of ζ_A (in $\mathbb{C}$) and the residue of ζ_A at $s = D$ are respectively given by

$$\begin{aligned} \mathcal{P}(\zeta_A) &= (D + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{0\}, \\ \operatorname{res}(\zeta_A, D) &= \frac{1-ma}{DT}\left(\frac{1-ma}{2(m-1)}\right)^{D-1}, \end{aligned} \tag{4.7}$$

where $\mathbf{p} := 2\pi/T$ is the oscillatory period of $C^{(m,a)}$ (in the sense of [Lap-vFr3]*). Finally, each pole in $\mathcal{P}(\zeta_A)$ is simple.*

Remark 4.3. The values of the upper and lower Minkowski contents of $C^{(m,a)}$ have been obtained earlier in [Žu2, Equations (3.12) and (3.13)]. In the case of the classical Cantor set, that is, for $m = 2$ and $a = 1/3$, we recover the values first obtained in [LapPo2, Theorem 2.4]. (See also [Lap-vFr2–3, Chapter 11] for further generalizations.) Finally, the tube formula (4.2) extends the one obtained in [Lap-vFr3, Equation (1.11)].

Definition 4.4. According to the terminology introduced in [Lap-vFr3], the value of $\mathbf{p} = 2\pi/\log(1/a)$, appearing in Proposition 4.2, is called the *oscillatory period* of the generalized Cantor set $A = C^{(m,a)}$.

As we see from Equation (4.5) and from the equivalence in (4.6), the set of all complex dimensions of the generalized Cantor set $A = C^{(m,a)}$ and the set of principal complex dimensions of A are given, respectively, by

$$\mathcal{P}(\zeta_A) = (D + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{0\} \quad \mathcal{P}_c(\zeta_A) = D + \mathbf{p}\mathbb{i}\mathbb{Z}.$$

4.2. **Construction of transcendentally 2-quasiperiodic sets.** In Example 4.5 below, we provide some basic ideas for further definitions and constructions. The main result of this subsection is obtained in Theorem 4.11.

Example 4.5. Let us define two generalized Cantor sets $A = C^{(a)} := C^{(2,a)} \subset [0,1]$, $a \in (0,1/2)$, and $B = C^{(3,b)} \subset [2,3]$, where $b \in (0,1/3)$. We choose b so that

$D := \log_{1/a} 2 = \log_{1/b} 3$. We may take, for example, $a = 1/3$ and $b = 3^{-\log_2 3}$. Note that we then have $3b = 3^{1-\log_2 3} < 1$. Also, we have

$$|A_t| = t^{1-D} G_1(\log t^{t-1}), \quad |B_t| = t^{1-D} G_2(\log t^{t-1}).$$

The functions G_1 and G_2 corresponding to A and B are T and S-periodic, respectively, with $T = \log(1/a) = \log 3$ and $S = \log(1/b)$. Furthermore, the quotient $T/S = \log 3/\log(1/b) = \log_3 2$ is transcendental, which is a well-known result going back to F. von Lindemann and K. Weierstrass; see [Ba, p. 4].

For our later needs, it will be convenient to introduce the following definition, which partly follows [Vin].

Definition 4.6. We say that a function $G = G(\tau) : \mathbb{R} \to \mathbb{R}$ is *transcendentally n-quasiperiodic* if it is of the form $G(\tau) = H(\tau, \dots, \tau)$, where $H : \mathbb{R}^n \to \mathbb{R}$ is a function which is nonconstant and T_k-periodic in its k-th component, for each $k = 1, \dots, n$, and the periods $T_1, \dots, T_n$ are *algebraically* independent (that is, linearly independent over the field of algebraic real numbers). The values of T_i are called the *quasiperiods of G*. The least positive integer n for which this definition is valid is called the *order of quasiperiodicity* of G.

Remark 4.7. It is possible to define analogously a class of *algebraically n-quasiperiodic functions*, but we do not study them here; see [LapRaŽu1].

Example 4.8. If $G(\tau) = G_1(\tau) + G_2(\tau)$, where the functions G_i are nonconstant and T_i-periodic (for $i = 1, 2$), such that T_1/T_2 is transcendental, then G is transcendentally 2-quasiperiodic (in the sense of Definition 4.6). In this case and in the notation of Definition 4.6, we have $H(\tau_1, \tau_2) = G_1(\tau_1) + G_2(\tau_2)$.

In the sequel, we shall need a classic result due to Gel'fond and Schneider (see [Gel]), proved independently by these two authors in 1934. We state it in a form that will be convenient for our purposes.

Theorem 4.9 (Gel'fond–Schneider, [Gel])**.** *Let m be a positive algebraic number different from one, and let x be an irrational algebraic number. Then m^x is transcendental.*

Definition 4.10. Given a bounded subset $A \subset \mathbb{R}^N$, we say that a function $G : \mathbb{R} \to \mathbb{R}$ *is associated with the set A* (or *corresponds to A*) if A has the following tube formula:

$$|A_t| = t^{N-D}(G(\log(1/t)) + o(1)) \text{ as } t \to 0^+, \tag{4.8}$$

where $0 < \liminf_{\tau\to\infty} G(\tau) \le \limsup_{\tau\to\infty} G(\tau) < \infty$. Note that it then follows that $\dim_B A$ exists and is equal to D.

In addition, we say that A is a *transcendentally n-quasiperiodic set* if the corresponding function $G = G(\tau)$ is transcendentally n-quasiperiodic.

Generalizing the idea of Example 4.5 above, we obtain the following result.

Theorem 4.11. *Let $A_1 = C^{(m_1,a_1)} \subset [0,1]$ and $A_2 = C^{(m_2,a_2)} \subset [2,3]$ be two generalized Cantor sets (see Definition 4.1) such that their box dimensions coincide, with the common value $D \in (0,1)$. Let $\{p_1, p_2, \dots, p_k\}$ be the set of all distinct prime factors of m_1 and m_2, and write*

$$m_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}, \quad m_2 = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}, \tag{4.9}$$

where $\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots, k$. If the exponent vectors

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \quad \text{and} \quad (\beta_1, \beta_2, \dots, \beta_k), \tag{4.10}$$

corresponding to m_1 and m_2, are linearly independent over the rationals, then the function $G = G_1 + G_2$, associated with $A = A_1 \cup A_2$, is transcendentally 2-quasiperiodic; that is, the quotient T_1/T_2 of the quasiperiods of G (i.e., of the periods of G_1 and G_2) is transcendental.

Moreover, we have that

$$\zeta_A(s) \sim \frac{1}{1 - m_1 a_1^s} + \frac{1}{1 - m_2 a_2^s}, \quad D(\zeta_A) = D, \quad D_{\mathrm{mer}}(\zeta_A) = -\infty,$$

and hence, the set $\dim_{PC} A = \mathcal{P}_c(\zeta_A)$ of principal complex dimensions of A coincides with the following nonarithmetic set:

$$\dim_{PC} A = D + \Big(\frac{2\pi}{T_1}\,\mathbb{Z} \cup \frac{2\pi}{T_2}\,\mathbb{Z}\Big)\mathbb{i}.$$

Besides $(\dim_{PC} A) \cup \{0\}$, there are no other poles of the distance zeta function ζ_A. In other words, $\mathcal{P}(\zeta_A) = \mathcal{P}_c(\zeta_A) \cup \{0\}$. Furthermore, all of the complex dimensions are simple.

Finally, exactly the same results hold for the tube zeta function $\tilde{\zeta}_A$ (instead of ζ_A).

Proof. First of all, using (4.2), applied to both A_1 and A_2, we conclude that for all $t \in (0, 1/2)$,

$$|(A_1 \cup A_2)_t| = t^{1-D} \left(G_1(\log 1/T) + G_2(\log 1/t)\right).$$

It thus suffices to show that the quotient T_1/T_2 of the quasiperiods T_1 and T_2 of the function $G(\tau) := G_1(\tau) + G_2(\tau)$ is transcendental.

From $D = \log_{1/a_1} m_1 = \log_{1/a_2} m_2$ and $T_i = \log m_i$, $i = 1, 2$, we deduce that $x := T_1/T_2$ satisfies the equation $(m_2)^x = m_1$. The exponent x cannot be an irrational algebraic number, since otherwise, by the Gel'fond-Schneider theorem (Theorem 4.9), $(m_2)^x$ would be transcendental. If x were rational, say, $x = b/a$, with $a, b \in \mathbb{N}$ (note that $x > 0$, since $m_1 \geq 2$), this would then imply that $(m_1)^a = (m_2)^b$; that is,

$$p_1^{a\alpha_1} p_2^{a\alpha_2} \dots p_k^{a\alpha_k} = p_1^{b\beta_1} p_2^{b\beta_2} \dots p_k^{b\beta_k}.$$

Therefore, using the fundamental theorem of arithmetic, we would have

$$a(\alpha_1, \alpha_2, \dots, \alpha_k) = b(\beta_1, \beta_2, \dots, \beta_k).$$

However, this is impossible due to the assumption of linear independence over the rationals of the above exponent vectors. Consequently, x is transcendental.

The claims about the zeta function $\zeta_{A_1 \cup A_2}$ follow from Proposition 4.2 applied to both A_1 and A_2. Indeed, since A_1 and A_2 are subsets of two disjoint compact intervals, then $\zeta_A(s) \sim \zeta_{A_1}(s) + \zeta_{A_2}(s)$, and on the other hand, $\zeta_{A_1}(s) + \zeta_{A_2}(s) \sim (1 - m_1 a_1^s)^{-1} + (1 - m_2 a_2^s)^{-1}$. This completes the proof of the theorem. □

Remark 4.12. Theorem 4.11 provides a construction of the set $A = A_1 \cup A_2$, such that the set $\dim_{PC} A := \mathcal{P}_c(\zeta_A)$ of principal complex dimensions of A is equal to the union of two (discrete) sets of complex dimensions, each of them composed of poles in infinite vertical arithmetic progressions, but with algebraically incommensurable *oscillatory quasiperiods* $\mathbf{p}_1 = 2\pi/T_1$ and $\mathbf{p}_2 = 2\pi/T_2$ of A_1 and A_2, respectively; that is, such that $\mathbf{p}_1/\mathbf{p}_2$ is transcendental. These oscillatory quasiperiods of A are equal to the oscillatory periods of A_1 and A_2, respectively.

4.3. **Transcendentally n-quasiperiodic sets and Baker's theorem.** The main result of this subsection is stated in Theorem 4.14 below, which extends Theorem 4.11 to any integer $n \geq 2$ and also provides further helpful information. In the sequel, we shall need the following important theorem from transcendental number theory, due to Baker [Ba, Theorem 2.1]. It represents a nontrivial extension of Theorem 4.9, due to Gel'fond and Schneider [Gel]. Recall that an algebraic number is a complex root of a polynomial with integer coefficients and that the field of algebraic numbers is isomorphic to the algebraic closure of $\mathbb{Q}$, the field of rational numbers.

Theorem 4.13 (Baker, [Ba, Theorem 2.1])**.** *Let $n \in \mathbb{N}$ with $n \geq 2$. If $m_1, \dots, m_n$ are positive algebraic numbers such that $\log m_1, \dots, \log m_n$ are linearly independent over the rationals, then*

$$1, \log m_1, \dots, \log m_n$$

are linearly independent over the field of all algebraic numbers.

We now state the main result of this section, which can be considered as a fractal set-theoretic interpretation of Baker's theorem. It extends Theorem 4.11 even in the case where $n = 2$.

Theorem 4.14. *Let $n \in \mathbb{N}$ with $n \geq 2$. Assume that $A_i = C^{(m_i, a_i)}$, $i = 1, \dots, n$, are generalized Cantor sets (in the sense of Definition 4.1) such that their box dimensions are all equal to a fixed number $D \in (0, 1)$. Assume that there is a disjoint family of closed unit intervals $I_1, \dots, I_n$ on the real line, such that $A_i \subset I_i$ for each $j = 1, \dots, n$. Let $T_i := \log(1/a_i)$ be the associated periods, and G_i be the corresponding (nonconstant) T_i-periodic functions, for $i = 1, \dots, n$. Let $\{p_j : j = 1, \dots, k\}$ be the union of all distinct prime factors which appear in the integers m_i, for $i = 1, \dots, n$; that is, $m_i = p_1^{\alpha_{i1}} \dots p_k^{\alpha_{ik}}$, where $\alpha_{ij} \in \mathbb{N} \cup \{0\}$.*

If the exponent vectors e_i of the numbers m_i,

$$e_i := (\alpha_{i1}, \dots, \alpha_{ik}), \quad i = 1, \dots, n, \tag{4.11}$$

are linearly independent over the rationals, then the numbers

$$\frac{1}{D}, T_1, \dots, T_n \tag{4.12}$$

are linearly independent over the field of all algebraic numbers. In particular, the set $A := A_1 \cup \dots \cup A_n \subset \mathbb{R}$ is transcendentally n-quasiperiodic; see Definition 4.10. Furthermore, in the notation of Definition 4.10, an associated function G is given by $G := G_1 + \dots + G_n$.

Moreover, we have that

$$\zeta_A(s) \sim \sum_{i=1}^{n} \frac{1}{1 - m_i a_i^s}, \quad D(\zeta_A) = D, \quad D_{\mathrm{mer}}(\zeta_A) = -\infty,$$

and hence, the set $\dim_{PC} A = \mathcal{P}_c(\zeta_A)$ of principal complex dimensions of A consists of simple poles and coincides with the following nonarithmetic set:

$$\dim_{PC} A = D + \Big(\bigcup_{i=1}^{n} \frac{2\pi}{T_i} \mathbb{Z} \Big) \mathbb{i}.$$

Besides $(\dim_{PC} A) \cup \{0\}$, there are no other poles of the distance zeta function ζ_A. That is, $\mathcal{P}(A) = \mathcal{P}_c(A) \cup \{0\}$. Furthermore, all of these complex dimensions are simple.

Finally, exactly the same results hold for the tube zeta function $\tilde{\zeta}_A$ (instead of ζ_A).

Proof. As in the proof of Theorem 4.11, using (4.2), applied to each A_i, for $i = 1, \dots, n$, we see that for all $t > 0$ small enough,

$$|A_t| = t^{1-D} \sum_{i=1}^{n} G_i\Big(\log \frac{1}{t}\Big),$$

and for each $i = 1, \dots, n$, $G_i = G_i(\tau)$ is T_i-periodic, where $T_i := \log 1/a_i$. We next proceed in three steps:

Step 1: It is easy to check that the numbers $\log p_j$ (for $j = 1, \dots, n$) are rationally independent. Indeed, if we had $\sum_{j=1}^{k} \lambda_j \log p_j = 0$ for some integers λ_j, then $\prod_{j=1}^{k} p_j^{\lambda_j} = 1$. This implies that $\lambda_j = 0$ for all j, since otherwise it would contradict the fundamental theorem of arithmetic.

Step 2: Let us show that $\log m_1, \dots, \log m_n$ are linearly independent over the rationals. Indeed, assume that for $i = 1, \dots, n$, $\mu_i \in \mathbb{Q}$ are such that $\sum_{i=1}^{n} \mu_i \log m_i = 0$. Then

$$\sum_{i=1}^{n} \mu_i \sum_{j=1}^{k} \alpha_{ij} \log p_j = 0. \tag{4.13}$$

Changing the order of summation, we have

$$\sum_{j=1}^{k} \left(\sum_{i=1}^{n} \mu_i \alpha_{ij} \right) \log p_j = 0. \tag{4.14}$$

Since, by Step 1, the numbers $\log p_j$ are rationally independent, we have that for all $j = 1, \dots, k$,

$$\sum_{i=1}^{n} \mu_i \alpha_{ij} = 0;$$

that is, $\sum_{i=1}^{n} \mu_i e_i = 0$, where the e_i's are the exponent vectors given by (4.11). According to the hypotheses of the theorem, the exponent vectors e_i are rationally independent, and we therefore conclude that $\mu_i = 0$ for all $i = 1, \dots, n$, as desired.

Step 3: Using [Ba, Theorem 2.1], that is, Theorem 4.13 above, we conclude that $1, \log m_1, \dots, \log m_n$ are linearly independent over the field of algebraic numbers. Since $T_i = \frac{1}{D} \log m_i$, for $i = 1, \dots, n$, it then follows that the numbers listed in (4.12) are also linearly independent over the field of algebraic numbers. Therefore, the function

$$G := G_1 + \cdots + G_n, \quad G(\tau) = G_1(\tau) + \cdots + G_n(\tau),$$

associated with A, is transcendentally n-quasiperiodic; that is, the set A is transcendentally n-quasiperiodic. Note that here, $H(\tau_1, \dots, \tau_n) := G_1(\tau_1) + \cdots + G_n(\tau_n)$, in the notation of Definition 4.6.

The last claim, about the distance zeta function ζ_A and its complex dimensions, now follows from Proposition 4.2 applied to each of the bounded sets A_i ($i = 1, \dots, n$). This concludes the proof of the theorem. □

Remark 4.15. In Theorem 4.14, we have constructed a class of bounded subsets of the real line possessing an arbitrary prescribed finite number of algebraically incommensurable quasiperiods. As will be further discussed in Section 4.4, this result will be extended in [LapRaŽu1,4], where we shall construct a bounded subset A_0 of the real line which is *transcendentally ∞-quasiperiodic set*; that is, A_0 contains infinitely many algebraically incommensurable quasiperiods.

In the following proposition, by a *quasiperiodic set* we mean a set which has one of the following *types of quasiperiodicity*: it is either n-transcendentally quasiperiodic (see Definition 4.10), or n-algebraically quasiperiodic (see Remark 4.7), for some $n \in \{2, 3, \dots\} \cup \{\infty\}$ (the case when $n = \infty$ is treated in [LapRaŽu1]). We adopt a similar convention for the quasiperiodic functions $G = G(\tau)$ appearing in Definition 4.6.

Proposition 4.16. *Assume that A is a quasiperiodic set in $\mathbb{R}^N$ of a given type, with an associated quasiperiodic function $G = G(\tau)$. If m is a positive integer and $L > 0$, then the subset $A \times [0, L]^m$ of $\mathbb{R}^{N+m}$ is also quasiperiodic of the same type, with the associated quasiperiodic function equal to $L^m \cdot G$. In particular, if $n \geq 2$ is an integer and A is the n-quasiperiodic subset of $\mathbb{R}$ constructed in Theorem 4.14, then the subset $A \times [0, L]^m$ of $\mathbb{R}^{1+m}$ is also n-quasiperiodic.*

Proof. Let us first prove the claim for $m = 1$. By assumption, we have that

$$|A_t|_N = t^{N-D}(G(1/t) + o(1)) \quad \text{as } t \to 0^+, \tag{4.15}$$

where $G = G(\tau)$ is a quasiperiodic function; see Equation (4.8). Much as in Equation (3.35), we can write

$$\begin{aligned} |(A \times [0, L])_t|_{N+1} &= |A_t|_N \cdot L + |A_t|_{N+1} \\ &= t^{(N+1)-(D+1)}(L \cdot G(1/t) + o(1)) + |A_t|_{N+1} \end{aligned} \tag{4.16}$$

as $t \to 0^+$. Since, obviously, $|A_t|_{N+1} \leq |A_t|_N \cdot t$, we have that

$$\begin{aligned} |A_t|_{N+1} &\leq t^{N+1-D}(G(1/t) + o(1)) = t^{(N+1)-(D+1)} \cdot t(G(1/t) + o(1)) \\ &= t^{(N+1)-(D+1)} \cdot O(t) \quad \text{as } t \to 0^+. \end{aligned} \tag{4.17}$$

Therefore,

$$\begin{aligned} |(A \times [0, L])_t|_{N+1} &= t^{(N+1)-(D+1)}(L \cdot G(1/t) + o(1) + O(t)) \\ &= t^{(N+1)-(D+1)}(L \cdot G(1/t) + o(1)) \quad \text{as } t \to 0^+. \end{aligned} \tag{4.18}$$

Hence, by Definition 4.10, the set $A \times [0, L]$ is quasiperiodic, with the associated quasiperiodic function $L \cdot G$. This completes the proof of the proposition for $m = 1$. The general case is easily obtained by induction on m. □

4.4. **Future applications and extensions: ∞-quasiperiodic sets, hyperfractals, and the notion of fractality.** The results of Section 4 and their various generalizations (and, especially, the construction of n-quasiperiodic sets carried out in Section 4.3 above, once it has been extended to the case where $n = \infty$) will play a key role in the later applications of the higher-dimensional theory of complex dimensions developed in the present paper and in [LapRaŽu1–7]. This will be so, in particular, in relation to the construction of (transcendentally) ∞-quasiperiodic, *maximally hyperfractal* sets (for which the associated fractal zeta functions have a natural boundary along the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$) and, in fact, have a

singularity at every point of that line). (See [LapRaŽu2–3], along with [LapRaŽu1, Section 4.6].) Such sets are as "fractal" as possible since, in some sense, they have a continuum of nonreal "complex dimensions" (interpreted here as singularities of the fractal zeta functions attached to A), in striking contrast with the more usual case where the fractal zeta functions can be meromorphically extended to an open connected neighborhood of the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$ and therefore have at most countably nonreal complex dimensions.

Recall that following [Lap-vFr3, Sections 12.1 and 13.4] (naturally extended to higher dimensions within the framework of our new theory), a bounded subset A of $\mathbb{R}^N$ is said to be "fractal" if its associated fractal zeta function (here, ζ_A or $\tilde{\zeta}_A$) has a *nonreal* complex dimension or else, if it has a natural boundary along a suitable curve (a screen S, in the sense of [Lap-vFr1–3] and of Section 2.4 above); that is, the tube zeta function $\tilde{\zeta}_A$ (or, equivalently, the distance zeta function ζ_A if $\overline{\dim}_B A < N$) cannot be meromorphically extended beyond S.

We close these comments by noting that throughout Section 4, we have worked with bounded subsets of the real line, $\mathbb{R}$. However, using the results of Section 3.4 (especially, Theorem 3.15), one can easily obtain corresponding constructions of transcendentally ∞-quasiperiodic compact sets A in $\mathbb{R}^N$ (for any $N \geq 1$), with $\overline{\dim}_B A \in (N-1, N)$. (See also Proposition 4.16 at the end of Section 4.3.) Likewise, using Theorem 3.15, one can construct ∞-quasiperiodic maximally hyperfractal compact subsets A of $\mathbb{R}^N$ (for any $N \geq 1$) such that $\overline{\dim}_B A \in (N-1, N)$. (Actually, by considering the Cartesian product of the original subset of $\mathbb{R}$ by $[0,1]^m$, with $0 \leq m \leq N-1$, one may assume that $\overline{\dim}_B A \in (m, N)$; the same comment can be made about all of the results obtained in Section 4.) Such a construction can also be extended to the more general setting of relative fractal drums (in the sense of [LapRaŽu3]); see [LapRaŽu1–7].

Finally, these results can also be applied in a key manner in order to establish the optimality of certain inequalities associated with the meromorphic continuations of the spectral zeta functions of (relative) fractal drums (see [LapRaŽu1, Section 4.3] and [LapRaŽu6, Section 6]).

Appendix A: Equivalence relation and extended Dirichlet-type integrals

One problem with the notion of "equivalence" provided in Definition 2.22 of Section 2.4 is that, strictly speaking, it is not an equivalence relation because, a priori, f and g do not belong to the same class of functions. (Indeed, f is a Dirichlet-type integral, abbreviated DTI in the sequel, while g is merely assumed to be meromorphic; in particular, the abscissa of convergence of g need not be well defined.) The situation is very analogous, in spirit, to the evaluation of the "leading part" ($g = g(s)$, in the present case) of a function ($f = f(s)$, here) in the theory of asymptotic expansions. In that situation, the "leading part" g belongs to a scale of typical functions (describing the possible asymptotic behaviors of the function f in the given asymptotic limit).

In our present situation, just as in the theory of asymptotic expansions, formally, the relation $\sim$ is both reflexive and (when it makes sense) transitive. Of course, it is also symmetric when it acts on the same class of functions (for example, DTIs).

However, it is also possible to modify both the definition of $\sim$ and the class of functions on which it acts so that it becomes a true equivalence relation on a single

space of functions, namely, the class of extended DTIs. The latter class of (tamed) extended DTIs contains the class of (tamed) DTIs (hence, all of the functions f we wanted to work with in Definition 2.22) and it also contains (essentially) all of the functions g occurring in practice (when applying Definition 2.22).

By definition, given $r \in (0,1)$, a DTI *of base* r is a function of the form

$$g(s) = \zeta_{E,\varphi,\mu}(r^{-s}), \tag{A.1}$$

where $f(s) := \zeta_{E,\varphi,\mu}(s)$ is a (standard) DTI defined by

$$\zeta_{E,\varphi,\mu}(s) := \int_E \varphi(x)^s \mathrm{d}\mu(x). \tag{A.2}$$

(See also Definition 2.12.) It is then easy to check (using the analogous result for ordinary DTIs) that if g is tamed (i.e., if f is tamed), then the abscissa of convergence $D(g)$ of g and the half-plane of convergence $\Pi(g) := \{\operatorname{Re} s > D(g)\}$ are not well defined. Indeed, note that

$$\varphi(x)^{r^s} = \varphi(x)^{r^{\operatorname{Re} s}(\cos((\log r)\operatorname{Im} s) + \mathrm{i}\sin((\log r)\operatorname{Im} s)},$$

so that the open set V of complex numbers s for which $\varphi(x)^{r^s}$ is Lebesgue integrable on E (typically) consists of countably many connected components, and, hence, does not have the form of a half-plane. The indicated open set V is analyzed in [LapRaŽu1, Appendix A, Section A.4].

Definition A.1. An *extended Dirichlet-type integral* (an *extended* DTI or EDTI, in short) $h = h(s)$ is either of the form

$$h(s) := \rho(s)\zeta_{E,\varphi,\mu}(s) \tag{A.3}$$

or of the form

$$h(s) := \rho(s)\zeta_{E,\varphi,\mu}(r^{-s}), \quad \text{for some } r \in (0,1), \tag{A.4}$$

where $\rho = \rho(s)$ is a nowhere vanishing entire function and $\zeta_{E,\varphi,\mu} = \zeta_{E,\varphi,\mu}(s)$ is a DTI. More generally, ρ can be a holomorphic function which does not have any zeros in the given domain $U \subseteq \mathbb{C}$ under consideration, where U contains the closed half-plane $\{\operatorname{Re} s > D(\zeta_{E,\varphi\mu})\}$.

If the extended DTI is of the form (A.3), it is said to be of *type* I, and if it is of the form (A.4), it is said to be of *type* II (or of type II_r if one wants to keep track of the underlying base r). Note that EDTIs of type I include all ordinary DTIs as a special case (by taking $\rho \equiv 1$).

Let us denote by $f(s) := \zeta_{E,\varphi,\mu}(s)$ the (standard) DTI and by $g(s) := \zeta_{E,\varphi,\mu}(r^{-s})$ the DTI of base r occurring in (A.4). Then, by definition (and in accordance with Definition A.1), if h is of the form (A.3), its *abscissa of convergence* $D(h)$ is given by $D(h) := D(f)$, while if h is of the form (A.4), then $D(h) = +\infty$, that is, $\Pi(h) = \emptyset$.

If the DTI $f(s) := \zeta_{E,\varphi,\mu}$ is tamed, then the extended DTI h from Definition A.1 (either in (A.3) or in (A.4)) is said to be *tamed*.

Finally, given any tamed extended DTI of type I, $h = h(s)$ (as in the first part of Definition A.1), we call

$$\Pi(h) := \{\operatorname{Re} s > D(h)\} \tag{A.5}$$

the *half-plane of convergence* of h (which is maximal, in an obvious sense), and (assuming that $D(h) \in \mathbb{R}$) we call $\{\operatorname{Re} s = D(h)\}$ the *critical line* of h. (The tameness condition enables us to show that this half-plane exists and is indeed,

maximal.) Using a classic theorem about the holomorphicity of integrals depending on a parameter, one can show that h is holomorphic on $\Pi(h)$. Hence, $D_{\mathrm{hol}}(h) \leq D(h)$.

Here, much as in Definition 2.12, $D(h)$ and $D_{\mathrm{hol}}(h)$ denote, respectively, the *abscissa of* (*absolute*) *convergence* and the *abscissa of holomorphic continuation* of h. Furthermore, if h is given by (A.3) above, we set $D(h) = D(\zeta_{E,\varphi,\mu})$ and $D_{\mathrm{hol}}(h) = D_{\mathrm{hol}}(\zeta_{E,\varphi,\mu})$, where $D(\zeta_{E,\varphi,\mu})$ and $D_{\mathrm{hol}}(\zeta_{E,\varphi,\mu})$ are defined in Definition 2.12.

Moreover, if $h = h(s)$ admits a meromorphic continuation to an open connected set U containing the closed half-plane $\{\operatorname{Re} s \geq D(h)\}$, we denote (much as was done in Definition 2.18 for the special case of DTIs) by $\mathcal{P}_c(h)$ the set of *principal complex dimensions* of h; that is, the set of poles of h (in U) located on the critical line $\{\operatorname{Re} s = D(h)\}$ of h:

$$\mathcal{P}_c(h) := \{\omega \in U : \omega \text{ is a pole of } h \text{ and } \operatorname{Re}\omega = D(h)\}. \tag{A.6}$$

Clearly, $\mathcal{P}_c(h)$ does not depend on the choice of the domain U satisfying the above condition.

We define similarly $\mathcal{P}(h) = \mathcal{P}(h, U)$, the *set of* (visible) *complex dimensions* of h, relative to U:

$$\mathcal{P}(h) := \{\omega \in U : \omega \text{ is a pole of } h\}. \tag{A.7}$$

Clearly, since h is of type I (i.e., is given as in (A.3)), then $\mathcal{P}_c(h) = \mathcal{P}_c(f)$ and $\mathcal{P}(h) = \mathcal{P}(f)$, where $f(s) := \zeta_{E,\varphi,\mu}(s)$.

We can now modify as follows the definition of the "equivalence relation" provided in Definition 2.22 of Section 2.4.

Definition A.2. Let h_1 and h_2 be arbitrary tamed, extended DTIs of type I (as in Definition A.1) such that $D(h_1) = D(h_2) =: D$, with $D \in \mathbb{R}$. Assume that each of h_1 and h_2 admits a (necessarily unique) meromorphic continuation to an open connected neighborhood U of the closed half-plane $\{\operatorname{Re} s \geq D\}$. Then the functions h_1 and h_2 are said to be *equivalent*, and we write $h_1 \sim h_2$, if the sets of poles of h_1 and h_2 on their common vertical line $\{\operatorname{Re} s = D\}$ (and the corresponding poles have the same multiplicities): $\mathcal{P}_c(h_1) = \mathcal{P}_c(h_2)$ (where the equality holds between multisets).

We conclude this appendix by providing a class of tamed extended DTIs which can be used to determine the "leading behavior" of most of the fractal zeta functions used in the present theory.

Theorem A.3. *Let $P \in \mathbb{C}[x]$ be a polynomial with complex coefficients. Then $f(s) := 1/P(s)$ is a tamed DTI of type I.*

More specifically, if $\deg P = n \geq 1$, *then*

$$f(s) := \frac{1}{P(s)} = \zeta_{E,\varphi,\mu}(s), \tag{A.8}$$

where $E := [1, +\infty)^n$, $\varphi(x) = (x_1 \cdots x_n)^{-1}$ for all $x \in E$, and

$$\mu(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) := c\, x_1^{a_1} \frac{\mathrm{d}x_1}{x_1} \ldots x_n^{a_n} \frac{\mathrm{d}x_n}{x_n}, \tag{A.9}$$

so that its total variation measure $|\mu|$ *(in the sense of local measures) is given by*

$$|\mu|(\mathrm{d}x_1,\dots,\mathrm{d}x_n) := c\, x_1^{\operatorname{Re} a_1}\frac{\mathrm{d}x_1}{x_1}\dots x_n^{\operatorname{Re} a_n}\frac{\mathrm{d}x_n}{x_n},$$

where $c := \frac{1}{n!}P^{(n)}(0)$ *and* $a_1,\dots,a_n$ *are the zeros of* $P = P(s)$ *(counted according to their multiplicities, so that* $P(s) = c\,\Pi_{m=1}^{n}(s-a_m)$*).*

Moreover, $D(f) = D(\zeta_{E,\varphi,\mu}) \le \max\{\operatorname{Re} a_1,\dots,\operatorname{Re} a_n\}$.

Remark 4.17. If, in Theorem A.3, we assume that $\deg P = 0$, i.e., if P is constant, say $P \equiv 1$, then clearly, $f(s) = 1/P(s) = 1 = \zeta_{E,\varphi,\mu}(s)$, where $E := [1,+\infty)$, $\varphi(t) := 1$ for all $x \in E$, and $\mu = \delta_1$ (the Dirac measure concentrated at 1). In particular, f is also tamed in this case.

Theorem A.3 is a consequence of the following two facts:

(*i*) If $f_a(s) := 1/(s-a)$, where $a \in \mathbb{C}$ is arbitrary, then f is a tamed DTI of type I, given by

$$f(s) := \frac{1}{s-a} = \zeta_{E,\varphi_a,\mu_a}(s), \tag{A.10}$$

where $E := [1,+\infty)$, $\varphi_a(x) := x^{-1}$ for all $x \in E$, and

$$\mu_a(\mathrm{d}x) := x^a\frac{\mathrm{d}x}{x}; \tag{A.11}$$

so that $|\mu_a|(\mathrm{d}x) := x^{\operatorname{Re} a}\mathrm{d}x/x$. Furthermore, $D(f_a) = \operatorname{Re} a$. Note that $f_a := \zeta_{E,\varphi_a,\mu_a}$ is obviously tamed because $\varphi_a(x) \le 1$ for all $x \ge 1$. An entirely analogous comment can be made about $f = \zeta_{E,\varphi,\mu}$ in the theorem.

(*ii*) The tensor product of two tamed DTIs is tamed. More specifically, if the DTIs $\zeta_{E,\varphi,\mu}$ and $\zeta_{F,\psi,\eta}$ are tamed, then their *tensor product* is given by the following tamed DTI:

$$h(s) := (\zeta_{E,\varphi,\mu}\otimes\zeta_{F,\psi,\eta})(s) = \zeta_{E\times F,\varphi\otimes\psi,\mu\otimes\eta}(s), \tag{A.12}$$

where the tensor product $\varphi\otimes\psi$ is defined by $(\varphi\otimes\psi)(x,y) := \varphi(x)\,\psi(y)$ for $(x,y) \in E\times F$ and the tensor product $\mu\otimes\eta$ is the product measure of μ and η (see, e.g., [Coh]). It is easy to check that the DTI h is tamed because (since $\zeta_{E,\varphi,\mu}$ and $\zeta_{F,\psi,\eta}$ are tamed), we have $0 \le \varphi(x) \le C(\varphi)$ $|\mu|$-a.e. on E and $0 \le \psi(x) \le C(\psi)$ $|\eta|$-a.e. on F, so that $0 \le (\varphi\otimes\psi)(x,y) \le C(\varphi)\,C(\psi)$ $|(\mu\otimes\eta)|$-a.e. on $E\times F$.

Furthermore, $D(h) \le \max\{D(\zeta_{E,\varphi,\mu}), D(\zeta_{F,\psi,\eta})\}$.

Statement (*i*) above follows from a direct computation, while statement (*ii*) is proved by an application of the Fubini–Tonelli theorem (for iterated integrals with respect to positive measures) combined with the inequality (between local positive measures) $|\mu\otimes\eta| \le |\mu|\otimes|\eta|$, followed by an application of the classic Fubini theorem (for iterated integrals with respect to possibly signed or complex measures).

Corollary A.4. *The meromorphic function (on all of* $\mathbb{C}$*) given by*

$$h_2(s) := \frac{\rho(s)}{P(r^{-s})}, \tag{A.13}$$

where $r \in (0,1)$*,* $P \in \mathbb{C}[x]$ *is an arbitrary polynomial with complex coefficients and* ρ *is a nowhere vanishing entire function, is a tamed extended DTI of type II.*

More specifically, $h_2(s) = \rho(s)\zeta_{E,\varphi,\mu}(r^{-s})$, where E, φ and μ are given in Theorem A.3 above.

As was alluded to earlier, in practice, when we apply the (modified) definition of the equivalence relation (see Definition A.2 above),

$$h_1 \sim h_2 \tag{A.14}$$

the meromorphic function h_1 is a fractal zeta function (an ordinary DTI of type I), as well as the function h_2 (which gives the "leading behavior" of h_1, to mimick the terminology of the theory of asymptotic expansions). Hence, the importance of Theorem A.3 in the theory developed in the present paper as well as in its sequels [LapRaŽu2–3], the survey article [LapRaŽu6], and the monograph [LapRaŽu1]. (See, however, Definition A.6 below and the comments surrounding it.)

We refer the interested reader to [LapRaŽu1, Appendix A] for more details about the topics discussed in the present appendix, along with detailed proofs of the main results.

Remark A.5. The two definitions of the notion of equivalence $\sim$ provided in Definition 2.22 and Definition A.2 are compatible in the sense that if, in Definition 2.22, we assume that f (denoted by h_1 in Definition A.2) is a DTI (as is the case in Definition 2.22), the meromorphic function g is an extended DTI, then $f \sim g$ in the sense of Definition A.2. Note that the functions f and g of Definition 2.22 are denoted by h_1 and h_2 in Definition A.2. (In particular, $D(g)$ and $\mathcal{P}_c(g)$ are well defined, $D(f) = D(g)$ and $\mathcal{P}_c(f) = \mathcal{P}_c(g)$.) The converse statement clearly holds as well.

Finally, it is possible, even likely, that in future applications of the current theory of fractal zeta functions developed in this paper and in [LapRaŽu1–7], we will need to deal with functions g which are no longer extended DTIs (of type I), but are meromorphic functions of a suitable kind. In that case, we propose to use the following definition, which is a suitable modification of Definition 2.22 and seems well suited to various applications. Strictly speaking, it no longer gives rise to an equivalence relation (since f and g belong to different classes of functions) but in this new sense, the statement $f \overset{\text{asym}}{\sim} g$ captures appropriately the idea that "f is asymptotic to g".

Definition A.6. Let f be a tamed EDTI and let g be a meromorphic function, both defined and meromorphic on an open and connected subset U of $\mathbb{C}$ containing the closed right half-plane $\{\operatorname{Re} s \geq D(f)\}$. Then, the function f is said to be *asymptotically equivalent* to g, and we write $f \overset{\text{asym}}{\sim} g$, if $D(f) = D_{\text{hol}}(g)$ (and this common value is a real number), and the poles of f and g located on the convergence critical line $\{\operatorname{Re} s > D(f)\}$ of f (which, by assumption, is also the holomorphy critical line of g) coincide (and have the same multiplicities).

More succinctly, and with the obvious notation (compare with Equation (2.34) in Definition 2.22 above), we have

$$f \overset{\text{asym}}{\sim} g \quad \overset{\text{def.}}{\iff} \quad D(f) = D_{\text{hol}}(g)\ (\in \mathbb{R}) \text{ and } \mathcal{P}_c(f) = \mathcal{P}_{c,\text{hol}}(g). \tag{A.15}$$

More specifically, we let

$$\mathcal{P}_{c,\text{hol}}(g) := \{\omega \in U : \omega \text{ is a pole of } g \text{ and } \operatorname{Re}\omega = D_{\text{hol}}(g)\}.$$

Furthermore, much as in Definition 2.22, $D(f)$ and $D_{\mathrm{hol}}(g)$ are viewed as multisets in Equation (A.15).

Remark A.7. Observe that even if g is assumed to be a tamed EDTI, Definition A.6 may differ from its counterpart used in the rest of this appendix (Definition A.2) or, in particular, in Definition 2.22. Indeed, there are examples of tamed DTIs g for which $D_{\mathrm{hol}}(g) < D(g)$. Therefore, strictly speaking, Definition A.6 does not extend Definition 2.22 (or Definition A.2). However, it is stated in the same spirit and seems to often be what is needed, in practice.

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