

# Anyon wave functions and probability distributions

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## Abstract

The problem of determining the ground state energy for a quantum gas of anyons in two dimensions is considered. A recent approach to this problem by means of lower bounds is here refined to bring out the dependence on the  $n$ -particle probability distributions encoded in the wave functions. Furthermore, a class of states which has been proposed in the context of upper bounds for a related many-anyon problem, is here considered from the point of view of these refined lower bounds. A numerical approach to determining their corresponding probability distributions is employed for a limited number of particles.

## 1 Introduction

Identical quantum particles in two spatial dimensions offer exciting possibilities beyond the traditional quantum statistics of fermions and bosons, so called fractional or anyon statistics. In its simplest (scalar/Abelian) formulation it is characterized by a purely complex phase shift  $e^{i\alpha\pi}$  for the wave function under the interchange of two particles, where  $\alpha \in [0, 2)$  is known as the statistics parameter. However, while the standard cases of non-interacting fermions ( $\alpha = 1$ ) and bosons ( $\alpha = 0$ ) are simply understood in terms of one-particle states and operators, and the properties of the corresponding ideal Bose and Fermi gases follow straightforwardly, the physics of the ideal anyon gas still to this date remains an open problem, despite possible relevance to the fractional quantum Hall effect (FQHE) in 2D electron gases [1, 15] as well as to other systems which are effectively two-dimensional [27, 13]. One way to appreciate the increased level of difficulty for the many-anyon model is to note that it is conveniently formulated as a

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system of bosons (or alternatively fermions) which are all interacting with each other via long-range magnetic vector potentials. Such fully interacting many-body systems are typically much more complicated to understand than to merely work with copies of a single-particle system as in the case of non-interacting bosons and fermions. Despite the lack of such fundamental understanding for the anyon gas, there has been considerable progress over the last 35 years on the quantum mechanics of many anyons (one of the more successful approaches being mean-field theory) and the topic has been summarized in several books and comprehensive reviews [9, 14, 17, 22, 23, 28].

Recently, a first set of rigorous lower bounds for the ground state energy for the ideal anyon gas were derived [18, 19]. Interestingly, these exhibit an unexpected and rather complicated dependence on the statistics parameter  $\alpha$ . More precisely, the bounds concerning the many-particle limit only become non-trivial for  $\alpha$  being an *odd numerator rational*, and they furthermore become weaker with the size of the denominator of  $\alpha$ . By dimensional considerations, the ground state energy  $E_0$  per unit area for a free gas of  $N$  anyons in the thermodynamic limit with fixed density  $\bar{\rho} := N/L^2$  is necessarily  $\bar{\rho}^2$  times a constant  $e(\alpha)$  dependent only on  $\alpha$  (we have chosen units s.t.  $m = \hbar = 1$ ):

$$\liminf_{N,L \rightarrow \infty} \frac{E_0}{L^2} = e(\alpha) \bar{\rho}^2.$$

Bosons saturate the trivial lower bound  $e(\alpha) \geq e(\alpha = 0) = 0$ , while for (spinless) fermions  $e(\alpha = 1) = \pi$ . Now, the results of [18, 19] imply that, if  $\alpha = \mu/\nu$  is a reduced fraction with  $\mu$  odd, then

$$e(\alpha) \geq C/\nu^2 > 0,$$

for some universal numerical constant  $0.021 \leq C \leq \pi$  (the upper bound following from the fermionic special case). This non-trivial bound for the energy originates from a local kind of exclusion principle, or degeneracy pressure, of the form of an effective repulsion between anyons caused by the mismatch between the interchange phase, which depends on  $\alpha$ , and the pairwise orbital angular momentum phase, which is quantized in *even* integer multiples of  $2\pi$ . However, although it is possible to reduce such a phase mismatch by assuming certain configurations of the particles, it can never be completely eliminated in the case of odd numerator rational  $\alpha$ , thus explaining the restriction on  $\alpha$  above. These facts, considered together with several other previously known peculiar features of many-anyon models<sup>1</sup>, led to the question whether the true ground state energy actually *is* lower for even numerator and/or large denominator rationals, and to a consideration of families of trial states which exhibit a structure that could help to minimize their energy for certain  $\alpha$  [20]. These families, also given explicitly below,

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<sup>1</sup>For example, several authors have pointed towards a non-analytic dependence on  $\alpha$  for the ground state energy in the large- $N$  limit of  $N$ -anyon models (see e.g. [4]).

are defined for rational  $\alpha$  with a strong dependence on the denominator, and are structurally different for even and odd numerator fractions.

With these considerations, we can formulate the following main open questions that need to be addressed concerning the anyon gas:

**Main questions.** *What is the true dependence  $e(\alpha)$  on  $\alpha$  for the ground state energy per unit area of the ideal anyon gas? Is it clear that  $e(\alpha) \leq e(1)$  for all  $\alpha$ ?<sup>2</sup> Does there appear a significant difference between even and odd numerator rational  $\alpha$ ? Does  $e(\alpha)$  depend in a non-trivial way on the denominator of  $\alpha$ ? Does the picture change significantly by the inclusion of interactions or an external magnetic field?*

In this work we approach this many-anyon problem from both directions of upper and lower bounds. We reconsider the previously derived lower bounds for the energy of the ideal anyon gas, following from a local form of the exclusion principle for anyons (Section 2), and show that they exhibit an even richer structure when the actual  $n$ -particle probability distribution of the wave function is taken into account (Section 3). We also conduct a numerical investigation (Section 4 and Appendix) of the distributions and the dependence of the corresponding bounds for a class of interesting states, including some of the above-mentioned trial states. Due to computational limitations, the considered total particle number unfortunately remains too low for any conclusions to be made concerning the thermodynamic limit, although some interesting features can be brought out.

## 2 The many-anyon problem

We model a quantum mechanical system of  $N$  anyons with statistics parameter  $\alpha \in \mathbb{R}$  by means of bosonic, i.e. completely symmetric, wave functions  $\psi \in L^2_{\text{sym}}((\mathbb{R}^2)^N)$  together with the free kinetic energy operator

$$\hat{T}_A := \frac{1}{2} \sum_{j=1}^N D_j^2, \quad D_j := -i\nabla_j + \mathbf{A}_j(x). \quad (1)$$

Here the purely topological magnetic vector potentials

$$\mathbf{A}_j(x) := \alpha \sum_{\substack{k=1 \\ k \neq j}}^N \frac{(\mathbf{x}_j - \mathbf{x}_k)I}{|\mathbf{x}_j - \mathbf{x}_k|^2} \quad (2)$$

are responsible for capturing the correct exchange symmetry of the formally defined (multivalued) anyonic wave function

$$\psi_A = \prod_{j < k} e^{i\alpha\phi_{jk}} \psi, \quad \phi_{jk} := \text{azimuthal angle of } \mathbf{x}_j - \mathbf{x}_k. \quad (3)$$

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<sup>2</sup>This second question is motivated by the fact that the ground state energy is known to peak for intermediate  $\alpha$  for certain  $N$ ; see e.g. [21, 25] and Fig. 1-2 below.

The case  $\alpha = 0$  then trivially corresponds to bosons, while for  $\alpha = 1$  the particles are fermions, although in a bosonic (albeit interacting) representation. It was shown in [19] that the operator (1) with the corresponding energy form ( $\psi$  will always be assumed to be normalized)

$$T_A = \langle \psi, \hat{T}_A \psi \rangle = \frac{1}{2} \int_{\mathbb{R}^{2N}} \sum_{j=1}^N |D_j \psi|^2 dx$$

uniquely<sup>3</sup> defines a self-adjoint operator  $\hat{T}_A$  for the free kinetic energy for  $N$  anyons on  $\mathbb{R}^2$ .

## 2.1 Recent bounds for the many-anyon ground state energy

In [18, 19] it was found that the energy per unit area for an ideal gas of  $N$  anyons is bounded from below by

$$\frac{T_A}{L^2} \geq c_A C_{\alpha,N}^2 \bar{\rho}^2, \quad \bar{\rho} := \frac{N}{L^2}, \quad (4)$$

where  $L^2$  is the volume (area) of the system,  $c_A$  is a universal numerical constant in the range  $[0.021, \pi]$ , and

$$C_{\alpha,N} := \min_{p \in \{0,1,\dots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|. \quad (5)$$

For fixed  $\alpha$ , this expression is non-increasing with  $N$  and such that, if  $\alpha = \mu/\nu$  is a reduced fraction with odd numerator  $\mu$ , then  $\lim_{N \rightarrow \infty} C_{\alpha,N} = 1/\nu$ , while  $\lim_{N \rightarrow \infty} C_{\alpha,N} = 0$  otherwise. The graph of the function  $\alpha \mapsto C_{\alpha,N}$  can be obtained by cutting out a symmetric wedge of slope  $\nu$  from the upper half plane at each even numerator rational  $\alpha = \mu/\nu$ ,  $\nu \leq 2N-3$ , on the horizontal axis (cp. with Fig. 2 in [18]). The origin of this peculiar statistics-dependent expression (5) is the following local energy bound, which we conveniently refer to as a local exclusion principle for anyons (drawing a parallel to a corresponding bound for fermions following exclusively from the Pauli principle and which was used by Dyson and Lenard in their proof of stability of matter [7]):

**Lemma 1** (Local exclusion for anyons). *Let  $\psi$  be a wave function of  $N$  anyons defined on some subdomain  $\Omega$  of  $\mathbb{R}^2$ . Then the contribution to the free kinetic energy with exactly  $n$  anyons on a square  $Q \subseteq \Omega$  with area  $|Q|$  and with all other anyons residing outside  $Q$  is*

$$\frac{1}{2} \int_{Q^n} \sum_{j=1}^n |D_j \psi|^2 dx \geq (n-1) C_{\alpha,n}^2 \frac{c}{|Q|} \int_{Q^n} |\psi|^2 dx, \quad (6)$$

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<sup>3</sup>The form  $T_A$  can be formally defined either minimally, taking its closure w.r.t. the space of smooth functions supported away from any two-particle diagonals, or maximally in the sense of distributions, and these two definitions can be shown [19] to coincide and hence naturally associate to one and the same non-negative self-adjoint operator  $\hat{T}_A$ .

where  $c = 0.056$ . The magnetic derivatives in the l.h.s. can be taken to be independent of the  $N - n$  anyons outside  $Q$  using a gauge transformation.

Note that the l.h.s. can become zero in the case of a single particle (with such Neumann b.c. on  $Q$  we can e.g. take  $\psi$  constant). The intuitive reason for a strictly positive energy for the two-particle case and beyond (if  $\alpha \notin 2\mathbb{Z}$ ), is the ‘mismatch’ between the phase contributed by the gauge potential under particle interchange, and the allowed phase of the wave function due to its imposed bosonic symmetry. Namely, contributing in (5) is the phase  $(2p + 1)\alpha\pi$  arising from the gauge potential under a pairwise continuous interchange of two anyons when  $p$  other anyons are being enclosed in the interchange loop (such a loop can equivalently be considered as a simple interchange of two anyons followed by a complete encircling of the other  $p$  anyons by one of the interchanged anyons), while  $-2q\pi$  is the pairwise orbital angular momentum phase of the wave function, for bosons being quantized in even integers. A non-zero residual phase  $\varphi = (2p + 1)\alpha\pi - 2q\pi$  gives rise to an effective angular momentum barrier, i.e. a local repulsive potential in the relative coordinate of each particle pair — in other words resulting in a statistics degeneracy pressure. In the case of  $\alpha$  being an odd numerator rational this phase mismatch can never be completely overcome regardless of the number and angular momenta of particles, while in the case of even numerators it actually can, for certain configurations of the particles. For example, for  $\alpha = 2/3$  and the situation that two such anyons symmetrically encircle a third one with relative angular momentum  $-2$ , there is an effective cancellation of the two phases:  $\varphi = 0$ .

In [18] another useful energy inequality for anyons was proved (from which (4) also follows but with a weaker constant  $c'_A$ ), namely the Lieb-Thirring inequality

$$T_A \geq c'_A C_{\alpha,N}^2 \int_{\mathbb{R}^2} \rho(\mathbf{x})^2 d\mathbf{x}, \quad (7)$$

where  $\rho$  is the one-particle density distribution (17) associated to  $\psi$ , and  $c'_A \in [10^{-4}, \pi]$ . An interesting application is to consider the model for  $N$  anyons in a harmonic oscillator potential, with the one-body Hamiltonian  $H_1 = \frac{1}{2}(-\Delta + \omega^2|\mathbf{x}|^2)$ , for which (7) yields the bound for the ground state energy [19]

$$E_0 \geq \frac{1}{3} \sqrt{\frac{8c'_A}{\pi}} C_{\alpha,N} \omega N^{3/2}. \quad (8)$$

Note that for the case  $N = 2$  one finds  $E_0 \geq \frac{8}{3} \sqrt{\frac{c'_A}{\pi}} \alpha \omega$  for  $\alpha \in [0, 1]$ , and in other words a *linear* dependence on  $\alpha$ . However, we also note that (7) does not capture the bosonic ground state energy and is comparatively good only for large values of  $N$ . On the other hand, for this simple special case of two anyons in a harmonic well the system is exactly solvable [16], and the

energy eigenvalues

$$E_{n,l} = \omega(2 + 2n + |l + \alpha|), \quad n = 0, 1, 2, \dots, \quad l = 0, 2, 4, \dots,$$

indeed exhibit a linear dependence on  $\alpha$ . In [4], a general lower bound was given for the ground state energy  $E_0$  for  $N$  anyons in a harmonic oscillator:

$$E_0 \geq \omega \left( N + \left| L + \alpha \binom{N}{2} \right| \right), \quad (9)$$

where  $L$  is the total angular momentum of the state. This expression is again linear in  $\alpha$  for fixed  $L$  and  $N$ , but allows for a more complicated dependence if one considers sequences of states with  $L \sim -\alpha \binom{N}{2}$ , which we observe is required in order for  $E_0 \sim O(N^{3/2})$ , and which was in [4] argued to be the case for the true ground state. Given (8) one can actually prove this energy dependence rigorously for odd numerator rational  $\alpha$  [20]. For fermions the correct asymptotics is  $E_0 \sim \frac{\sqrt{8}}{3} \omega N^{\frac{3}{2}}$  (and note that  $L \sim -\binom{N}{2}$  in the bosonic representation (12)).

## 2.2 Proposed trial states

Because of the peculiar dependence on the numerator and denominator of  $\alpha$  in the bound (4) following from the local exclusion bound (6), considered together with a number of other interesting facts such as the bound (9), the question was raised concerning the validity of such a non-trivial bound or Lieb-Thirring inequality for general intermediate  $\alpha$ . To test this question the following families of anyonic trial states were introduced in [20].

Consider a one-body Hamiltonian in two dimensions  $H_1 = -\frac{1}{2}\Delta + V$  with confining external potential  $V$  and a corresponding set of eigenstates  $\varphi_k$ . The corresponding  $N$ -anyon Hamiltonian is  $H = \sum_{j=1}^N (\frac{1}{2}D_j^2 + V(\mathbf{x}_j))$ . Denote by  $z_j := x_{j,1} + ix_{j,2}$  the particle coordinates represented in the complex plane, and by  $z_{jk} := z_j - z_k$  the pairwise relative coordinates. Assuming that the total particle number  $N$  is given as a multiple of  $\nu$ ,  $N = K\nu$ , we arrange the particles into  $\nu$  collections of  $K$  particles. Over each such particle collection denoted  $V_q$ ,  $q = 1, \dots, \nu$ , we form a complete graph  $(V_q, E_q)$ , the set of edges  $E_q$  of which consists of all unordered pairs  $(j, k)$  of particles in  $V_q$ . For even numerator rational  $\alpha = \mu/\nu \in [0, 1]$  we then define the  $N$ -particle wave function

$$\psi_\alpha(z) := \prod_{j < k} |z_{jk}|^{-\alpha} \mathcal{S} \left[ \prod_{q=1}^{\nu} \prod_{(j,k) \in E_q} (\bar{z}_{jk})^\mu \right] \prod_{k=1}^N \varphi_0(z_k), \quad (10)$$

while for odd numerator  $\mu$ ,

$$\psi_\alpha(z) := \prod_{j < k} |z_{jk}|^{-\alpha} \mathcal{S} \left[ \prod_{q=1}^{\nu} \prod_{(j,k) \in E_q} (\bar{z}_{jk})^\mu \bigwedge_{k=0}^{K-1} \varphi_k(z_{j \in V_q}) \right], \quad (11)$$

where  $\mathcal{S}$  denotes the action of symmetrization of all particle labels. Both expressions in brackets are symmetric w.r.t. particle relabeling within each collection  $V_q$  thanks to the parity of  $\mu$  and the total antisymmetry of the Slater determinant. Note that for the simple special case  $\nu = \mu = 1$ , the expression (11) reduces to the fermionic ground state

$$\psi_{\alpha=1}(z) = \prod_{j < k} \frac{\bar{z}_{jk}}{|z_{jk}|} \bigwedge_{k=0}^{N-1} \varphi_k(z) \quad (12)$$

expressed in the bosonic representation (3). The expressions (10) and (11) also generalize to the correct gauge copies of the ground states for arbitrary integer  $\alpha \in \mathbb{Z}$ . On the other hand, the symmetrized states in the brackets in (10) with  $\nu > 1$  actually coincide with the Read-Rezayi states in the FQHE [24, 3] (up to complex conjugation). However, because of the highly singular Jastrow factor in (10) and (11), these wave functions need to be regularized for  $\nu > 1$ . We consider regularized states of the form  $\psi = \Phi_r \psi_\alpha$ , with

$$\Phi_r(z) := \prod_{j < k} \frac{|z_{jk}|^{2\alpha}}{(r^2 + |z_{jk}|^2)^\alpha},$$

which results in the local pairwise dependence  $\psi(z) \sim |z_{jk}|^\alpha$  on short scales  $|z_{jk}| \ll r$  (from considering the two-anyon model we know that this is the expected local pairwise dependence).

The structure of these trial states  $\psi$  is s.t. in each term of the symmetrized expression (10) resp. (11) at most  $\nu$  particles can be selected without necessarily involving two particles in the same collection  $V_q$ . As soon as  $\nu + 1$  particles are selected, there must be one collection  $V_q$  with two particles selected, say  $j, k \in V_q$ , and hence with an edge  $(j, k) \in E_q$  connecting them. This amounts therefore to an associated factor  $(\bar{z}_{jk})^\mu$  in  $\psi_\alpha$ , i.e. both a repulsive factor  $|z_{jk}|^\mu$  as well as an orbital phase  $e^{-i\mu\phi_{jk}}$ . The Jastrow factor acts to attract all particles but balances with this repulsion in such a way that groups of  $\nu$  particles can form, but any additional particle  $z_k$ , say at a large distance  $r$  from the group, sees apart from the attractive factor  $\sim (r^{-\alpha})^\nu = r^{-\mu}$  also an equally strong repulsion  $r^\mu$ . This balance could act to distribute the particles in such groups of  $\nu$ . At the same time, while the contribution from each such group to the magnetic potential  $\mathbf{A}_k$  at  $z_k$ , say with a relative position vector  $\mathbf{r}$  from the group, is approximately  $\sim \mu \mathbf{r}^{-1} I$ , there is also an orbital angular momentum in  $\psi$  with corresponding velocity  $\sim -\nabla \phi_{jk} \sim -\mu \mathbf{r}^{-1} I$ , again leading to a cancellation. The attractive feature of these trial states is hence this groupwise cancellation of terms appearing in the magnetic derivatives  $D_j \psi$  and which otherwise are expected to produce a rapid growth in the energy with  $N$  (and indeed this is seen to be the case in (9) if the angular momentum  $L$  would remain fixed).

These trial states were originally considered in the context of harmonic oscillator confinement, with the one-body operator

$$H_1 = \frac{1}{2}(-\Delta + \omega^2|\mathbf{x}|^2),$$

and its corresponding eigenstates  $\varphi_k$ . In this case (10) actually turns out to be an exact but singular eigenfunction [5, 2] with energy  $E = \omega(N + \deg \psi_\alpha)$ , where  $\deg \psi_\alpha = -\alpha \binom{N}{2} + \nu \mu \binom{K}{2} = -\alpha \frac{\nu-1}{2} N$  is the degree of the non-Gaussian part of the wave function. Note that in the case of odd numerator  $\alpha$  the degree increases to  $\deg \psi_\alpha \sim \nu K^{3/2} = \frac{1}{\sqrt{\nu}} N^{3/2}$  due to the Slater determinants of harmonic oscillator eigenstates. Similarly, we have for the total angular momentum  $L$  of the states  $\psi$  that  $L = -\alpha \binom{N}{2} + \alpha \frac{\nu-1}{2} N$  for (10) and for certain magic numbers  $K$  in (11). This means that for these particular sequences of states the r.h.s. of the lower bound (9) grows only linearly with  $N$ .

Here we will consider the generalized hypothesis that these expressions remain valid for other cases of one-body operators, such as the free Neumann Laplacian  $H_1 = \frac{1}{2}(-\Delta_{Q_0})$  on a domain  $Q_0$ . Because of computational advantages, it is under this assumption that these states will be considered in the present work. We remark that in [4] it was argued that also in the case of confinement on a disk (for which  $L$  is also a good quantum number) there is the requirement that for the ground state  $L \sim -\alpha \binom{N}{2}$  to leading order in  $N$ .

One may ask whether a grouping or clustering of the form (10)-(11) is necessary for an energy of the lowest order in  $N$ . However, one can show (cp. [20]) that by separating the particles one-by-one on disjoint supports and then gauge transforming away the magnetic potentials, that the ground state energy must be bounded by  $\sim N^2/|Q_0|$  for arbitrary  $\alpha$  (and  $\lesssim \omega N^{3/2}$  for the harmonic oscillator problem). We may still ask about even numerator rational and irrational  $\alpha$  however, where the current lower bounds are not sufficient to settle this question. Another question concerns the actual value of the constant, which perhaps could be lowered using clustering states.

### 3 Bounds depending on probability distributions

Proceeding at first in some generality, assume that we have a collection of  $N$  quantum particles (either distinguishable or indistinguishable) confined to a square  $Q_0$  in the plane. The particles could either be ‘rigidly’ confined with zero (Dirichlet) boundary conditions, or satisfy free (Neumann) boundary conditions<sup>4</sup>. The quantum wave function is in any case denoted by  $\psi \in L^2(Q_0^N)$  and assumed to be normalized  $\|\psi\| = 1$  on  $Q_0$ . The total expected

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<sup>4</sup>Note that periodic b.c. leads to some technicalities in the case of anyons [8, 12].



kinetic energy is given by

$$T_\psi = \langle \psi, \hat{T} \psi \rangle = \frac{1}{2} \sum_{j=1}^N \int_{Q_0^N} |D_j \psi|^2 dx, \quad \hat{T} = \frac{1}{2} \sum_{j=1}^N D_j^2,$$

where  $D_j$  denotes, depending on the particle types, either an anyonic or an ordinary momentum operator  $-i\nabla_j$  for the  $j$ :th particle.

Consider now an arbitrary smaller square  $Q \subseteq Q_0$ , and denote by  $Q^c = Q_0 \setminus Q$  its complement in  $Q_0$ . For any given subset  $A \subseteq \{1, \dots, N\}$  of the particles we then have the probability

$$\tilde{p}_A(Q) := \int_{(Q^c)^{N-|A|}} \int_{Q^{|A|}} |\psi|^2 \prod_{k \in A} d\mathbf{x}_k \prod_{k \notin A} d\mathbf{x}_k \quad (13)$$

of finding exactly those particles in the square  $Q$  upon measurement of the state  $\psi$ . Note that

$$\sum_{A \subseteq \{1, \dots, N\}} \tilde{p}_A(Q) = \int_{Q_0^N} |\psi|^2 \prod_{k=1}^N (\chi_Q(\mathbf{x}_k) + \chi_{Q^c}(\mathbf{x}_k)) d\mathbf{x} = \|\psi\|^2 = 1, \quad (14)$$

in other words, some subset (possibly empty) of particles must always be found on  $Q$ . The probability of finding exactly  $n$  particles on  $Q$  *irrespective of their labels*, is

$$p_n(Q) := \sum_{A \subseteq \{1, \dots, N\} \text{ s.t. } |A|=n} \tilde{p}_A(Q), \quad (15)$$

again satisfying the normalization  $\sum_{n=0}^N p_n(Q) = 1$ . The expected number of particles to be found on  $Q$  is therefore

$$\begin{aligned} \rho(Q) &:= \sum_{n=0}^N n p_n(Q) = \int_{Q_0^N} |\psi|^2 \sum_{j=1}^N \chi_Q(\mathbf{x}_j) \prod_{k=1}^N (\chi_Q(\mathbf{x}_k) + \chi_{Q^c}(\mathbf{x}_k)) d\mathbf{x} \\ &= \int_Q \rho(\mathbf{x}) d\mathbf{x}, \end{aligned} \quad (16)$$

where  $\mathbf{x} \mapsto \rho(\mathbf{x}) \in L^1(Q_0)$  is the one-particle density distribution function

$$\rho(\mathbf{x}) := \sum_{j=1}^N \int_{Q_0^{N-1}} |\psi(\mathbf{x}_1, \dots, \mathbf{x}_j = \mathbf{x}, \dots, \mathbf{x}_N)|^2 \prod_{k \neq j} d\mathbf{x}_k. \quad (17)$$

Now, consider for simplicity the case of identical particles (anyons with statistics parameter  $\alpha$ ) and the contribution to the free kinetic energy when exactly  $n$  particles reside on  $Q$ ,

$$\frac{1}{2} \int_{Q^n} \sum_{j=1}^n |D_j \psi|^2 dx \geq \frac{e_n}{|Q|} \int_{Q^n} |\psi|^2 dx, \quad (18)$$

where we denote by  $e_n = e_n(\alpha)$  the infimum of the spectrum of the corresponding  $n$ -particle kinetic energy operator  $\hat{T}_Q^n := \frac{1}{2} \sum_{j=1}^n D_j^2$  on  $Q^n$  (where  $D_j$  now only depend on these  $n$  particles in  $Q$ ; cp. Lemma 1):

$$e_n := \inf \left\{ \frac{|Q|}{2} \int_{Q^n} \sum_{j=1}^n |D_j \psi|^2 dx \Big/ \int_{Q^n} |\psi|^2 dx : \psi \neq 0 \text{ in domain of } T_A \right\}.$$

Note that  $e_n$  is independent of  $|Q|$  by scale invariance and that we consider free/Neumann b.c. on  $Q$  (i.e. in a generalized sense for  $D_j \psi$ ), so  $e_1 = 0$  for all  $\alpha$ . For bosons we then have  $e_n \equiv 0$  for all  $n$ , while for fermions we have a sum of one-particle eigenvalues  $\lambda_k = \frac{\pi^2}{4} (n_{k,x}^2 + n_{k,y}^2)$ ,  $n_{k,x}, n_{k,y} \in \mathbb{Z}_{\geq 0}$ , for the Neumann Laplacian on  $[-1, 1]^2$ :

$$e_n = 2 \sum_{k=0}^{n-1} \lambda_k \geq \frac{\pi^2}{2} (n-1)_+ \quad \text{and} \quad e_n \sim \pi n^2 + o(n^2) \text{ as } n \rightarrow \infty.$$

For the case of anyons with statistics parameter  $\alpha \in \mathbb{R}$ , the local exclusion principle (6) provides the lower bound

$$e_n(\alpha) \geq c (n-1)_+ C_{\alpha,n}^2, \quad c = 0.056. \quad (19)$$

We consider it to be a very important question to understand the true dependence of  $e_n(\alpha)$  on  $n$  and  $\alpha$ , and we will return to this question below.

Assuming Dirichlet b.c. on  $\partial Q_0$  and given an arbitrarily sized square  $Q$  centered at the origin, we can, following [7, 19], write the total expected kinetic energy for  $N$  particles as

$$\begin{aligned} 2T_\psi &= \int_{\mathbb{R}^{2N}} \sum_{j=1}^N |D_j \psi|^2 \frac{1}{|Q|} \int_{Q_0+Q} \chi_{\mathbf{y}+Q}(\mathbf{x}_j) d\mathbf{y} d\mathbf{x} \\ &= \frac{1}{|Q|} \int_{Q_0+Q} \sum_{A \subseteq \{1, \dots, N\}} \int_{((\mathbf{y}+Q)^c)^{N-|A|}} \int_{(\mathbf{y}+Q)^{|A|}} \sum_{j \in A} |D_j \psi|^2 \prod_{k \in A} d\mathbf{x}_k \prod_{k \notin A} d\mathbf{x}_k d\mathbf{y}, \end{aligned}$$

where in the second step we used a similar partition of unity as in (14) on the square  $\mathbf{y}+Q$ . Applying the local energy bound (18) on  $\mathbf{y}+Q$  then yields

$$\begin{aligned} T_\psi &\geq \frac{1}{|Q|} \int_{Q_0+Q} \sum_{A \subseteq \{1, \dots, N\}} \frac{e_{|A|}}{|Q|} \tilde{p}_A(\mathbf{y}+Q) d\mathbf{y} \\ &= \frac{1}{|Q|^2} \sum_{n=0}^N e_n \int_{Q_0+Q} p_n(\mathbf{y}+Q) d\mathbf{y}, \end{aligned}$$

and hence

$$\frac{T_\psi}{|Q_0|} \geq \bar{\rho}^2 \frac{1}{\rho_{|Q|}^2} \sum_{n=0}^N e_n \frac{1}{|Q_0|} \int_{Q_0+Q} p_n(\mathbf{y}+Q) d\mathbf{y}, \quad (20)$$

where  $\bar{\rho} := N/|Q_0|$  is the mean density, and  $\rho_{|Q|} := N|Q|/|Q_0|$ . Now, assuming homogeneity of  $|\psi|^2$  on  $Q_0$  (i.e. translation invariance; this is a natural assumption for an ideal gas in the thermodynamic limit) up to some small error close to the boundary of  $Q_0$ , we have that the integrand  $p_n(\mathbf{y}+Q)$  is constant and that  $\rho(\mathbf{y}+Q) \approx \rho_{|Q|}$ . If furthermore  $|Q| \ll |Q_0|$  so that any boundary effects can be neglected, then approximately

$$\frac{1}{|Q_0|} \int_{Q_0+Q} p_n(\mathbf{y}+Q) d\mathbf{y} \approx p_n(Q') \quad \text{and} \quad \rho_{|Q|} \approx \rho(Q'),$$

where  $Q'$  is any square in the interior of  $Q_0$  s.t.  $|Q'| = |Q|$ . Hence (20) simplifies to

$$\frac{T_\psi}{|Q_0|} \gtrsim \bar{\rho}^2 \frac{1}{\rho_{|Q'|}^2} \sum_{n=0}^N e_n p_n(Q'). \quad (21)$$

An alternative approach to bounding the energy, also applicable in the case of Neumann boundary conditions, is to split  $Q_0$  into exactly  $M$  equally sized squares  $Q_m$ ,  $|Q_m| = |Q_0|/M$ . Applying the bound (cp. above, or e.g. [18, 19])

$$T_Q := \int_{Q_0^N} \sum_{j=1}^N |D_j \psi|^2 \chi_Q(\mathbf{x}_j) dx \geq \frac{1}{|Q|} \sum_{n=0}^N e_n p_n(Q)$$

on each such square yields, again assuming homogeneity on  $Q_0$ ,

$$\frac{T_\psi}{|Q_0|} = \frac{1}{|Q_0|} \sum_{m=1}^M T_{Q_m} \geq \frac{1}{|Q_0|} \frac{M}{|Q_1|} \sum_{n=0}^N e_n p_n(Q_1) = \bar{\rho}^2 \frac{1}{\rho_{|Q_1|}^2} \sum_{n=0}^N e_n p_n(Q_1), \quad (22)$$

where  $\rho_{|Q_1|} = \rho(Q_1) = N/M$ .

The above observations lead us to define for a general probability vector  $\mathbf{p} = (p_n) \in (\mathbb{R}_{\geq 0})^{N+1}$ :

$$\mathcal{E}_\alpha[\mathbf{p}] := \frac{1}{\rho[\mathbf{p}]^2} \sum_{n=0}^N e_n(\alpha) p_n, \quad \rho[\mathbf{p}] := \sum_{n=0}^N n p_n,$$

as well as the explicit lower bound energy functional for anyons with statistics parameter  $\alpha$  following from (19):

$$E_\alpha[\mathbf{p}] := \frac{1}{\rho[\mathbf{p}]^2} \sum_{n=0}^N C_{\alpha,n}^2 (n-1)_+ p_n. \quad (23)$$

Hence, it follows under assumptions of homogeneity and  $|Q|/|Q_0| \ll 1$  that the kinetic energy per unit area for an  $N$ -anyon state  $\psi$  is bounded as

$$\frac{T_\psi}{|Q_0|} \gtrsim \bar{\rho}^2 \mathcal{E}_\alpha[\mathbf{p}_\psi(Q)] \geq c \bar{\rho}^2 E_\alpha[\mathbf{p}_\psi(Q)], \quad (24)$$

where  $\mathbf{p}_\psi(Q)$  is the probability distribution (15) induced from  $\psi$  on  $Q \subseteq Q_0$ . Given a wave function  $\psi$ , we can hence try to find as good a lower bound (24) as possible by choosing the size of  $Q$  to maximize  $E_\alpha[\mathbf{p}_\psi(Q)]$ . Note that by using the monotonicity of  $n \mapsto C_{\alpha,n}$  for fixed  $\alpha$  we have a simple lower bound for  $E_\alpha$ :

$$E_\alpha[\mathbf{p}] \geq \frac{1}{\rho[\mathbf{p}]^2} \left( \sum_{n=0}^N C_{\alpha,N}^2 (n-1) p_n + C_{\alpha,N}^2 p_0 \right) = \frac{C_{\alpha,N}^2}{\rho[\mathbf{p}]^2} (\rho[\mathbf{p}] - 1 + p_0).$$

Hence, for a probability distribution  $\mathbf{p}$  s.t. the expected number of particles is  $\rho[\mathbf{p}] = 2$ ,

$$E_\alpha[\mathbf{p}] \geq \frac{C_{\alpha,N}^2}{4} (1 + p_0) \geq \frac{C_{\alpha,N}^2}{4}, \quad \text{for } \rho[\mathbf{p}] = 2. \quad (25)$$

It follows therefore by choosing  $|Q|$  s.t.  $\rho(Q) = 2$  that  $T_\psi/|Q_0| \gtrsim \frac{c}{4} C_{\alpha,N}^2 \bar{\rho}^2$ , which is actually nothing but the bound (4) (but with the slightly better constant associated to a disk geometry). Furthermore, note that there is also a simple upper bound for  $E_\alpha[\mathbf{p}]$  depending on  $\rho = \rho[\mathbf{p}]$ :

$$E_\alpha[\mathbf{p}] \leq \frac{1}{\rho^2} \sum_{n=0}^N C_{\alpha,2n}^2 p_n = \frac{C_{\alpha,2}^2}{\rho}, \quad (26)$$

and that then in particular  $E_\alpha[\mathbf{p}] \rightarrow 0$  as  $\rho \rightarrow \infty$ .

It is interesting to point out that for probability distributions  $\mathbf{p}$  with a relatively high weight on  $p_2$ , it is actually the *two*-particle exclusion constant  $C_{\alpha,2}$  (which is *monotone* in  $\alpha \in [0, 1]$ ) which governs the behavior of the energy bound in  $E_\alpha[\mathbf{p}]$ . This is something we will see explicitly in some of our examples below.

### 3.1 Bosonic ground state

Consider as a concrete example the ground state wave function  $\psi$  for  $N$  bosons on  $Q_0$  with Neumann boundary conditions, i.e. the constant function

$$\psi(x) := |Q_0|^{-N/2}.$$

In this case we can easily compute the probability distributions:

$$\tilde{p}_A(Q) = \frac{|Q|^{|A|} |Q^c|^{N-|A|}}{|Q_0|^N} = p^{|A|} (1-p)^{N-|A|},$$

with  $p := |Q|/|Q_0|$ , and

$$p_n(Q) = \binom{N}{n} p^n (1-p)^{N-n}, \quad \sum_{n=0}^N n p_n(Q) = pN = N \frac{|Q|}{|Q_0|},$$

i.e. a binomial distribution with  $p = \rho/N$ . In the limit  $N \rightarrow \infty$  we obtain a Poisson distribution

$$p_n(Q) = \frac{1}{n!} \rho^n e^{-\rho}, \quad \rho = N \frac{|Q|}{|Q_0|}.$$

Due to the quadratic dependence on  $\rho$  in  $p_2$ , we note that with such bosonic probability distribution  $\mathbf{p}_\psi(Q)$  in (24) we have

$$\begin{aligned} E_\alpha[\mathbf{p}_\psi(Q)] &= \frac{1}{\rho^2} \sum_{n=2}^N C_{\alpha,n}^2 (n-1) \binom{N}{n} \left(\frac{\rho}{N}\right)^n \left(1 - \frac{\rho}{N}\right)^{N-n} \\ &\rightarrow C_{\alpha,2}^2 \binom{N}{2} \frac{1}{N^2} = C_{\alpha,2}^2 \frac{N-1}{2N}, \quad \text{as } \rho \rightarrow 0, \end{aligned} \quad (27)$$

and hence

$$\frac{T_\psi}{|Q_0|} \gtrsim c C_{\alpha,2}^2 \frac{N-1}{2N} \bar{\rho}^2.$$

We conclude that it is actually the *two*-particle exclusion constant  $C_{\alpha,2}$  which dictates a lower bound for such bosonic probability distributions, as well as for any wave functions with a similar *quadratic* small- $\rho$  dependence in  $p_2$ . This could e.g. include states which have been obtained by a regularization on very short scales in order to have finite energy, i.e. to actually belong to the domain of the quadratic form  $T_\psi$  (which the bosonic ground state *does not* for  $\alpha \notin 2\mathbb{Z}$ ; see [19]).

### 3.2 Fermionic ground state

The situation is immediately much more complicated for the fermionic  $N$ -particle ground state wave function on  $Q_0$ ,

$$\psi(x) := \frac{1}{\sqrt{N!}} \bigwedge_{k=0}^{N-1} \varphi_k(\mathbf{x}_1, \dots, \mathbf{x}_N),$$

where the Slater determinant ranges over the  $N$  lowest eigenfunctions  $\varphi_k$  for the Neumann Laplacian  $-\Delta_{Q_0}$ , ordered by non-decreasing eigenvalue  $\lambda_k$ . In this case we can only expect homogeneity in the limit  $N \rightarrow \infty$ . If we consider the behavior of the probability distribution  $\mathbf{p}_\psi(Q)$  for small  $|Q|$  then we find that, thanks to the pairwise antisymmetry of the wave function,

$$p_2(Q) \sim \int_{Q^2} |\psi(\mathbf{x}_1, \mathbf{x}_2, x')|^2 d\mathbf{x}_1 d\mathbf{x}_2 \sim \int_{Q^2} |(\mathbf{x}_1 - \mathbf{x}_2) \cdot \Psi_1(x')|^2 d\mathbf{x}_1 d\mathbf{x}_2 \sim |Q|^3,$$

and in general

$$p_n(Q) \sim |Q|^{\lfloor 3n/2 \rfloor},$$

hence  $\rho(Q) \sim |Q|$  to leading order, and  $E_\alpha[\mathbf{p}_\psi(Q)] \sim \rho \rightarrow 0$  as  $|Q| \rightarrow 0$ . The optimal bound following from  $E_\alpha$  must therefore in this case be obtained for  $|Q|$  s.t.  $\rho \sim 1$ .

### 3.3 Anyonic trial states

Interestingly, we observe that for even numerator (reduced) rational  $\alpha = \mu/\nu$  we have

$$C_{\alpha,n} \equiv 0 \quad \text{for } n \geq \nu, \quad (28)$$

and hence the energy bound  $E_\alpha[\mathbf{p}]$  truncates to

$$E_\alpha[\mathbf{p}] = \frac{1}{\rho[\mathbf{p}]^2} \sum_{n=2}^{\nu-1} C_{\alpha,n}^2 (n-1) p_n. \quad (29)$$

In order to verify this claim (28), note that since  $\nu \geq 3$  ( $\nu = 1$  is trivial) is necessarily odd it must also be contained in the set  $\{1, 3, 5, \dots, 2\nu - 3\}$  and hence there is a  $p \in \{0, 1, \dots, \nu - 2\}$  s.t.  $\nu = 2p + 1$ . Taking  $q = \mu/2$  we then have

$$C_{\alpha,n} \leq C_{\alpha,\nu} \leq \left| (2p+1) \frac{\mu}{\nu} - 2q \right| = 0 \quad \text{for } n \geq \nu.$$

Of course these observations only apply in the context of the lower bound (19), and the situation for the functional  $\mathcal{E}_\alpha[\mathbf{p}]$  could certainly be very different. Indeed, we can actually show that *all*  $e_n$  are bounded from below in terms of  $C_{\alpha,2}$ , although this bound rapidly beomes weaker with  $n$ :

**Proposition 2.** *We have  $e_2(\alpha) \geq c C_{\alpha,2}^2$  and  $e_n(\alpha) \geq \frac{n(n-1)}{8} \left(\frac{3}{4}\right)^{n-2} c C_{\alpha,2}^2$  for  $n \geq 3$ .*

*Proof.* We consider in the l.h.s. of (18) the square  $Q$  split into four smaller squares  $Q_q$ ,  $|Q_q| = |Q|/4$ , and insert a corresponding partition of unity

$$1 = \prod_{k=1}^n \sum_{q=1}^4 \chi_{Q_q}(\mathbf{x}_k) = \sum_{\mathbf{q} \in \{1,2,3,4\}^n} \chi_{Q_{q_1} \times Q_{q_2} \times \dots \times Q_{q_n}}(\mathbf{x}_1, \dots, \mathbf{x}_n). \quad (30)$$

We keep only the terms  $\chi_{Q_{q_1} \times Q_{q_2} \times \dots \times Q_{q_n}}$  where two of the  $q_i$  are equal and all the others are different, and for each such term we can apply the corresponding 2-particle exclusion bound involving  $e_2 \geq \tilde{e}_2 := c C_{\alpha,2}^2$ , i.e.

$$\hat{T}_Q^n \geq \sum_{q_i=1}^4 \sum_{\mathbf{q}} \frac{\tilde{e}_2}{|Q_{q_i}|} \chi_{Q_{q_1} \times Q_{q_2} \times \dots \times Q_{q_n}} =: f,$$

where there are in total  $4 \cdot \binom{n}{2} \cdot 3^{n-2}$  terms in the sum  $f$  (note that some terms in (30) contribute several terms in  $f$  thanks to the summation over all  $n$  particles in the energy). Now we apply the uncertainty principle as in Lemma 7 in [18], splitting

$$\hat{T}_Q^n = \kappa \hat{T}_Q^n + (1 - \kappa) \hat{T}_Q^n \geq \frac{\kappa}{2} (-\Delta_{Q^n}) + (1 - \kappa) f$$

into two terms with  $\kappa \in (0, 1)$  and applying the diamagnetic inequality on the first. We separate these operators in terms of the projection  $P_0 = u_0 \langle u_0, \cdot \rangle$  onto the ground state  $u_0 := |Q|^{-n/2}$  of the Neumann Laplacian,

$$-\Delta_{Q^n} \geq \frac{\pi^2}{|Q|} P_0^\perp,$$

$$f = (P_0 + P_0^\perp) f (P_0 + P_0^\perp) \geq (1 - \varepsilon) P_0 f P_0 + (1 - \varepsilon^{-1}) P_0^\perp f P_0^\perp,$$

with  $\varepsilon \in (0, 1)$ , hence

$$|Q| \hat{T}_Q^n \geq \left( \frac{\kappa \pi^2}{2} - (1 - \kappa)(\varepsilon^{-1} - 1) 4\tilde{e}_2 \right) P_0^\perp + (1 - \kappa)(1 - \varepsilon) \frac{4 \binom{n}{2} 3^{n-2}}{4^n} 4\tilde{e}_2 P_0,$$

using  $\|P_0^\perp f P_0^\perp\| \leq \|f\|_\infty \leq \tilde{e}_2/|Q_{q_i}|$  and

$$\|P_0 f P_0\| = \int_{Q_0} \bar{u}_0 f u_0 dx = |Q|^{-n} \cdot 4 \binom{n}{2} 3^{n-2} \cdot \frac{\tilde{e}_2}{|Q_1|} |Q_1|^n.$$

Taking  $\kappa = \varepsilon = 0.5$  and using that  $2\tilde{e}_2 \ll \pi^2/4$  we obtain the claimed inequality.  $\square$

We have thus obtained non-zero bounds for all  $e_{n \geq 2}(\alpha)$  for  $\alpha \notin 2\mathbb{Z}$ , although they weaken rapidly with  $n$  and cannot compete with (19) for odd numerator rationals. However, there is also a different reason why the true Neumann energy  $e_n$  might be lower for even numerator rationals and  $n = k\nu$  being a multiple of  $\nu$ . Namely, as discussed in [6], the collective motion of the  $n - 1$  relative variables  $\mathbf{y}_j$  of an  $n$ -anyon system can be partially separated out in terms of a radial variable  $R = \sqrt{\sum_j |\mathbf{y}_j|^2}$  and an overall angular variable  $\theta$ :

$$\mathbf{y}_j := \frac{1}{j(j+1)} \sum_{k=1}^j \mathbf{x}_k - \sqrt{\frac{j}{j+1}} \mathbf{x}_{j+1} = R \boldsymbol{\xi}_j e^{\theta I}, \quad (\boldsymbol{\xi}_j) \in S^{2(N-1)-1}, \quad \boldsymbol{\xi}_1 = \mathbf{e}_1.$$

In terms of these, the kinetic energy operator can be written

$$\hat{T}_Q^n = \frac{1}{2} \hat{P}_R^2 + \frac{1}{2R^2} \left( \hat{P}_\theta - \alpha \frac{n(n-1)}{2} \right)^2 + \frac{1}{2R^2} \hat{H}(\theta, \boldsymbol{\xi}).$$

The first two terms correspond to a two-anyon system with statistics parameter  $\alpha' = \alpha \binom{n}{2}$ , while solutions to  $\hat{H}\psi = 0$  classically would correspond to a pure collective motion of particles and frozen internal motion,  $\dot{\boldsymbol{\xi}}_j = 0$  [6]. We note that for even numerator  $\alpha = \mu/\nu$  and  $n = k\nu$ , the effective statistics parameter  $\alpha' = \alpha \binom{n}{2} = \mu \frac{k(k\nu-1)}{2}$  is always an even integer, while for e.g.  $\mu$  odd,  $\nu$  even, and  $n = k\nu$  an odd multiple of  $\nu$ ,  $\alpha' = \alpha \binom{n}{2}$  is always a half-integer.

Summing up the discussion so far, the main questions concerning the anyon gas are now being approached along the following lines:

**Question 1.** *What is the true dependence on  $\alpha$  for the Neumann energies  $e_{n \geq 3}$ ? Is  $e_n$  significantly higher/lower for  $\alpha = \mu/\nu$  with  $\mu$  odd/even at  $n = \nu$  or  $n = k\nu$  for some  $k > 1$ ?*

*Partial answers.* The above analytical results are certainly not sufficient to settle this question. We have also addressed this question by means of numerical studies using a cut (non-orthonormal) basis approach (cp. [21, 25, 26]) — see Figure 1 — as well as by means of a lattice approximation (cp. [11]) — see Figure 2. Both of these suggest (and also comparing with the harmonic oscillator case [21, 25]) that  $e_3$  rather has a peak around  $\alpha = 2/3$ , where there is a crossing of two energy levels corresponding to the bosonic resp. fermionic ground states. Unfortunately we have not been able to make any conclusions concerning larger  $n$  due to computational limitations, although the lattice study showed no clear such peak for 6 particles (see Figure 2).  $\square$

**Question 2.** *What conclusions can be made by understanding the limit of  $e_n$  as  $n \rightarrow \infty$ ?*

*Partial answers.* If a superlinear dependence does emerge at some  $n$ , e.g.  $e_n \geq c'(n - k)_+$  for some  $k$ , then we must have  $\mathcal{E}[\mathbf{p}] \geq \frac{c'}{4k}$  at  $\rho[\mathbf{p}] = 2k$ . Indeed,

$$\begin{aligned} \mathcal{E}[\mathbf{p}] &\geq \frac{1}{\rho[\mathbf{p}]^2} c' \sum_{n=k+1}^N (n - k)p_n = \frac{c'}{\rho[\mathbf{p}]^2} \left( \sum_{n=0}^N (n - k)p_n + \sum_{n=0}^{k-1} (k - n)p_n \right) \\ &\geq \frac{c'}{\rho[\mathbf{p}]^2} (\rho[\mathbf{p}] - k + 0) = \frac{c'}{4k}. \end{aligned}$$

It then follows by (22) and choosing  $M \sim N/(2k)$  that actually  $e_n \gtrsim \frac{c'}{4k} n^2$  as  $n \rightarrow \infty$  (assuming homogeneity for the Neumann problem).

On the other hand, if there exists a subsequence s.t.  $e_k/k \rightarrow 0$  as  $k \rightarrow \infty$ , then for every  $\varepsilon > 0$  there is a family of probability distributions  $[0, N] \ni \rho \mapsto \mathbf{p}(\rho)$  s.t.  $\mathcal{E}[\mathbf{p}(\rho)] < \varepsilon$  for all  $\rho$  (if  $N$  large enough). Indeed, we can e.g. choose  $\mathbf{p}$  s.t.  $p_0 = 1 - \rho/k$ ,  $p_k = \rho/k$ , and zero otherwise, implying  $\rho[\mathbf{p}] = kp_k = \rho$  and  $\mathcal{E}[\mathbf{p}] = \frac{1}{\rho} e_k/k$ . We can then take  $k$  large enough so that  $\mathcal{E}[\mathbf{p}(\rho)] < \varepsilon$  (note that for  $0 \leq \rho \leq 1$  we could also take e.g.  $p_0 = 1 - \rho$ ,  $p_1 = \rho$  and zero otherwise, implying  $\mathcal{E}[\mathbf{p}(\rho)] = 0$ ).

We can thereby conclude that (19) can be improved for odd numerator rational  $\alpha$  and large  $n$ :

$$e_n(\alpha) \gtrsim \frac{c}{4} C_{\alpha,n}^2 n^2, \quad \text{as } n \rightarrow \infty,$$

while for even numerator  $\alpha$  we ask if  $e_n(\alpha)/n \rightarrow 0$  as  $n \rightarrow \infty$  or if this holds for some subsequence of  $n$  and if wavefunctions can be constructed with probability distributions supported on such a specific sequence of  $p_n$ .  $\square$



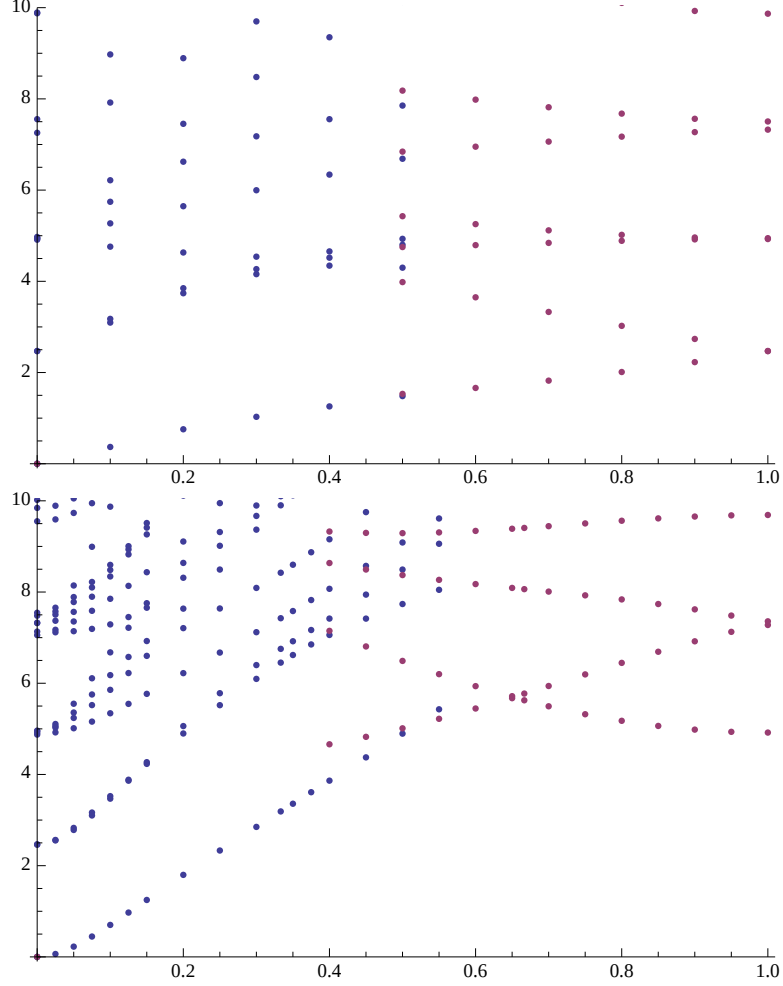


Figure 1: Approximative numerical diagonalization for the Neumann problem with  $N = 2$  (upper plot) resp.  $N = 3$  (lower plot) anyons on a square  $Q = [-1, 1]^2$ . The vertical axis shows the energy and the horizontal axis the statistics parameter, ranging from  $\alpha = 0$  to  $\alpha = 1$ . A cut approximating basis has been used, consisting of symmetric states times a Jastrow factor (blue, starting from the bosonic end) resp. antisymmetric states (red, starting from the fermionic end) involving the  $K$  lowest-energy one-particle eigenstates  $\varphi_k$ . For  $N = 2$  we have used  $K = 5$  (15 resp. 10  $N$ -particle states), while for  $N = 3$  we have taken  $K = 4$  (20 states) for the symmetric (blue) states, and  $K = 6$  (15 states) for the antisymmetric (red) states. The lowest range of points in these plots constitute approximative upper bounds for the ground state energy  $e_N(\alpha)$  (the integrals were evaluated numerically using Monte Carlo methods).

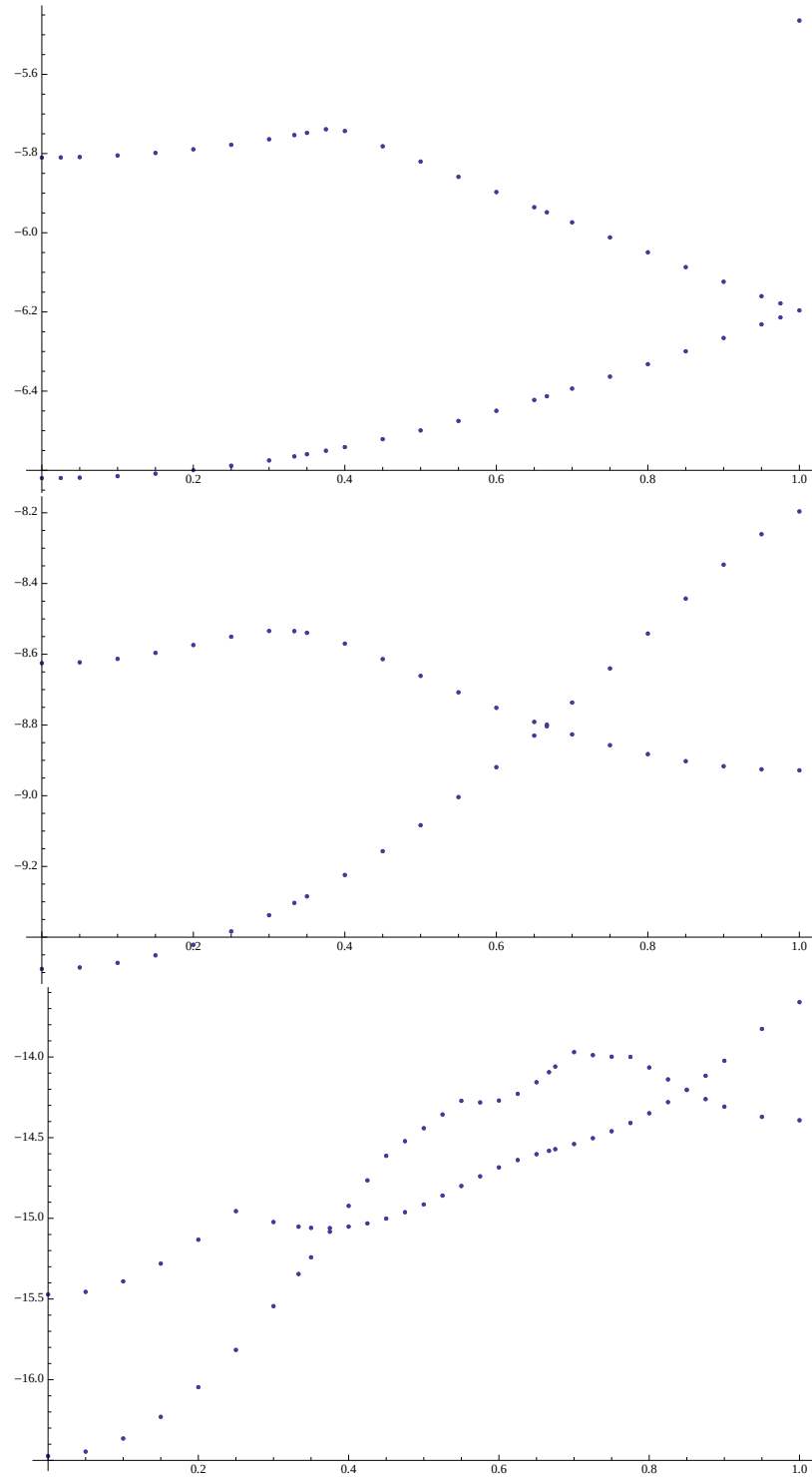


Figure 2: Spectra for  $N = 2$ ,  $N = 3$  resp.  $N = 6$  anyons on a  $5 \times 5$  lattice, using the prescription in [11] for the board geometry. Only the two lowest eigenvalues have been plotted, with  $\alpha$  on the horizontal axis.

**Question 3.** *Is it possible to realize such a specialized probability distribution  $\mathbf{p}$  as arising from a wave function  $\psi$ , i.e.  $\mathbf{p} = \mathbf{p}_\psi$ ?*

Let us investigate the approximate small- $|Q|$  behavior for the regularized anyonic trial states  $\psi = \Phi_r \psi_\alpha$ . Because of the regularization parameter, there are two possible scales involved;  $|Q| \ll r^2$  resp.  $r^2 \ll |Q| \ll |Q_0|$ . On the very shortest scales the attraction from the Jastrow factor in  $\psi_\alpha$  is suppressed and we are left with a purely repulsive expression: for  $\mathbf{x}_1, \dots, \mathbf{x}_n \in Q$  and  $x' \in (Q^c)^{N-n}$  with  $|Q| \ll r^2$ ,

$$|\psi(\mathbf{x}_1, \dots, \mathbf{x}_n, x')| \sim \prod_{1 \leq j < k \leq n} |z_{jk}|^\alpha \prod_{(j,k) \in \cup_q E_q} |z_{jk}|^\mu |\Lambda| \sim |Q|^{\frac{\alpha}{2} \binom{n}{2} + \frac{\mu}{2} m + \frac{\lambda}{2}}.$$

Here we are considering only an approximate leading order term in the symmetrized expression for  $\psi$ , where  $\Lambda$  denotes the remaining regular factor involving  $\varphi_k$  with short-scale dependence  $|\Lambda(x)| \sim |Q|^{\frac{\lambda}{2}}$  for some  $\lambda = \lambda(n) \geq 0$ , and  $m = m(n)$  denotes the minimal number of edges in  $\cup_q E_q$  connecting  $n$  particles. Hence, after integrating over  $Q^n$ ,

$$p_n(Q) \sim |Q|^{n + \alpha \binom{n}{2} + \mu m + \lambda}, \quad \text{for } |Q| \ll \min\{r^2, |Q_0|\},$$

where we have  $m = 0$  for  $n \leq \nu$ ,  $m = k$  for  $n = \nu + k$ ,  $1 \leq k \leq \nu$ , and so on. In particular,  $\rho(Q) \sim |Q|$  and  $p_2(Q) \sim \rho^{2+\alpha}$  for all considered trial states on the smallest scales.

On the other hand, on relatively large scales  $r^2 \ll |Q| \ll |Q_0|$  the regularizing factor becomes suppressed and leaves a pure attraction,

$$|\psi(\mathbf{x}_1, \dots, \mathbf{x}_n, x')| \sim \prod_{1 \leq j < k \leq n} |z_{jk}|^{-\alpha} \prod_{(j,k) \in (E_q)} |z_{jk}|^\mu |\Lambda| \sim |Q|^{-\frac{\alpha}{2} \binom{n}{2} + \frac{\mu}{2} m + \frac{\lambda}{2}},$$

so that

$$p_n(Q) \sim |Q|^{n - \alpha \binom{n}{2} + \mu m + \lambda}, \quad \text{for } r^2 \ll |Q| \ll |Q_0|.$$

In the particularly interesting case of  $n = k\nu$  being a multiple of  $\nu$ , we have  $m = \nu \binom{k}{2}$  and find for the even numerator case ( $\lambda = 0$ )

$$n - \alpha \binom{n}{2} + \mu m + \lambda = \frac{k}{2}(\mu - (\mu - 2)\nu). \quad (31)$$

Hence, for  $\mu = 2$  we find  $p_{n=k\nu}(Q) \sim |Q|^k$ , reminiscent of the bosonic probability distribution but now in multiples of  $k$ , while for even  $\mu \in [4, \nu - 1]$  the r.h.s. of (31) is smaller than  $\frac{k}{2}(\nu - 2\nu) = -k\nu/2 < 0$  and we arrive at the somewhat startling conclusion  $p_{n=k\nu}(Q) \gtrsim |Q|^{-k\nu/2}$ . In the case of odd numerators there is an additional power  $\lfloor k/2 \rfloor$  from each Slater determinant in  $\Lambda$ , producing

$$n - \alpha \binom{n}{2} + \mu m + \lambda = \frac{k}{2}(\mu - (\mu - 2)\nu) + \nu \lfloor k/2 \rfloor = \frac{k}{2}(\mu - (\mu - 3)\nu) \quad (-\nu/2), \quad (32)$$

where the additional parenthesis concerns the case when  $k$  is odd. For  $\mu = 1$  we then have  $p_{n=k\nu}(Q) \sim |Q|^{\frac{k}{2}(2\nu+1) (-\nu/2)}$ , and for  $\mu = 3$ ,  $p_{n=k\nu}(Q) \sim |Q|^{\frac{3}{2}k (-\nu/2)}$ , but for odd  $\mu \in [5, \nu - 1]$  we again find  $p_{n=k\nu}(Q) \gtrsim |Q|^{-k\nu/2}$ . Consider as a concrete example the  $\alpha = 2/3$  state for which

$$p_0 \sim 1, p_1 \sim |Q|, p_2 \sim |Q|^{\frac{4}{3}}, p_3 \sim |Q|, p_4 \sim |Q|^2, p_5 \sim |Q|^{\frac{7}{3}}, p_6 \sim |Q|^2,$$

while for  $\alpha = 1/3$

$$p_0 \sim 1, p_1 \sim |Q|, p_2 \sim |Q|^{\frac{5}{3}}, p_3 \sim |Q|^2, p_4 \sim |Q|^4, p_5 \sim |Q|^{\frac{17}{3}}, p_6 \sim |Q|^7.$$

In the even numerator case we note that  $p_2$  is of a lower order than  $p_3 \sim p_1 \sim \rho$ , possibly with the result of bringing a higher weight to the 3-particle term in  $\mathcal{E}_\alpha$  or  $E_\alpha$ . We do not know how to interpret the less well-behaved cases where  $p_n$  involve negative powers of  $|Q|$  (and, interestingly, peaking for  $n$  at multiples of  $\nu$ ), other than to conclude the breakdown of this approximation scheme concerning  $r^2 \ll |Q| \ll |Q_0|$ . The interesting question is what actually happens for any of these anyonic trial states around  $|Q| \sim r^2$ , and for this we will resort to numerical studies.

## 4 Numerical studies

We wish to investigate the forms of the probability distributions  $\mathbf{p}_\psi(Q)$  induced from  $N$ -anyon wave functions  $\psi$  such as the above trial states, and the corresponding strength of local exclusion for anyons given by the explicit bound  $E_\alpha[\mathbf{p}]$  for such  $\mathbf{p} = \mathbf{p}_\psi(Q)$  (however, as we have pointed out, the stronger bound concerning  $\mathcal{E}_\alpha[\mathbf{p}]$  might actually be very different). Our preliminary results are given in Appendix A-C. We have evaluated the probability distributions

$$p_n(Q) = \binom{N}{n} \tilde{p}_{A=\{1,\dots,n\}}(Q)$$

numerically to a reasonable precision (based on the complexity of the state and the corresponding speed of convergence) using Monte Carlo based integration methods. We used Mathematica for the construction of the explicit integrands, and for multi-dimensional numerical integration we employed the Vegas algorithm which is part of the Cuba library for Mathematica [10].

For computational simplicity we choose the squares  $Q_0 = [-1, 1]^2$  with Neumann boundary conditions and  $Q = [-\ell, \ell]^2$ ,  $\ell < 1$ , centered at the origin. Some of the states we consider typically cannot be expected to be homogeneous, and it is therefore also important to estimate their degree of inhomogeneity. For this purpose we have plotted the corresponding expected number of particles  $\rho(Q) = \rho[\mathbf{p}_\psi(Q)]$  as a function of the expected number of particles under the assumption of homogeneity,  $\rho_{|Q|} := N|Q|/|Q_0|$ .

Depending on the complexity and homogeneity, we have chosen a range of sizes  $|Q|$  of squares (or, equivalently, densities  $\rho_{|Q|}$ ) to evaluate the distributions for. The corresponding distributions  $\mathbf{p}_\psi(Q)$  have been plotted (both in a linear and a logarithmic scale; see below), as well as the resulting values of  $E_\alpha[\mathbf{p}_\psi(Q)]$  for the full range  $\alpha \in [0, 1]$ . For convenience, we have indicated by light-gray curves the canonical lower bound (25) for the energy as well as the limiting lower bound (27) for bosons.

We have chosen to study the simplest even numerator trial state  $\alpha = 2/3$  in order to more easily compare the effects of clustering of particles using a small total particle number  $N$ . It is known that the Read-Rezayi type states lead to an effective clustering of particles even after symmetrization, however, here we would also like to study the interplay of this effect with the repulsion due to the regularizing factor  $\Phi_r$ . We therefore also consider the behavior of the pure Jastrow factors  $\Phi_r^*(z) := \Phi_r(z) \prod_{j < k} |z_{jk}|^{-\alpha}$  in our trial state wave functions, i.e. the completely symmetric states

$$\Phi_r^*(z) = \prod_{j < k} \frac{|z_{jk}|^\alpha}{(r^2 + |z_{jk}|^2)^\alpha}, \quad \Phi_{r \rightarrow \infty}^*(z) = \prod_{j < k} |z_{jk}|^\alpha.$$

#### 4.1 $N = 6$ particles

The case  $N = 6$ , considered in Appendix A, is the first non-trivial case where an even numerator trial state,  $\alpha = 2/3$ , can be investigated. We have plotted probability distributions and energy bounds  $E_\alpha$  for this state  $\psi = \Phi_r \psi_{\alpha=2/3}$  for  $r = 0.03, 0.3, 1$  and  $r = 3$ , as well as for  $\Phi_r^*$  for  $r = 1$  and  $r = \infty$ . For comparison, we have also plotted corresponding values for the bosonic and the fermionic ground states. We note that the trial states appear to be more homogeneous for smaller  $r$ .

#### 4.2 $N = 9$ particles

In the case  $N = 9$ , considered in Appendix B, the precision is worse because of the increased complexity and consequently evaluation time. We have again plotted distributions and bounds for  $\psi = \Phi_r \psi_{\alpha=2/3}$  with  $r = 0.1, 0.3, 1$ , and for  $\Phi_r^*$  with  $r = 1$  and  $r = \infty$ . It is clear that for the trial states there is a push of the weight of the distributions towards the 3-particle probability  $p_3$  (for small  $\rho$ , where the contribution of  $p_2$  and  $p_3$  are of similar order in  $E_\alpha$ , and logarithmic scale), e.g. comparing the  $r = 1$  states  $\Phi_{r=1} \psi_{\alpha=2/3}$  resp.  $\Phi_{r=1}^*$ , i.e. with and without the ‘Read-Rezayi factor’ responsible for the grouping of particles.

#### 4.3 $N = 12$ particles

The  $N = 12$  case is considered in Appendix C. We have succeeded in evaluating some approximate distributions for the  $\alpha = 2/3$  trial state for  $r = 0.3$

and  $r = 1$ . There is also here a strong indication that there is a push of the weight towards  $p_3$  for the trial state.

## 5 Conclusions and outlook

We have introduced a refined approach to bounding the ground state energy for the anyon gas from below, based on the  $n$ -particle probabilities  $p_n$  of the wave function and the energies  $e_n$  for the  $n$ -anyon Neumann problems. Whether this would improve the currently available bounds or not depends on details of the actual large- $n$  dependence of  $e_n$  which we have not yet been able to determine, despite some improved understanding. However, if  $e_n$  turn out to be particularly low for certain values or sequences of  $n$  then these bounds may not be able to improve the situation (or possibly rather be replaced by upper bounds) if wave functions can be constructed with suitable properties for their probability distributions. Motivated by this question, we also investigated some families of anyonic trial states which appear to have some clustering properties, possibly resulting in their probability distributions  $p_n$  being peaked at certain  $n$ . Some preliminary numerical investigations were carried out for the simplest trial state at  $\alpha = 2/3$  with particle numbers up to  $N = 12$ , and these indeed showed indications of a readjustment of the weight of the distributions towards  $p_3$  for small densities. Evidently, much more work is needed in order to make any firm conclusions, however one possible approach to gaining new valuable insight into this difficult problem has been outlined.

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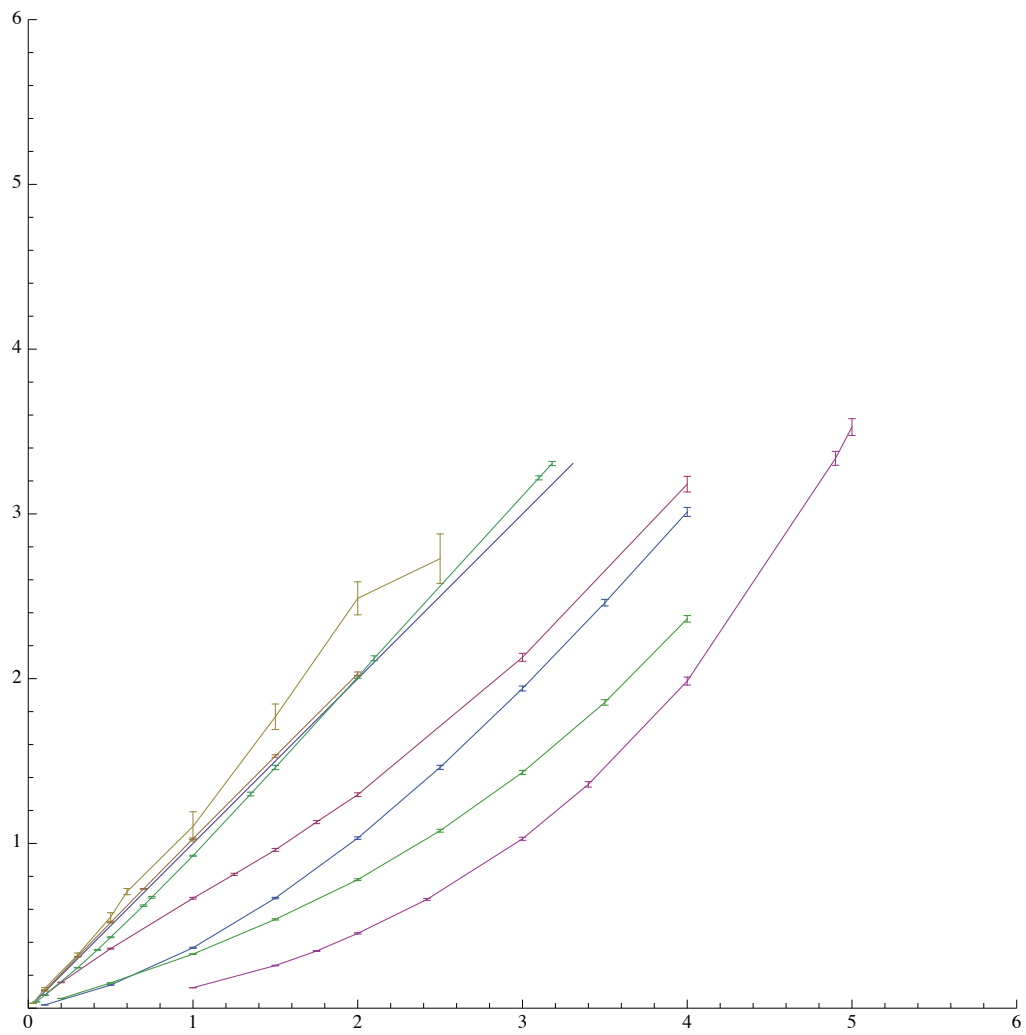
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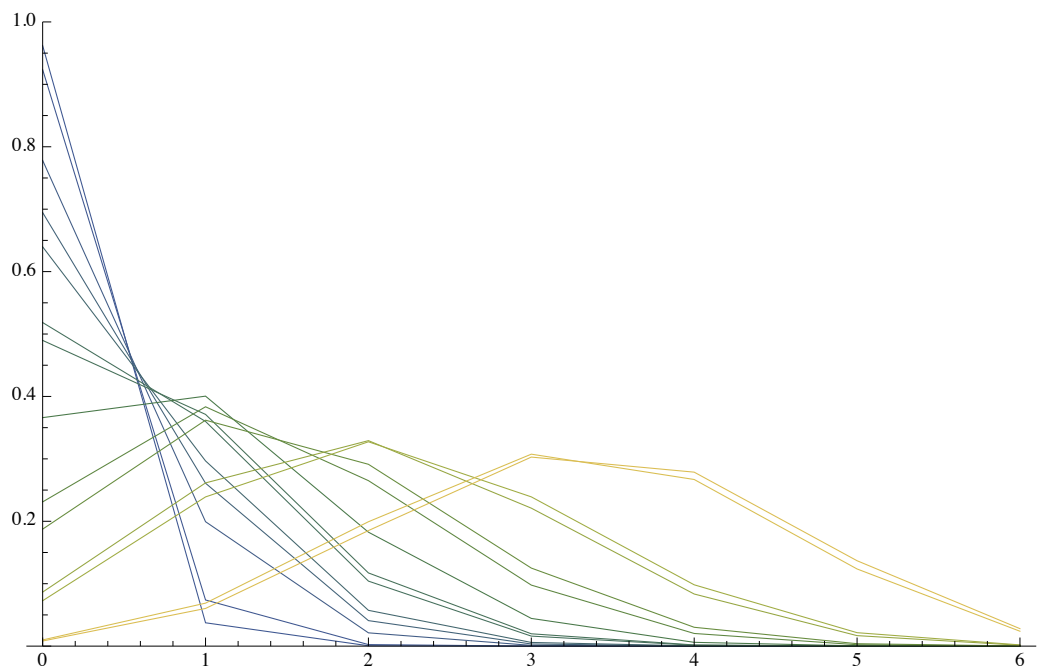
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(\* homogeneity: \*)

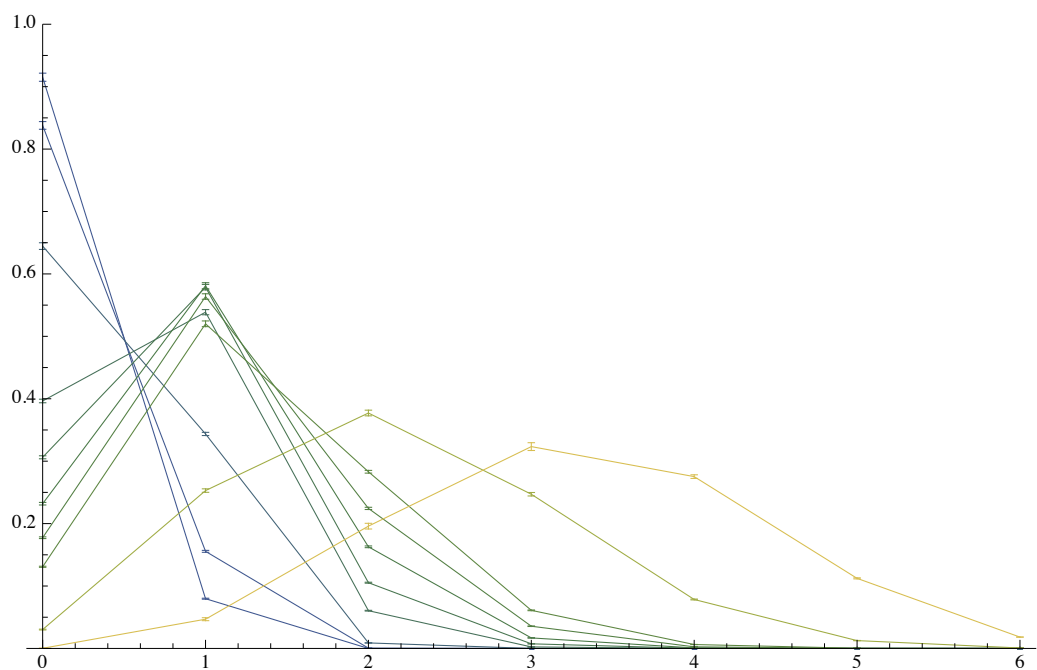


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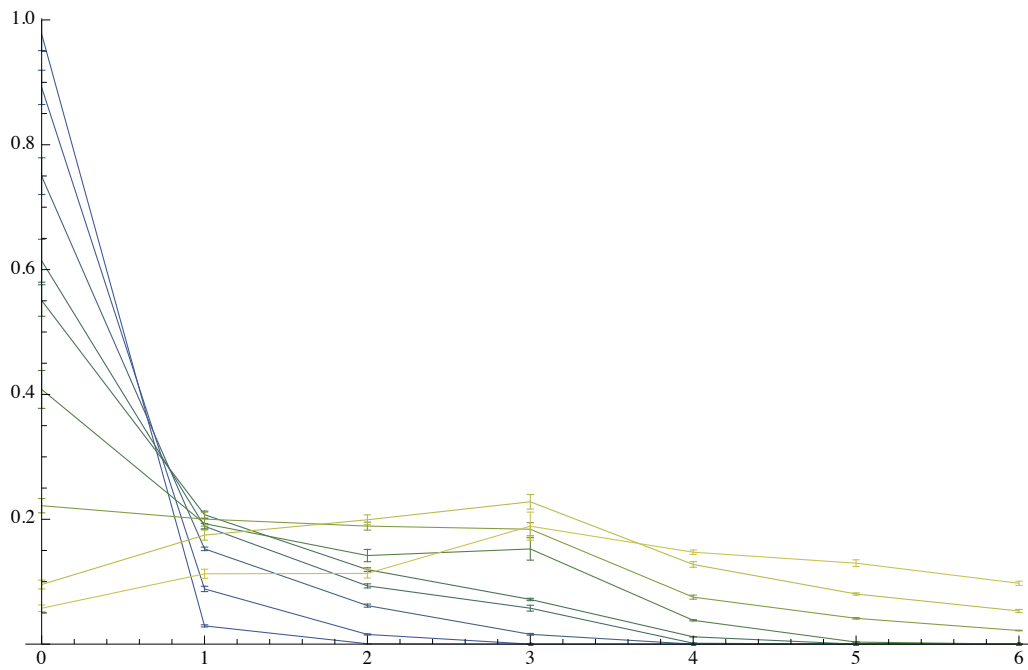
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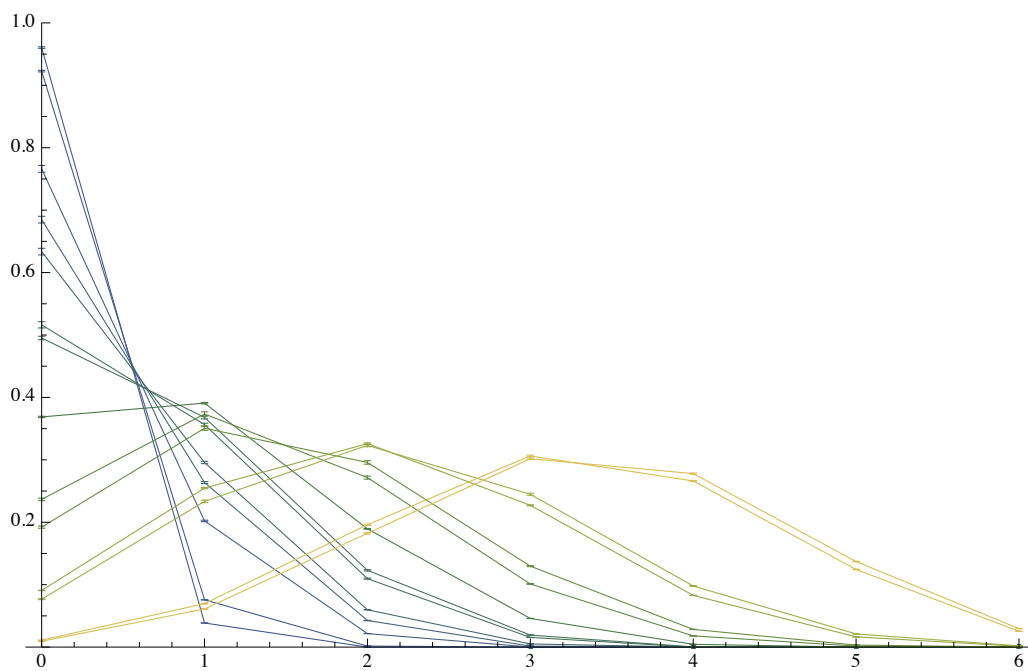
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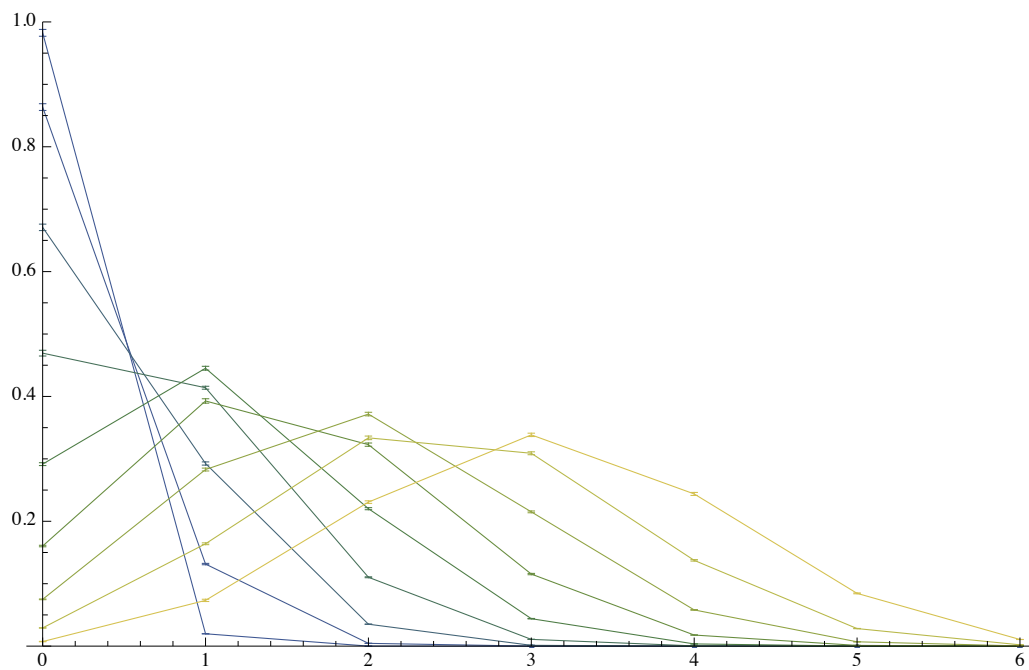
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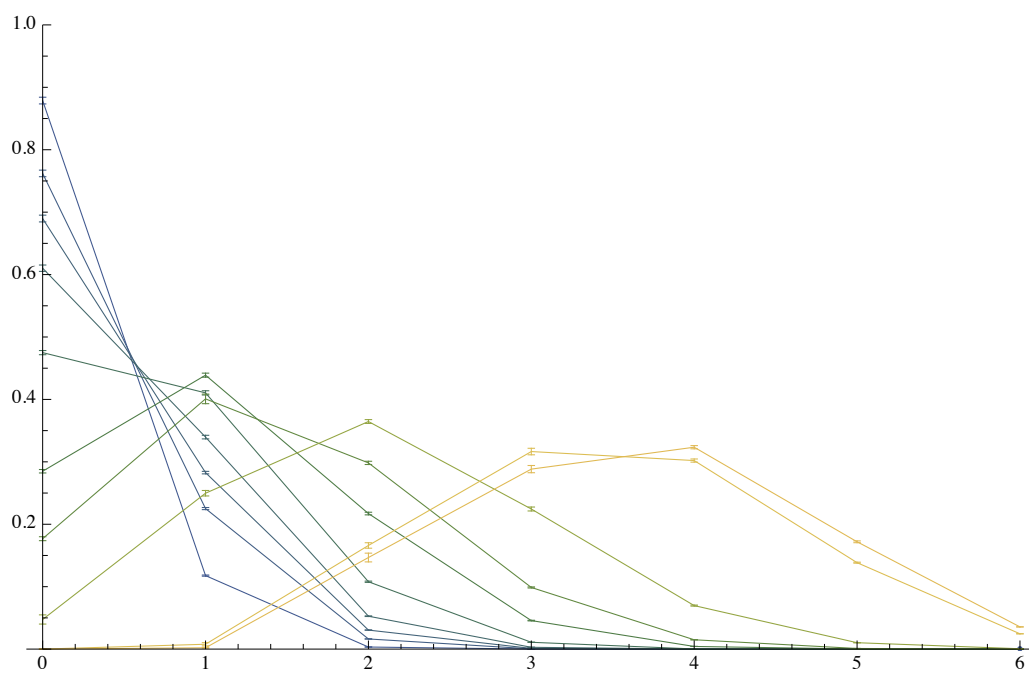
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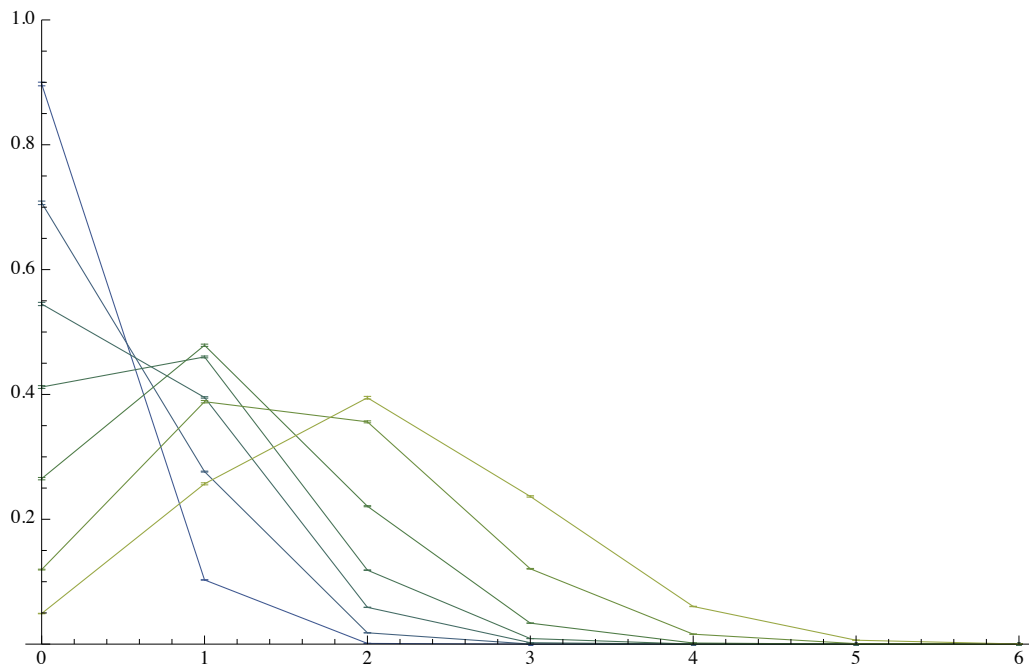
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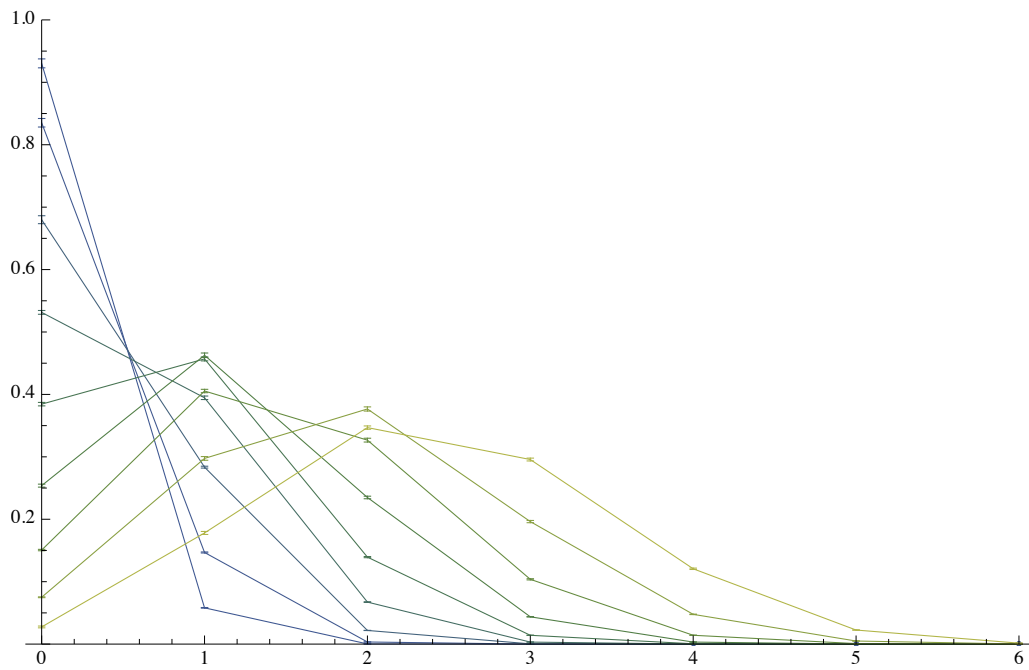
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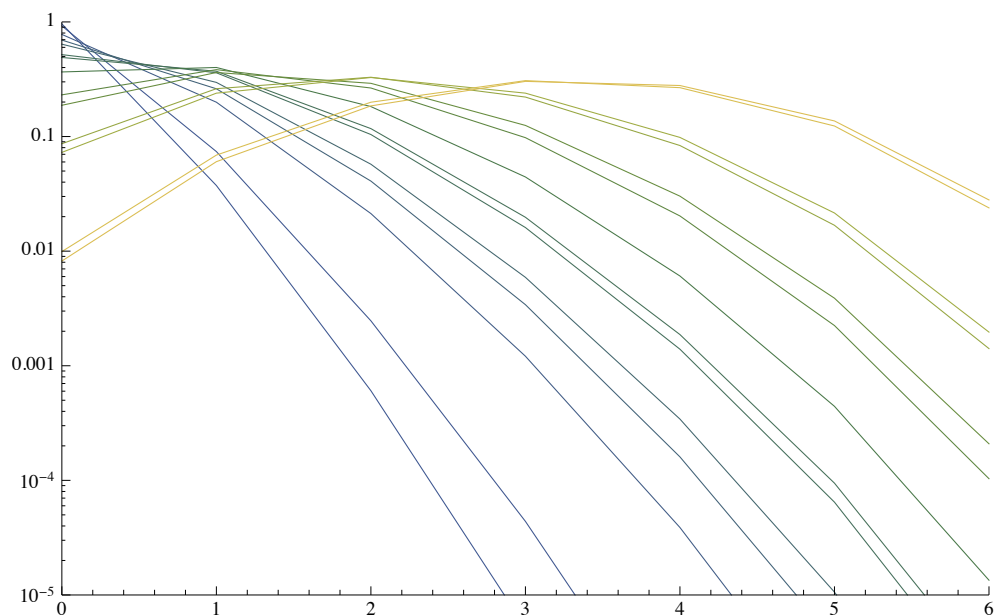


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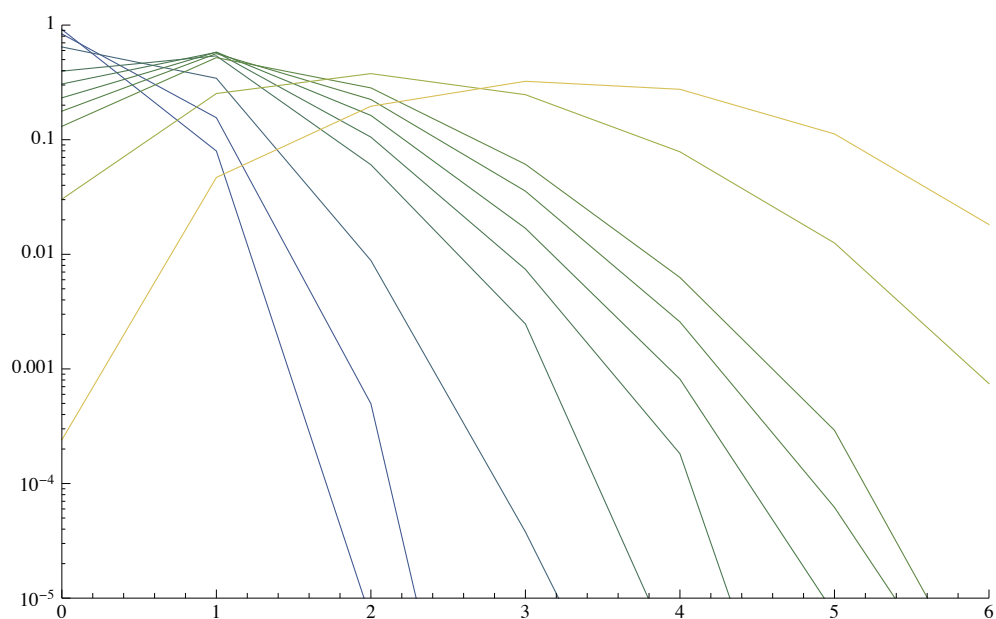


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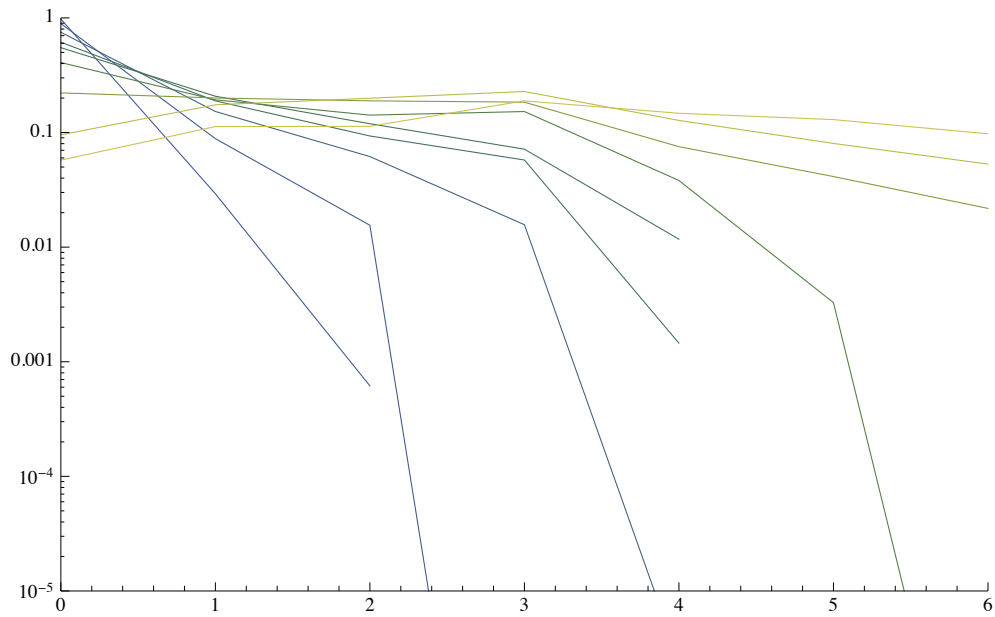
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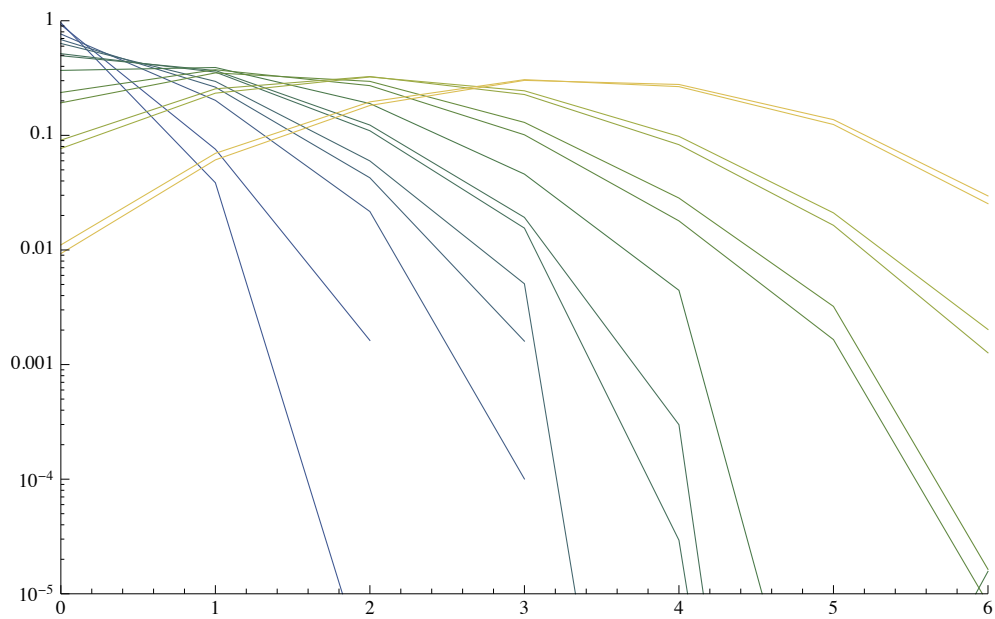
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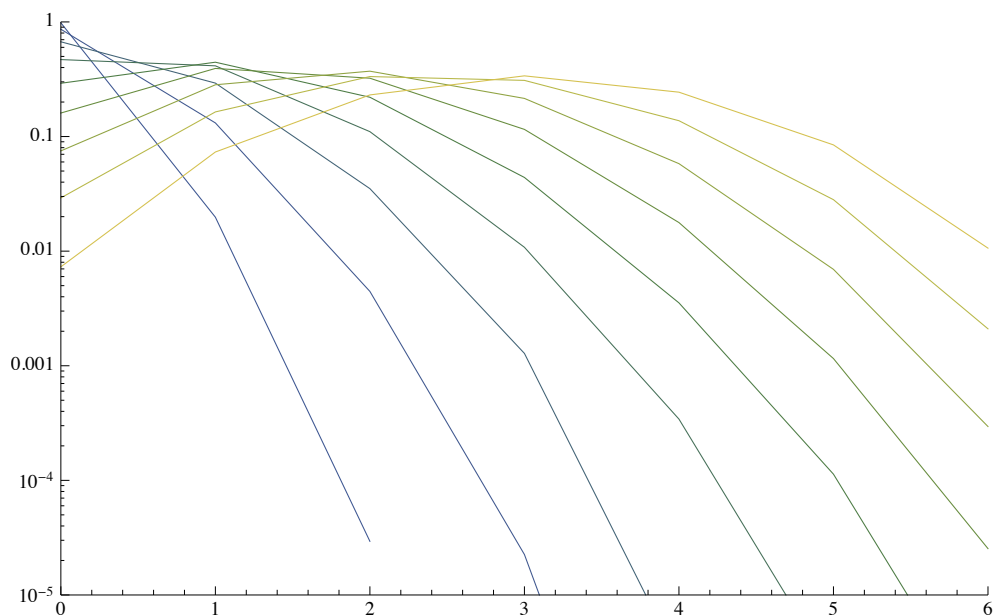
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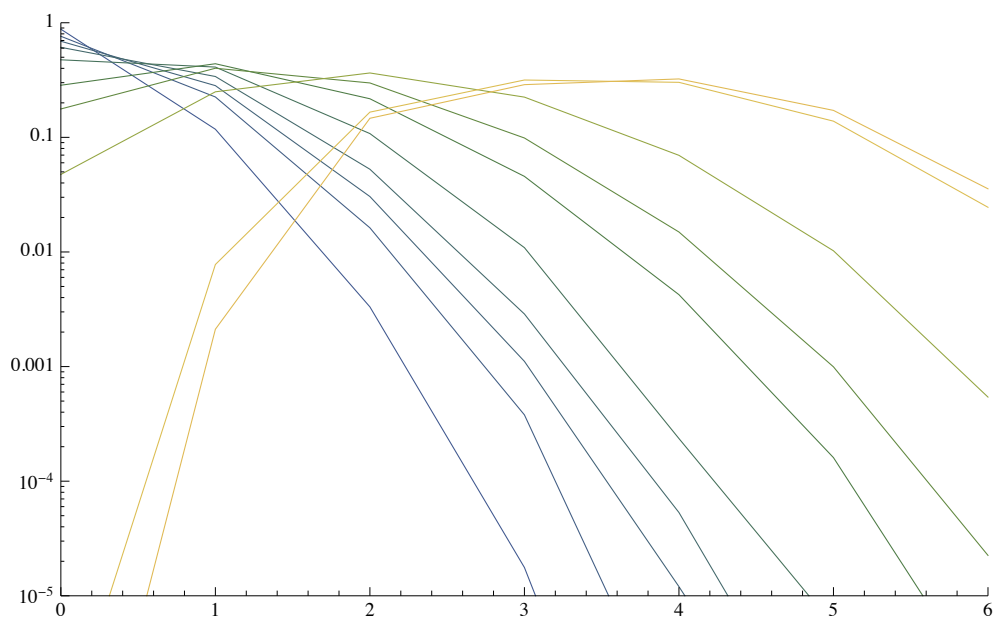
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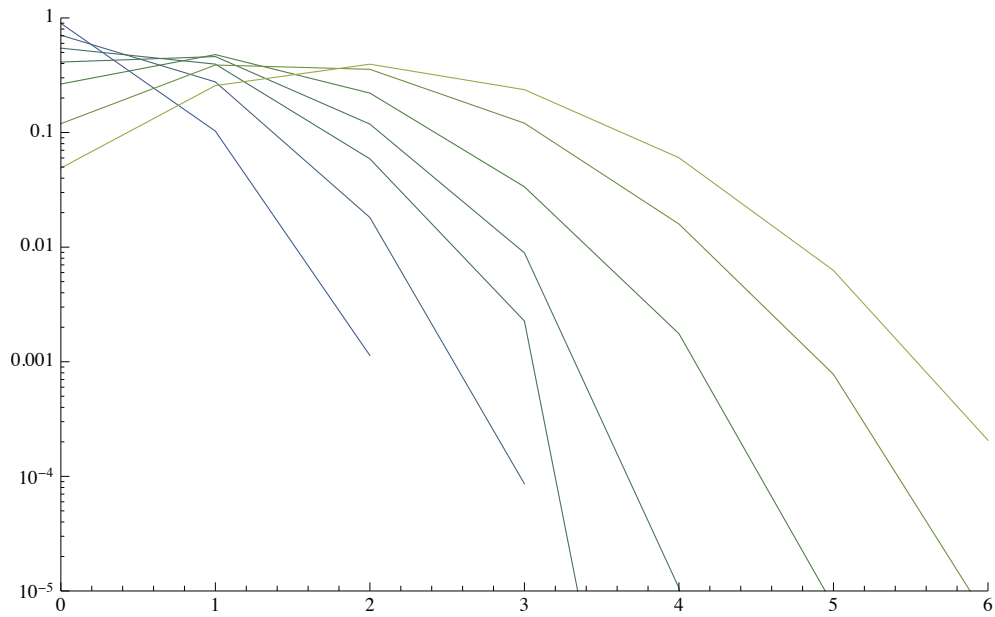


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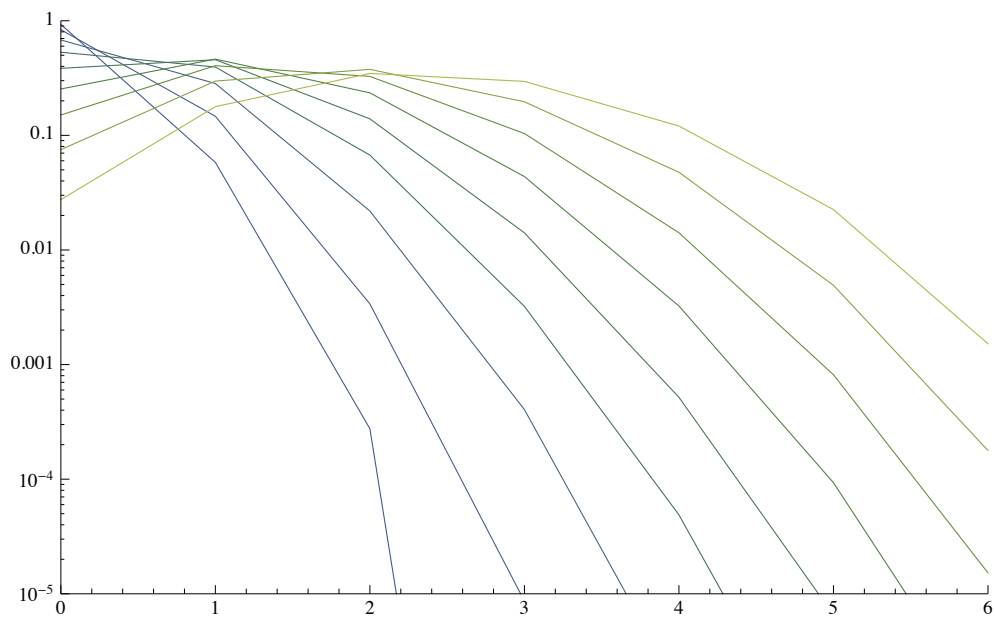


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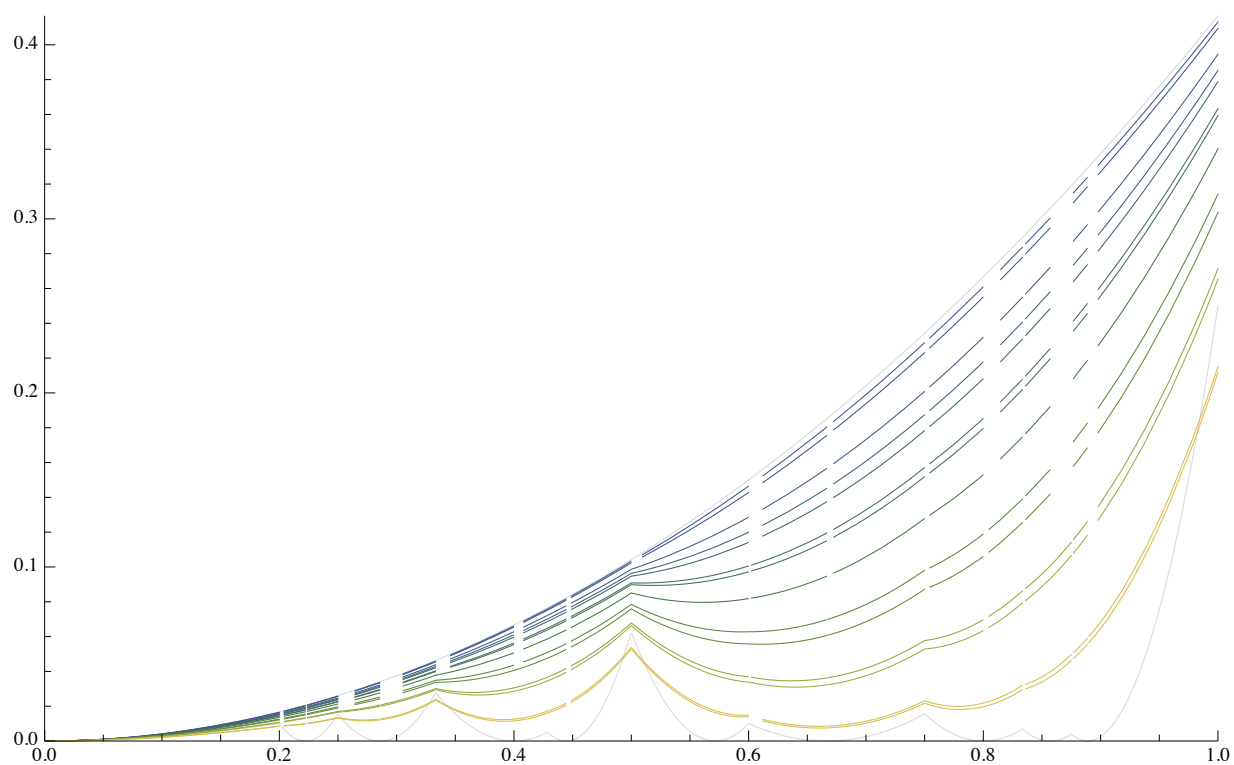


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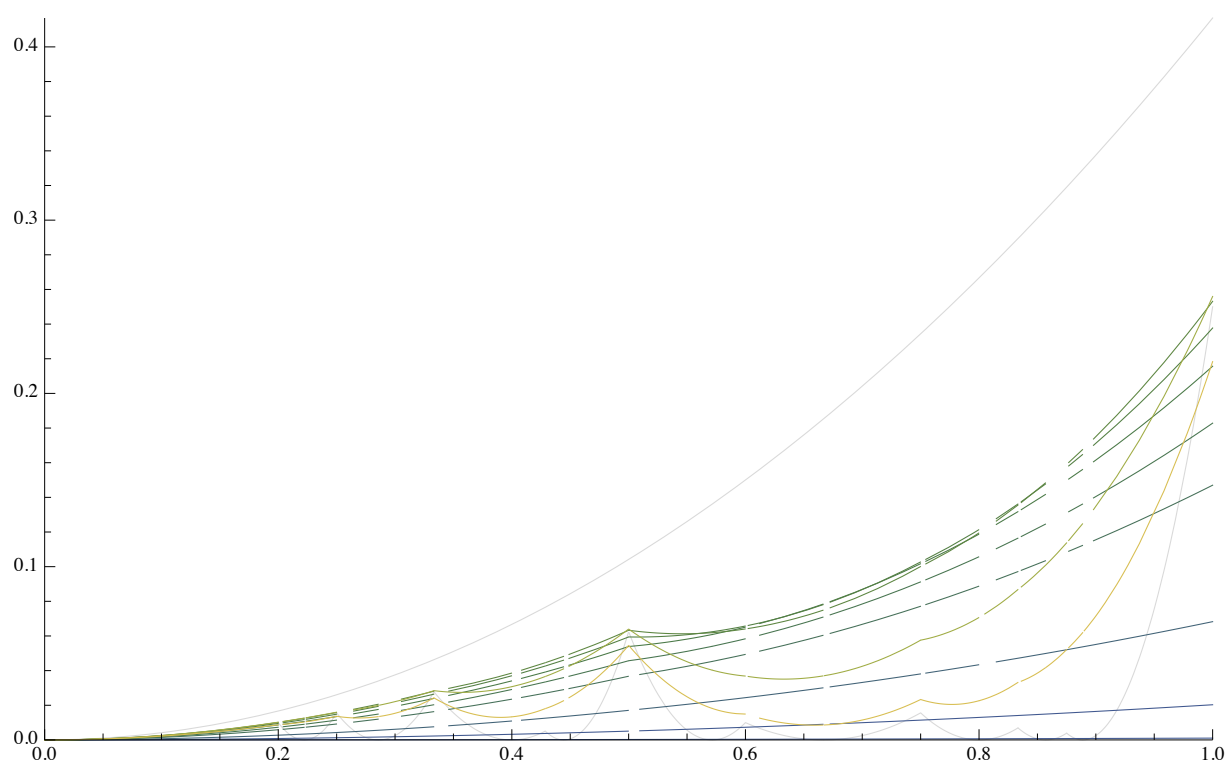


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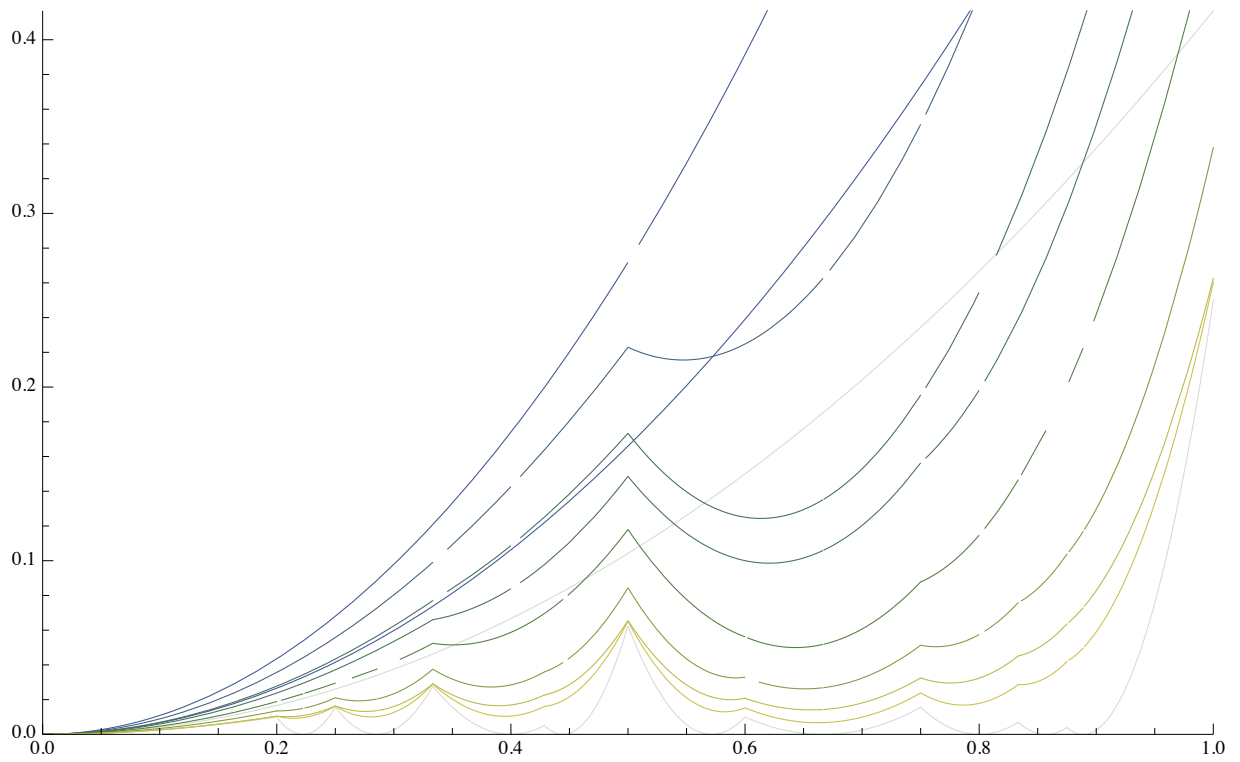
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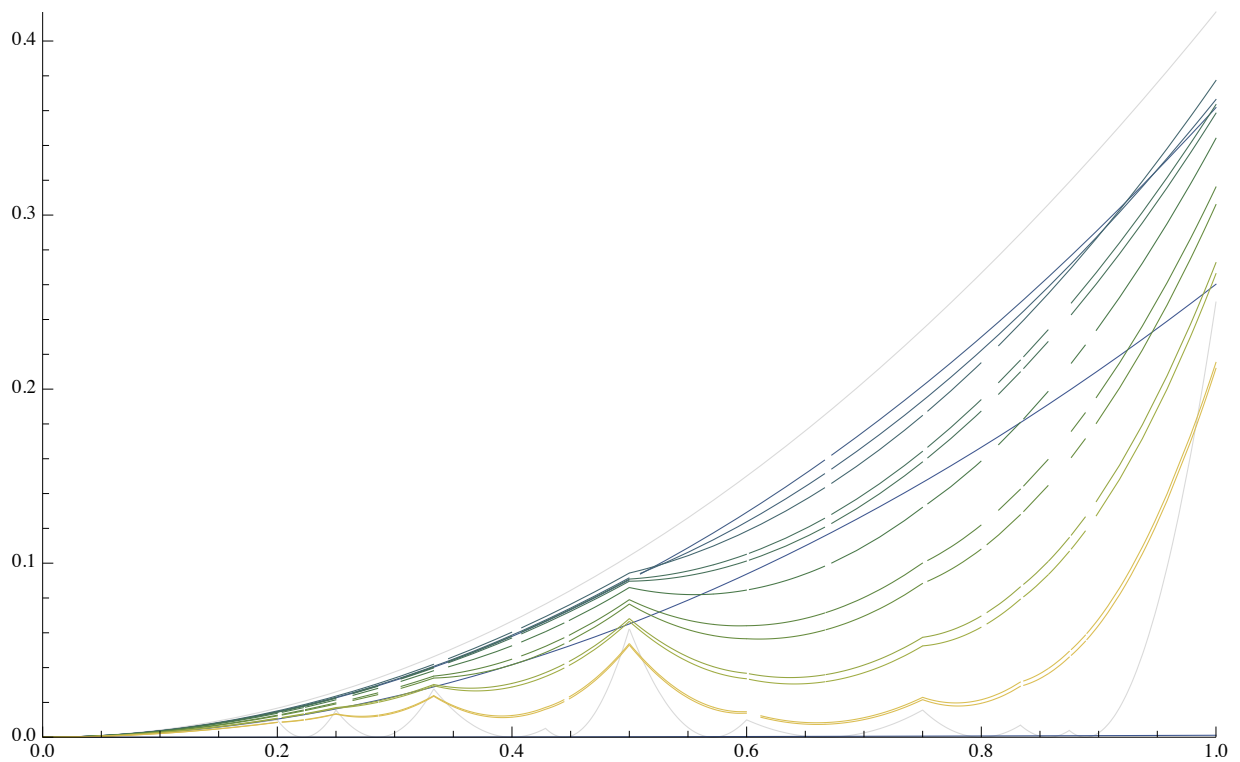
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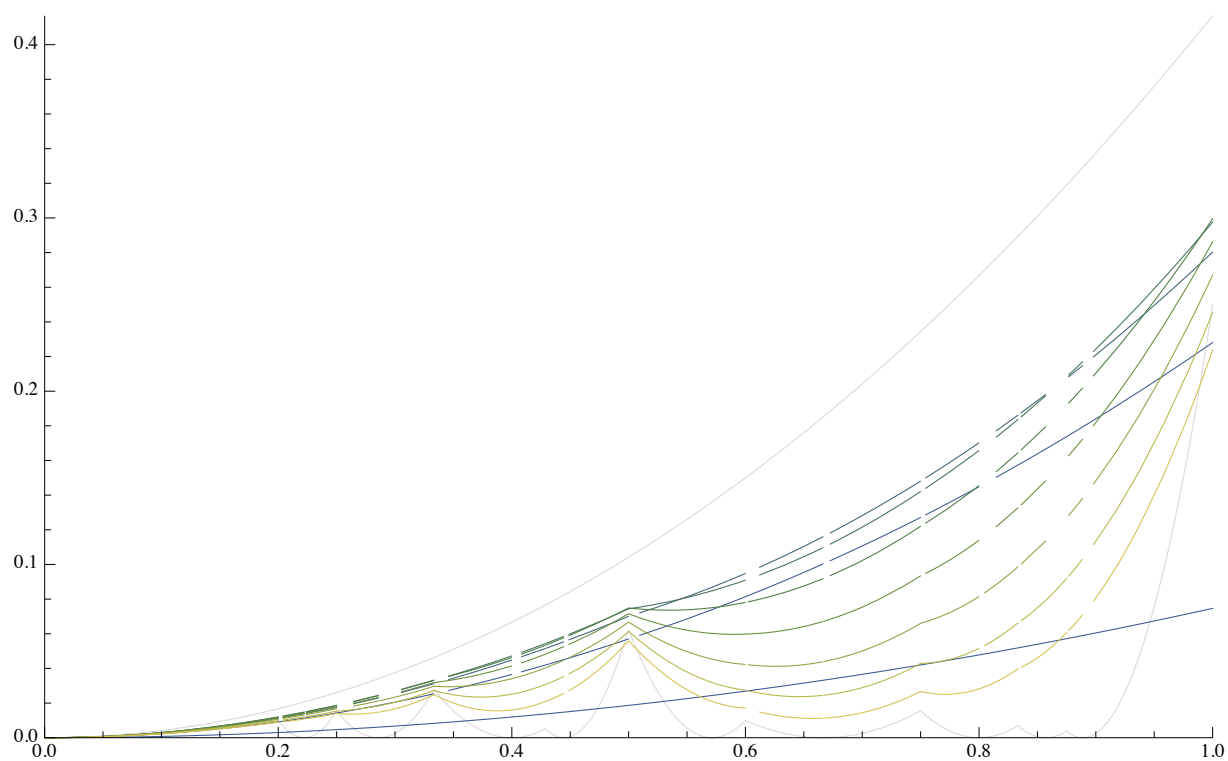
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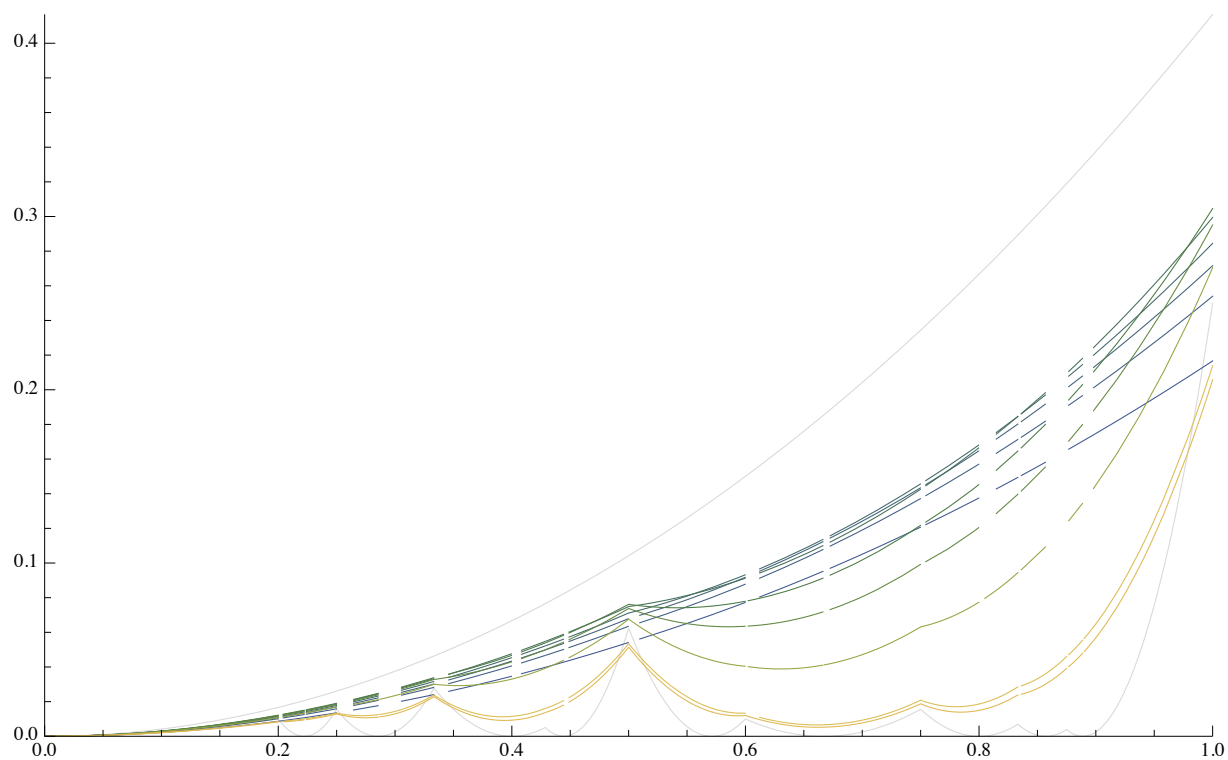
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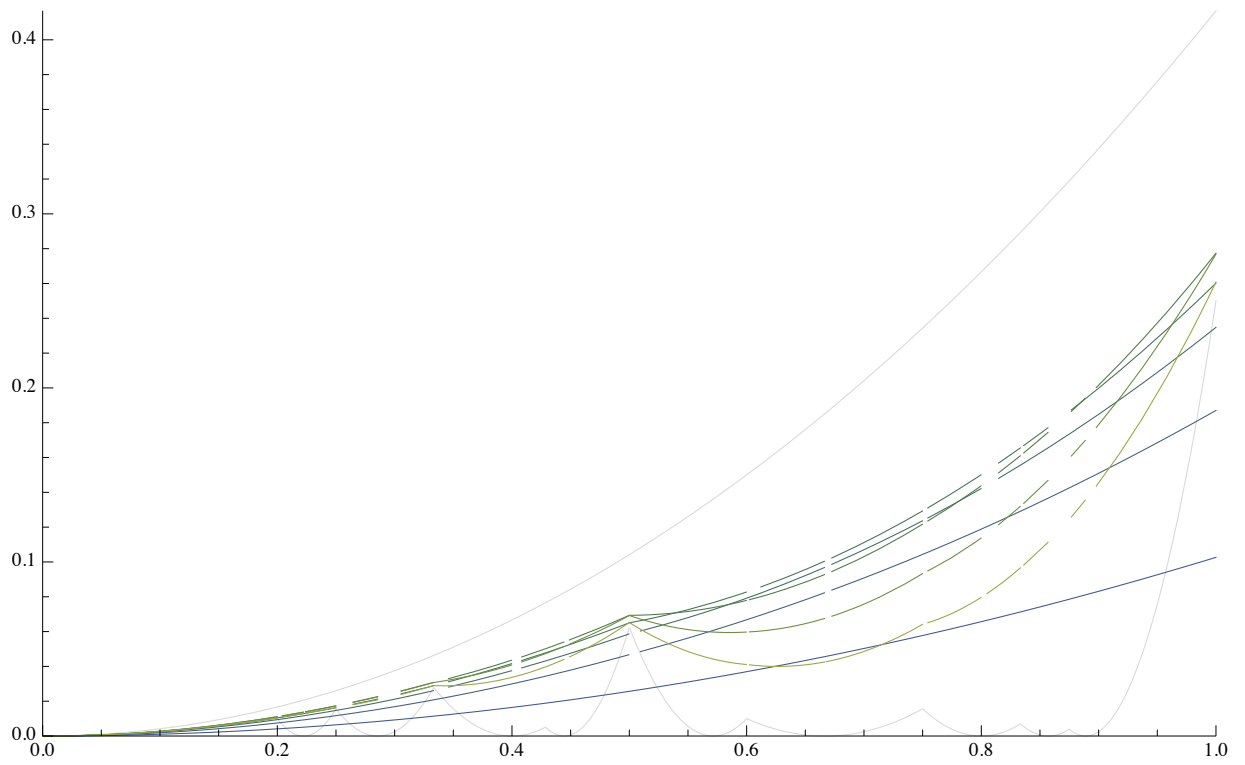
{Trial 2/3 r=0.3, rho=0.0386155, rho=0.0790005, rho=0.245689, rho=0.353612,  
rho=0.430539, rho=0.622338, rho=0.672995, rho=0.925311, rho=1.29991,  
rho=1.46108, rho=2.00949, rho=2.12355, rho=3.21895, rho=3.30581}



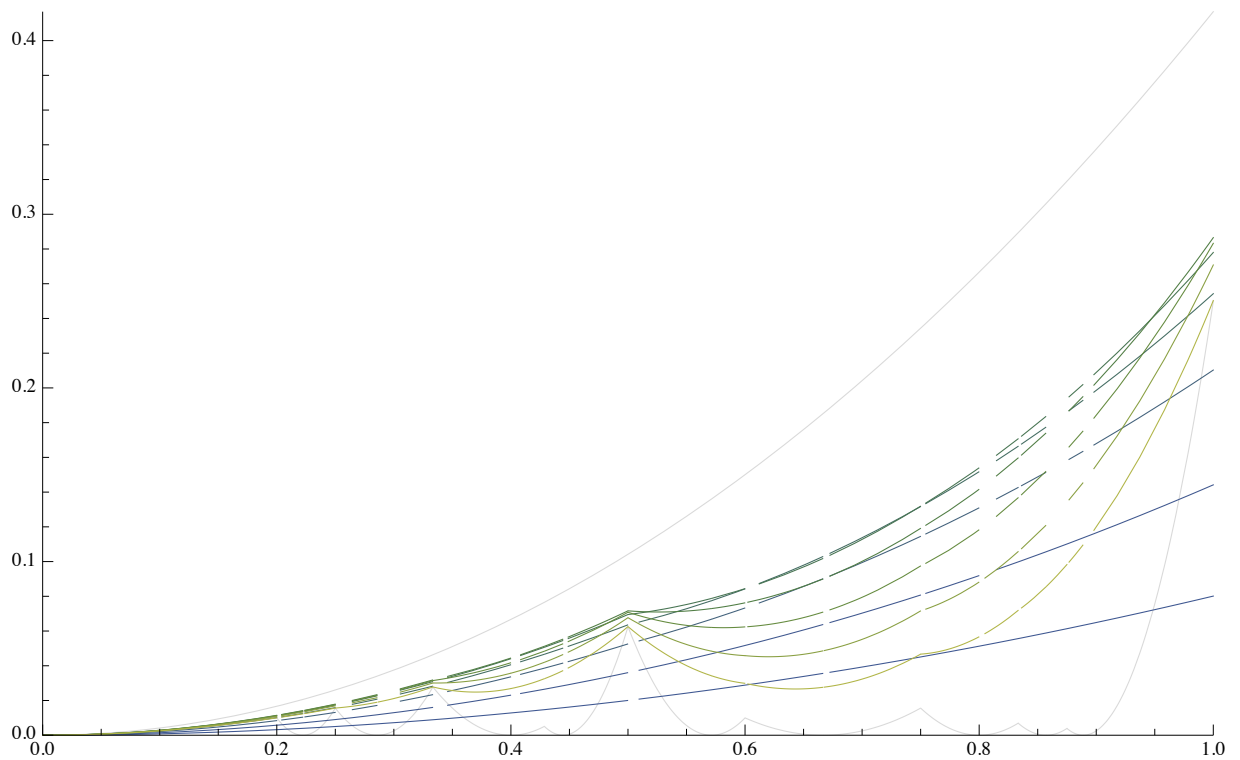
{Trial 2/3 r=1,  $\rho=0.0198397$ ,  $\rho=0.140437$ ,  $\rho=0.366651$ ,  $\rho=0.668377$ ,  
 $\rho=1.03216$ ,  $\rho=1.4616$ ,  $\rho=1.9402$ ,  $\rho=2.46115$ ,  $\rho=3.01171$ }



{Trial 2/3 r=3,  $\rho=0.12442$ ,  $\rho=0.258754$ ,  $\rho=0.347023$ ,  $\rho=0.453875$ ,  
 $\rho=0.659974$ ,  $\rho=1.02796$ ,  $\rho=1.35856$ ,  $\rho=1.98542$ ,  $\rho=3.33688$ ,  $\rho=3.52716$ }



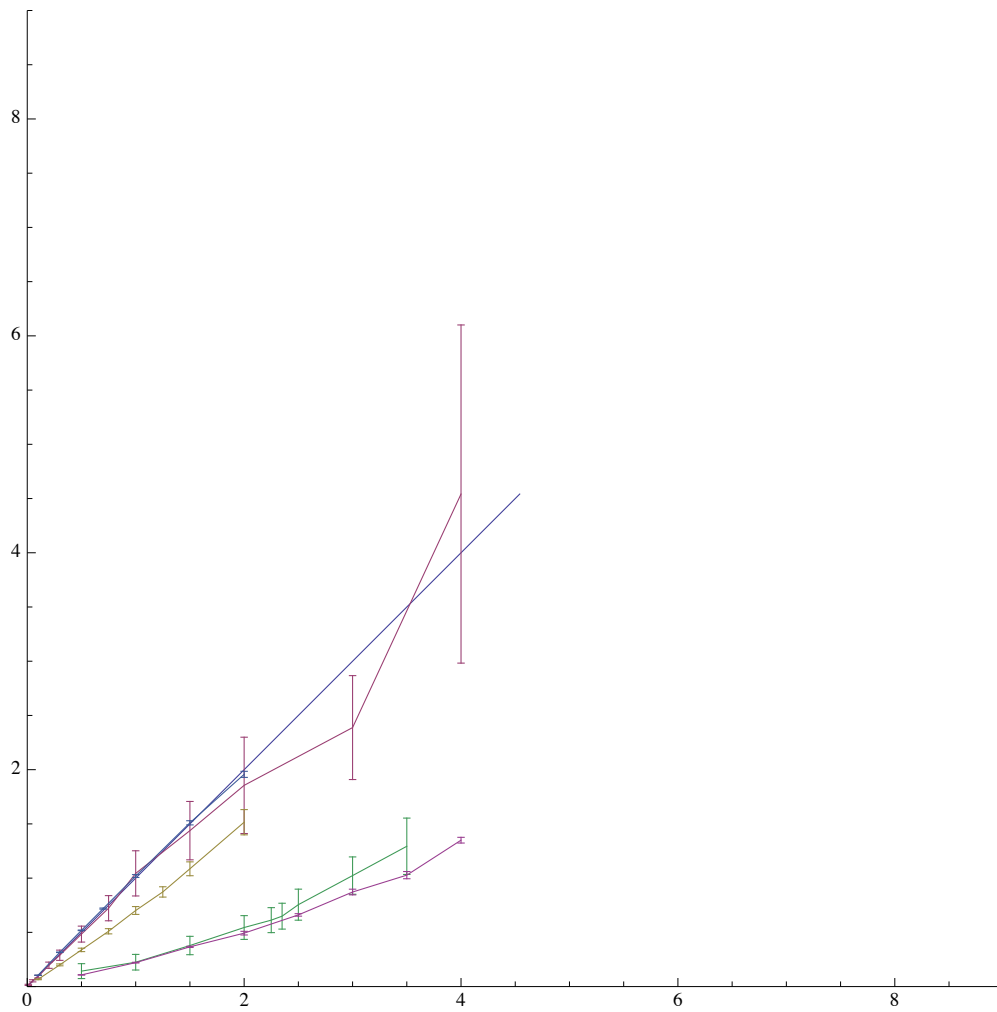
{Phi\* 2/3 r=1, rho=0.105118, rho=0.312937, rho=0.519832,  
rho=0.723843, rho=1.02853, rho=1.53025, rho=2.02992}



{Phi\* 2/3 r= $\infty$ , rho=0.0586291, rho=0.153581, rho=0.328449, rho=0.539031,  
rho=0.779993, rho=1.07711, rho=1.431, rho=1.85595, rho=2.36315}

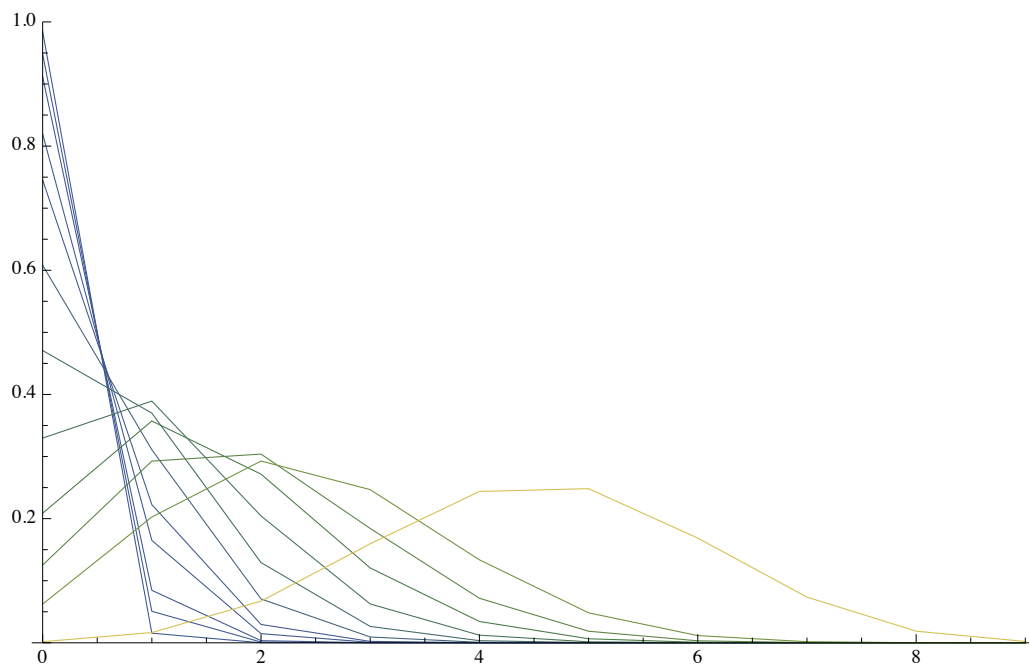
(\* Appendix B: N=9 \*)

(\* homogeneity: \*)

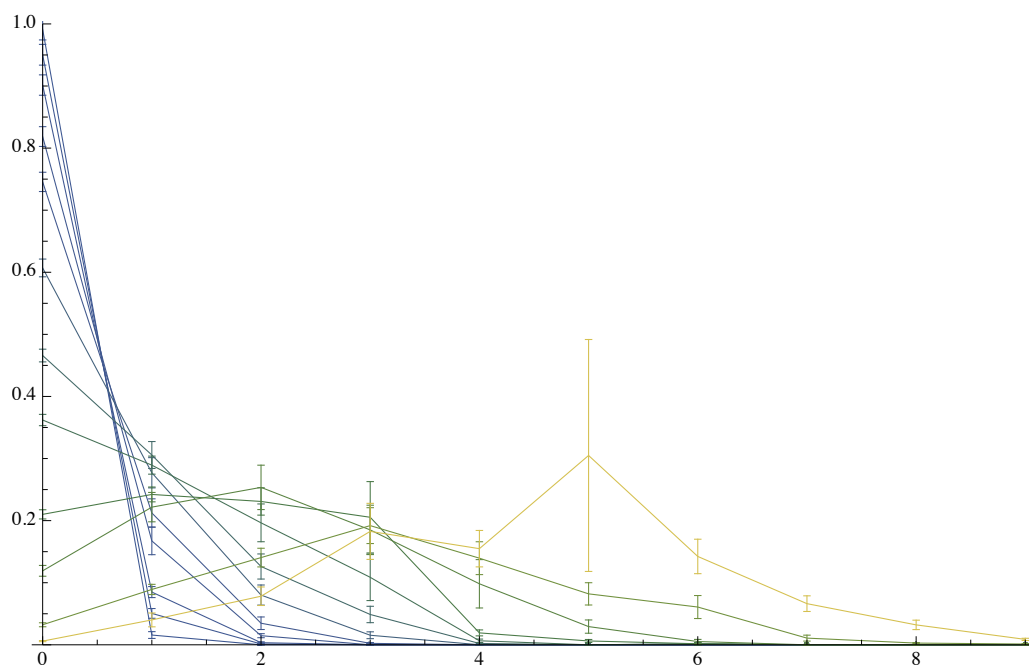


{Bose, Trial 2/3 r=0.1, Trial 2/3 r=0.3,  
Trial 2/3 r=1, Phi\* 2/3 r=1, Phi\* 2/3 r= $\infty$ }

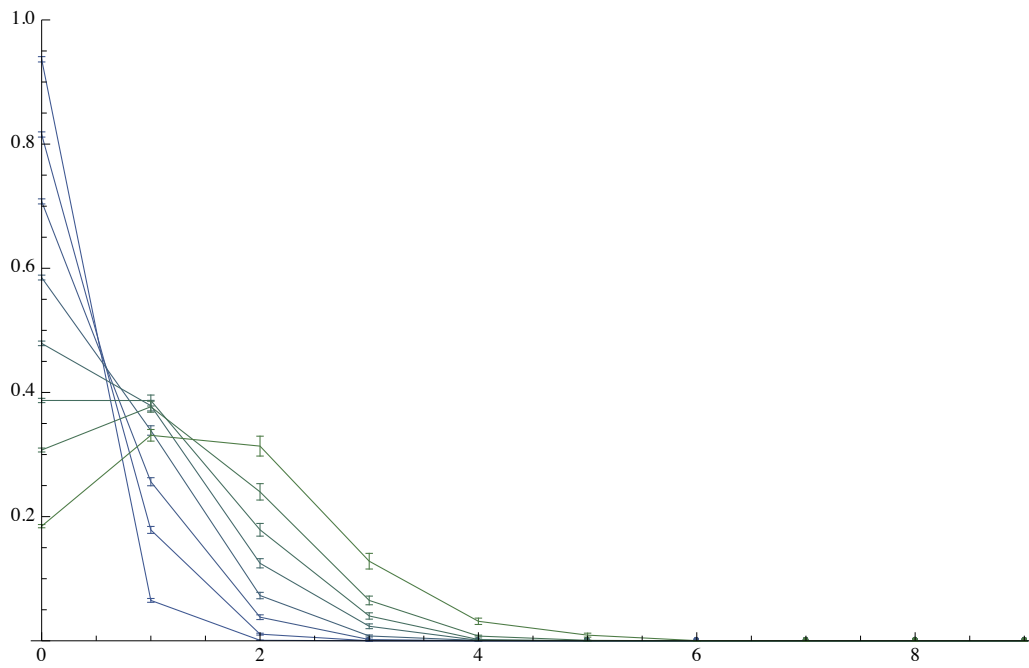
(\* probability distributions, linear scale: \*)



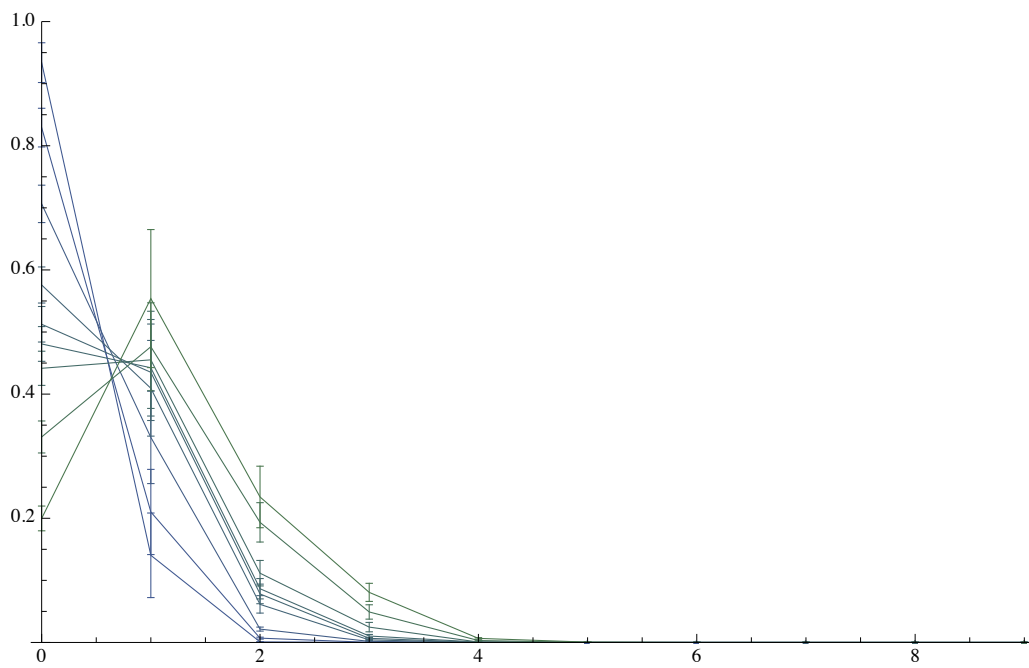
```
{Bose, rho=0.0157224, rho=0.0531926, rho=0.0917516,
rho=0.196825, rho=0.288883, rho=0.483844, rho=0.722692,
rho=1.04385, rho=1.43788, rho=1.85531, rho=2.38806, rho=4.54153}
```



```
{Trial 2/3 r=0.1, rho=0.0157224, rho=0.0531926, rho=0.0917516,
rho=0.196825, rho=0.288883, rho=0.483844, rho=0.722692,
rho=1.04385, rho=1.43788, rho=1.85531, rho=2.38806, rho=4.54153}
```

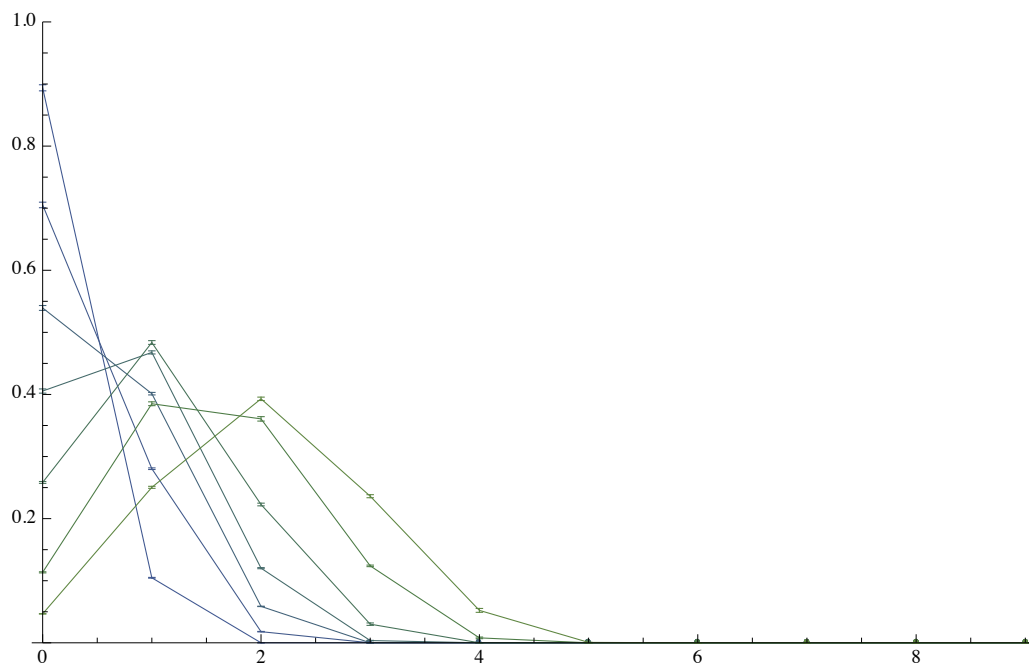


{Trial 2/3  $r=0.3$ ,  $\rho=0.066692$ ,  $\rho=0.200146$ ,  $\rho=0.33829$ ,  
 $\rho=0.510032$ ,  $\rho=0.701594$ ,  $\rho=0.872781$ ,  $\rho=1.08606$ ,  $\rho=1.51552$ }

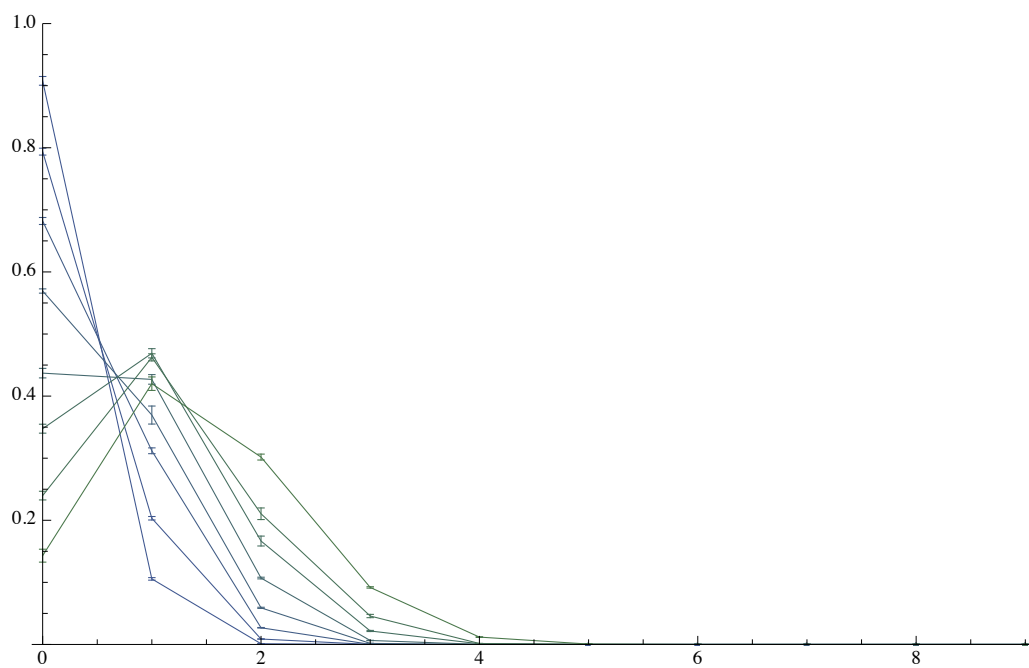


{Trial 2/3  $r=1$ ,  $\rho=0.142511$ ,  $\rho=0.224671$ ,  $\rho=0.378589$ ,  $\rho=0.544604$ ,  
 $\rho=0.612614$ ,  $\rho=0.649183$ ,  $\rho=0.755251$ ,  $\rho=1.02342$ ,  $\rho=1.2937$ }



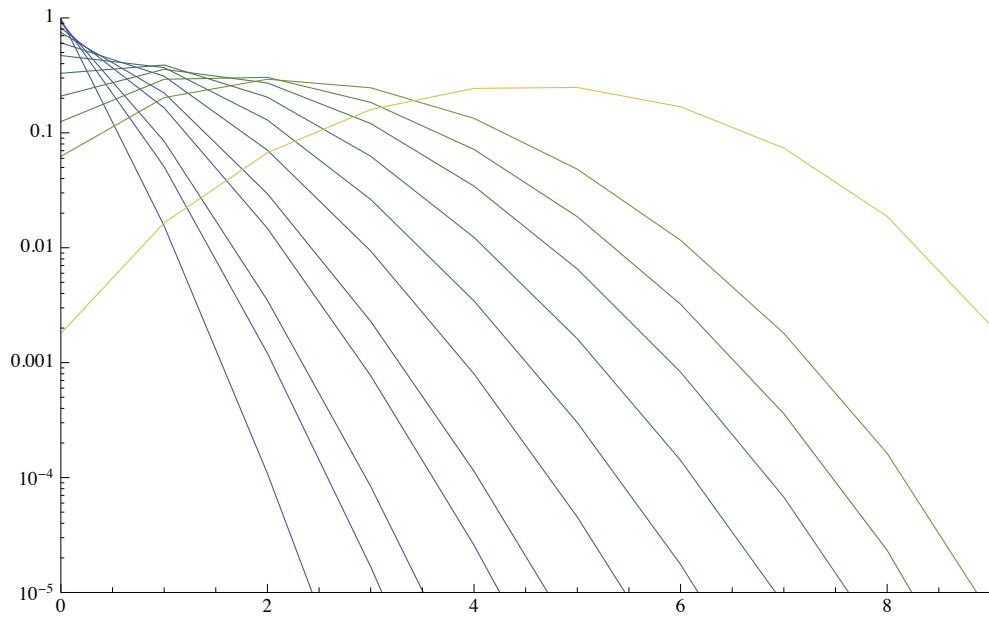


{Phi\* 2/3 r=1, rho=0.104726, rho=0.316299, rho=0.518583,  
rho=0.719014, rho=1.01922, rho=1.50994, rho=1.95685}

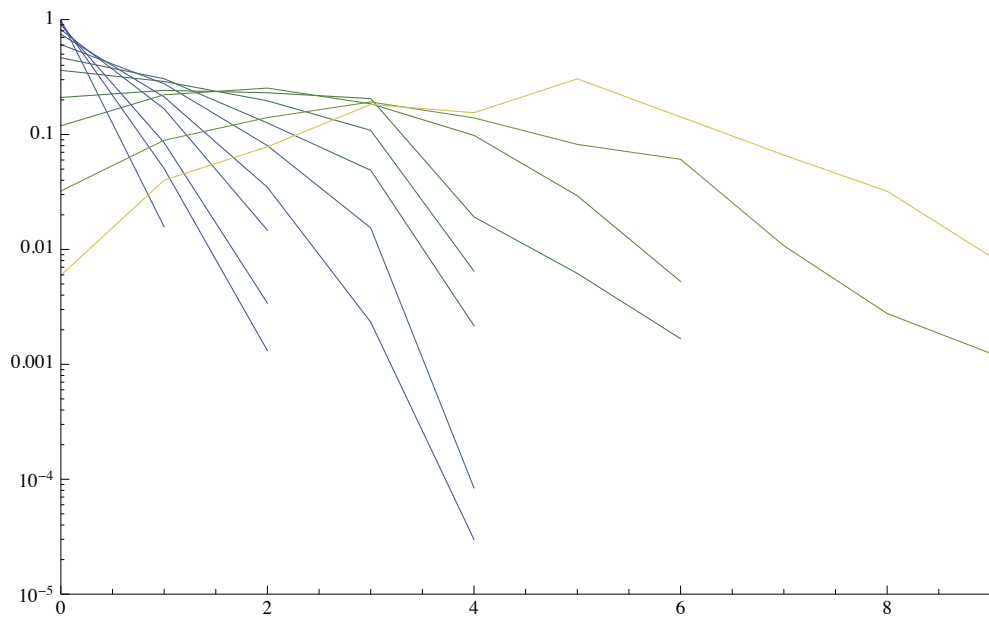


{Phi\* 2/3 r=infinity, rho=0.107912, rho=0.220794, rho=0.366225,  
rho=0.492635, rho=0.660826, rho=0.871802, rho=1.02751, rho=1.34986}

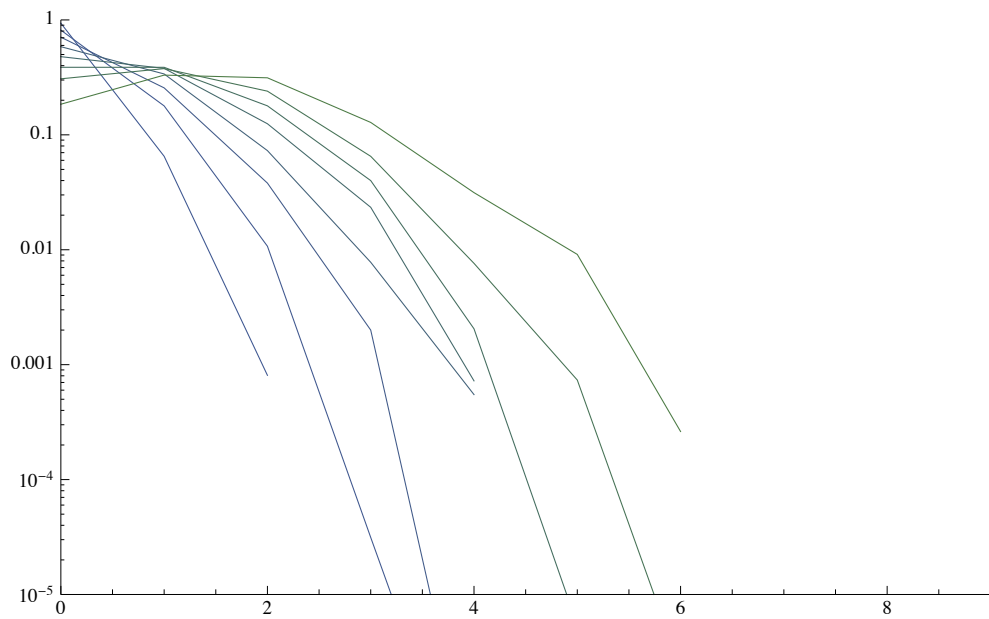
(\* probability distributions, logarithmic scale: \*)



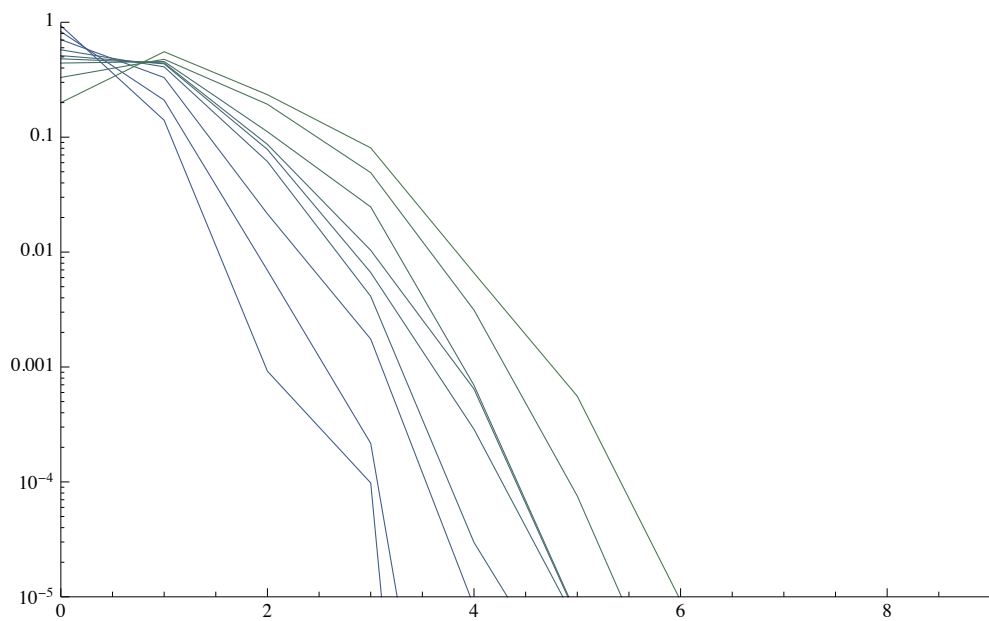
```
{Bose, rho=0.0157224, rho=0.0531926, rho=0.0917516,
rho=0.196825, rho=0.288883, rho=0.483844, rho=0.722692,
rho=1.04385, rho=1.43788, rho=1.85531, rho=2.38806, rho=4.54153}
```



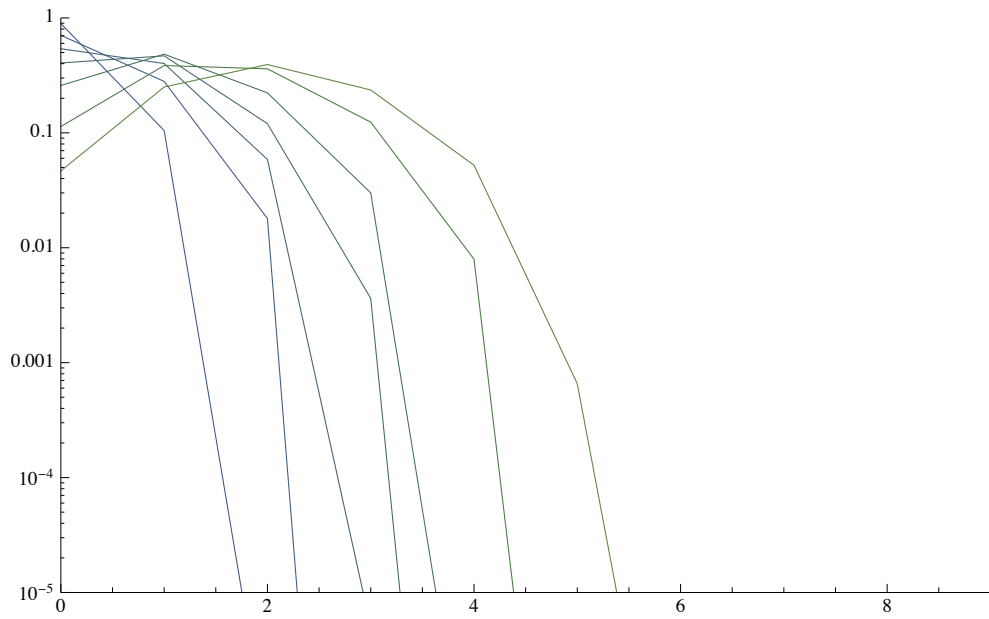
```
{Trial 2/3 r=0.1, rho=0.0157224, rho=0.0531926, rho=0.0917516,
rho=0.196825, rho=0.288883, rho=0.483844, rho=0.722692,
rho=1.04385, rho=1.43788, rho=1.85531, rho=2.38806, rho=4.54153}
```



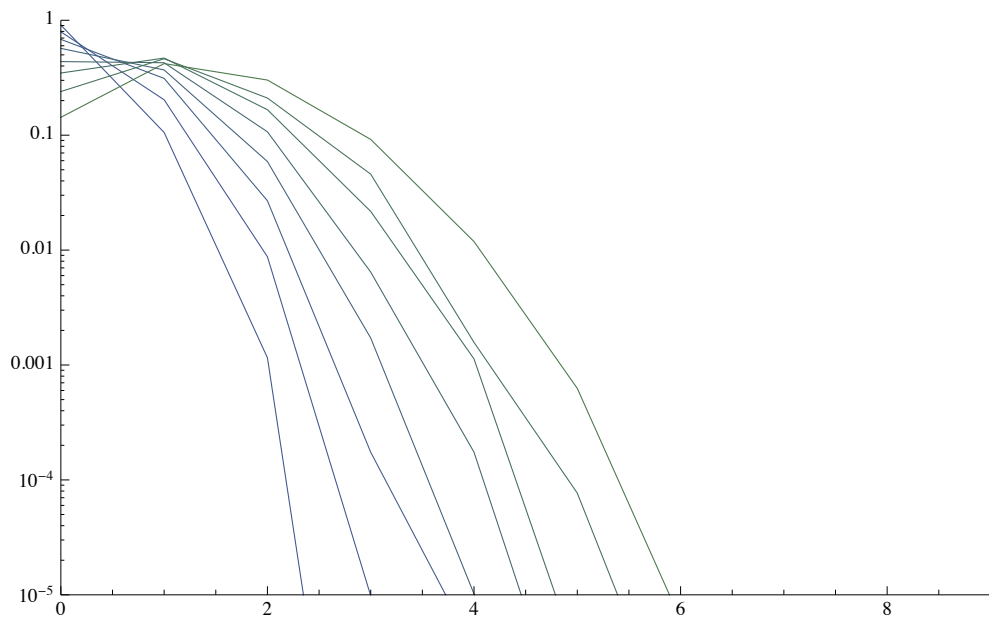
```
{Trial 2/3 r=0.3, rho=0.066692, rho=0.200146, rho=0.33829,
  rho=0.510032, rho=0.701594, rho=0.872781, rho=1.08606, rho=1.51552}
```



```
{Trial 2/3 r=1, rho=0.142511, rho=0.224671, rho=0.378589, rho=0.544604,
  rho=0.612614, rho=0.649183, rho=0.755251, rho=1.02342, rho=1.2937}
```

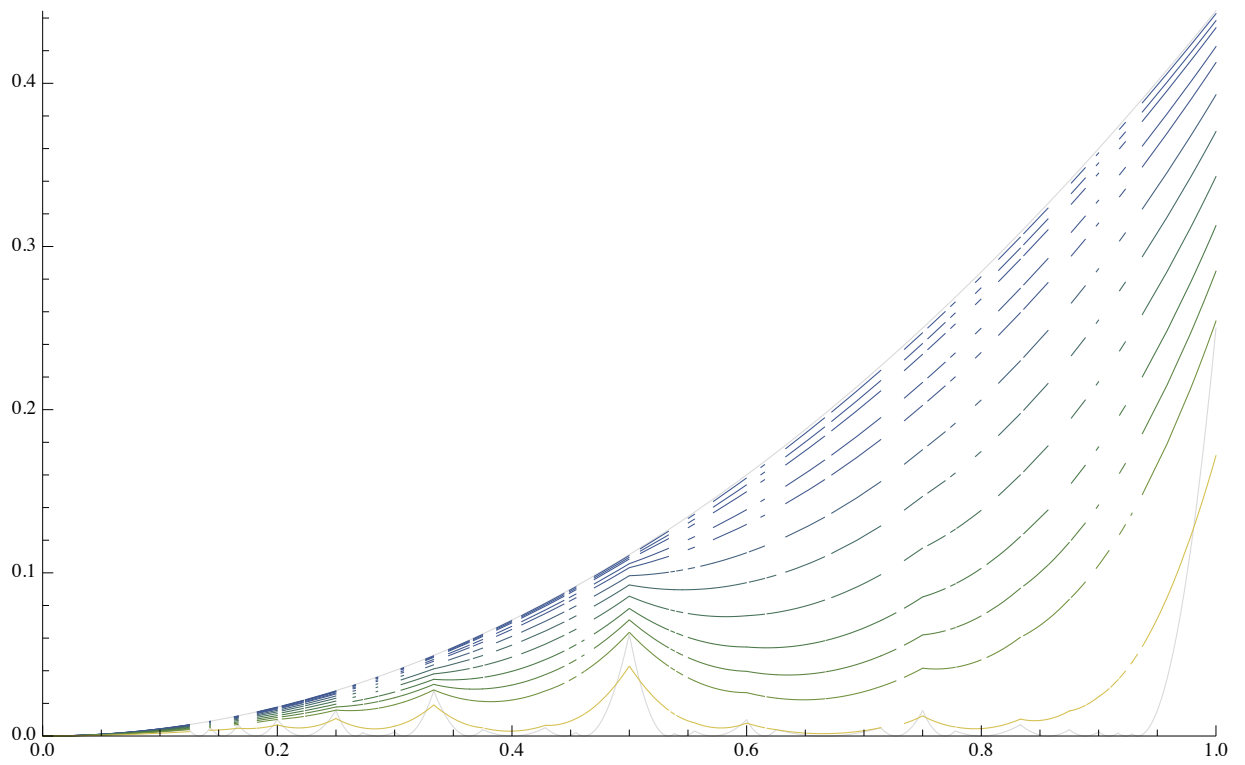


{Phi\* 2/3 r=1, rho=0.104726, rho=0.316299, rho=0.518583,  
rho=0.719014, rho=1.01922, rho=1.50994, rho=1.95685}

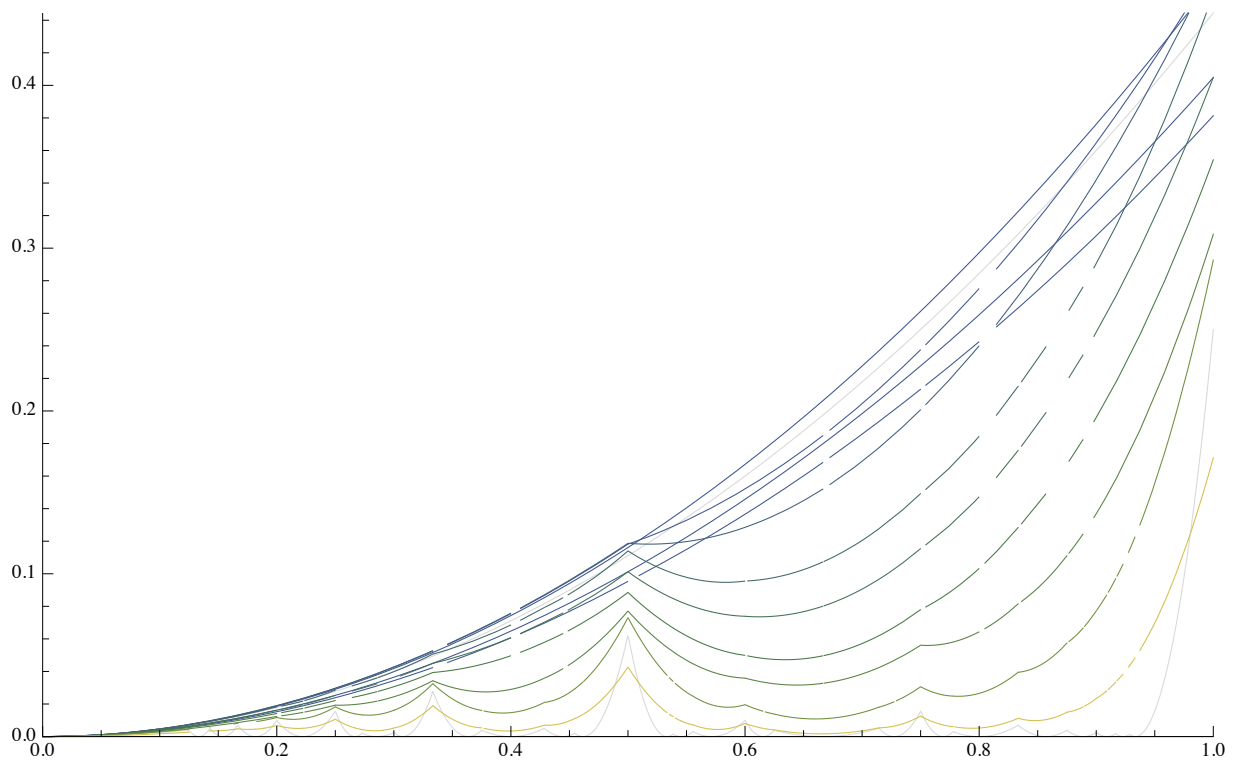


{Phi\* 2/3 r=infinity, rho=0.107912, rho=0.220794, rho=0.366225,  
rho=0.492635, rho=0.660826, rho=0.871802, rho=1.02751, rho=1.34986}

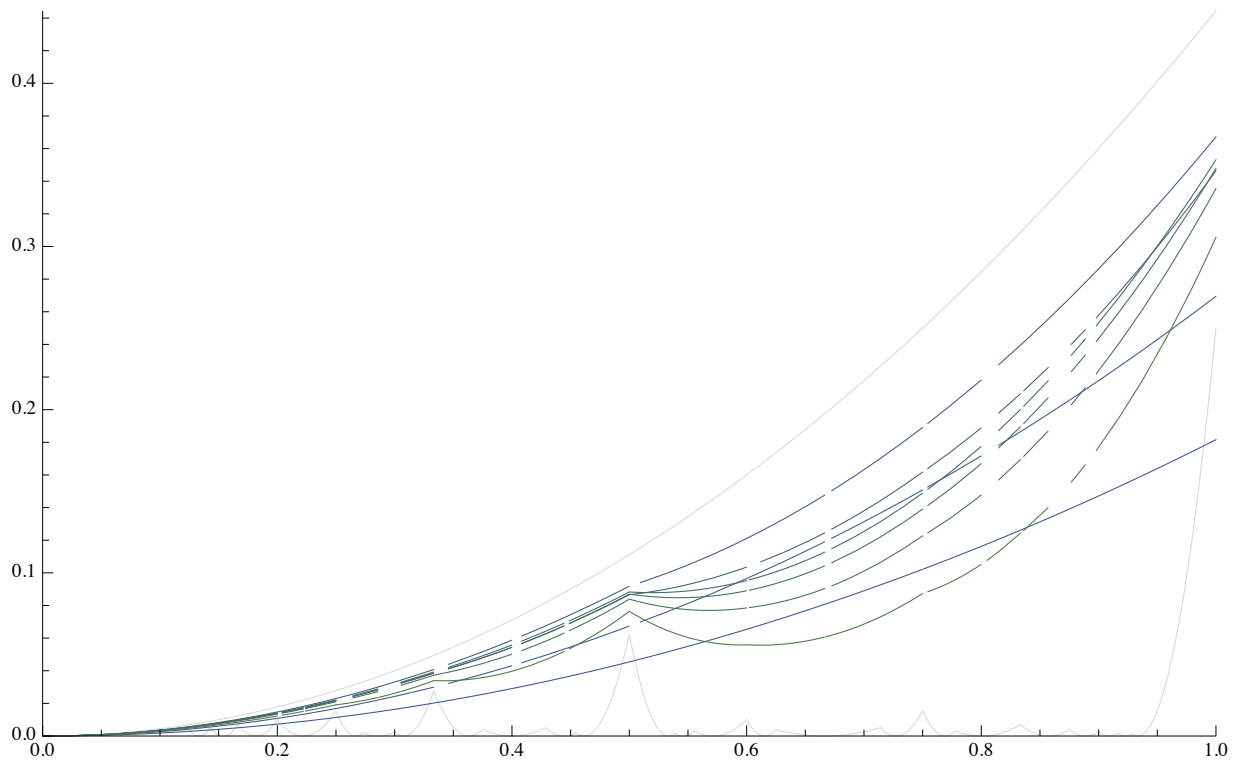
(\* bounds in terms of E\_alpha: \*)



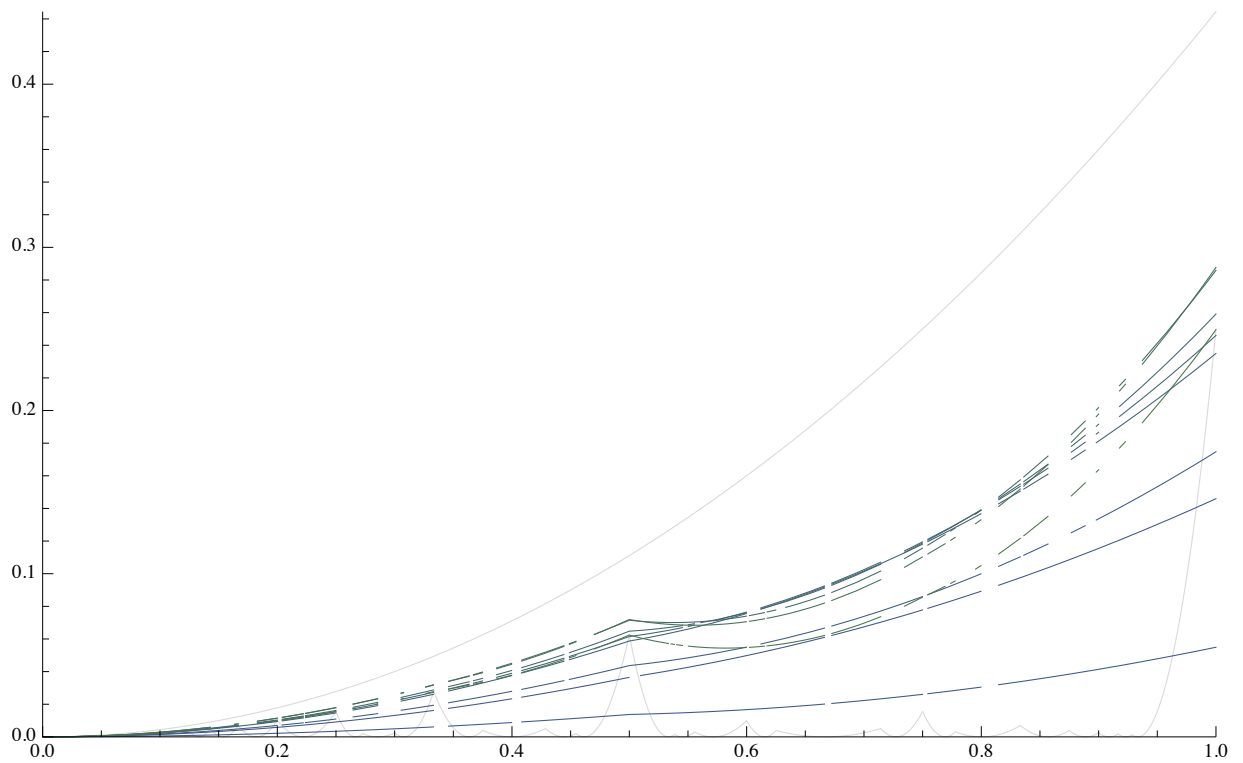
```
{Bose, rho=0.0157224, rho=0.0531926, rho=0.0917516,
rho=0.196825, rho=0.288883, rho=0.483844, rho=0.722692,
rho=1.04385, rho=1.43788, rho=1.85531, rho=2.38806, rho=4.54153}
```



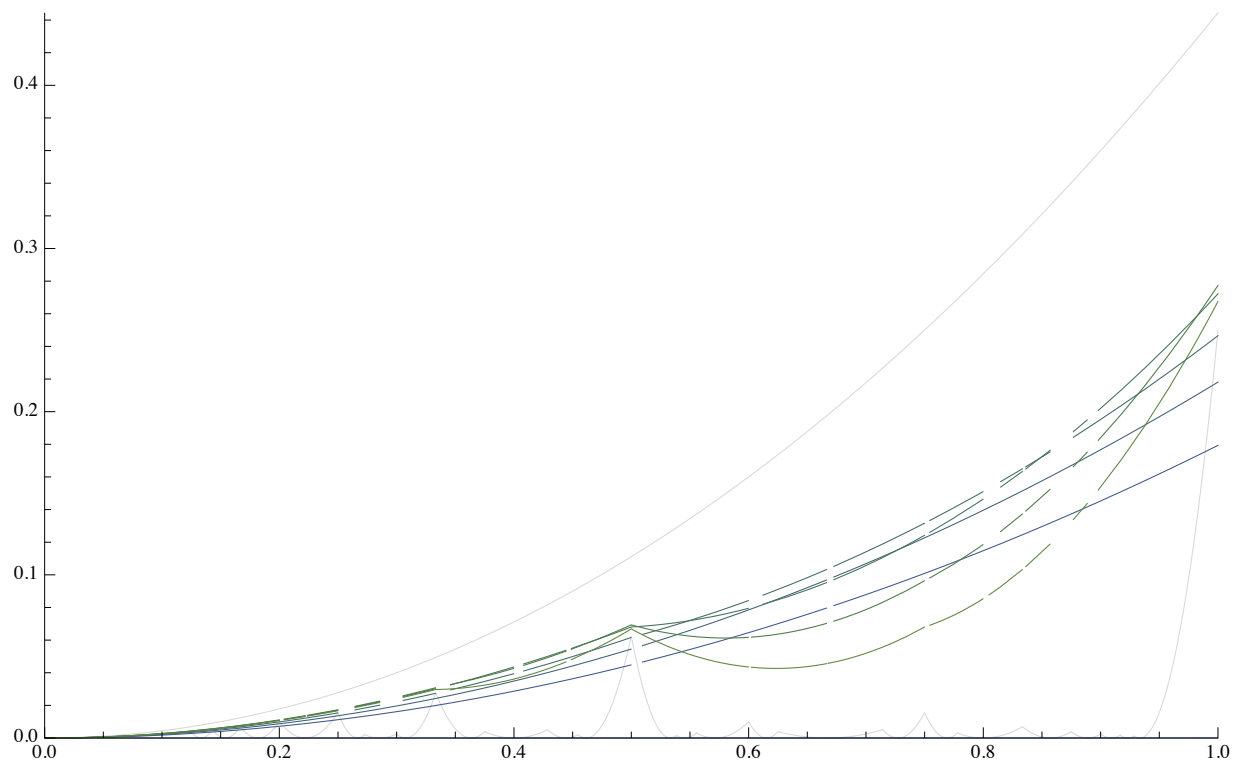
```
{Trial 2/3 r=0.1, rho=0.0157224, rho=0.0531926, rho=0.0917516,
rho=0.196825, rho=0.288883, rho=0.483844, rho=0.722692,
rho=1.04385, rho=1.43788, rho=1.85531, rho=2.38806, rho=4.54153}
```



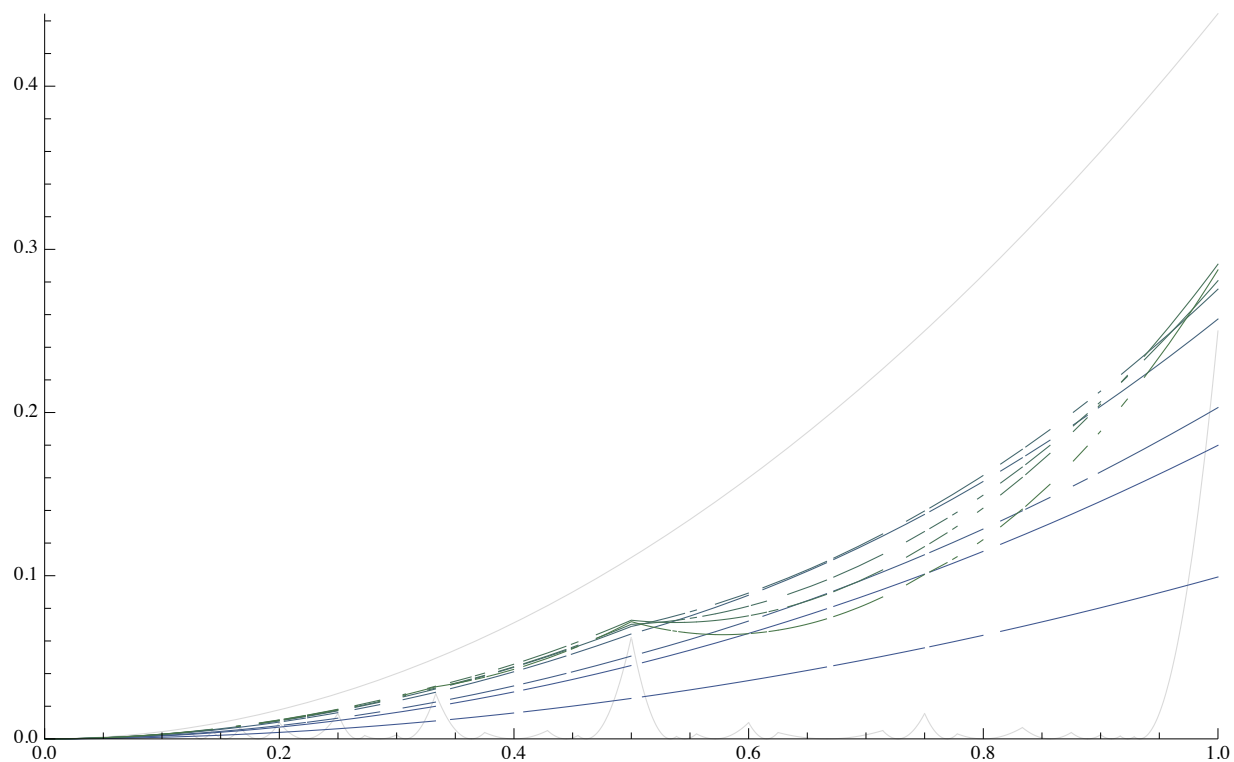
{Trial 2/3 r=0.3, rho=0.066692, rho=0.200146, rho=0.33829,  
rho=0.510032, rho=0.701594, rho=0.872781, rho=1.08606, rho=1.51552}



{Trial 2/3 r=1, rho=0.142511, rho=0.224671, rho=0.378589, rho=0.544604,  
rho=0.612614, rho=0.649183, rho=0.755251, rho=1.02342, rho=1.2937}



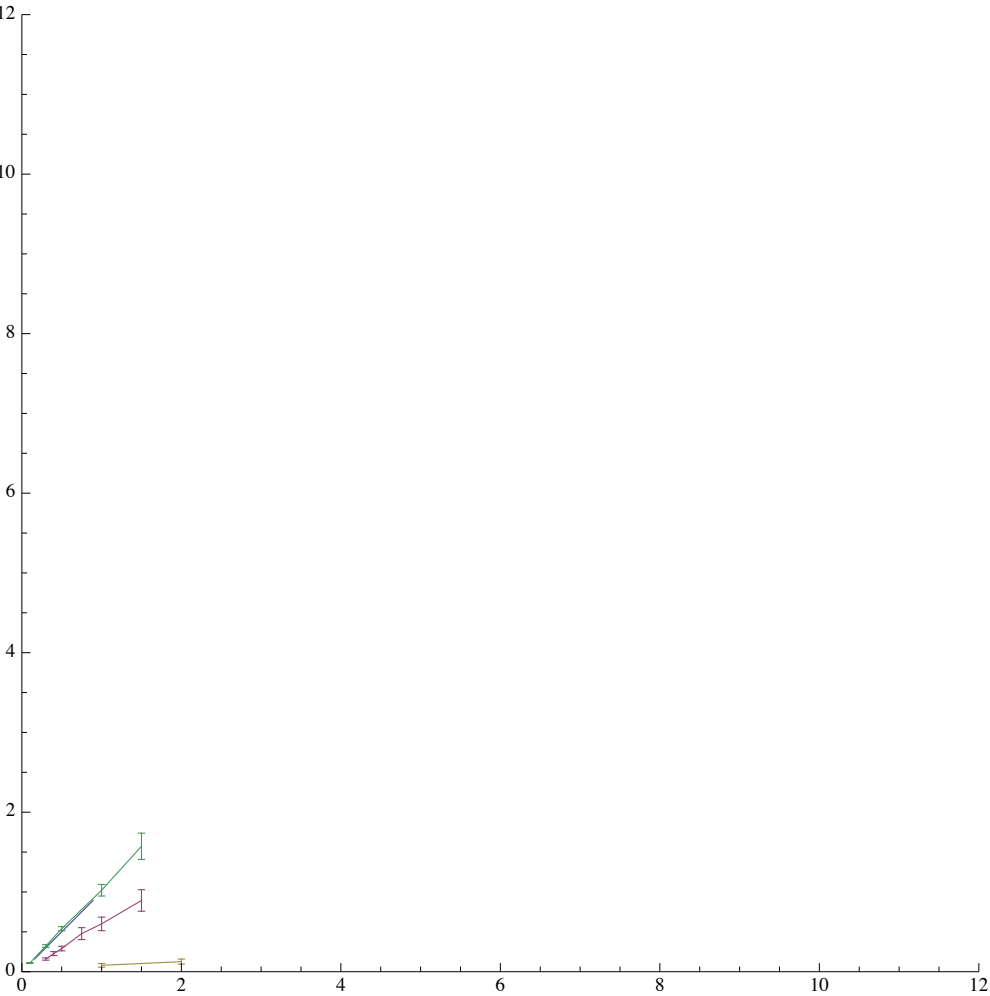
{Phi\*  $\frac{2}{3}$  r=1, rho=0.104726, rho=0.316299, rho=0.518583,  
rho=0.719014, rho=1.01922, rho=1.50994, rho=1.95685}



{Phi\*  $\frac{2}{3}$  r= $\infty$ , rho=0.107912, rho=0.220794, rho=0.366225,  
rho=0.492635, rho=0.660826, rho=0.871802, rho=1.02751, rho=1.34986}

(\* Appendix C: N=12 \*)

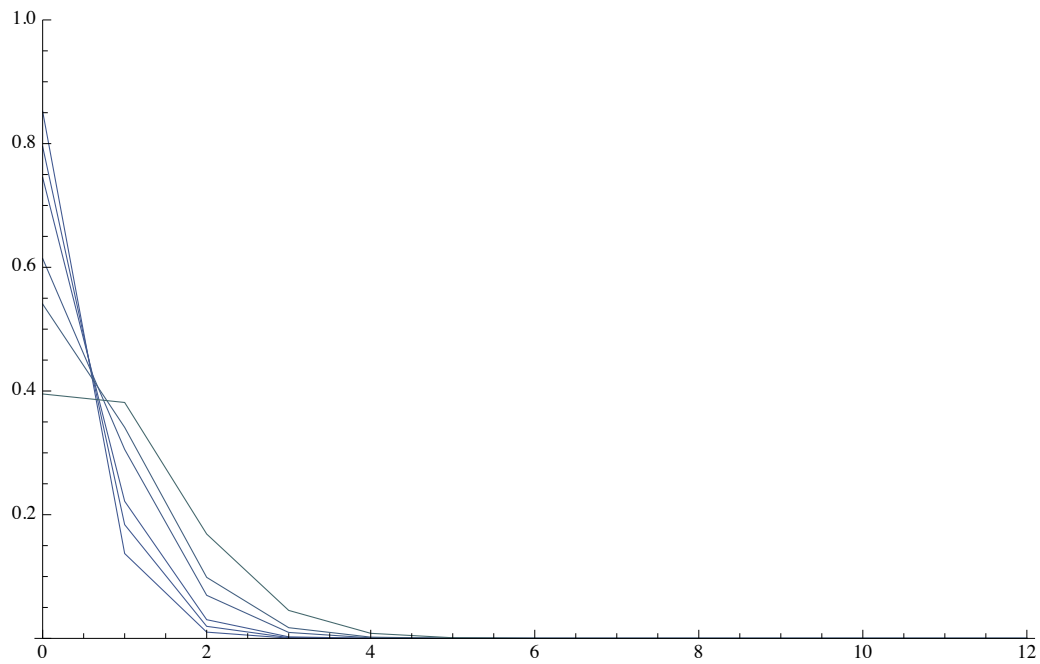
(\* homogeneity: \*)



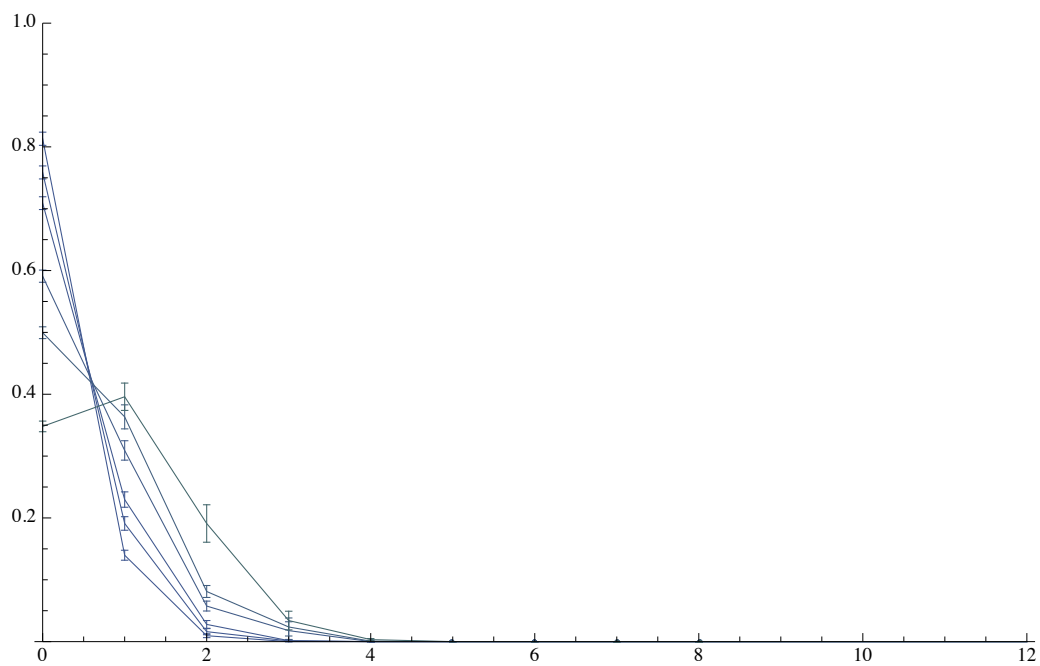
{Bose, Trial 2/3 r=0.3, Trial 2/3 r=1, Phi\* 2/3 r=1}

(\* probability distributions, linear scale: \*)

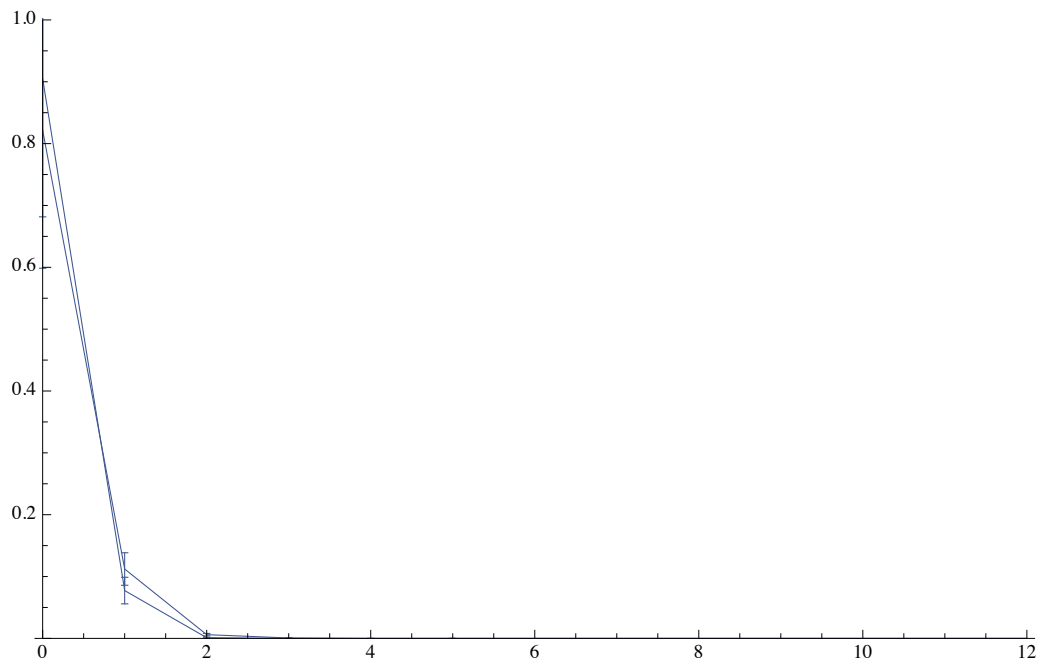




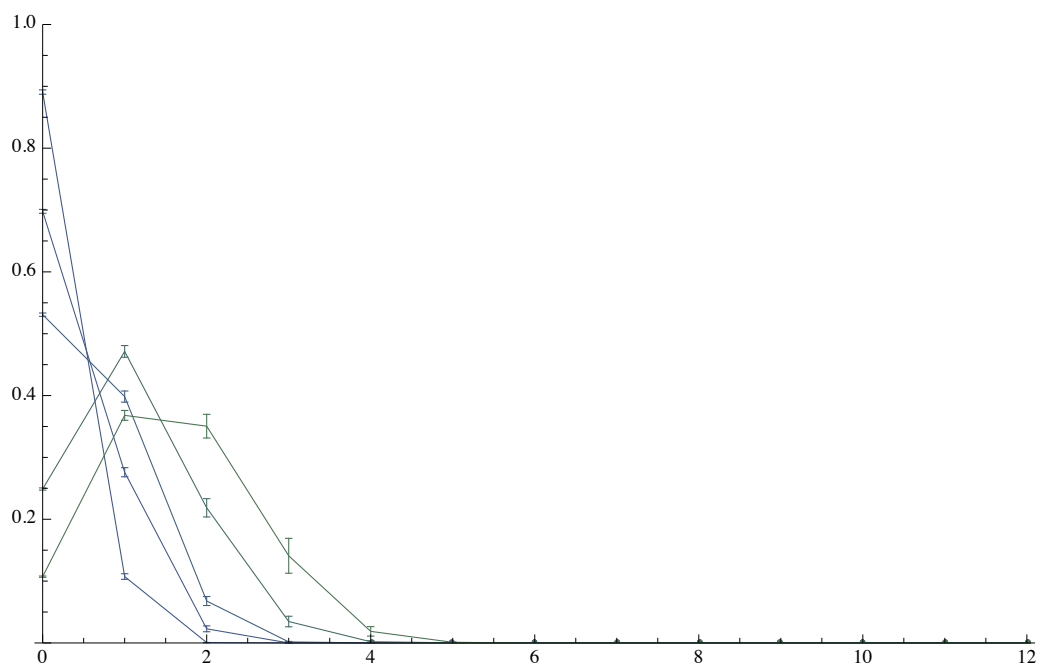
```
{Bose, rho=0.158854, rho=0.22687,
  rho=0.290331, rho=0.477664, rho=0.599564, rho=0.893327}
```



```
{Trial 2/3 r=0.3, rho=0.158854, rho=0.22687,
  rho=0.290331, rho=0.477664, rho=0.599564, rho=0.893327}
```

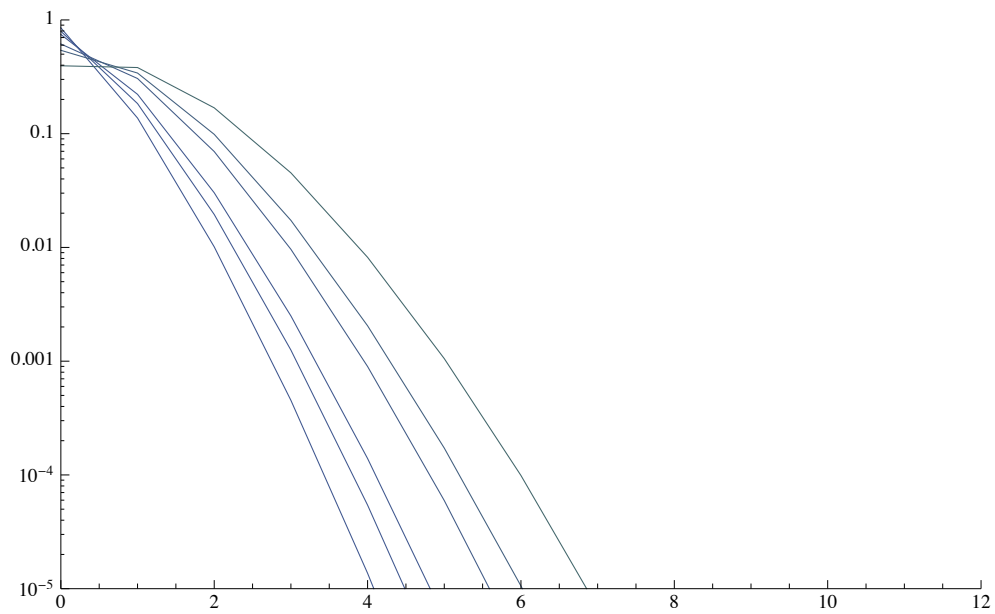


{Trial 2/3 r=1, rho=0.0796426, rho=0.126296}

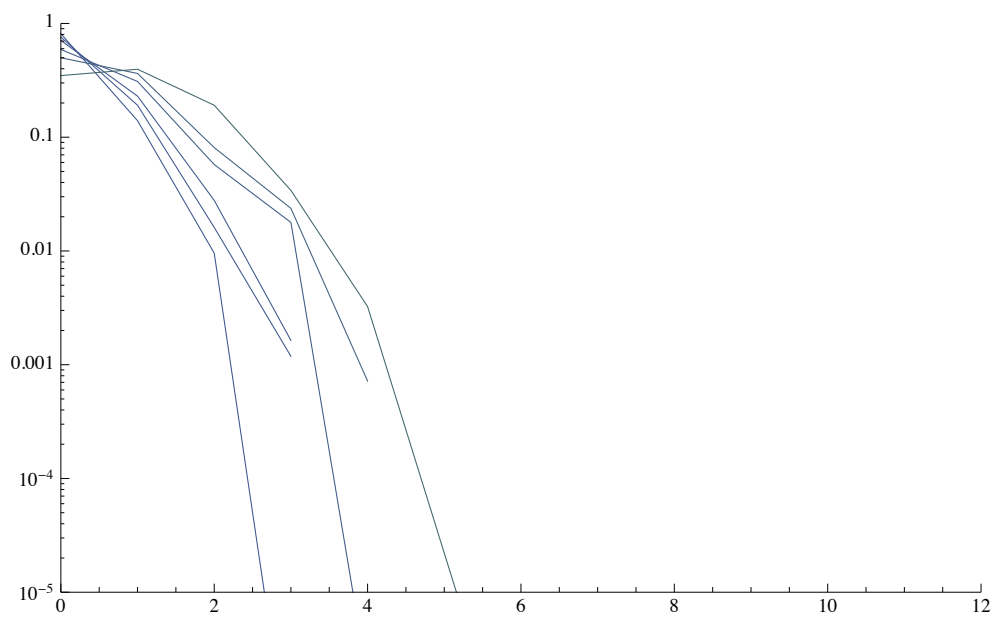


{Phi\* 2/3 r=1, rho=0.108509,  
rho=0.321778, rho=0.539109, rho=1.02154, rho=1.5722}

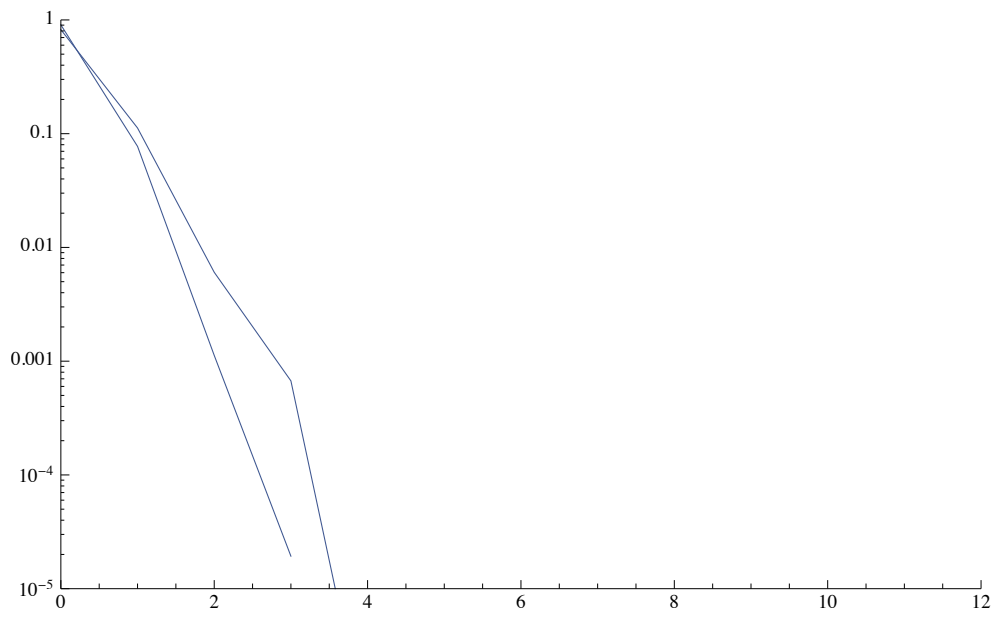
(\* probability distributions, logarithmic scale: \*)



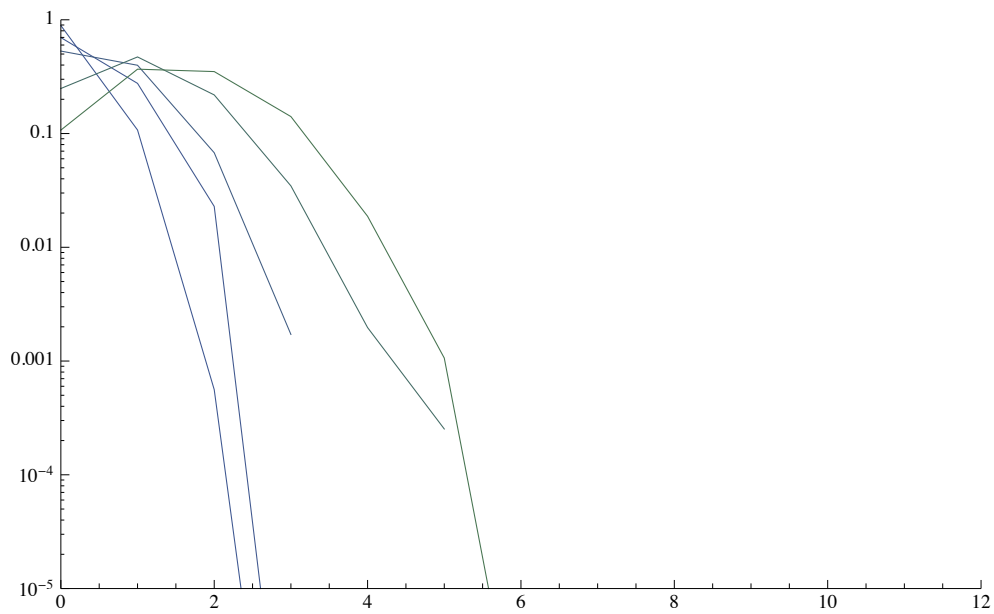
```
{Bose, rho=0.158854, rho=0.22687,
  rho=0.290331, rho=0.477664, rho=0.599564, rho=0.893327}
```



```
{Trial 2/3 r=0.3, rho=0.158854, rho=0.22687,
  rho=0.290331, rho=0.477664, rho=0.599564, rho=0.893327}
```

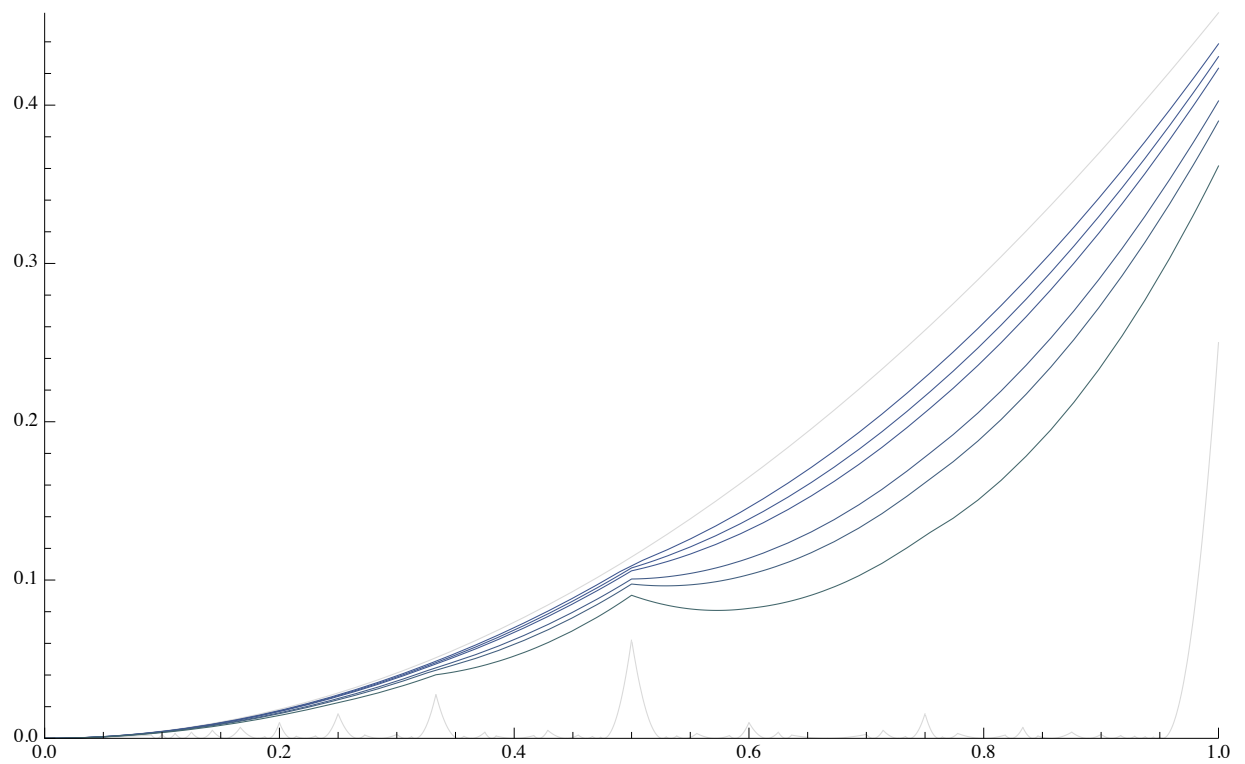


{Trial 2/3 r=1, rho=0.0796426, rho=0.126296}

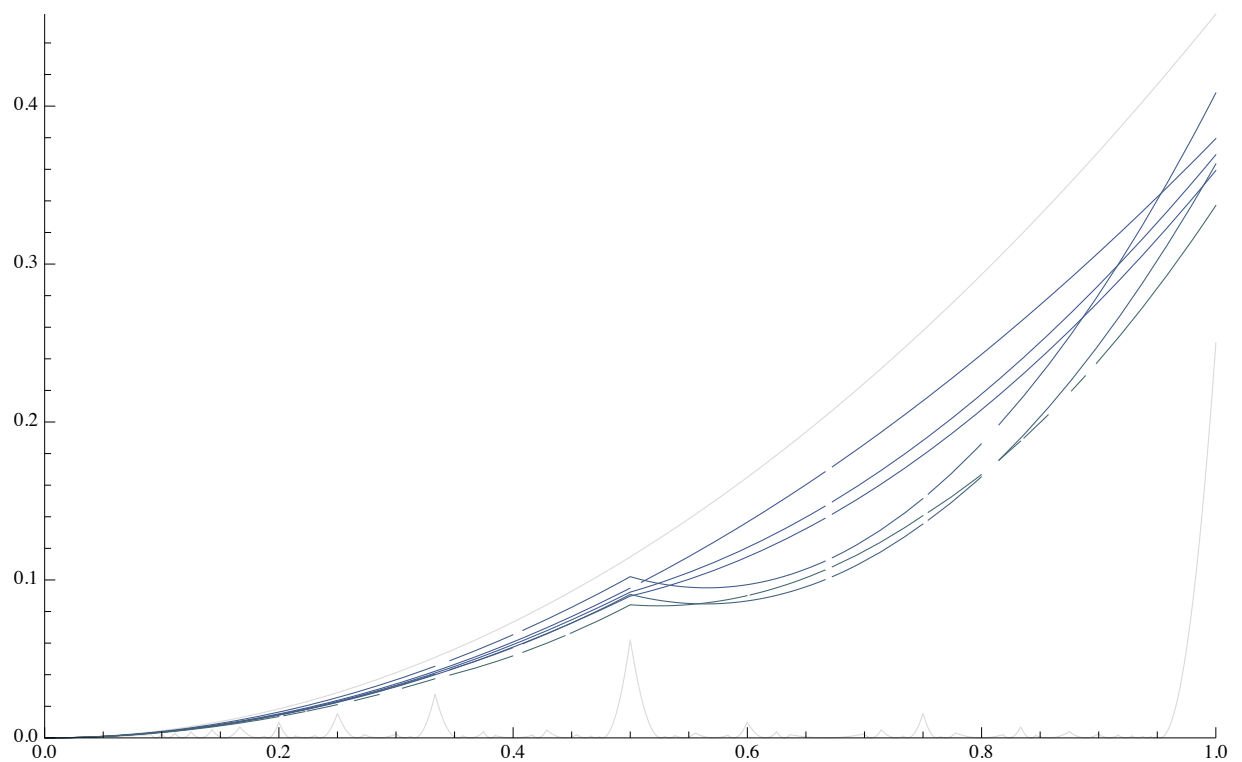


{Phi\* 2/3 r=1, rho=0.108509,  
rho=0.321778, rho=0.539109, rho=1.02154, rho=1.5722}

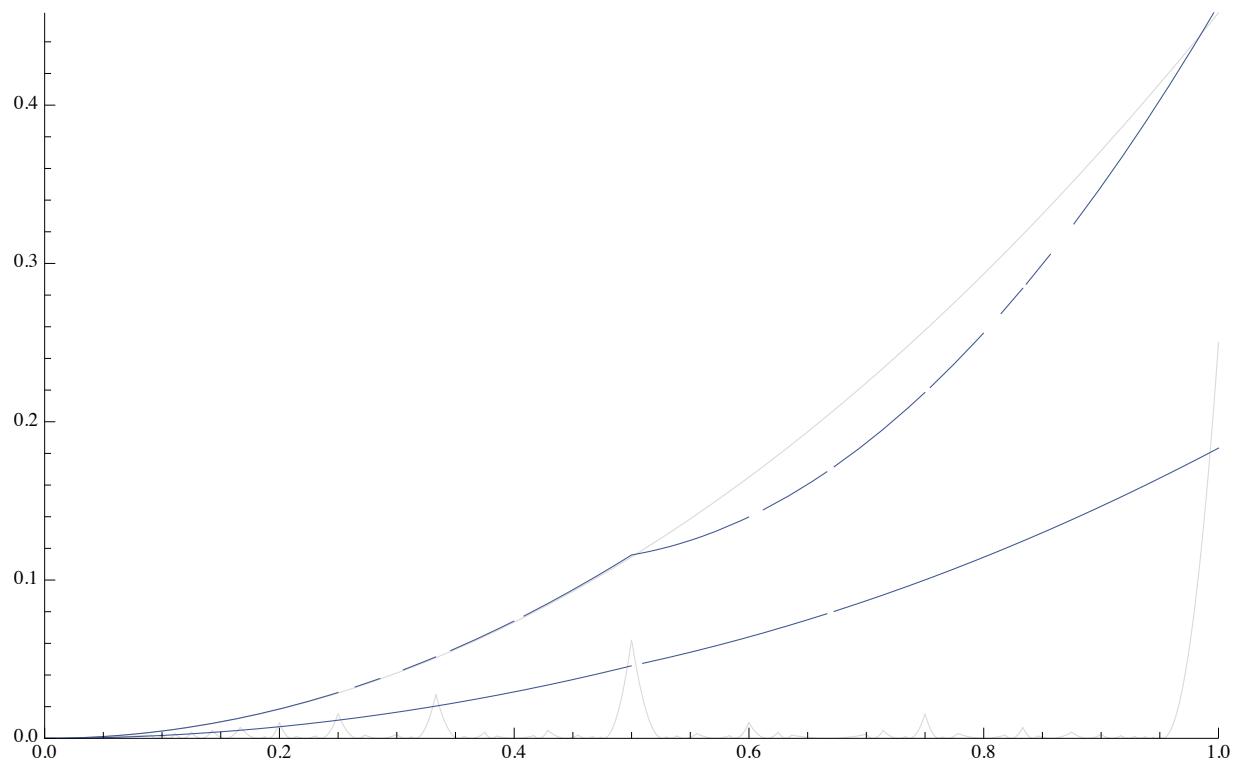
(\* bounds in terms of E\_alpha: \*)



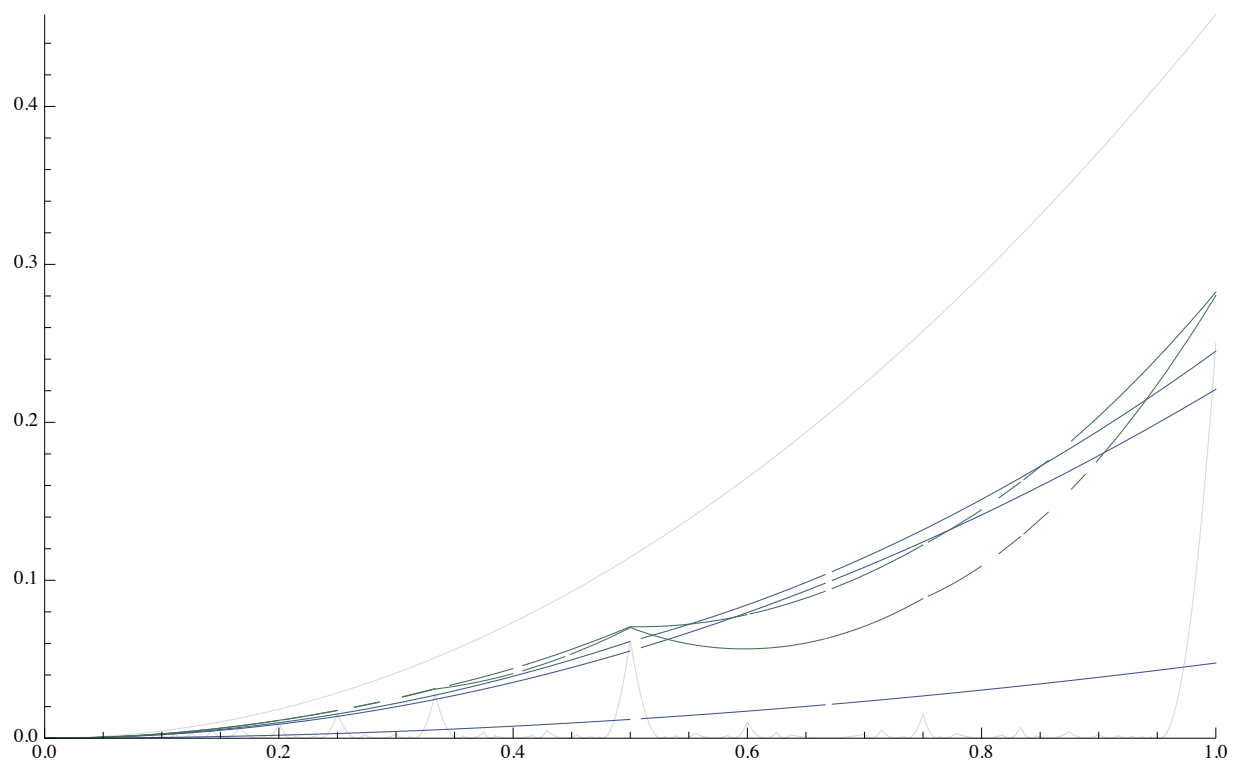
```
{Bose, rho=0.158854, rho=0.22687,
  rho=0.290331, rho=0.477664, rho=0.599564, rho=0.893327}
```



```
{Trial 2/3 r=0.3, rho=0.158854, rho=0.22687,
  rho=0.290331, rho=0.477664, rho=0.599564, rho=0.893327}
```



{Trial 2/3 r=1, rho=0.0796426, rho=0.126296}



{Phi\* 2/3 r=1, rho=0.108509,  
rho=0.321778, rho=0.539109, rho=1.02154, rho=1.5722}