

q -deformed quadrature operator and optical tomogram

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Abstract

In this paper, we define the homodyne q -deformed quadrature operator and find its eigenstates in terms of the deformed Fock states. We find the quadrature representation of q -deformed Fock states in the process. Furthermore, we calculate the explicit analytical expression for the optical tomogram of the q -deformed coherent states.

Keywords: Quantum tomography, quadrature operator, deformed coherent states

1. Introduction

The general principle behind quantum tomography is that instead of extracting a particular property of a quantum state (e.g. quantum entanglement), it aims to extract all possible information about the state that are contained in the density operator. Quantum tomography characterizes the complete quantum state of a particle or particles through a series of measurements in different

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quantum systems described by identical density matrices, much like its classical counterpart, which aims at reconstructing three-dimensional images via a series of two-dimensional projections along various directions. In optical phase space, the position and momentum of a quantum particle are determined by the quadratures. By measuring one of the quadratures of a large number of identical quantum states, one obtains a probability density corresponding to that particular quadrature, which characterizes the particle's quantum state. Thus, the quantum tomogram is defined as the probability that the system is in the eigenstate of the quadrature operator[1].

Quantum tomography is often used for analyzing optical signals, including measuring the signal gain and loss of optical devices [2], as well as in quantum computing and quantum information theory to reliably determine the actual states of the qubits [3]. As for instance, one can imagine a situation in which a person Bob prepares some quantum states and then sends the states to Alice to look at. Not being confident with Bob's description of the states, Alice may wish to do quantum tomography to classify the states herself. Balanced homodyne detection provides an experimental technique to study the quantum tomogram [3, 4], which is a probability distribution of homodyne quadrature depending on an extra parameter of local oscillator phase θ . When θ is varied over a whole cycle, it becomes the tomogram and, thus, tomogram contains complete information about the system. Quasi-probabilistic distributions describing the state of the system can be reconstructed from the tomogram via transformations like inverse Radon transformations[5]. In [6], the authors deal with the tomography of photon-added coherent states, even and odd coherent states, thermal states etc. The tomogram of coherent states as well as the evolution of tomogram of a state in a nonlinear medium was studied in [7], which essentially demonstrated the signatures of revivals, fractional revivals and decoherence effects (both amplitude decay and phase damping) in the tomogram. Recently, the signatures of entanglement was observed theoretically in the optical tomogram of the quantum state without reconstructing the density matrix of the system [8]. The squeezed state of light [9] and time evolution of quantum states [10] were exper-

imentally observed by measuring the quadrature amplitude distributions using balanced homodyne detection technique. A detailed discussion on the formulation of quantum mechanics using tomographic probabilities has been reported in [11, 12].

On the other hand, q -deformed oscillator algebras have been very famous in various subjects during last few decades, which were introduced through a series of articles [13, 14, 15, 16]. The q -deformed oscillator algebras are specific cases of f -deformed algebras discussed in [17]. There are mainly two kinds of deformed algebras, namely, maths type [18, 19, 20] and physics type [13, 14, 21]. Algebras of both types have been utilized to construct q -deformed bosons having applications in many different contexts, in particular, in the construction of coherent states [13, 22, 20], cat states [23, 24], photon-added coherent states [25, 26], atom laser [27], nonideal laser [28], etc. Besides, they are frequently used on the study of quantum gravity [29], string theory [30], non-Hermitian Hamiltonian systems [31, 20, 24], etc. The principal motivation of the present article is to study a method of quantum tomography for q -deformed coherent states by considering the maths type deformed canonical variables studied in [31, 20]. We also introduce the q -deformed homodyne quadrature related to the above mentioned deformed algebra, which is one of the principal requirements for the study of quantum tomography.

Our paper is organized as follows: In Sec. 2, we define the q -deformed homodyne quadrature operator. The eigenstates of the deformed quadrature have been found analytically in Sec. 3. In the process, we also find the quadrature representation of the deformed Fock states. In Sec. 4, we provide a short review of the optical tomography followed by the tomography of q -deformed coherent states. Finally, our conclusions are stated in Sec. 5.

2. q -deformed quadrature operator

Let us commence with a brief discussion of a q -deformed oscillator algebra introduced in [19, 31, 20]

$$AA^\dagger - q^2 A^\dagger A = 1, \quad |q| < 1, \quad (1)$$

which is often known as the math type q -deformation in the literature. As obviously, in the limit $q \rightarrow 1$, the q -deformed algebra (1) reduces to the standard canonical commutation relation $[a, a^\dagger] = 1$. The deformed algebra has been used before in describing plenty of physical phenomena [20, 24, 26]. Moreover, a concrete Hermitian representation of the corresponding algebra was derived in [20] by utilizing the Rogers-Szëgo polynomial [21] with the operators $A, A^\dagger$ being bounded on the region of unit circle. The deformed algebra given in Eq. (1) can be defined on the q -deformed Fock space forming a complete orthonormal basis provided that there exists a deformed number operator $[n]$ of the form

$$[n] = \frac{1 - q^{2n}}{1 - q^2}, \quad (2)$$

such that the action of the annihilation and creation operators on the Fock states $|n\rangle_q$ are given by

$$A |n\rangle_q = \sqrt{[n]} |n-1\rangle_q, \quad A |0\rangle_q = 0, \quad (3)$$

$$A^\dagger |n\rangle_q = \sqrt{[n+1]} |n+1\rangle_q. \quad (4)$$

In the limit $q \rightarrow 1$, the deformed Fock state $|n\rangle_q$ reduces to the Fock state, $|n\rangle$, which is an eigenstate of the operator $a^\dagger a$ with eigenvalue n . It is possible to define a set of canonical variables X, P in terms of the q -deformed oscillator algebra generators

$$X = \alpha(A^\dagger + A), \quad P = i\beta(A^\dagger - A), \quad (5)$$

with $\alpha = \beta = \frac{\sqrt{1+q^2}}{2}$ satisfying the deformed commutation relation [20]

$$[X, P] = i \left[1 + \frac{q^2 - 1}{q^2 + 1} (X^2 + P^2) \right]. \quad (6)$$

Let us now define the homodyne q -deformed quadrature operator

$$\hat{X}_\theta = \frac{\sqrt{1+q^2}}{2}(Ae^{-i\theta} + A^\dagger e^{i\theta}), \quad (7)$$

with θ being the phase of the local oscillator associated with the homodyne detection setup such that $0 \leq \theta \leq 2\pi$. Clearly at $\theta = 0$ and $\pi/2$, one obtains the dimensionless canonical observables X and P , respectively. The definition given in Eq. (7) is consistent with the homodyne detection theory [32, 33, 34]. In the limit $q \rightarrow 1$, the quadrature operator $\hat{X}_\theta$ reduces to the quadrature operator,

$$\hat{x}_\theta = \frac{1}{\sqrt{2}}(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}), \quad (8)$$

in the non-deformed algebra $[a, a^\dagger] = 1$.

3. Eigenstates of the q -deformed quadrature operator

This section contains the explicit calculation of the eigenstate of the q -deformed quadrature operator $\hat{X}_\theta$:

$$\hat{X}_\theta |X_\theta\rangle_q = X_\theta |X_\theta\rangle_q, \quad (9)$$

with X_θ being the eigenvalue. By using Eqs. (3), (4) and (7), we obtain

$$\begin{aligned} {}_q\langle n | \hat{X}_\theta |X_\theta\rangle_q &= X_\theta \bar{\Psi}_{n_q}(X_\theta) = \frac{\sqrt{1+q^2}}{2} {}_q\langle n | (Ae^{-i\theta} + A^\dagger e^{i\theta}) |X_\theta\rangle_q \quad (10) \\ &= \frac{\sqrt{1+q^2}}{2} \left(\sqrt{[n+1]} e^{-i\theta} {}_q\langle n+1 |X_\theta\rangle_q + \sqrt{[n]} e^{i\theta} {}_q\langle n-1 |X_\theta\rangle_q \right) \\ &= \frac{\sqrt{1+q^2}}{2} \left(\sqrt{[n+1]} e^{-i\theta} \bar{\Psi}_{n+1_q}(X_\theta) + \sqrt{[n]} e^{i\theta} \bar{\Psi}_{n-1_q}(X_\theta) \right), \quad (11) \end{aligned}$$

where we denote ${}_q\langle n |X_\theta\rangle_q$, ${}_q\langle n+1 |X_\theta\rangle_q$ and ${}_q\langle n-1 |X_\theta\rangle_q$ by $\bar{\Psi}_{n_q}(X_\theta)$, $\bar{\Psi}_{n+1_q}(X_\theta)$ and $\bar{\Psi}_{n-1_q}(X_\theta)$, respectively. The complex conjugate of $\bar{\Psi}_{n_q}(X_\theta)$ gives the quadrature representation of the deformed Fock state $|n\rangle_q$:

$$\Psi_{n_q}(X_\theta) = {}_q\langle X_\theta |n\rangle_q. \quad (12)$$

When $\theta = 0$, the wave function $\Psi_{n_q}(X_{\theta=0})$ corresponds to the position representation of the deformed Fock state. Henceforth, we use $\Psi_{n_q}(X_\theta)$ in the

calculation instead of $\bar{\Psi}_{n_q}(X_\theta)$ because the former is directly the quadrature representation of the deformed Fock state to obtain. After taking the complex conjugate of the Eq. (11) and rearranging the terms in it, we get a three term recurrence relation for $\Psi_{n_q}(X_\theta)$:

$$\Psi_{n+1_q}(X_\theta) = \frac{e^{-i\theta}}{\sqrt{[n+1]}} \left[\frac{2}{\sqrt{1+q^2}} X_\theta \Psi_{n_q}(X_\theta) - \sqrt{[n]} \Psi_{n-1_q}(X_\theta) e^{-i\theta} \right]. \quad (13)$$

First few terms of which are

$$\Psi_{1_q} = \frac{e^{-i\theta}}{\sqrt{[1]}} \frac{2X_\theta}{\sqrt{1+q^2}} \Psi_{0_q}(X_\theta) \quad (14)$$

$$\Psi_{2_q} = \frac{e^{-2i\theta}}{\sqrt{[2]}} \left[\frac{2X_\theta}{\sqrt{1+q^2}} \left(\frac{2X_\theta}{\sqrt{[1]}(1+q^2)} \right) - \sqrt{[1]} \right] \Psi_{0_q}(X_\theta) \quad (15)$$

$$\Psi_{3_q} = \frac{e^{-3i\theta}}{\sqrt{[3]}} \left[\frac{2X_\theta}{\sqrt{1+q^2}} \frac{1}{\sqrt{[2]}} \left(\frac{2X_\theta}{\sqrt{1+q^2}} \frac{2X_\theta}{\sqrt{[1]}(1+q^2)} - \sqrt{[1]} \right) - \sqrt{[2]} \frac{2X_\theta}{\sqrt{[1]}(1+q^2)} \right] \Psi_{0_q}(X_\theta). \quad (16)$$

Using Eqs. (13-16), we find the analytical expression for q -deformed Fock state $|n\rangle_q$ in the quadrature basis as

$$\Psi_{n_q}(X_\theta) = J_{n_q}(X_\theta) e^{-in\theta} \Psi_{0_q}(X_\theta). \quad (17)$$

Here, we introduce the new polynomial $J_{n_q}(X_\theta)$ which is defined by the following recurrence relation

$$J_{n+1_q}(X_\theta) = \frac{1}{\sqrt{[n+1]}} \left[\frac{2X_\theta}{\sqrt{1+q^2}} J_{n_q}(X_\theta) - \sqrt{[n]} J_{n-1_q}(X_\theta) \right], \quad (18)$$

with $J_{0_q}(X_\theta) = 1$ and $J_{1_q}(X_\theta) = 2X_\theta/\sqrt{[1](1+q^2)}$. In order to check the consistency, we take the limit $q \rightarrow 1$ and, indeed in the limiting condition the wavefunction $\Psi_{n_q}(X_\theta)$ given in Eq. (17) reduces to the quadrature representation of the Fock state $|n\rangle$:

$$\Psi_{n_{q \rightarrow 1}}(X_\theta \rightarrow x_\theta) = \frac{H_n(x_\theta)}{\pi^{1/4} 2^{n/2} \sqrt{n!}} e^{-in\theta} e^{-x_\theta^2/2}, \quad (19)$$

with $H_n(x_\theta)$ being the Hermite polynomial of order n and identifying

$$\Psi_{0_{q \rightarrow 1}}(X_\theta \rightarrow x_\theta) = \frac{e^{-x_\theta^2/2}}{\pi^{1/4}}. \quad (20)$$

Correspondingly, the recurrence relation in Eq. (18) merges with the recurrence relation of the Hermite polynomials

$$H_{n+1}(x_\theta) = 2x_\theta H_n(x_\theta) - 2nH_{n-1}(x_\theta). \quad (21)$$

Next, we calculate the eigenstate of the q -deformed quadrature operator. By using Eqs. (12) and (17), we derive the explicit expression for the eigenstates of q -deformed quadrature operator $\hat{X}_\theta$ as follows:

$$|X_\theta\rangle_q = \sum_{n=0}^{\infty} |n\rangle_q {}_q\langle n|X_\theta\rangle_q = \bar{\Psi}_{0_q}(X_\theta) \sum_{n=0}^{\infty} J_{n_q}(X_\theta) e^{in\theta} |n\rangle_q, \quad (22)$$

with $\Psi_{0_q}(X_\theta)$ being the ground state wavefunction in the deformed quadrature basis such that

$${}_q\langle X'_\theta|X_\theta\rangle_q = \delta(X_\theta - X'_\theta) = \bar{\Psi}_{0_q}(X_\theta) \Psi_{0_q}(X'_\theta) \sum_{n=0}^{\infty} J_{n_q}(X_\theta) J_{n_q}(X'_\theta). \quad (23)$$

When we take the limit $q \rightarrow 1$ in the expression given in Eq. (22), we get the eigenstates of the quadrature operator $\hat{x}_\theta$ [1]:

$$|x_\theta\rangle = \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{e^{in\theta}}{\sqrt{n!}} \frac{1}{2^{n/2}} H_n(x_\theta) e^{-x_\theta^2/2} |n\rangle. \quad (24)$$

In the following section, we use the eigenstates $|X_\theta\rangle_q$ obtained in Eq. (22) to calculate the optical tomogram of the q -deformed coherent state.

4. q -deformed optical tomography

In order to find the optical tomogram of the q -deformed coherent states, let us first briefly recall the notions of the optical tomography. For a state of the system represented by the density matrix $\hat{\rho}$, the optical tomogram $\omega(X_\theta, \theta)$ is given by the expression

$$\omega(X_\theta, \theta) = \langle X_\theta | \hat{\rho} | X_\theta \rangle, \quad (25)$$

with the normalization condition

$$\int \omega(X_\theta, \theta) dX_\theta = 1, \quad (26)$$

where $|X_\theta\rangle$ is the eigenstate of homodyne quadrature operator $\hat{X}_\theta$ with eigenvalue X_θ . Thus, the tomogram of a pure state represented by the density matrix $\hat{\rho} = |\Phi\rangle\langle\Phi|$ is given by the expression $\omega(X_\theta, \theta) = |\langle X_\theta|\Phi\rangle|^2$ [7, 6, 35]. Here, we are interested to compute the tomogram of the deformed Fock state and coherent states. Using the above equation for $\omega(X_\theta, \theta)$ and the eigenstates (22) of the q -deformed quadrature operator $\hat{X}_\theta$, we calculate the optical tomogram of the deformed Fock state $|n\rangle_q$ as

$$\omega_1(X_\theta, \theta) = |\langle X_\theta|n\rangle_q|^2 = \Psi_{0_q}^2(X_\theta) J_{n_q}^2(X_\theta). \quad (27)$$

In the limit $q \rightarrow 1$, the optical tomogram $\omega_1(X_\theta, \theta)$ goes to the optical tomogram of the Fock state $|n\rangle$:

$$\omega_1(x_\theta, \theta) = \frac{e^{-x_\theta^2} H_n^2(x_\theta)}{\sqrt{\pi} 2^n n!}. \quad (28)$$

Next, we calculate the tomogram of the q -deformed coherent states [19]

$$|\Phi\rangle_q = \frac{1}{\sqrt{E_q(|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} |n\rangle_q, \quad [n]! = \prod_{k=1}^n [k], \quad [0]! = 1, \quad (29)$$

where $\alpha \in \mathbb{C}$ and

$$E_q(|\alpha|^2) = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!}. \quad (30)$$

Given the eigenstates (22) of the q -deformed quadrature operator $\hat{X}_\theta$, we find the tomogram of the above q -deformed coherent states $|\Phi\rangle_q$ as follows

$$\omega_2(X_\theta, \theta) = \left| \sum_{n=0}^{\infty} \frac{\alpha^n J_{n_q}(X_\theta) e^{-in\theta} \Psi_{0_q}(X_\theta)}{\sqrt{E_q(|\alpha|^2)} \sqrt{[n]!}} \right|^2. \quad (31)$$

In the limit $q \rightarrow 1$, the above tomogram $\omega(X_\theta, \theta)$ become the tomogram of the Glauber coherent states $|\alpha\rangle$ [1]

$$\omega_2(x_\theta, \theta) = \frac{1}{\pi^{1/4}} \exp \left(-\frac{x_\theta^2}{2} - \frac{|\alpha|^2}{2} - \frac{\alpha^2 e^{-i2\theta}}{2} + \sqrt{2} \alpha x_\theta e^{-i\theta} \right), \quad (32)$$

which corroborate the expression given in Eq. (31) for the tomogram of q -deformed coherent state. Figures 1(a)-1(c) show the optical tomogram of q -deformed coherent state for $q = 0.9, 0.95$, and 0.999 with $|\alpha|^2 = 5$, respectively. Figure 1(d) displays the tomogram of coherent state $|\alpha\rangle$ with the same value of

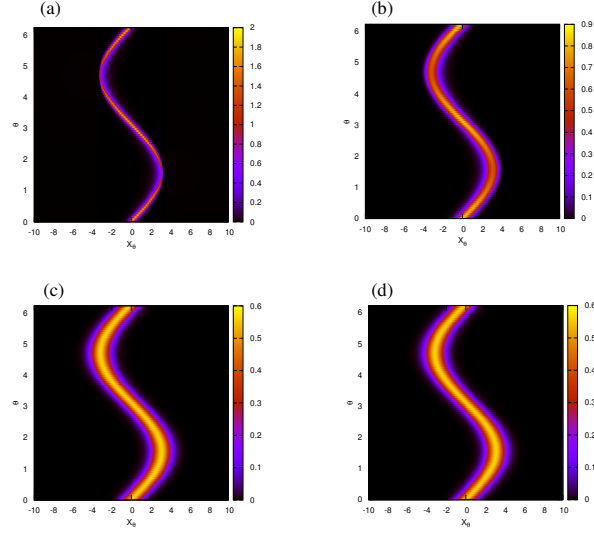


Figure 1: Optical tomogram of q -deformed coherent states for various values of q : (a) 0.9, (b) 0.95, (c) 0.999 with $|\alpha|^2 = 5$. The plot (d) shows the optical tomogram of Glauber coherent state $|\alpha\rangle$ with the same value of $|\alpha|^2$.

$|\alpha|^2$ used in Figs. 1(a)-1(c). It is evident from the Figs. 1(c) and 1(d) that the q -deformed coherent state goes to coherent state $|\alpha\rangle$ in the limit $q \rightarrow 1$.

5. Conclusions

We defined a q -deformed quadrature operator and found its eigenstates in terms of a new q -deformed polynomial. This q -deformed quadrature operator reduces to the standard one in the limit $q \rightarrow 1$. In this limit, the existing homodyne detection technique can be used to measure the quadrature operator eigenvalues but it has to be verified that the same technique can be used for measuring the q -deformed quadrature amplitudes for $q \neq 1$, which is not attempted in this study. The eigenstates of the quadrature operator obtained in this paper are very important because they enable us to find the quadrature representation of any q -deformed state. These eigenstates are also required to find theoretically the optical tomogram of the quantum states. We found

the quadrature representation of the deformed Fock states and confirmed it by checking the limiting case. These quadrature representations can be used to find easily the quasi-probability distributions of deformed quantum states. Finally, the q -deformation of the quantum tomography has been found by utilizing the expression for the eigenstates of the q -deformed quadrature operator.

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