

OPEN-CLOSED MODULI SPACES AND RELATED ALGEBRAIC STRUCTURES

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ABSTRACT. We set up a Batalin-Vilkovisky Quantum Master Equation (QME) for open-closed string theory and show that the corresponding moduli spaces give rise to a solution, a generating function for their fundamental chains. The equation encodes the topological structure of the compactification of the moduli space of bordered Riemann surfaces. The moduli spaces of bordered J -holomorphic curves are expected to satisfy the same equation, and from this viewpoint, our paper treats the case of the target space equal to a point. We also introduce the notion of a symmetric Open-Closed Topological Conformal Field Theory (OC TCFT) and study the L_∞ and A_∞ algebraic structures associated to it.

CONTENTS

1. Introduction	1
1.1. Convention on Chains	2
2. The Open-Closed Moduli Space	3
3. Orientation	5
4. BV Structure for the Open-Closed Moduli Spaces	6
5. Quantum Master Equation	10
6. An Algebraic Counterpart	13
7. Symmetric Open-Closed TCFTs	15
8. Algebraic Structures	16
8.1. The L_∞ structure coming from spheres	16
8.2. The cyclic A_∞ structure coming from disks	17
References	18

1. INTRODUCTION

The string-field theoretic formulation of open-closed string theory was developed by Zwiebach in [Zwi98]. In his seminal work a suitable BV algebra was introduced and a solution to the QME was obtained from so-called string vertices. String vertices are cycles in the infinite dimensional moduli spaces of nonsingular bordered Riemann surfaces with a choice of parameterization of the boundary. These cycles must have a certain topological type, basically, the one governed by the QME, and their construction is far from being complete. An alternative approach to this theory involves the use of certain real compactifications of moduli spaces of Riemann surfaces with boundary. This was achieved by Costello in [Cos05] for the closed case using the compactifications introduced in [KSV95]. We obtain an analogous result for the open-closed case in this paper using the open-closed moduli spaces of [Liu02]. It is interesting to note the existence of an interaction between closed and open strings that does not seem to have appeared before in the literature.

Date: June 5, 2008.

Partially supported by a University of Minnesota Doctoral Dissertation Fellowship.

Such interaction arises from the intrinsic nature of degenerations of surfaces with boundary and is expressed in our work by the component Δ_{co} of the BV operator Δ . In the work [Zwi98] of Zwiebach, this interaction was present implicitly, via the antibracket with the unstable moduli space of disks with one interior puncture.

As pointed out by Sullivan in [Sul05], his QME satisfied by the Gromov-Witten potential in the purely closed case is expected to generalize to an open-closed version of his sigma model. Fukaya has a solution for the punctured disk case in [Fuk06]. The goal of our work is to set out foundations for the study of the full-blown, arbitrary genus, arbitrary number of closed and open strings Gromov-Witten theory.

Another accomplishment of this paper is a new treatment of the algebraic counterpart (the state space) of OC TCFT (Section 6), a new notion of a symmetric OC TCFT, and a description of algebraic structures associated with an OC TCFT in Section 8.

Acknowledgments. We are very grateful to Kevin Costello, Anton Kapustin, Andrey Lazarev, Melissa Liu, Andrei Losev, Albert Schwarz, and Scott Wilson for helpful discussions. The third author also thanks IHÉS for hospitality during the final stage of writing the paper.

1.1. Convention on Chains. Throughout the paper we will use a notion of geometric chains (P, f) based on continuous maps $f : P \rightarrow X$ from oriented smooth orbifolds P with corners to a given topological space X . The reason for the use of this notion is technical: we need the unit circle S^1 to have a canonical fundamental cycle and the moduli spaces which we consider to have canonical fundamental chains, irrespective of the choice of a triangulation. On the other hand, we believe it is generally better to work at the more basic level of chains rather than that of homology. Similar, though different theories have been used by Gromov [Gro83], Sen and Zwiebach [Zwi93, SZ96, Zwi98], Fukaya, Oh, Ohta, and Ono [Fuk96, FOOO00], Jakob [Jak00], Chataur [Cha05], and Sullivan [Sul05]. Our notion of chains leads to a version of oriented bordism theory via passing to homology. If we impose extra equivalence relations, such as some kind of a suspension isomorphism, as in [Jak00], or work with piecewise smooth geometric chains and treat them as currents, in the spirit of [FOOO00], we may obtain a complex whose homology is isomorphic to the ordinary real homology of X .

Given a topological space X , a (*geometric*) *chain* is a formal linear combination over $\mathbb{Q}$ of continuous maps

$$f : P \rightarrow X,$$

where P is a compact connected oriented (smooth) orbifold with corners, modulo the equivalence relation induced by isomorphisms between the source orbifolds P . Here, an *orientation* on an orbifold with corners is a trivialization of the determinant of its tangent bundle. Geometric chains form a graded $\mathbb{Q}$ -vector space $C_{\bullet}^{\text{geom}}(X)$, graded by the dimension of P . The boundary of a chain is given by $(\partial P, f|_{\partial P})$, where ∂P is the sum of codimension one faces of P with the induced orientation (Locally, in positively oriented coordinates near ∂P , the manifold P is given by the equation “the last coordinate is nonnegative.”) and $f|_{\partial P}$ is the restriction of f to the boundary ∂P of P . Since the standard simplex has the structure of a compact oriented manifold with corners, the singular chain complex of X admits a natural morphism to the geometric chain complex $C_{\bullet}^{\text{geom}}(X)$. This morphism is not a homotopy equivalence, in general, even when X is a point, as there are closed manifolds P giving nontrivial classes of oriented cobordisms. However, any geometric chain (P, f) with a boundary constraint $f(\partial P) \subset A$ for some $A \subset X$ produces a relative homology class $f_*[P] \in H_{\bullet}(X, A; \mathbb{Q})$, where $[P] \in H_{\bullet}(P, \partial P; \mathbb{Q})$ is the relative fundamental class (sometimes called the “fundamental chain”) of P .

We will also need to consider geometric chains with local coefficients. If $\mathcal{F}$ is a locally constant sheaf of $\mathbb{Q}$ -vector spaces on X , a *geometric chain with coefficients*

in $\mathcal{F}$ will be a (finite) formal sum $c = \sum_i (P_i, f_i, c_i)$, where f_i 's are continuous maps from compact connected oriented orbifold P_i 's with corners to X and c_i 's are global sections: $c_i \in \Gamma(P_i; f_i^* \mathcal{F})$. The differential is defined as $dc := \sum_i (\partial P_i, f_i|_{\partial P_i}, c_i|_{\partial P_i})$. We will use $C_{\bullet}^{\text{geom}}(X; \mathcal{F})$ to denote this complex. Since we will only consider geometric chains, we will take the liberty to call them chains.

If M is a compact connected oriented orbifold with corners, then its *fundamental chain* $[M]$ is by definition the identity map $\text{id} : M \rightarrow M$, understood as a geometric chain $(M, \text{id}) \in C_d^{\text{geom}}(M; \mathbb{Q}) = C_0^{\text{geom}}(M; \mathbb{Q}[d])$, where $d = \dim M$ and $\mathbb{Q}[d]$ is the constant sheaf $\mathbb{Q}$ shifted by d in degree, regarded as a graded local system concentrated in degree $-d$. If M is not necessarily oriented and $p : M^* \rightarrow M$ is the orientation cover, then we define the *fundamental chain* $[M] \in C_0^{\text{geom}}(M; \mathbb{Q}^\epsilon)$ of M to be $(M^*, p, \frac{\text{or}}{2})$, where $\mathbb{Q}^\epsilon = \mathbb{Q} \times_{\mu_2} M^*[d]$ is the *orientation local system* (in particular, a locally constant sheaf of rational graded vector spaces of rank one, concentrated in degree $-d$) on M , with M^* thought of as a principle bundle over the multiplicative group $\mu_2 = \{\pm 1\}$ of changes of orientation and $\text{or} \in \Gamma(M^*; p^* \mathbb{Q}^\epsilon)$ being the canonical orientation on M^* . If $M = M'/G$, where M' is an oriented compact connected orbifold with corners and G a finite group acting on M , then the fundamental chain of M may be obtained from the natural projection $\pi : M' \rightarrow M$ as $(M', \pi, \frac{\text{or}}{|G|}) \in C_0^{\text{geom}}(M; \mathbb{Q}^\epsilon)$, where $\mathbb{Q}^\epsilon$ is the orientation local system of M . Note that a *geometric chain with coefficients in the orientation local system* on an orbifold M may be understood as a linear combination of geometric chains $f : P \rightarrow M$ with a (continuous) choice of *local orientation on M along P* .

2. THE OPEN-CLOSED MODULI SPACE

In this section, we will describe a certain moduli space $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$, which will be used to set up and solve the Quantum Master Equation (QME).

That space $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ will be closely related to the moduli space $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$, introduced by Melissa Liu [Liu02], of stable bordered Riemann surfaces of type (g, b) with $(n, \vec{m})$ punctures (marked points). The space $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ generalizes the *Deligne-Mumford moduli space* $\overline{\mathcal{M}}_{g,n}$, parameterizing isomorphism classes of stable algebraic curves of genus g with n punctures, to the open-closed case, i.e., $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ parameterizes isomorphism classes of stable bordered Riemann surfaces with b boundary components, n punctures in the interior, and m_i punctures on the i th boundary component, if $\vec{m} = (m_1, \dots, m_b)$, see more details in the next paragraph. All the boundary components are considered labeled by numbers 1 through b , and all the punctures must be distinct and labeled 1 through n for the interior punctures and 1 through m_i for the punctures on the i th boundary component, $i = 1, \dots, b$. We require that labels on the boundary punctures must be placed in a way compatible with the induced orientation on the boundary, i.e., the cyclic, $\text{mod } m_i$ order of punctures on the i th boundary component must increase when moving along the boundary *counterclockwise*, that is to say, keeping the surface on the right-hand side. In the ideal world conveniently provided by string theory, interior and boundary punctures allegedly correspond to evolving closed and open strings, respectively, whose world sheet is the Riemann surface.

A *bordered Riemann surface* here means a complex curve with real boundary, i.e., a compact, connected, Hausdorff topological space, locally modeled on the upper half-plane $H = \{z \in \mathbb{C} \mid \text{Im} z \geq 0\}$ using analytic maps. A *prestable bordered Riemann surface* is a bordered Riemann surface with at most a finite number of *singularities of nodal type* at points other than the punctures. The *allowed types of nodes* are denoted X, E, and H, where X means an interior node (locally isomorphic to a neighborhood of 0 on $\{xy = 0\} \subset \mathbb{C}^2$), E a boundary node, when the whole boundary component is shrunk to a point (locally modeled on a neighborhood of

0 on $\{x^2 + y^2 = 0 \subset \mathbb{C}^2\}/\sigma$, where $\sigma(x, y) = (\bar{x}, \bar{y})$ is the complex conjugation), and H a boundary node at which a boundary component intersects itself or another boundary component (locally modeled on a neighborhood of 0 on $\{x^2 - y^2 = 0 \subset \mathbb{C}^2\}/\sigma$).

A prestable bordered Riemann surface is *stable*, if its automorphism group is discrete. Here an automorphism must map the boundaries to the boundaries and the punctures to the punctures, respecting the labels. The stability condition is equivalent to the condition that the Euler characteristic (in a certain generalized sense, see below) of each component of the surface obtained by removing all the punctures is negative. Note that the *Euler characteristic* is by definition one half of the Euler characteristic of the double. Thus, for a nondegenerate bordered surface Σ , its Euler characteristic is given by

$$(1) \quad \chi(\Sigma) = 2 - 2g - b - n - m/2.$$

The stability condition thereby excludes a finite number of types $(g, b; n, \vec{m})$, namely $g = b = 0$ with $n \leq 2$; $g = 1, b = 0$ with $n = 0$; $g = 0, b = 1$ with $n \leq 1$, $m = 0$ or $n = 0, m \leq 2$; and $g = 0, b = 2$ with $m = n = 0$. The spaces $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ have been thoroughly studied by M. Liu in [Liu02]. They are compact, Hausdorff topological spaces with the structure of a smooth orbifold with corners of dimension $6g - 6 + 2n + 3b + m$, where $m = \sum_{i=1}^b m_i$ is the total number of boundary punctures.

Our space $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ is the moduli space of isomorphism classes of stable bordered Riemann surfaces of type (g, b) with $(n, \vec{m})$ punctures and certain extra data, namely, decorations by a real tangent direction, i.e., a ray, in the complex tensor product of the tangent spaces on each side of each interior node. The space $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ can be obtained by performing real blowups along the divisors of $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ corresponding to the interior nodes, as in [KSV95]. The dimension of $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ is the same as that of $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$: $\dim \underline{\mathcal{M}}_{g,b}^{n,\vec{m}} = \dim \overline{\mathcal{M}}_{g,b}^{n,\vec{m}} = 6g - 6 + 2n + 3b + m$.

We will concentrate on the moduli space

$$\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S} = \left(\prod_{\vec{m}: \sum m_i = m} \underline{\mathcal{M}}_{g,b}^{n,\vec{m}}/\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_b} \right) / \mathfrak{S}_b \times \mathfrak{S}_n$$

of stable bordered Riemann surfaces as above with *unlabeled* boundary components and punctures, that is, the quotient of the disjoint union $\underline{\mathcal{M}}_{g,b}^{n,m} = \prod_{\vec{m}: \sum m_i = m} \underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$

of moduli spaces with labeled boundaries and punctures by an appropriate action of the permutation group $\mathfrak{S} = \left(\prod_{\vec{m}: \sum m_i = m} \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_b} \right) \rtimes \mathfrak{S}_b \times \mathfrak{S}_n$. To set

up the QME, we will work with geometric chains of this moduli space with twisted coefficients, i.e., a one-dimensional local system $\mathbb{Q}^\epsilon$ obtained from a certain sign representation $\rho: \mathfrak{S} \rightarrow \text{End} L = \mathbb{Q}^*$ of the permutation group $\mathfrak{S}$ in a one-dimensional rational graded vector space L concentrated in degree $-d := -\dim \underline{\mathcal{M}}_{g,b}^{n,m}$. This representation is defined as follows: ρ is a trivial representation of $\mathfrak{S}_n$;

$$\rho(\zeta_i) = (-1)^{m_i-1}$$

for the generator $\zeta_i(p) = p + 1 \pmod{m_i}$ of the group $\mathbb{Z}_{m_i}$ of cyclic permutations of the punctures on the i th boundary component (where $m_i \geq 1$); and

$$\rho(\tau_{ij}) = (-1)^{(m_i-1)(m_j-1)}$$

for the transposition $\tau_{ij} \in \mathfrak{S}_b$ interchanging (the labels of) the i th and j th boundary components. Then $\mathbb{Q}^\epsilon$ is the locally constant sheaf $\underline{\mathcal{M}}_{g,b}^{n,m} \times_{\mathfrak{S}} L$ over $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$.

Note that an ordering of the boundary components and an ordering of the boundary punctures on each boundary component compatible with the cyclic ordering thereof on a given Riemann surface determines a section of the local system over the point in the moduli space $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$ corresponding to that Riemann surface. A change of these orderings will change this section by a sign factor as defined by the representation ρ .

3. ORIENTATION

Here we claim that the space $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ is an orientable orbifold with corners. Recall that an orientation on an orbifold with corners is the choice of a nowhere vanishing section of the orbifold determinant tangent (or, equivalently, cotangent) bundle up to a positive real function factor. An orientation on $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ may be defined similarly to an orientation on the Deligne-Mumford-Liu space $\overline{\mathcal{M}}_{g,b}^{n,\vec{m}}$, see [Liu02, Theorem 4.14], as follows.

The interior (i.e., pre-compactification) moduli space $\mathcal{M}_{g,b}^{n,(1,\dots,1)}$ is an orbifold with a natural complex structure, see [IS01], and thereby has a natural orientation. The reason is that this moduli space is isomorphic to the moduli space of complex algebraic curves with n labeled punctures with a *holomorphic* involution of a certain topological type (namely, with b invariant closed curves), which is naturally complex as a moduli space of complex objects. We will orient the spaces $\mathcal{M}_{g,b}^{n,\vec{m}}$ for $\vec{m} = (m_1, \dots, m_b)$, $m_i \geq 0$, inductively, by lifting or projecting orientation along the following fiber bundles:

$$(2) \quad \mathcal{M}_{g,b}^{n,(m_1, \dots, m_i+1, \dots, m_b)} \rightarrow \mathcal{M}_{g,b}^{n,(m_1, \dots, m_i, \dots, m_b)},$$

which have naturally oriented fibers, identified with the open arc of the i th boundary component B_i of the Riemann surface between the first and the m_i th punctures. As usual, we use the counterclockwise orientation on the boundary components under which the surface is always on the right-hand side. We say that the orientations on $\mathcal{M}_{g,b}^{n,(m_1, \dots, m_i+1, \dots, m_b)}$ and $\mathcal{M}_{g,b}^{n,(m_1, \dots, m_i, \dots, m_b)}$ agree, if the frame of tangent vectors to $\mathcal{M}_{g,b}^{n,(m_1, \dots, m_i+1, \dots, m_b)}$ obtained by appending a counterclockwise tangent vector to B_i to a positively oriented frame of tangent vectors to $\mathcal{M}_{g,b}^{n,(m_1, \dots, m_i, \dots, m_b)}$ is positively oriented. To endow all the spaces $\mathcal{M}_{g,b}^{n,\vec{m}}$, $m_i \geq 0$ for all i , with orientation, start with the orientation on $\mathcal{M}_{g,b}^{n,(1,\dots,1)}$ coming from the complex structure and order the total of $m \geq 0$ open punctures in a linear fashion: 1 through m_1 (in the order in which they are labeled) on the first boundary component, then $m_1 + 1$ through $m_1 + m_2$ on the second boundary component, and so on. Then use the fiber bundles (2), one by one, in the resulting order, to induce orientation on all spaces $\mathcal{M}_{g,b}^{n,\vec{m}}$ and thereby on its compactification $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ to an orbifold with corners.

The choice of orientation determines uniquely a fundamental chain $[\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}] \in C_{6g-6+2n+3b+m}^{\text{geom}}(\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}; \mathbb{Q})$ of the orbifold $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ with corners. We are rather interested in the unlabeled moduli space $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$, though.

Proposition 3.1. *The local system $\mathbb{Q}^\epsilon$ is the orientation sheaf for the orbifold $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$ with corners.*

Proof. The (real) orientation bundle of an orbifold M with corners is by definition given by the determinant tangent bundle of M , regarded as an orbifold real vector bundle. Here by the determinant tangent bundle we mean the top graded symmetric power $\det TM := S^d(TM[1])$ of the tangent bundle TM placed in degree -1 , where $d = \dim M$. The real orientation bundle is induced from a unique (up

to isomorphism) locally constant sheaf of graded $\mathbb{Q}$ -vector spaces by extension of scalars. Thus, it is enough to talk about the real orientation bundle in the proof.

The fact that $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ is orientable means the determinant bundle $\det T\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ of the tangent bundle of its interior part $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ is trivial. The orientation bundle over $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$ will then be $\det T(\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S})$. Looking at how orientation was defined on $\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$, observe that on the union $\coprod_{\vec{m}: \sum m_i = m} \det T\underline{\mathcal{M}}_{g,b}^{n,\vec{m}}$ of bundles, the permutation group $\mathfrak{S}_n$ of the set of interior punctures acts trivially, while transposition of the i th and the j th boundary components acts by $(-1)^{(m_i-1)(m_j-1)}$, and the basic cyclic permutation ζ_i acts by $(-1)^{m_i-1}$, if $m_i \geq 1$. $\square$

Thus, the *fundamental chain* $[\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}] \in C_0^{\text{geom}}(\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}; \mathbb{Q}^\epsilon)$ is well defined, see Section 1.1.

4. BV STRUCTURE FOR THE OPEN-CLOSED MODULI SPACES

A *differential graded (dg) Batalin-Vilkovisky (BV) algebra* structure on a complex V (i.e., a vector space with a differential $d : V \rightarrow V$, $d^2 = 0$, of degree 1) consists of a (graded) commutative *dot product* $V \otimes V \rightarrow V$, $a \otimes b \mapsto ab$, and a *BV operator* $\Delta : V \rightarrow V$ which is a second-order differential of degree 1. Both operations must be compatible with the differential d , which must be a (graded) derivation of the dot product (assuming annihilation of constants) and (graded) commute with the BV operator: $[d, \Delta] := d\Delta + \Delta d = 0$. Secondary to this basic structure is an *antibracket*

$$\{a, b\} := (-1)^{|a|} \Delta(ab) - (-1)^{|a|} \Delta(a)b - a\Delta b,$$

which turns out to be a Lie bracket of degree 1 on which both differentials d and Δ act as derivations. In fact, the derivation property of the BV operator with respect to the antibracket is equivalent to its second-order derivation property with respect to the dot product.

Consider the space

$$U = \bigoplus_{g,b,m,n} C_{\bullet}^{\text{geom}}(\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}; \mathbb{Q}^\epsilon)$$

of geometric chains. Since the coefficient system carries a degree shift by the dimension of $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$, the space U carries a natural grading by the negative codimension of the chain in $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$. We will take the opposite grading on U , i.e., the *grading by the codimension* of the chain in $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$. The differential d of geometric chains will then have degree 1 and make U into a complex of rational vector spaces.

The space V on which we will introduce a dg BV algebra structure will be defined as follows:

$$V := \bigoplus_{b,m,n} C_{\bullet}^{\text{geom}}(\underline{\mathcal{M}}_b^{n,m}/\mathfrak{S}; \mathbb{Q}^\epsilon),$$

where $\underline{\mathcal{M}}_b^{n,m}/\mathfrak{S}$ is the moduli space of stable bordered Riemann surfaces with b boundary components, n interior punctures, and m boundary punctures, just like the Riemann surfaces in $\underline{\mathcal{M}}_{g,b}^{n,m}/\mathfrak{S}$, but in general having multiple connected components of various genera. The grading on V is given by codimension in the corresponding connected component of $\underline{\mathcal{M}}_b^{n,m}/\mathfrak{S}$, and the dot product is induced by disjoint union of Riemann surfaces. We formally add a copy of the ground field $\mathbb{Q}$ to V , and the unit element $1 \in \mathbb{Q} \subset V$ might be interpreted as the fundamental chain of the one-point moduli space comprised by the empty Riemann surface. The algebra V is also isomorphic to the graded symmetric algebra $S(U)$. We will be using this observation when discussing the L_∞ structure on U later.

To define a dg BV algebra structure on V , it remains to define a BV operator $\Delta : V \rightarrow V$ satisfying required properties. It will consist of three components:

$$\Delta = \Delta_c + \Delta_o + \Delta_{\text{co}},$$

each of degree 1, square zero, and (graded) commuting with each other.

The operator Δ_c is induced on chains by twist-attaching at each pair of interior punctures. To achieve this, we define a bundle $ST\mathcal{M}_b^{n,m}/\mathfrak{S}$ over $\mathcal{M}_b^{n,m}/\mathfrak{S}$ of triples (Σ, P, r) , where $\Sigma \in \mathcal{M}_b^{n,m}/\mathfrak{S}$, P is a choice of an unordered pair of interior punctures on Σ , and r is one of the S^1 ways of attaching them (i.e., a real ray in the tensor product (over $\mathbb{C}$) of the tangent spaces of Σ at these two punctures). So the fiber is homeomorphic to $\coprod^{n(n-1)/2} S^1$. Then we have a diagram

$$(3) \quad \mathcal{M}_b^{n,m}/\mathfrak{S} \xleftarrow{\pi} ST\mathcal{M}_b^{n,m}/\mathfrak{S} \xrightarrow{a_c} \mathcal{M}_b^{n-2,m}/\mathfrak{S},$$

where π is the bundle projection map and a_c is obtained by attaching the two chosen punctures P on Σ and decorating the resulting node with the chosen real ray r . (One can view this diagram as a morphism realizing twist-attaching in the category of correspondences.) Then *twist-attaching* for chains is defined as the corresponding “push-pull,” giving us the “closed” part Δ_c of the BV operator:

$$\Delta_c := (a_c)_* \pi^! : C_{\bullet}^{\text{geom}}(\mathcal{M}_b^{n,m}/\mathfrak{S}; \mathbb{Q}^\epsilon) \rightarrow C_{\bullet+1}^{\text{geom}}(\mathcal{M}_b^{n-2,m}/\mathfrak{S}; \mathbb{Q}^\epsilon),$$

Here the pullback $\pi^!$ for geometric chains is simply the geometric pre-image. More precisely, to define the pullback of a geometric chain (P, f, c) , we take the pullback f^*ST of the fiber bundle ST along f and the chain $(f^*ST, \tilde{f})$, where f^*ST is the total space and $\tilde{f} : f^*ST \rightarrow ST$ is the pullback of f (if f^*ST is disconnected we regard $\tilde{f}$ as a sum of maps). To define what $\pi^!$ does to a section c of $\mathbb{Q}^\epsilon$, lift the diagram (3) to

$$\mathcal{M}_b^{n,m} \xleftarrow{\pi} ST\mathcal{M}_b^{n,m} \xrightarrow{\Delta_c} \mathcal{M}_b^{n-2,m},$$

defined before taking the quotient by the symmetric groups. Here, $ST\mathcal{M}_b^{n,m}$ is the bundle whose fiber over $\Sigma \in \mathcal{M}_b^{n,m}$ consists of all the $n(n-1)/2$ possible choices of unordered pairs $\{i, j\}$ of labeled punctures along with the S^1 ways of attaching them. The fiber of π (isomorphic to $n(n-1)/2$ copies of S^1) has a natural orientation coming from the counterclockwise orientation on the tensor product (over $\mathbb{C}$) of the tangent spaces at the punctures i and j . The orientation on the total space $ST\mathcal{M}_b^{n,m}$ is then defined (locally) as the orientation on the base $\mathcal{M}_b^{n,m}$ times the orientation of the fiber of π , and the orientation sheaf on $ST\mathcal{M}_b^{n,m}/\mathfrak{S}$ determines a local system. Now recall that a section c of $\mathbb{Q}^\epsilon$ is a rational number c' multiplied by the orientation or of the moduli space $\mathcal{M}_b^{n,m}$. Since an orientation of $\mathcal{M}_b^{n,m}$ determines an orientation on $ST\mathcal{M}_b^{n,m}$, as we have just described, we use that orientation, multiplied by the same number c' , to get a section of the local system on $ST\mathcal{M}_b^{n,m}/\mathfrak{S}$.

The operator Δ_o is induced on geometric chains by attaching at each pair of boundary punctures. To describe this procedure precisely, we form the bundle $B'\mathcal{M}_b^{n,m}/\mathfrak{S}$ where the fiber over a point $\Sigma \in \mathcal{M}_b^{n,m}/\mathfrak{S}$ consists of all possible choices of pairs of punctures on Σ which both lie on the same boundary component. Similarly, form the bundle $B''\mathcal{M}_b^{n,m}/\mathfrak{S}$ whose fibers are all possible pairs of punctures lying on different boundary components. Then we get the following diagrams:

$$\mathcal{M}_b^{n,m}/\mathfrak{S} \xleftarrow{\pi'} B'\mathcal{M}_b^{n,m}/\mathfrak{S} \xrightarrow{a'_o} \mathcal{M}_{b+1}^{n,m-2}/\mathfrak{S}$$

and

$$\mathcal{M}_b^{n,m}/\mathfrak{S} \xleftarrow{\pi''} B''\mathcal{M}_b^{n,m}/\mathfrak{S} \xrightarrow{a''_o} \mathcal{M}_{b-1}^{n,m-2}/\mathfrak{S},$$

where a'_o and a''_o are the obvious attaching maps.

We perform the push-pull again to obtain chain maps Δ'_o and Δ''_o . The bundles before quotienting, $B'\mathcal{M}_b^{n,m}$ and $B''\mathcal{M}_b^{n,m}$, are just direct products of $\mathcal{M}_b^{n,m}$ with finite discrete sets and are thus orientable, and we define the pullback (in this case also known as the “transfer homomorphism”) of geometric chains as in the

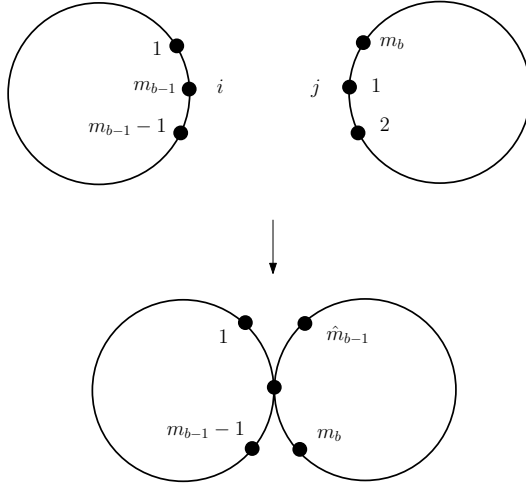


FIGURE 1. Attaching two punctures on different boundary components. The resulting new boundary component has $\hat{m}_{b-1} = m_{b-1} + m_b - 2$ punctures.

closed case. The pushforward of sections of the local system is defined in the next paragraph. We will then define the corresponding component of the BV operator as

$$\Delta_o := \Delta'_o + \Delta''_o.$$

Now let us define the pushforward of sections of the local system via a'_o and a''_o . Recall that an ordering of the boundary components and a cyclic ordering of the punctures on each boundary, for a given surface $\Sigma \in \underline{\mathcal{M}}_b^{n,m}/\mathfrak{S}$, gives a section of the local system over the point Σ . The same can be said for $(\Sigma, P) \in B'\underline{\mathcal{M}}_b^{n,m}/\mathfrak{S}$ or $B''\underline{\mathcal{M}}_b^{n,m}/\mathfrak{S}$, where P is the choice of a pair of boundary punctures on Σ . Thus to define the pushforward it suffices to explain how attaching acts on the labeling of boundaries and boundary punctures. If the punctures i and j in the pair P lie on different boundary components, see Figure 1, first change the ordering of the boundary components and boundary punctures in a way that the puncture i is the last puncture on the $b-1$ st boundary component and the puncture j is the first puncture on the b th boundary component. Then, after the punctures are attached to form a single boundary component, order the boundary components so that this new one goes last (i.e., becomes number $b-1$), with the same ordering of the old boundary components. Order the punctures on the new boundary component by placing the punctures coming from the old $b-1$ st boundary component first, preserving their order, followed by the punctures coming from the old b th boundary component, in their old order. Keep the old ordering of punctures on the boundary components not affected by attaching.

In the case when the punctures i and j happen to be on the same boundary component, see Figure 2, change the ordering of the boundary components so that this component goes last and the puncture j is the last puncture on that boundary component. Out of the two boundary components obtained by the pinching, order the one following j counterclockwise first, the other new boundary component next, preceded by the old boundary components in the old order. Order the punctures on the two new boundary components, declaring the puncture going after the double point on the first new boundary component to be first, followed by the other punctures in the counterclockwise order, and the puncture going after the double point on the second boundary component in the counterclockwise manner first on that

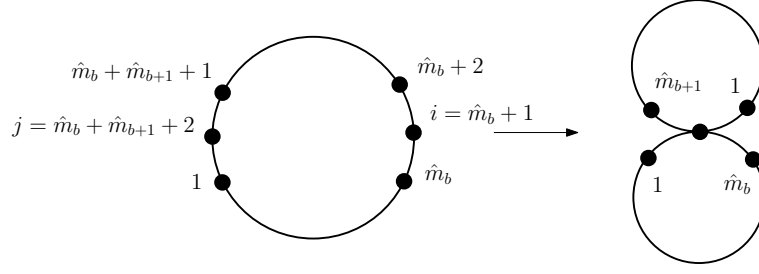


FIGURE 2. Attaching two punctures on the same boundary component with $m_b = \hat{m}_b + \hat{m}_{b+1} + 2$ punctures.

boundary component. Again, keep the old ordering of punctures on the boundary components not affected by attaching.

Finally, define Δ_{co} as follows. Let $B\mathcal{M}_b^{n,m}/\mathfrak{S}$ be the bundle over $\mathcal{M}_b^{n,m}/\mathfrak{S}$ whose fiber over Σ consists of all choices of an interior puncture on Σ . Then we get:

$$\mathcal{M}_b^{n,m}/\mathfrak{S} \xleftarrow{\pi} B\mathcal{M}_b^{n,m}/\mathfrak{S} \xrightarrow{a_{\text{co}}} \mathcal{M}_{b+1}^{n-1,m}/\mathfrak{S}$$

where a_{co} is the map induced by declaring the chosen interior puncture to be a degenerate boundary component with no “open” punctures on it. Then the chain map Δ_{co} is defined as the corresponding push-pull where the pushforward of the section of our local system is defined by putting this boundary component after the other boundary components in the ordering.

Theorem 4.1. *The operator $\Delta = \Delta_c + \Delta_o + \Delta_{\text{co}}$ is a graded second-order differential on the dg graded commutative algebra V and thereby defines the structure of a dg BV algebra on V .*

Proof. The fact that $\Delta(1) = 0$ follows tautologically from the definition of the three components of Δ . It will be enough to check the following identities:

$$\begin{aligned} [\Delta_c, d] &= [\Delta_o, d] = [\Delta_{\text{co}}, d] = 0, \\ \Delta_c^2 &= \Delta_o^2 = \Delta_{\text{co}}^2 = 0, \\ [\Delta_c, \Delta_o] &= [\Delta_c, \Delta_{\text{co}}] = [\Delta_o, \Delta_{\text{co}}] = 0, \\ \Delta_c \text{ and } \Delta_o &\text{ are second-order differential operators,} \\ \Delta_{\text{co}} &\text{ is a derivation.} \end{aligned}$$

The operator Δ_c and the differential d commute, because the copy of S^1 acquired by twist-attaching is a closed manifold. The commutation of d with the other components of Δ is more obvious.

The fact that Δ_c is a differential, i.e., $\Delta_c^2 = 0$, comes from our definition of orientation: we place the extra component S^1 last in orientation, so that each term in $\Delta_c^2 C$ will have two extra twists S^1 , as compared to the original chain C , and will be canceled by another term in $\Delta_c^2 C$, in which the two twists come in the opposite order. The property $\Delta_o^2 = 0$ is also true because of our choice of orientation. Each term in $\Delta_o^2 C$ is obtained by attaching one pair of boundary punctures together and then another pair. This term will be canceled by the term in which those two pairs of punctures are attached in the opposite order. It is a straightforward calculation to see that the signs coming from the our local system work out to cancel those pairs of terms in $\Delta_o^2 C$.

A more conceptual explanation of the same phenomenon may be done using the interpretation of the local system as the orientation sheaf of our orbifold. Note that the choice of orientation under attaching a pair of open punctures is performed in a way that we remove one factor in the top wedge power of the tangent bundle to the

moduli space $\underline{\mathcal{M}}_b^{n,m}$, leaving the other factors intact. The corresponding pairs of terms in $\Delta_o^2 C$ in which the same two pairs of punctures are attached in the opposite order will cancel each other, because the orders in which the corresponding factors are removed from the wedge product will be opposite. The same argument applies to showing $\Delta_{co}^2 = 0$ and, in fact, the graded commutation

$$[\Delta_c, \Delta_o] = [\Delta_c, \Delta_{co}] = [\Delta_o, \Delta_{co}] = 0.$$

The fact that Δ_{co} is a graded derivation of the dot product,

$$\Delta_{co}(a \cdot b) = \Delta_{co}(a) \cdot b + (-1)^{|a|} a \cdot \Delta_{co}(b),$$

is obvious: transformation of an interior puncture into a degenerate boundary component on a disjoint union of two Riemann surfaces happens on either one surface or the other.

The fact that Δ_c is a second-order derivation is equivalent to the following statement. Define a bracket

$$\{a, b\}_c := (-1)^{|a|} \Delta_c(ab) - (-1)^{|a|} \Delta_c(a)b - a\Delta_c b.$$

Then this bracket is a graded derivation in each (or equivalently, one) of its variables, that is

$$(4) \quad \{a, bc\}_c = \{a, b\}_c c + (-1)^{(|a|+1)|b|} b \{a, c\}_c.$$

What is clear from the definition, the geometric meaning of the bracket $\{a, b\}_c$ is $(-1)^{|a|}$ multiplied by the alternating sum of twist-attachments over all pairs of closed punctures for the chains a and b , respectively. Given three geometric chains a , b , and c , Equation (4) is obvious, as it just says that twist-attaching of closed punctures in a with those in the disjoint union of b and c breaks into twist-attaching with punctures in b and c and then taking the disjoint union. The signs come out right, because of our definition of orientation under twist-attaching and disjoint union.

The same argument applies to Δ_o . Consider a bracket

$$\{a, b\}_o := (-1)^{|a|} \Delta_o(ab) - (-1)^{|a|} \Delta_o(a)b - a\Delta_o b$$

and show it satisfies the derivation property

$$\{a, bc\}_o = \{a, b\}_o c + (-1)^{(|a|+1)|b|} b \{a, c\}_o.$$

□

Remark. Note that the part Δ'_o of Δ_o corresponding to attaching punctures lying on the same boundary component is actually a derivation, and therefore the open part of the antibracket comes only from Δ''_o corresponding to attaching punctures lying on different boundary components:

$$\{a, b\}'_o = 0, \quad \{a, b\}_o = \{a, b\}''_o.$$

5. QUANTUM MASTER EQUATION

In any dg BV algebra, it makes sense to set up the *Quantum Master Equation (QME)*:

$$(5) \quad dS + \Delta S + \frac{1}{2} \{S, S\} = 0$$

for $S \in V$ of degree zero, in which case all the terms will be in the same degree. If we allow the formal power series

$$e^S = 1 + S + S^2/2! + S^3/3! + \dots,$$

then, using the second-order differential operator property of Δ , the QME may be written equivalently as follows:

$$(6) \quad (d + \Delta)e^S = 0.$$

In our context, we will need an extra formal variable λ of degree 0, called the *string coupling constant*, and set up the QME in the space:

$$V[[\lambda]] := \left\{ \sum_{n=0}^{\infty} v_n \lambda^n \mid v_n \in V \right\},$$

which inherits a dg BV algebra structure from V by linearity in λ . More generally, we will consider the following modification of the QME in the dg BV algebra $V[[\lambda, \sqrt{\hbar}]]$, where $\hbar$ is another formal variable of degree 0, called the *Planck constant*:

$$(7) \quad dS + \hbar \Delta S + \frac{1}{2} \{S, S\} = 0,$$

or, equivalently, in $V[[\lambda, \sqrt{\hbar}, (\sqrt{\hbar})^{-1}]]$

$$(8) \quad (d + \hbar \Delta) e^{S/\hbar} = 0.$$

These last equations turn into the QMEs (5) and (6), respectively, under the specification $\hbar = 1$, so that if $S(\lambda, \hbar) \in V[[\lambda, \sqrt{\hbar}]]$ is a solution of (7) and $S(\lambda, 1)$ makes sense, then it will automatically satisfy (5). We will deal with the more general QME of the form (7) or (8) in this paper. However, we will modify the BV operator on $V[[\lambda, \sqrt{\hbar}]]$ in our case, when V is the space of geometric chains of the open-closed moduli space, in the following way:

$$\Delta := \Delta_c + \Delta_o + \sqrt{\hbar} \Delta_{co}.$$

Note that since Δ_{co} is a first-order differential operator, the new Δ will still be a BV operator on the algebra $V[[\lambda, \sqrt{\hbar}]]$ over the ring $\mathbb{Q}[[\lambda, \sqrt{\hbar}]]$, and the modification does not change the antibracket. Note also the change does not affect Δ after we make the evaluation $\hbar = 1$.

Take

$$(9) \quad S := \sum_{g,b,m,n} S_{g,b}^{n,m} \lambda^{-2\chi} \hbar^{p-\chi},$$

where $S_{g,b}^{n,m} := [\mathcal{M}_{g,b}^{n,m} / \mathfrak{S}] \in C_0^{\text{geom}}(\mathcal{M}_{g,b}^{n,m} / \mathfrak{S}; \mathbb{Q}^\epsilon)$ denotes the fundamental chain, χ is the Euler characteristic (1) of the bordered Riemann surface representing a point in $\mathcal{M}_{g,b}^{n,m}$, the summation runs over the indices corresponding to stable moduli spaces, i.e., $\chi = 2 - 2g - b - n - m/2 < 0$, and

$$p = 1 - (m + n)/2.$$

Theorem 5.1. *S is a solution of the QME, i.e.,*

$$(d + \hbar \Delta) e^{S/\hbar} = 0.$$

Proof. We will prove the equation in the equivalent form (7). This equation in fact encodes the structure of the boundary $\partial \mathcal{M}_{g,b}^{n,m} / \mathfrak{S}$ as the result of the operation on lower-dimensional moduli spaces induced by attaching open-closed Riemann surfaces at punctures and regarding interior punctures as degenerated boundary components. We will start with representing the boundary of a single moduli space $\mathcal{M}_{g,b}^{n,m} / \mathfrak{S}$ in this way and then pass to the generating series (9) to get the QME.

The boundary of the open-closed moduli space $\mathcal{M}_{g,b}^{n,m}$ is given by the locus of Riemann surfaces with at least one node. There are three types of nodes, X, E, and H, see Section 2. Type X and E nodes may also be subdivided into two types depending on whether the node separates two irreducible components of the Riemann surface. Likewise, nonseparating type E nodes split into two types, depending on whether they are obtained by pinching together two punctures on the same or different boundary components. This implies that the boundary of the moduli space $\mathcal{M}_{g,b}^{n,m} / \mathfrak{S}$ with unlabeled punctures is given exactly as follows, see also Figures 3 and 4:

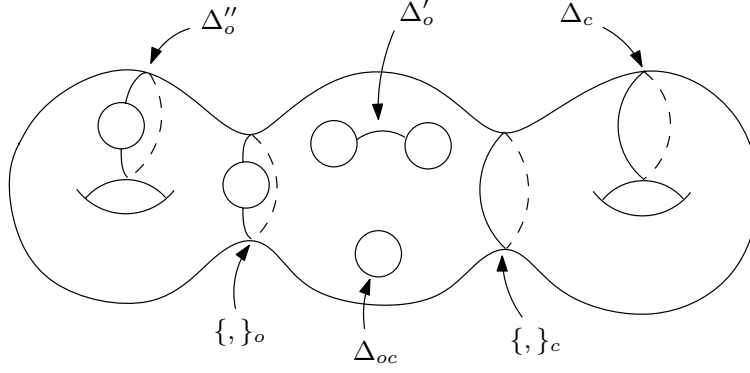


FIGURE 3. Different types of degeneration, resulting from contraction of curves on the surface.

$$\begin{aligned}
 d \begin{bmatrix} n, m \\ g, b \end{bmatrix} = & - \begin{bmatrix} n+2, m \\ g-1, b \end{bmatrix} \text{ (solid loop) } - \begin{bmatrix} n, m+2 \\ g, b-1 \end{bmatrix} \text{ (dashed loop) } - \begin{bmatrix} n, m+2 \\ g-1, b+1 \end{bmatrix} \text{ (dashed loop) } - \begin{bmatrix} n+1, m \\ g, b-1 \end{bmatrix} \text{ (circle on pole) } \\
 & - \frac{1}{2} \begin{bmatrix} n_1, m_1 \\ g_1, b_1 \end{bmatrix} \text{---} \begin{bmatrix} n_2, m_2 \\ g_2, b_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} n_1, m_1 \\ g_1, b_1 \end{bmatrix} \text{---} \begin{bmatrix} n_2, m_2 \\ g_2, b_2 \end{bmatrix} \\
 & \begin{matrix} g_1 + g_2 = g \\ n_1 + n_2 = n + 2 \\ m_1 + m_2 = m \\ b_1 + b_2 = b \end{matrix} \qquad \begin{matrix} g_1 + g_2 = g \\ n_1 + n_2 = n \\ m_1 + m_2 = m + 2 \\ b_1 + b_2 = b + 1 \end{matrix}
 \end{aligned}$$

FIGURE 4. The boundary of the moduli space: the solid connector means twist-attaching at interior punctures, the dotted connector means attaching at boundary punctures, the circle on a pole means turning an interior puncture into a degenerated boundary component.

$$\begin{aligned}
 (10) \quad dS_{g,b}^{n,m} = & -\Delta_c S_{g-1,b}^{n+2,m} - \Delta'_o S_{g,b-1}^{n,m+2} - \Delta''_o S_{g-1,b+1}^{n,m+2} - \Delta_{co} S_{g,b-1}^{n+1,m} \\
 & - \frac{1}{2} \sum_{\substack{g_1+g_2=g, \, b_1+b_2=b, \\ n_1+n_2=n+2, \, m_1+m_2=m}} \{S_{g_1,b_1}^{n_1,m_1}, S_{g_2,b_2}^{n_2,m_2}\}_c \\
 & - \frac{1}{2} \sum_{\substack{g_1+g_2=g, \, b_1+b_2=b+1, \\ n_1+n_2=n, \, m_1+m_2=m+2}} \{S_{g_1,b_1}^{n_1,m_1}, S_{g_2,b_2}^{n_2,m_2}\}_o.
 \end{aligned}$$

The -1 and $-1/2$ factors can be explained in the following way. The negative signs on the right-hand side are due to our choice of orientation. For example, the component of the boundary $\partial \underline{\mathcal{M}}_{g,b}^{n,m}$ corresponding to contracting a closed real curve in the interior of the Riemann surface contributes to $\Delta_c S_{g-1,b}^{n+2,m}$ on the right-hand side with a negative sign, because the corresponding angular Fenchel-Nielsen coordinate goes last in the orientation defined for Δ_c , while the boundary on the left-hand side is oriented in the way that the last coordinate, which is the circumference Fenchel-Nielsen coordinate, is nonnegative. Otherwise, all the remaining coordinates may be chosen with the same order on both sides. Since in the complex

structure in Fenchel-Nielsen coordinates, the perimeter coordinate goes just before the angular coordinate for the same real curve, the perimeter coordinate is actually the second-to-last in the orientation of $\underline{\mathcal{M}}_{g,b}^{n,m}$, whence the negative sign.

Let us look at another example, the $\Delta'_o S_{g,b-1}^{n,m+2}$ term in Equation (10). Again, locally on the moduli space, if the last two boundary components (if not the last two, we will change the ordering, changing orientation on the moduli space, if necessary), number $b-1$ and number b , get close together and touch, the moduli space will be given by the equation $r \geq r_0$, where r is the distance between the centers of the two circles. This r is the first real coordinate in the polar coordinate system (r, θ) , where θ is the angle of the direction from the center of the $b-1$ st circle to the center of the b th circle. This coordinate system has the same orientation as the one coming from the complex coordinate given by the position of the center of the b th circle near the $b-1$ st one on the Riemann surface. When the orderings of the punctures on the circles are chosen in a way that the circles touch at points between the first and the last punctures on each circle (we can always do that by changing the orderings and adjusting orientation), this ordering will be the same as the one coming from the operation Δ'_o of attaching two punctures on the $b-1$ st boundary component of the moduli space $\underline{\mathcal{M}}_{g,b-1}^{n,m+2}$, as described by Figure 2, except that the figure refers to Δ'_o on $\underline{\mathcal{M}}_{g,b}^{n,m}$. This all gives the same orientation as on the boundary of $\underline{\mathcal{M}}_{g,b}^{n,m}$, but since an oriented orbifold with positively oriented boundary must be given by the last coordinate being nonnegative and r was the second-to-last coordinate in a positively oriented coordinate system, we get a negative sign in front of $\Delta'_o S_{g,b-1}^{n,m+2}$ in Equation (10).

The factor of $1/2$ is due to the fact that each term $\{S_{g_1,b_1}^{n_1,m_1}, S_{g_2,b_2}^{n_2,m_2}\}_c$ in the sum for the closed part of the antibracket, is counted twice (in the “orbi” sense), the second time as $\{S_{g_2,b_2}^{n_2,m_2}, S_{g_1,b_1}^{n_1,m_1}\}_c$, even though this term is present only once in the boundary on the left-hand side.

Now as we have checked Equation (10), let us sum up these equations over different g, b, m , and n with weights $\lambda^{-2\chi} \hbar^{p-\chi}$ in a single equation for the generating series. Then use the fact that the Euler characteristic of an open-closed Riemann surface does not change under degeneration of the surface, or equivalently, the equations $\chi(S_{g,b}^{n,m}) = \chi(S_{g-1,b}^{n+2,m}) = \chi(S_{g,b-1}^{n,m+2}) = \chi(S_{g-1,b+1}^{n,m+2}) = \chi(S_{g,b-1}^{n+1,m}) = \chi(S_{g_1,b_1}^{n_1,m_1}) + \chi(S_{g_2,b_2}^{n_2,m_2})$, while $p(S_{g,b}^{n,m}) = p(S_{g-1,b}^{n+2,m}) + 1 = p(S_{g,b-1}^{n,m+2}) + 1 = p(S_{g-1,b+1}^{n,m+2}) + 1 = p(S_{g,b-1}^{n+1,m}) + 1/2 = p(S_{g_1,b_1}^{n_1,m_1}) + p(S_{g_2,b_2}^{n_2,m_2})$ to obtain (7). Here in both sequences of equations, the last one is considered under the assumption of the summation in (10). $\square$

Remark. According to [MMS06], solutions to the quantum master equation in geometry are in bijection with the set of wheeled representations of a certain wheeled PROP. It would be interesting to see which modular operad or, more generally, wheeled PROP would be responsible for the above dg BV algebra structure.

6. AN ALGEBRAIC COUNTERPART

First of all, consider a linear version of the BV structure of Section 4. Suppose we have a pair of complexes H_c and H_o of $\mathbb{C}$ -vector spaces, whose physical meaning is the state spaces, including ghosts, of the closed and the open string, respectively. Suppose these spaces are provided with symmetric bilinear forms

$$\begin{aligned} H_c \otimes H_c &\rightarrow \mathbb{C}[1], \\ a \otimes b &\mapsto (a, b) \in \mathbb{C}, \end{aligned}$$

for a, b in H_c ,

$$\begin{aligned} H_o[1] \otimes H_o[1] &\rightarrow \mathbb{C}, \\ a \otimes b &\mapsto (a, b)' \in \mathbb{C}, \end{aligned}$$

for a, b in H_o , and

$$\begin{aligned} H_o \otimes H_o &\rightarrow \mathbb{C}, \\ a \otimes b &\mapsto (a, b)'' \in \mathbb{C}, \end{aligned}$$

for a, b in H_o , which are assumed to be morphisms of complexes. Suppose also a morphism

$$\Delta_{co} : H_c \rightarrow \mathbb{C}$$

of complexes is given. Form the following space

$$A := S(H_c) \otimes S(C^\lambda(H_o)),$$

where S stands for graded symmetric algebra and C^λ denotes the *reduced cyclic complex* (considered without the standard differential b , with λ having nothing to do with the string coupling constant):

$$C^\lambda(H_o) := \bigoplus_{n=-1}^{\infty} H_o[1]^{\otimes(n+1)}[-1]/(1-t),$$

where the grading shifts result in placing $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ in degree $|a_0| + |a_1| + \cdots + |a_n| - n$, the operator t is the cyclic-permutation generator:

$$t(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := (-1)^{(|a_n|-1)(|a_0|+\cdots+|a_{n-1}|-n)} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

for any $a_0, a_1, \dots, a_n \in H_o$, and the quotient is taken with respect to the image of the operator $1 - t$, i.e., C^λ is the space of coinvariants of t .

The graded vector space A may be endowed with the following structure of a dg BV algebra. The differential d is just the internal differential, i.e., the one coming from the differentials on H_c and H_o . The BV operator Δ is the sum of three components:

$$\Delta := \Delta_c + \Delta_o + \Delta_{co},$$

which are defined as follows:

$$\Delta_c(a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \sum_{1 \leq i < j \leq n} (-1)^{\epsilon_{a_1} \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_n} (a_i, a_j)$$

for $a_1, a_2, \dots, a_n \in H_c$ with $(-1)^\epsilon$ being the sign coming from taking a_i and a_j over to the back of the tensor product; then Δ_c is extended by $S(C^\lambda(H_o))$ -linearity to the whole algebra A .

$$\Delta_o := \Delta'_o + \Delta''_o$$

with the following components:

$$\begin{aligned} &\Delta'_o(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &:= \sum_{0 \leq i < j \leq n} (-1)^{\epsilon_{[a_{j+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}]}} (a_i, a_j)' [a_{i+1} \otimes \cdots \otimes a_{j-1}] \\ &\hspace{15em} \in S^2(C^\lambda(H_o)) \end{aligned}$$

for a representative $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ of a cyclic chain in $C^\lambda(H_o)$ with $(-1)^\epsilon$ being the sign coming from permuting $a_0 \otimes \cdots \otimes a_n$ cyclically to $a_j \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1}$ and taking $a_j \in H_o[1]$ over to the place right after a_i . The operator Δ'_o is extended

to the symmetric algebra $S(C^\lambda(H_o))$ as a graded derivation and to the whole tensor product A by $S(H_c)$ -linearity.

$$\begin{aligned} \Delta_o''(A_1 \otimes A_2 \otimes \cdots \otimes A_n) \\ := \sum_{1 \leq i < j \leq n} (-1)^\epsilon A_1 \otimes \cdots \otimes \hat{A}_i \otimes \cdots \otimes \hat{A}_j \otimes \cdots \otimes A_n \otimes \{A_i, A_j\}'', \end{aligned}$$

where $A_1, \dots, A_n \in C^\lambda(H_o)$ and $(-1)^\epsilon$ is the sign coming from taking A_i and A_j to the right in $S(C^\lambda(H_o))$, and

$$\begin{aligned} (11) \quad & \{a_0 \otimes \cdots \otimes a_k, b_0 \otimes \cdots \otimes b_l\}'' \\ := & \sum_{\substack{0 \leq i \leq k \\ 0 \leq j \leq l}} (-1)^\epsilon a_{i+1} \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes (a_i, b_j)'' b_{j+1} \otimes \cdots \otimes b_l \otimes b_0 \otimes \cdots \otimes b_{j-1} \end{aligned}$$

for $[a_0 \otimes \cdots \otimes a_k]$ and $[b_0 \otimes \cdots \otimes b_l] \in C^\lambda(H_o)$ and $(-1)^\epsilon$ being the sign coming from permuting $a_0 \otimes \cdots \otimes a_k$ into $a_{i+1} \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i$ and $b_0 \otimes \cdots \otimes b_l$ into $b_j \otimes b_{j+1} \otimes \cdots \otimes b_l \otimes b_0 \otimes \cdots \otimes b_{j-1}$ in $C^\lambda(H_o)$. The operator Δ_o'' is extended to A by $S(H_c)$ -linearity.

Finally, let

$$\Delta_{co} : S(H_c) \otimes S(C^\lambda(H_o)) \rightarrow S(H_c) \otimes S(C^\lambda(H_o))[1]$$

be the extension of the map $\Delta_{co} : H_c \rightarrow \mathbb{C} \subset C^\lambda(H_o)[1] \subset S(C^\lambda(H_o))[1]$, as before, as an $S(C^\lambda(H_o))$ -linear derivation.

Proposition 6.1. *Together with the standard graded commutative multiplication, the operator Δ defines the structure of a dg BV algebra on the complex $A = S(H_c) \otimes S(C^\lambda(H_o))$. If we consider the formal power series $A[[\lambda, \sqrt{\hbar}]]$ and modify Δ to be $\Delta := \Delta_c + \Delta_o + \sqrt{\hbar} \Delta_{co}$, we will get the structure of a dg BV algebra on $A[[\lambda, \sqrt{\hbar}]]$ over $\mathbb{C}[[\lambda, \sqrt{\hbar}]]$.*

Remark. Note that the “closed-sector” component Δ_c is just another form of the standard odd Laplacian $\sum_i \partial^2 / \partial x^i \partial \xi^i$, see [Sch93], in the case when the form (a, b) defines the structure of an odd symplectic manifold on H_c^* with Darboux coordinates (x^i, ξ^i) . Also, the antibracket (11) on the reduced cyclic complex is the one coming from the Gerstenhaber bracket on the Hochschild cochain complex, while the antibracket induced by Δ_o' is zero.

7. SYMMETRIC OPEN-CLOSED TCFTs

Definition 7.1. A *symmetric OC TCFT* is a morphism of dg BV algebras $\phi : V[[\lambda, \sqrt{\hbar}]] \rightarrow A[[\lambda, \sqrt{\hbar}]]$ over $\mathbb{C}[[\lambda, \sqrt{\hbar}]]$, where $V[[\lambda, \sqrt{\hbar}]]$ is the one from Sections 4 and 5, based on chains in the OC moduli spaces, and $A[[\lambda, \sqrt{\hbar}]]$ is the algebraic one from Section 6, based on the state spaces H_c and H_o . The morphism must also respect some extra gradings: the component with n interior punctures and $m_1, \dots, m_b$ punctures located on b boundary components must map to the component of $S(H_c) \otimes S(C^\lambda(H_o))[[\lambda, \sqrt{\hbar}]]$ with the number of factors from H_c equal to n , the number of factors from $C^\lambda(H_o)$ equal to b , and the numbers of those from H_o in these factors equal to $m_1, \dots, m_b$, irrespective of the order.

Remark. Here the word “symmetric” refers to the fact that we symmetrize over all inputs and outputs (not mixing up open and closed strings, though). Mathematically, this means that we consider the moduli space with unlabeled interior and boundary punctures, as in Section 2. It is quite obvious how to “unsymmetrize” the above definition, which will then turn into a variation of the standard one, see [Cos07, Zwi98]. The idea of Definition 7.1 is that a map ϕ respecting the dot products behaves well under disjoint union of Riemann surfaces, while the condition of

respecting the BV operators means that the map ϕ behaves well under attaching Riemann surfaces at punctures, i.e., the correlators of the theory satisfy a *factorization axiom*, as well as ϕ behaves well with respect to turning interior punctures into degenerated boundary components, some sort of open-closed string interaction. The idea that an OC TCFT gives rise to a morphism from a BV algebra similar to V to another BV algebra is due to Zwiebach, see [Zwi98].

Thus, by definition, the dg BV algebra V constructed in Section 4 is the *universal symmetric OC TCFT*. In particular, if $S \in V[[\lambda, \sqrt{\hbar}]]$ is a solution to the QME, then its image $\phi(S) \in A[[\lambda, \sqrt{\hbar}]]$ is a solution to the QME as well. On the other hand, the dg BV algebra V is a particular case of the dg BV algebra of the sigma model, when the target is a one-point space.

It is widely expected, see [Cos05, FOOO00, Sul05], that the OC sigma model (A model or Gromov-Witten theory) produces a Gromov-Witten potential satisfying QME. We summarize this expectation in the language of symmetric OC TCFT as follows.

Conjecture 7.2. *Sigma model produces an example of a symmetric OC TCFT.*

8. ALGEBRAIC STRUCTURES

Suppose we have a symmetric OC TCFT $V[[\lambda, \sqrt{\hbar}]] \rightarrow A[[\lambda, \sqrt{\hbar}]]$, as in the previous section. We claim that this data implies certain algebraic structures on H_c and H_o , as well as on the space U generated by the geometric cycles in the connected OC moduli spaces.

8.1. The L_∞ structure coming from spheres. Let us consider the Riemann sphere with punctures and the corresponding part of the solution (9) to the QME:

$$S_c := S_{0,0}^{\bullet,0} := \sum_{n \geq 3} S_{0,0}^{n,0} \lambda^{2n-4} \hbar^{n/2-1}.$$

Since the image of operator Δ_c corresponds to strictly positive genus and the image of Δ_{co} to strictly positive number of boundary components, the QME rewrites in this case as

$$(12) \quad dS_c + \frac{1}{2}\{S_c, S_c\} = 0,$$

which is a *Maurer-Cartan equation*, also known as the *Classical Master Equation (CME)*. This equation implies that $d + \{S_c, -\}$ is a differential of degree one on the graded commutative algebra $V[[\lambda, \sqrt{\hbar}]]$.

If we have a symmetric OC TCFT ϕ , then, since ϕ preserves grading by b , the image $\phi(S_c)$ of S_c , whose b -grading is zero, must lie in $S(H_c)[[\lambda, \sqrt{\hbar}]]$ and the operator $d + \{\phi(S_c), -\}$ is a differential on $S(H_c)[[\lambda, \sqrt{\hbar}]]$. This translates into the structure of an L_∞ coalgebra on $H_c[1]$, according to the following definition.

Definition 8.1. The structure of an L_∞ coalgebra on a dg vector space $\mathfrak{g}$ is a (continuous) degree-one differential D on the completed graded symmetric algebra $\hat{S}(\mathfrak{g}[-1])$, the completion being taken with respect to the $(\mathfrak{g}[-1])$ -adic topology.

Remark. Since such a derivation is determined by its value on the subspace $\mathfrak{g}[-1]$ of generators of the symmetric algebra, this structure gives rise to a collection of linear maps $D_k : \mathfrak{g}[-1] \rightarrow \mathfrak{g}^{\otimes k}[-k]$, $k \geq 1$, interpreted as higher cobrackets, satisfying identities dual to those satisfied by brackets in an L_∞ algebra. These identities come from breaking the equation $D^2 = 0$ into components D_k . In the finite-dimensional case, $\dim \mathfrak{g} < \infty$, this structure is equivalent to the structure of an L_∞ algebra on the dual space $\mathfrak{g}^*$.

Proposition 8.2. *In a symmetric OC TCFT, the space $H_c[1]$ carries the structure of an L_∞ coalgebra.*

Proof. The degree-one differential $D(\lambda, \hbar) = d + \{\phi(S_c), -\}$ on $S(H_c)[[\lambda, \sqrt{\hbar}]]$ expands as follows:

$$D(\lambda, \hbar) = d + \sum_{n \geq 3} \{\phi(S_{0,0}^{n,0}), -\} \lambda^{2n-4} \hbar^{n/2-1} = \sum_{k \geq 1} D_k \lambda^{2(k-1)} \hbar^{(k-1)/2},$$

where $D_1 := d$ and $D_k := \{\phi(S_{0,0}^{k+1,0}), -\}$ for $k \geq 2$ are operators $D_k : H_c \rightarrow S^k(H_c)$. The fact that $D(\lambda, \hbar)$ is a differential on $S(H_c)[[\lambda, \sqrt{\hbar}]]$ implies that $D = D(1, 1)$ is a differential on the completed symmetric algebra $\hat{S}(H_c)$ and thereby defines the structure of an L_∞ coalgebra on $H_c[1]$. $\square$

Note also that the relative homology classes of the fundamental chains in $\underline{\mathcal{M}}_{0,0}^{k+1,0}$ generate the L_∞ operad, while their boundaries are generated by twist-attachments of lower-dimensional fundamental chains, producing the defining relations of the L_∞ operad, see [KSV95]. These classes are stable with respect to permutations of punctures, and their symmetrizations satisfy the CME (12), therefore, the L_∞ coalgebra structure on $H_c[1]$ coincides, up to duality, with the L_∞ algebra structure on H_c constructed in [Zwi93], see [KSV95].

8.2. The cyclic A_∞ structure coming from disks. Kontsevich introduced in [Kon94] the notion of a cyclic A_∞ algebra as an A_∞ algebra with an invariant inner product. This notion has different variations, called symplectic A_∞ algebras [Kon93, HL06], Calabi-Yau A_∞ algebras [Kon04, Cos07], and Sullivan-Wilson's homotopy open Frobenius algebras. The following definition is motivated by deformation theory of cyclic A_∞ algebras of [PS95].

Definition 8.3. A *cyclic A_∞ algebra* structure on a (dg) vector space H with a symmetric bilinear form $H \otimes H \rightarrow \mathbb{C}$ is a solution $M = \sum_{k \geq 2} m_{k+1} \lambda^{k-1}$ for $m_{k+1} \in H[1]^{\otimes(k+1)}[-1]/(1-t) \subset C^\lambda(H)$ to the CME

$$dM + \frac{1}{2}\{M, M\} = 0$$

in the reduced cyclic chain complex $C^\lambda(H)[[\lambda]]$ provided with the antibracket given by (11) and the internal differential d coming from that on H .

Before describing this structure in the presence of an OC TCFT, let us show that this definition is equivalent to the notion of an A_∞ algebra with an invariant inner product, under the assumption that the underlying graded vector space has finite-dimensional graded components and the inner product $(,) : H \otimes H \rightarrow \mathbb{C}$ is nondegenerate componentwise. A straightforward computation shows that the CME $dM + \frac{1}{2}\{M, M\} = 0$ is equivalent to the equation

$$(13) \quad d\hat{M} + \frac{1}{2}\{\hat{M}, \hat{M}\} = 0,$$

where $\hat{M} = \sum \hat{m}_k$ with $\hat{m}_k \in \text{Hom}(H^{\otimes k}, H)$ being the operator defined by

$$(14) \quad \hat{m}_k(v_1 \otimes \cdots \otimes v_k) := \sum_{i=0}^k (-1)^i m_{k+1}^{(i)}(m_{k+1}^{(i+1)}, v_1) \cdots (m_{k+1}^{(i+k)}, v_k),$$

where $m_{k+1} = m_{k+1}^{(0)} \otimes m_{k+1}^{(1)} \otimes \cdots \otimes m_{k+1}^{(k)} \in H^{\otimes(k+1)}[k]$ in the standard notation skipping the summation sign and $(-1)^i$ is the sign coming from permuting $m_{k+1}^{(0)} \otimes m_{k+1}^{(1)} \otimes \cdots \otimes m_{k+1}^{(k)} \otimes v_1 \otimes \cdots \otimes v_k$ into $m_{k+1}^{(i)} \otimes m_{k+1}^{(i+1)} \otimes v_1 \otimes \cdots \otimes m_{k+1}^{(i+k)} \otimes v_k$. The equation $d\hat{M} + \frac{1}{2}\{\hat{M}, \hat{M}\} = 0$ takes place in the Hochschild complex $\text{Hom}(H[1]^{\otimes \bullet}, H[1])$, with just the internal differential d and the antibracket being the Gerstenhaber bracket. If we think of that differential as an element $d \in \text{Hom}(H[1], H[1])$, then the CME (13) is equivalent to $\{d + \hat{M}, d + \hat{M}\} = 0$, known to be equivalent to the fact that the formal series $d + \hat{M}$ defines an A_∞ structure on H , as the Gerstenhaber

bracket on the Hochschild complex is the same as the commutator of Hochschild cochains identified with derivations of the tensor coalgebra on H , [Sta93]. Equation (14) makes it obvious that the higher products $\hat{m}_k$ are invariant with respect to the inner product.

Now, if we take the part of our solution S to the QME corresponding to the disk with boundary punctures,

$$S_o := S_{0,1}^{0,\bullet} := \sum_{m \geq 3} S_{0,1}^{0,m} \lambda^{m-2},$$

it will satisfy the CME

$$dS_o + \frac{1}{2}\{S_o, S_o\} = 0.$$

If $\phi : V[[\lambda, \sqrt{\hbar}]] \rightarrow A[[\lambda, \sqrt{\hbar}]]$ is an OC TCFT, then by grading arguments, the image $\phi(S_o)$ will be contained in $C^\lambda(H_o)[[\lambda]]$ and also satisfy the CME. This implies the following result.

Proposition 8.4. *In a symmetric OC TCFT, the space H_o carries the structure of a cyclic A_∞ algebra.*

Remark. Note that the solution (9) of the QME modulo $\sqrt{\hbar}$ takes into account only the terms with $p - \chi = 0$, which implies $g = n = 0$, $b = 1$. Those terms correspond to the disk with punctures on the boundary, and one can think of S_o as $S \bmod \sqrt{\hbar}$. The appearance of an A_∞ structure is geometrically due to the fact that the spaces $\underline{M}_{0,1}^{0,m}$ are homeomorphic to the Stasheff associahedra K_{m-1} , [Kon94].

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