

Universal cocycles and the graph complex action on homogeneous Poisson brackets by diffeomorphisms

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Abstract

The graph complex acts on the spaces of Poisson bi-vectors $\mathcal{P}$ by infinitesimal symmetries. We prove that whenever a Poisson structure is homogeneous, i.e. $\mathcal{P} = L_{\vec{V}}(\mathcal{P})$ w.r.t. the Lie derivative along some vector field $\vec{V}$, but not quadratic (the coefficients of $\mathcal{P}$ are not degree-two homogeneous polynomials), and whenever its velocity bi-vector $\dot{\mathcal{P}} = Q(\mathcal{P})$, also homogeneous w.r.t. $\vec{V}$ by $L_{\vec{V}}(Q) = nQ$ whenever $Q(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P}^{\otimes n})$ is obtained using the orientation morphism Or from a graph cocycle γ on n vertices and $2n - 2$ edges, then the 1-vector $\tilde{\mathcal{X}} = \text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$ is a Poisson cocycle. Its construction is uniform for all Poisson bi-vectors $\mathcal{P}$ satisfying the above assumptions, on all finite-dimensional affine manifolds M . Still, if the bi-vector $Q \neq 0$ is exact in the respective Poisson cohomology, so there exists a vector field $\vec{Y}$ such that $Q(\mathcal{P}) = \llbracket \vec{Y}, \mathcal{P} \rrbracket$, then the universal cocycle $\tilde{\mathcal{X}}$ does not belong to the coset of $\vec{Y}$ mod $\ker \llbracket \mathcal{P}, \cdot \rrbracket$. We illustrate the construction using two examples of cubic-coefficient Poisson brackets associated with the R -matrices for the Lie algebra $\mathfrak{gl}(2)$.

Introduction. Bi-vector cocycles $Q(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P}^{\otimes n}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ are obtained by Kontsevich's graph orientation morphism Or from graph cocycles γ on n vertices and $2n - 2$ edges in a way which is uniform for all finite-dimensional affine Poisson manifolds $(M^r, \mathcal{P})$. The (non)triviality of cocycles $Q(\mathcal{P})$ in the second Poisson cohomology w.r.t. the differential $\partial_{\mathcal{P}} = \llbracket \mathcal{P}, \cdot \rrbracket$ remains an open problem, twenty-five years after the discovery of the graph complex and orientation morphism (see [11]). In all the Poisson geometries probed so far, the known infinitesimal symmetries $\dot{\mathcal{P}} = Q(\mathcal{P})$ of the Jacobi identity $\frac{1}{2} \llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$ are $\partial_{\mathcal{P}}$ -exact: there always exists a vector field $\vec{Y}$ such that $Q(\mathcal{P}) = \llbracket \vec{Y}, \mathcal{P} \rrbracket$. The evolution $\mathcal{P}(\varepsilon = 0) \mapsto \mathcal{P}(\varepsilon > 0)$ of the tensor $\mathcal{P}$ then amounts to its reparametrisations under the diffeomorphisms of Poisson manifold which are induced by the shifts along the integral trajectories of the vector field $\vec{Y}$. This is why, instead of producing new Poisson brackets from a given one, the Kontsevich graph flows on the spaces of Poisson bi-vectors induce (non)linear

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diffeomorphisms of the base manifold M , although no more than its affine structure was the initial assumption and no possibility of smooth coordinate reparametrizations was presumed.

For a much used class of (scaling-)homogeneous Poisson bi-vectors $\mathcal{P} = L_{\vec{V}}(\mathcal{P})$, we obtain an explicit formula, $\vec{\mathcal{X}} = \text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$, of a 1-vector cocycle $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P}) \in \ker[\mathcal{P}, \cdot]$ which is built from the graph cocycles γ uniformly for all homogeneous Poisson bi-vectors $\mathcal{P}$ on affine manifolds $M^{r < \infty}$. The cocycle $\vec{\mathcal{X}}$ is however not necessarily a 1-vector representative of the coset $\vec{\mathcal{Y}} \bmod \{\vec{\mathcal{Z}} \in \ker[\mathcal{P}, \cdot]\}$ which would trivialise the value $\mathcal{Q}(\mathcal{P}) = [\vec{\mathcal{Y}}, \mathcal{P}]$ of Kontsevich's symmetries at homogeneous Poisson structures. Indeed, the Poisson cocycle $\mathcal{Q}(\mathcal{P})$ can be, we show, a nonzero bi-vector on M^r , whereas the bi-vector $[\vec{\mathcal{X}}, \mathcal{P}]$ is identically zero on M^r by construction. We contrast the formulas of universal cocycles $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P})$ and trivialising vector fields $\vec{\mathcal{Y}}$ for nonzero symmetries $\dot{\mathcal{P}} = \text{Or}(\gamma)(\mathcal{P})$ by two examples, namely, using cubic-coefficient Poisson brackets associated with the R -matrices for $\mathfrak{gl}(2)$.

This paper is organized as follows. In §1 we recall elements of Poisson cohomology theory in the context of Kontsevich's universal deformations of bi-vectors by using the unoriented graph cocycles. In §2 we phrase the notion of structures which are homogeneous w.r.t. a 1-vector field, and we prove the main theorem. Finally, we illustrate the result (cf. [10]).

1. Poisson cohomology and the graph complex. A Poisson bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ on a real manifold M is a bi-linear skew-symmetric bi-derivation which takes $C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ and satisfies the Jacobi identity $\frac{1}{2} \sum_{\sigma \in S_3} \{\{\sigma(f), \sigma(g)\}_{\mathcal{P}}, \sigma(h)\}_{\mathcal{P}} = 0$ for any $f, g, h \in C^\infty(M)$. The fact that both the arguments f, g and their bracket $\{f, g\}_{\mathcal{P}}$ are scalars dictates the tensor transformation law of the components $\mathcal{P}^{ij}$ of a bi-vector $\mathcal{P} = \sum_{i,j} \mathcal{P}^{ij}(\mathbf{x}) \partial_i \otimes \partial_j = \frac{1}{2} \sum_{i,j} \mathcal{P}^{ij}(\mathbf{x}) (\partial_i \otimes \partial_j - \partial_j \otimes \partial_i) = \frac{1}{2} \sum_{i,j} \mathcal{P}^{ij} \partial_i \wedge \partial_j$ whenever the structure is referred to a system of coordinates $\mathbf{x} = (x^1, \dots, x^r)$ and $\partial_i = \partial/\partial x^i$ is a shorthand notation.

The calculus on the space of multivectors $\Gamma(\bigwedge^\bullet TM) \cong C^\infty(\Pi T^*M)$ is simplified if one uses the parity-odd coordinates ξ_i along the directions dx^i in the fibres of the cotangent bundle T^*M over points $\mathbf{a} \in M$ (which are parametrized by x^i). The symbol ξ_i thus corresponds to $\partial/\partial x^i$ dual to dx^i , and bi-vectors are $\mathcal{P} = \frac{1}{2} \sum_{i,j} \mathcal{P}^{ij} \xi_i \xi_j$, so that $\{f, g\}_{\mathcal{P}}(\mathbf{a}) = (f) \vec{\partial}/\partial x^\mu \cdot \vec{\partial}/\partial \xi_\mu (\mathcal{P}) \vec{\partial}/\partial \xi_\nu \cdot \vec{\partial}/\partial x^\nu (g)$; here, both the coefficients $\mathcal{P}^{ij}$ and derivatives $\partial/\partial x^k$ are evaluated at the point $\mathbf{a} \in M$ as in the left-hand side.¹

The space of multivectors is endowed with the parity-odd Poisson bracket $[\![\cdot, \cdot]\!]$ (the Schouten bracket, or antibracket) of own degree -1 . For arbitrary multivectors $\mathcal{P}, \mathcal{Q}$, the formula is $[\![\mathcal{P}, \mathcal{Q}]\!] = (\mathcal{P}) \vec{\partial}/\partial \xi_i \cdot \vec{\partial}/\partial x^i (\mathcal{Q}) - (\mathcal{Q}) \vec{\partial}/\partial x^i \cdot \vec{\partial}/\partial \xi_i (\mathcal{P})$; in particular, $[\![\vec{\mathcal{X}}, \vec{\mathcal{Y}}]\!] = [\vec{\mathcal{X}}, \vec{\mathcal{Y}}]$ is the usual commutator of vector fields $\vec{\mathcal{X}}, \vec{\mathcal{Y}}$ on M . The Schouten bracket $[\![\cdot, \cdot]\!]$ is shifted-graded skew-symmetric: $[\![\mathcal{Q}, \mathcal{P}]\!] = -(-)^{(|\mathcal{P}|-1) \cdot (|\mathcal{Q}|-1)} [\![\mathcal{P}, \mathcal{Q}]\!]$ for $\mathcal{P}$ and $\mathcal{Q}$ grading-homogeneous. This is why, unlike the tautology $[\![\vec{\mathcal{X}}, \vec{\mathcal{X}}]\!] \equiv 0$, the equation $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ is a nontrivial restriction for bi-vectors $\mathcal{P}$, containing the tri-vector in the l.h.s. of the Jacobi identity $\frac{1}{2} [\![\mathcal{P}, \mathcal{P}]\!](f, g, h) = 0$ for the bracket $\{f, g\}_{\mathcal{P}} = [\![f, \mathcal{P}]\!], g]$. The Schouten bracket itself satisfies the graded Jacobi identity $[\![\mathcal{P}, [\![\mathcal{Q}, \mathcal{R}]\!]] - (-)^{(|\mathcal{P}|-1) \cdot (|\mathcal{Q}|-1)} [\![\mathcal{Q}, [\![\mathcal{P}, \mathcal{R}]\!]] = [\![[\![\mathcal{P}, \mathcal{Q}]\!], \mathcal{R}]\!]$ with $\mathcal{P}$ and $\mathcal{Q}$ grading-homogeneous. This identity implies that for Poisson bi-vectors $\mathcal{P}$, their adjoint action by $\partial_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ is a differential of degree $+1$ on the space of multivectors on M . The Poisson differential $\partial_{\mathcal{P}}$ gives rise to the Poisson cohomology $H_{\mathcal{P}}^i(M)$ of the manifold M (see [13]).²

¹The dot $\cdot$ denotes the coupling of iterated variations of the objects $f, \mathcal{P}$, and g with respect to the canonically conjugate variables x^i and ξ_i , see [9] and references therein.

²The group $H_{\mathcal{P}}^0(M)$ spans the Casimirs, i.e. the functions which Poisson-commute with any $f \in C^\infty(M)$; the group $H_{\mathcal{P}}^1(M)$ consists of vector fields which preserve the Poisson structure but do not amount to the Hamiltonian vector fields $\vec{\mathcal{X}}_h = [\![\mathcal{P}, h]\!]$; the second group $H_{\mathcal{P}}^2(M) \ni \mathcal{Q}$ contains infinitesimal symmetries $\mathcal{P} \mapsto \mathcal{P} + \varepsilon \mathcal{Q} + \bar{o}(\varepsilon)$ of Poisson bi-vectors, whereas the next group $H_{\mathcal{P}}^3(M)$ stores the obstructions to formal

If a bi-vector $\mathcal{Q} = \llbracket \vec{\mathcal{X}}, \mathcal{P} \rrbracket$ is a trivial Poisson cocycle, then it certainly is an infinitesimal symmetry of the Jacobi identity $\frac{1}{2} \llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$. But the infinitesimal change $\llbracket \vec{\mathcal{X}}, \mathcal{P} \rrbracket$ of the tensor $\mathcal{P}$ then amounts to its reparametrisation under the infinitesimal change of coordinates $\mathbf{x}'(\mathbf{x}) \rightleftharpoons \mathbf{x}(\mathbf{x}')$ along the integral trajectories of the vector field $\vec{\mathcal{X}}$ on the manifold M . The following fact is true for all multivectors (regardless of the concept of Poisson cohomology).

Proposition 1. *Let $\mathbf{a} \in M$ be a point of an r -dimensional manifold and $\vec{\mathcal{X}} \in \Gamma(TM)$ be a vector field on it. For every $\varepsilon \in \mathcal{I} \subseteq \mathbb{R}$ such that there is the integral trajectory bringing $\mathbf{b}(-\varepsilon) := \exp(-\varepsilon \vec{\mathcal{X}})(\mathbf{a})$ to $\mathbf{a}$ by the $(+\varepsilon)$ -shift, and for any choice of the r -tuple $\mathbf{x} = (x^1, \dots, x^r)$ of local coordinates in a chart U_α around $\mathbf{a} \in M$ (and for $|\varepsilon|$ small enough for the points $\mathbf{b}(-\varepsilon)$ to not yet run out of the chart U_α), introduce a new parametrization³ for the point $\mathbf{a}$ by using the new r -tuple $\mathbf{x}'$. By definition, put $\mathbf{x}'(\mathbf{a}) := \mathbf{x}(\mathbf{b}(-\varepsilon))$. Let Ω be any multi-vector field near $\mathbf{a}$ on M . Under the reparametrization $\mathbf{x}'(\mathbf{x})$, the speed at which the components of Ω at the point $\mathbf{a}$ change in ε , as $\varepsilon \rightarrow 0$, equals $\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \Omega(\mathbf{a}) = \llbracket \vec{\mathcal{X}}, \Omega \rrbracket(\mathbf{a})$. In particular, a 1-vector field $\vec{\mathcal{Y}}$ near $\mathbf{a}$ would change at $\mathbf{a}$ as fast as its commutator with the vector field $\vec{\mathcal{X}}$: $\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \vec{\mathcal{Y}}(\mathbf{a}) = [\vec{\mathcal{X}}, \vec{\mathcal{Y}}](\mathbf{a})$.*

The geography of the set of Poisson structures near a given bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ on a given manifold M^r is, generally speaking, unknown. All the more it was a priori unclear whether Poisson bi-vectors $\mathcal{P}$, irrespective of the dimension $r \geq 3$, topology of M^r , etc., can be infinitesimally shifted by Poisson 2-cocycles $\mathcal{Q}(\mathcal{P})$, the construction of which would be universal for all $\mathcal{P}$. The discovery of the graph complex in 1993–94 allowed Kontsevich to state (in [11]) the affirmative answer to the above question. Namely, the graph orientation morphism $\text{Or}(\cdot)(\mathcal{P}) : \ker d \ni \gamma \mapsto \mathcal{Q}(\mathcal{P}) \in \ker \partial_{\mathcal{P}}$ takes graph cocycles on n vertices and $2n - 2$ edges in each term (e.g., the tetrahedron, cf. [1, 3, 5, 6]) to Poisson cocycles whenever the bi-vector $\mathcal{P}$ itself is Poisson. Willwacher [15] revealed that the generators of Drinfeld’s Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ are source of at least countably many such cocycles in the vertex-edge bi-grading $(n, 2n - 2)$; these cocycles are marked by the $(2\ell + 1)$ -wheel graphs (e.g., see [6, 7]). Brown proved in [2] that, under the Willwacher isomorphism $\mathfrak{grt} \cong H^0(\text{GRA})$ these graph cocycles with wheels generate a free Lie subalgebra in $\mathfrak{grt}$, which means effectively that the iterated commutators of already known cocycles – under the bracket in the differential graded Lie algebra GRA of graphs – would never vanish. The commutator of two cocycles is a cocycle by the Jacobi identity. All of them again being of the bi-grading $(n, 2n - 2)$, these graph cocycles determine countably many infinitesimal symmetries of a given Poisson bi-vector $\mathcal{P}$; the construction is uniform for all the geometries $(M^r, \mathcal{P})$.

Lemma 2. *For a given Poisson bi-vector $\mathcal{P}$, the graph orientation mapping $\text{Or}(\cdot)(\mathcal{P}) : \ker d \ni \gamma \mapsto \mathcal{Q}(\mathcal{P}) \in \ker \partial_{\mathcal{P}}$ is a Lie algebra morphism that takes the bracket of two cocycles in bi-grading $(n, 2n - 2)$ to the commutator $[\frac{d}{d\varepsilon_1}, \frac{d}{d\varepsilon_2}](\mathcal{P})$ of two symmetries $\frac{d}{d\varepsilon_i}(\mathcal{P}) = \mathcal{Q}_i(\mathcal{P})$.⁴*

By construction, the components of universal symmetry bi-vectors $\mathcal{Q}(\mathcal{P})$ are differential polynomials w.r.t. the components $\mathcal{P}^{ij}$ of the Poisson bi-vector $\mathcal{P}$ that evolves. It can of course be that a graph flow $\dot{\mathcal{P}} = \text{Or}(\gamma)(\mathcal{P})$ vanishes identically over the manifold M^r

integration $\mathcal{P} \mapsto \mathcal{P}(\varepsilon) = \mathcal{P} + \sum_{k \geq 1} \varepsilon^k \mathcal{Q}_{(k)}$ of infinitesimal symmetries $\mathcal{Q} = \mathcal{Q}_{(1)}$ to Poisson bi-vector formal power series satisfying $\llbracket \mathcal{P}(\varepsilon), \mathcal{P}(\varepsilon) \rrbracket = 0$.

³Actually, this is a way to construct new coordinates for *all* points of M near $\mathbf{a}$ in U_α , i.e. not only those which lie on a piece of the integral trajectory of $\vec{\mathcal{X}}$ passing through $\mathbf{a}$.

⁴By Brown [2], the commutator does in general not vanish for Willwacher’s odd-sided wheel cocycles.

whenever $\mathcal{Q}$ is evaluated at a particular class of Poisson structures $\mathcal{P}$.⁵ Nevertheless, there is no mechanism which would force a given Kontsevich's graph flow to vanish at all Poisson structures on all manifolds of all dimensions.⁶ Independently, it remains an open problem (cf. [10]) whether there is a Poisson manifold $(M^r, \mathcal{P})$ and a graph cocycle γ such that the Poisson cohomology class of $\mathcal{Q}(\mathcal{P}) := \text{Or}(\gamma)(\mathcal{P})$ would be *nontrivial* in $H_{\mathcal{P}}^2(M)$. In other words, for all the shifts $\mathcal{Q} = \text{Or}(\gamma)$ and all Poisson bi-vectors tried so far, the Poisson coboundary equation $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{X}}, \mathcal{P} \rrbracket$ did have vector field solutions $\vec{\mathcal{X}}$ on the manifolds M .

Remark 1. Obtained from the graphs $\gamma \in \ker d$, the symmetries $\mathcal{Q}(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ are independent of a choice of local coordinates x^i (hence ξ_i) on a chart if, the Kontsevich construction requires, the manifold M^r is endowed with an *affine* structure: all the coordinate transformations amount to $\mathbf{x}' = A\mathbf{x} + \vec{b}$ with a constant (over the intersection of charts) Jacobian matrix A . The parity-odd fibre variables are transformed using the inverse Jacobian matrix, $\xi_i = A_i^{i'} \xi'_{i'}$, making sense of the couplings $\vec{\partial}/\partial \xi_i \cdot \vec{\partial}/\partial x^i$ which decorate the oriented edges of Kontsevich's graphs after the morphism Or works (see [3, 11]). The problem of Poisson cohomology class (non)triviality for the Kontsevich infinitesimal symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ thus acquires two diametrically opposite interpretations:

1 (as in [11]). The Poisson manifold $M^{r<\infty}$ is equipped with *both* the smooth and affine structures.⁷ By definition, two Poisson bi-vectors are equivalent, $\mathcal{P}_1 \sim \mathcal{P}_2$, if they are related by a diffeomorphism of the manifold M : using its smooth structure, the diffeomorphism identifies points in two copies of M , then relating the Poisson tensors by local coordinate reparametrizations near the respective points. The affine structure on M is now used to run the Kontsevich flows in two initial value problems $\dot{\mathcal{P}}_i(\varepsilon) = \mathcal{Q}(\mathcal{P}_i(\varepsilon))$, $\mathcal{P}_i(\varepsilon = 0) = \mathcal{P}_i$. The Poisson triviality $\mathcal{Q}(\mathcal{P}(\varepsilon)) = \llbracket \vec{\mathcal{X}}(\varepsilon), \mathcal{P}(\varepsilon) \rrbracket$ would relate either of bi-vectors $\mathcal{P}_i(\varepsilon)$ back to the Cauchy datum $\mathcal{P}_i$ by diffeomorphisms (as long as $|\varepsilon|$ is small enough). Consequently, the Poisson bi-vectors $\mathcal{P}_1(\varepsilon) \sim \mathcal{P}_2(\varepsilon)$ do not run out of the old equivalence class. In conclusion, the goal is to produce essentially new Poisson brackets by using a nontrivial cocycle $\mathcal{Q}$, two given structures on the manifold M^r , and its diffeomorphism. No examples of nontrivial action, so that $\mathcal{P}_2(\varepsilon) \not\sim \mathcal{P}_i \not\sim \mathcal{P}_1(\varepsilon)$ at $\varepsilon > 0$, have ever been produced since 1996 (see [7, 11]).

2 (as in [10]). The Poisson manifold $M^{r<\infty}$ is equipped only with an affine structure. The countably many **grt**-related graph cocycles on n vertices and $2n - 2$ edges in every term (the tetrahedron, the pentagon-wheel cocycle, etc., see [6, 15]) generate a noncommutative Lie algebra of infinitesimal symmetries $\mathcal{Q}(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P})$ for a given Poisson structure $\mathcal{P}$. Consider the extreme case when *all* the cocycles $\mathcal{Q}(\mathcal{P}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ are exact in the coho-

⁵**Example.** So it is for the Kontsevich tetrahedral flow ([11] and [1]) evaluated at the Kirillov–Kostant linear Poisson brackets on the duals $\mathfrak{g}^*$ of Lie algebras because in every term within the cocycle $\mathcal{Q}(\mathcal{P})$ under study, at least one copy of $\mathcal{P}$ is differentiated at least twice with respect to the global coordinates on $\mathfrak{g}^*$.

⁶**Example.** The Poisson bi-vectors $\mathcal{P} = da_1 \wedge \dots \wedge da_m / \text{dvol}(\mathbb{R}^{m+2})$ of Nambu type with arbitrary Casimirs $a_1, \dots, a_m \in C^\infty(\mathbb{R}^{m+2})$ and an arbitrary density in the volume element can have polynomial components $\mathcal{P}^{ij} \in \mathbb{R}[x^1, \dots, x^{m+2}]$ of degrees as high as need be w.r.t. the global Cartesian coordinates x^α on the vector space $\mathbb{R}^{m+2}$. The universal symmetries $\dot{\mathcal{P}} = \text{Or}(\gamma)(\mathcal{P})$ obtained from Kontsevich's graph cocycles deform the symplectic foliation (which is given in $\mathbb{R}^{m+2}$ by the intersections of the level sets for the Casimirs $a_1, \dots, a_m$) in a regular way on an open dense subset of $\mathbb{R}^{m+2}$, so that the symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ preserve this Nambu class of Poisson brackets: the flows force the evolution of the Casimirs and the volume density. Its integrability is an open problem; by Lemma 2 and [2], the evolutions induced by different graph cocycles do not commute.

⁷On the circle $\mathbb{S}^1$, the affine coordinate ‘angle’ is obvious whereas the smooth structure is used in the realm of Poincaré topology. A smooth atlas is always available for the spheres $\mathbb{S}^r$, but not for all $r \in \mathbb{N}$ would the r -dimensional sphere admit an affine structure.

mology group $H^2_{\mathcal{P}}(M)$ w.r.t. the Poisson differential $\partial_{\mathcal{P}}$. This assumption gives rise to the countable set of vector fields $\vec{\mathcal{Y}}(\gamma, \mathcal{P})$ on M such that $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$. (Some of these vector fields can be identically zero over M .) But if at least one such vector field is not constant w.r.t. the affine structure on M , then the shifts along its integral trajectories are nonlinear diffeomorphisms of M . The evolution of bi-vector $\mathcal{P}$ is $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$ or similarly, $\dot{\Omega} = \llbracket \vec{\mathcal{Y}}, \Omega \rrbracket$ for any multi-vector Ω on M (see Proposition 1); this evolution is now seen as multivectors' response to the diffeomorphism whose construction refers only to the simple, affine local portrait of M . Summarizing, the store of flows $\text{Or}(\gamma)(\mathcal{P})$ from the **grt**-related graph cocycles γ could be enough to approximate arbitrary smooth vector fields on M^r , that is, imitate its smooth structure. Whether this theoretical possibility is actually realised in relevant Poisson models is an open problem.

The Kontsevich symmetry construction is, therefore, either a generator of new Poisson brackets or the mechanism that provides diffeomorphisms of the underlying manifold.

2. Homogeneous Poisson structures. By definition, a bi-vector $\mathcal{P}$ on a manifold M is called *homogeneous* (of scale λ) with respect to a vector field $\vec{\mathcal{V}}$ on M if $\llbracket \vec{\mathcal{V}}, \mathcal{P} \rrbracket = \lambda \cdot \mathcal{P}$.

Example 1. Let $M = \mathbb{R}^r$ be a vector space (only linear reparametrizations $\mathbf{x}' = A\mathbf{x}$ are allowed, so that the polynomial degrees of monomials in the ring $\mathbb{R}[x^1, \dots, x^r]$ is well defined). Introduce the Euler vector field $\vec{\mathcal{E}} = \sum_{i=1}^r x^i \partial/\partial x^i$, and let all the components $\mathcal{P}^{ij}$ of a bi-vector $\mathcal{P}$ be homogeneous polynomials of degree d in the variables x^i . Then we have that $\llbracket \vec{\mathcal{E}}, \mathcal{P} \rrbracket = (d-2) \cdot \mathcal{P}$, which means that $\mathcal{P}$ is homogeneous of scale $d-2$ w.r.t. the Euler vector field $\vec{\mathcal{E}}$. In particular, if $d \neq 2$ (i.e. if the coefficients of bi-vector $\mathcal{P}$ are not quadratic), then we set $\vec{\mathcal{V}} = (d-2)^{-1} \cdot \vec{\mathcal{E}}$ and from the equality $\mathcal{P} = \llbracket \vec{\mathcal{V}}, \mathcal{P} \rrbracket$ we obtain that the same bi-vector $\mathcal{P}$ has homogeneity scale $\lambda = 1$ w.r.t. the multiple $\vec{\mathcal{V}}$ of the Euler vector field $\vec{\mathcal{E}}$ on $\mathbb{R}^r$.

Example 2. Under the same assumptions, suppose further that $\gamma = \sum_a c_a \gamma_a$ is a graph cocycle with n vertices and $2n-2$ edges in every term γ_a (e.g., take the tetrahedron). Orient the ordered (by First $\prec \dots \prec$ Last) edges in every γ_a using the edge decoration operators $\vec{\Delta}_{ij} = \sum_{\mu=1}^r (\vec{\partial}/\partial \xi_{\mu}^{(i)} \otimes \vec{\partial}/\partial x_{(j)}^{\mu} + \vec{\partial}/\partial \xi_{\mu}^{(j)} \otimes \vec{\partial}/\partial x_{(i)}^{\mu})$. By placing a copy of bi-vector $\mathcal{P} = \frac{1}{2} \mathcal{P}^{kl}(\mathbf{x}) \xi_k \xi_l$ in each vertex $v^{(i)}$ of γ_a and taking the sum (over the graph index a) of products of the content of vertices in γ_a after all the edge operators $\vec{\Delta}_{ij}$ work, we obtain⁸ the bi-vector $\mathcal{Q}(\mathcal{P}) := \text{Or}(\gamma)(\mathcal{P})$. Then the coefficients of the bi-vector $\mathcal{Q}(\mathcal{P})$ are homogeneous polynomials of degree $n \cdot d - (2n-2)$ with respect to $x^1, \dots, x^r$, so that $\llbracket \vec{\mathcal{E}}, \mathcal{Q}(\mathcal{P}) \rrbracket = n(d-2) \mathcal{Q}(\mathcal{P})$. In particular, if $d \neq 2$, then $\llbracket \vec{\mathcal{V}}, \mathcal{Q}(\mathcal{P}) \rrbracket = n \cdot \mathcal{Q}(\mathcal{P})$, whereas quadratic-coefficient bi-vectors $\mathcal{P}$ (with $d = 2$) are deformed within their subspace by the quadratic bi-vectors $\mathcal{Q}(\mathcal{P})$ which are obtained from the Kontsevich graph cocycles.

Lemma 3. *If a Poisson bi-vector $\mathcal{P} = \llbracket \vec{\mathcal{V}}, \mathcal{Q}(\mathcal{P}) \rrbracket$ is homogeneous and $\mathcal{Q}(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P}^{\otimes n})$ is built from a graph cocycle γ on n vertices, now containing a copy of $\mathcal{P}$ in each vertex, then the bi-vector $\mathcal{Q}(\mathcal{P})$ is also homogeneous: $\llbracket \vec{\mathcal{V}}, \mathcal{Q}(\mathcal{P}) \rrbracket = n \cdot \mathcal{Q}(\mathcal{P})$, so that its scale is n .*⁹

Remark 2 ([14, Rem. 4.9]). Consider a Nambu-type Poisson bi-vector $\mathcal{P} = da/dx dy dz$ on $\mathbb{R}^3$ with Cartesian coordinates x, y, z ; here $a \in \mathbb{R}[x, y, z]$ is a weight-homogeneous polynomial

⁸We refer to the original paper [11] and to [3] for illustrations and discussion how the graph orientation morphism works in practice.

⁹The proof amounts to the Leibniz rule: let us inspect how fast the bi-vector $\mathcal{Q}(\mathcal{P})$, which by construction is a homogeneous differential polynomial of degree n in $\mathcal{P}$, evolves along the vector field $\vec{\mathcal{V}}$.

with an isolated singularity at the origin¹⁰, so that $(w_{(x)} \cdot x\partial/\partial x + w_{(y)} \cdot y\partial/\partial y + w_{(z)} \cdot z\partial/\partial z)(a) = w_{(a)} \cdot a$. Then a vector field $\vec{V}$ with polynomial components satisfying the first-order PDE $\mathcal{P} = \llbracket \vec{V}, \mathcal{P} \rrbracket$ exists if and only if¹¹ the weight degree $w_{(a)}$ of the polynomial a is not equal to the sum $w_{(x)} + w_{(y)} + w_{(z)}$ of weight degrees for the variables x, y, z .¹²

Summarizing, the homogeneity assumption about bi-vectors $\mathcal{P}$ is restrictive; it is not always satisfied in Poisson models.

Theorem 4. *Let $(M, \mathcal{P})$ be an affine finite-dimensional real Poisson manifold with $\mathcal{P} = \llbracket \vec{V}, \mathcal{P} \rrbracket$ homogeneous. Let $\gamma = \sum_a c_a \cdot \gamma_a$ be a graph cocycle consisting of unoriented graphs γ_a over n vertices and $2n-2$ edges (with a fixed ordering of edges in each γ_a). Then the 1-vector $\vec{X}(\gamma, \vec{V}, \mathcal{P}) = \text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$, which is obtained by representing each edge $i-j$ with the operator $\vec{\Delta}_{ij}$ and by (graded-)symmetrizing over all the ways $\sigma \in \mathbb{S}_n$ to send the n -tuple $\vec{V} \otimes \mathcal{P}^{\otimes n-1}$ into the n vertices in each γ_a , is a Poisson cocycle: $\vec{X} \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$.¹³*

The vector field $\vec{X}$ is defined up to adding arbitrary Poisson 1-cocycles $\vec{Z} \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$.

Proof. The expansion $0 = \text{Or}(d\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n})$ for $\gamma \in \ker d$ goes along the lines of [11] and [3, 7, 8], but the $(n+1)$ -tuple of multivectors now contains one 1-vector and only n copies of the Poisson bi-vector $\mathcal{P}$. By assumption, $d\gamma = \mathbf{0} \in \text{GRA}$; recall that $\text{Or}(\mathbf{0})(\text{any multivectors}) = 0 \in \Gamma(\wedge^\bullet TM)$. This zero l.h.s. equates $0 = (\pi_S \vec{\sigma} \text{Or}(\gamma) - (-)^{(-1) \cdot (-N)} \text{Or}(\gamma) \vec{\sigma} \pi_S)(\vec{V} \otimes \mathcal{P}^{\otimes n})$.¹⁴

The appointment of graded (multi)vectors into the vertices of $d\gamma$ (hence, into the argument slots of the endomorphism $\text{Or}(d\gamma)$) is achieved by the graded symmetrization using $((n+1)!)^{-1} \text{Or}(d\gamma)(\pm \sigma(\vec{V} \otimes \mathcal{P}^{\otimes n}))$. Fortunately, the field $\vec{V}$ is the only parity-odd object, so its transpositions with the parity-even bi-vectors $\mathcal{P}$ produce no sign factor: these $\pm$ are all $+$. Likewise, the $n!$ permutations of n indistinguishable copies of $\mathcal{P}$ leave only $n+1$ from $(n+1)!$ in the denominator; to get rid of it, let us multiply by $n+1$ both sides of the equality $0 = \text{Or}(d\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n})$. The symmetrization thus amounts, by the linearity of $\text{Or}(\gamma)$, to its evaluation at the sum of arguments, $\vec{V} \cdot \mathcal{P}^n + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-1} + \dots + \mathcal{P}^n \cdot \vec{V}$, in which the ordering of (multi)vectors now matches an arbitrary fixed enumeration of the vertices.

The rest of the proof is standard.¹⁵ There remains $0 = \text{Or}(\gamma)(\pi_S(\vec{V}, \mathcal{P}) \cdot \mathcal{P}^{n-1}) + \mathcal{P} \cdot$

¹⁰The Milnor number is the dimension $\dim_{\mathbb{R}} \mathbb{R}[x, y, z]/(\partial a/\partial x, \partial a/\partial y, \partial a/\partial z)$ – here, $< \infty$ by assumption.

¹¹This means that not all Nambu-type Poisson bi-vectors $\mathcal{P} = da/dxdydz$ are homogeneous w.r.t. a vector field $\vec{V}$ with polynomial components; the PDE $\mathcal{P} = \llbracket \vec{V}, \mathcal{P} \rrbracket$ with polynomial coefficients and unknown $\vec{V}$ can in principle admit non-polynomial solutions.

¹²**Example.** If the weights of (x, y, z) are $(1, 1, 1)$ and $a = \frac{1}{3}(x^3 + y^3 + z^3)$ is cubic-homogeneous, then the components of Poisson bi-vector $\mathcal{P}$ are quadratic and (by the above and also by [12, Exerc. 4.5.7c]) not of the form $\mathcal{P} = \llbracket \vec{V}, \mathcal{P} \rrbracket$ for any polynomial-coefficient vector field $\vec{V}$. The non-existence of a solution $\vec{V}$ with smooth non-polynomial coefficients is a separate problem.

¹³**Open problem** (for $\mathcal{P}$ homogeneous and Poisson). Is the universal 1-vector field $\vec{X}(\gamma, \vec{V}, \mathcal{P}) \in \ker \partial_{\mathcal{P}}$ Hamiltonian, i.e. $\vec{X} = \llbracket \mathcal{P}, h \rrbracket$ for $h \in C^\infty(M)$ or at least, $\vec{X} = \mathcal{P} \lrcorner \eta$ for a maybe not exact 1-form η on M ?

¹⁴Here, π_S is the graded-symmetric Schouten bracket (so $\pi_S(F, G) = (-)^{|F|-1} \llbracket F, G \rrbracket$), the graph insertion $\vec{\sigma}$ into vertices is now the endomorphism insertion into argument slots, $|\pi_S| = -1$, and $N = 2n - 2$ is the even number of edges in γ , hence minus the even number of $\partial/\partial \xi_\mu$ in the edge operators $\vec{\Delta}_{ij}$ making $\text{Or}(\gamma)$.

¹⁵We have $0 = \text{Or}(\gamma)(\pi_S(\vec{V}, \mathcal{P}), \mathcal{P}^{n-1}) + \text{Or}(\gamma)(\pi_S(\mathcal{P}, \vec{V}), \mathcal{P}^{n-1}) + \text{Or}(\gamma)(\pi_S(\mathcal{P}, \mathcal{P}), \vec{V}, \mathcal{P}^{n-2}) + \dots + \text{Or}(\gamma)(\pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-2}, \vec{V}) + \text{Or}(\gamma)(\vec{V}, \pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-2}) + \text{Or}(\gamma)(\mathcal{P}, \pi_S(\vec{V}, \mathcal{P}), \mathcal{P}^{n-2}) + \text{Or}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \vec{V}), \mathcal{P}^{n-2}) + \text{Or}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \mathcal{P}), \vec{V}, \mathcal{P}^{n-3}) + \dots + \text{Or}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-3}, \vec{V}) + \dots$ (the Schouten bracket π_S passes along the slots towards the end) $+ \text{Or}(\gamma)(\vec{V}, \mathcal{P}^{n-2}, \pi_S(\mathcal{P}, \mathcal{P})) + \dots + \text{Or}(\gamma)(\mathcal{P}^{n-2}, \vec{V}, \pi_S(\mathcal{P}, \mathcal{P})) + \text{Or}(\gamma)(\mathcal{P}^{n-1}, \pi_S(\vec{V}, \mathcal{P})) + \text{Or}(\gamma)(\mathcal{P}^{n-1}, \pi_S(\mathcal{P}, \vec{V})) - (-)^N \cdot [\pi_S(\text{Or}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}), \mathcal{P}) + \pi_S(\text{Or}(\gamma)(\mathcal{P}^n), \vec{V}) + \pi_S(\vec{V}, \text{Or}(\gamma)(\mathcal{P}^n)) + \pi_S(\mathcal{P}, \text{Or}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}))]$. For $\mathcal{P}$ Poisson, $\pi_S(\mathcal{P}, \mathcal{P}) = 0$, so we exclude all such terms ([4]). The remaining graded-symmetric Schouten brackets π_S contain a bi-vector as one of the arguments, hence those can be swapped at no sign factor; all doubles, so let us divide by 2.

$\pi_S(\vec{V}, \mathcal{P}) \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \pi_S(\vec{V}, \mathcal{P})) - (-)^N [\pi_S(\text{Or}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}), \mathcal{P}) + \pi_S(\text{Or}(\gamma)(\mathcal{P}^n, \vec{V})]$. By the homogeneity assumption, $\pi_S(\vec{V}, \mathcal{P}) = (-)^{1-1} \llbracket \vec{V}, \mathcal{P} \rrbracket = \mathcal{P}$, and by construction, $\text{Or}(\gamma)(\mathcal{P}^n) = \mathcal{Q}(\mathcal{P})$, whence the minuend equals $n \cdot \mathcal{Q}(\mathcal{P})$. By Lemma 3, the graph flow is also homogeneous: $\llbracket \vec{V}, \mathcal{Q}(\mathcal{P}) \rrbracket = \lambda \cdot \mathcal{Q}(\mathcal{P})$ with the vertex count $\lambda = n$. We obtain the equality

$$\begin{aligned} (-)^{2n-2} \cdot \llbracket \text{Or}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}), \mathcal{P} \rrbracket &= \\ &= n \cdot \mathcal{Q}(\mathcal{P}) - (-)^{2n-2} \lambda \cdot \mathcal{Q}(\mathcal{P}) = (n - (-)^{\text{even}} n) \cdot \mathcal{Q}(\mathcal{P}) \equiv 0. \end{aligned}$$

We conclude that the 1-vector $\vec{\mathcal{X}} := \text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$ lies in $\ker \llbracket \mathcal{P}, \cdot \rrbracket$.¹⁶ $\square$

Example 3. Take the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ with its four-dimensional vector space structure; denote by x, y, z, v the Cartesian coordinates. Consider the R -matrix $\begin{pmatrix} x & y \\ z & v \end{pmatrix} \mapsto \begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix}$ known from [12]; the standard construction then yields the Poisson bi-vector in the algebra of coordinate functions, $\mathcal{P} = (x^2y + y^2z) \partial_x \wedge \partial_y + (x^2z + yz^2) \partial_x \wedge \partial_z + (2xyz + 2yzv) \partial_x \wedge \partial_v + (y^2z + yv^2) \partial_y \wedge \partial_v + (yz^2 + zv^2) \partial_z \wedge \partial_v$. This bracket has cubic-nonlinear homogeneous polynomial coefficients, hence $d = 3$. The vector field $\vec{V} = (d - 2)^{-1} \cdot \vec{\mathcal{E}}$ is the (multiple of the) Euler vector field on $\mathbb{R}^4$. As the graph cocycle γ , we take the tetrahedron (see [1, 11]); then the symmetry flow is $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = (-48x^5y - 288x^3y^2z - 240xy^3z^2 + 192y^3z^2v - 384xy^2zv^2 - 192y^2zv^3) \partial_x \wedge \partial_y + (-48x^5z - 288x^3yz^2 - 240xy^2z^3 + 192y^2z^3v - 384xyz^2v^2 - 192yz^2v^3) \partial_x \wedge \partial_z + (-336x^4yz - 480x^2y^2z^2 - 576x^3yzv + 480y^2z^2v^2 + 576xyzv^3 + 336yzv^4) \partial_x \wedge \partial_v + (192x^3y^2z - 192xy^3z^2 + 288y^2zv^3 + 48yv^5 + 48(8x^2y^2z + 5y^3z^2)v) \partial_y \wedge \partial_v + (192x^3yz^2 - 192xy^2z^3 + 288yz^2v^3 + 48zv^5 + 48(8x^2yz^2 + 5y^2z^3)v) \partial_z \wedge \partial_v$. We detect that this bi-vector is a coboundary, $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{Y}, \mathcal{P} \rrbracket$ with the vector $\vec{Y} = (-24x^4 + 120y^2z^2 - 96yzv^2) \partial_x + (96x^3y - 96yv^3) \partial_y + (96x^3z - 96zv^3) \partial_z + (96x^2yz - 120y^2z^2 + 24v^4) \partial_v \pmod{\ker \llbracket \mathcal{P}, \cdot \rrbracket}$. The vector field $\vec{Y} \notin \ker \partial_{\mathcal{P}}$ cannot be Poisson-exact (clearly, $\mathcal{Q}(\mathcal{P}) \not\equiv 0$), hence $\vec{Y}$ does not mark the Poisson cocycle of zero 1-vector.¹⁷ But the universal vector field $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P}) \in \ker \partial_{\mathcal{P}}$ is identically zero on $\mathbb{R}^4$. Indeed, the Euler field $\vec{\mathcal{E}} = \vec{V}$ is linear, yet it is readily seen from the figures in [1] that in every orgraph from the 1-vector $\text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$, the vertex with $\vec{V}$ is differentiated at least twice (and at most thrice), so $\vec{\mathcal{X}} \equiv 0$.

Proposition 5. *The flow $\dot{\mathcal{P}} = \text{Or}(\text{tetrahedron } \gamma_3)(\mathcal{P})$ preserves the Nambu class of Poisson brackets, $\{f, g\}_{\mathcal{P}} = \varrho(x, y, z) \cdot \det(\partial(a, f, g)/\partial(x, y, z))$ with arbitrary ϱ and global Casimir a on $\mathbb{R}^3$: the flow forces the nonlinear evolution $\dot{a}, \dot{\varrho}$ with differential-polynomial r.-h.s.*

• *This flow $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ is not Poisson-exact in terms of any vector field $\vec{Y}$ with differential-polynomial coefficients (cubic in both a and ϱ , of total differential order eight).*

¹⁶**Exercise.** Extend the proof to the case $n = 1$, $\gamma = \bullet$, $d\gamma = -\bullet\bullet$ (so that the l.-h.s. was nonzero).

¹⁷Likewise, by using another R -matrix for $\mathfrak{gl}_2(\mathbb{R})$, namely $\begin{pmatrix} x & y \\ z & v \end{pmatrix} \mapsto \begin{pmatrix} x & y \\ -z & v \end{pmatrix}$ also from [12], we obtain the Poisson bi-vector $\mathcal{P} = 2x^2y\partial_x \wedge \partial_y + 2yz^2\partial_x \wedge \partial_z + (2xyz + 2yzv)\partial_x \wedge \partial_v + (-2xyz + 2yzv)\partial_y \wedge \partial_z + 2yv^2\partial_y \wedge \partial_v + 2yz^2\partial_z \wedge \partial_v$ on $\mathbb{R}^4$ with Cartesian coordinates x, y, z, v . The tetrahedral flow then equals $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = (-384x^5y - 384x^3y^2z - 1536xy^2zv^2 + 384(x^2y^2z - 4y^3z^2)v) \partial_x \wedge \partial_y + (-384x^3yz^2 - 2688xy^2z^3 + 1152xyz^2v^2 + 384yz^2v^3 - 384(3x^2yz^2 - 7y^2z^3)v) \partial_x \wedge \partial_z + (-384x^4yz - 2688x^2y^2z^2 - 1536x^3yzv + 2688y^2z^2v^2 + 1536xyzv^3 + 384yzv^4) \partial_x \wedge \partial_v + (384x^4yz + 384x^2y^2z^2 + 1536y^3z^3 - 384xyzv^3 + 384yzv^4 + 384(x^2yz + y^2z^2)v^2 - 384(x^3yz - 2xy^2z^2)v) \partial_y \wedge \partial_z + (1536xy^3z^2 + 1536x^2y^2zv - 384xy^2zv^2 + 384y^2zv^3 + 384yv^5) \partial_y \wedge \partial_v + (-384x^3yz^2 - 2688xy^2z^3 + 1152xyz^2v^2 + 384yz^2v^3 - 384(3x^2yz^2 - 7y^2z^3)v) \partial_z \wedge \partial_v$. It is Poisson-trivial: $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{Y}, \mathcal{P} \rrbracket$ with a representative $\vec{Y} = (-96x^4 + 576y^2z^2 - 384yzv^2) \partial_x + (-192xy^2z + 192y^2zv - 384yv^3) \partial_y + (-96x^3z - 96xzv^2 + 96zv^3 + 96(x^2z - 4yz^2)v) \partial_z + (-576y^2z^2 - 384xyzv + 96v^4) \partial_v$. These explicit examples of Poisson-exact bi-vector flows $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = \llbracket \vec{Y}, \mathcal{P} \rrbracket$ will be useful in the future study of the mechanism $\vec{Y} = \vec{Y}(\gamma, \vec{V}, \mathcal{P})$ of their observed $\partial_{\mathcal{P}}$ -triviality.

The cocycle equation at hand, $\mathcal{E}(\gamma_3, a, \varrho) = \{\dot{\mathcal{P}} = [\vec{\mathcal{Y}}, \mathcal{P}]\}$, is a first-order PDE with differential-polynomial coefficients (their skew-symmetry under permutations of x, y, z is inherited from the property of the Jacobian determinant and from the transformation law for the density ϱ in $\mathcal{P}$). Whether this equation $\mathcal{E}$ does not admit any non-polynomial solutions $\vec{\mathcal{Y}}(a_{|\sigma| \leq 3}, \varrho_{|\tau| \leq 2})$ is an open problem.

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References

- [1] Bouisaghoulane A., Buring R., Kiselev A. The Kontsevich tetrahedral flow revisited // J. Geom. Phys. 2017. V. 119. P. 272–285.
- [2] Brown F. Mixed Tate motives over $\mathbb{Z}$ // Annals of Math. 2012. V. 175. P. 949–976.
- [3] Buring R., Kiselev A. V. The orientation morphism: from graph cocycles to deformations of Poisson structures // J. Phys.: Conf. Ser. 2019. V. 1194. Paper 012017 P. 1–10; Kiselev A. V., Buring R. The Kontsevich graph orientation morphism revisited. 2019. Preprint arXiv:1904.13293 [math.CO]. P. 1–11.
- [4] Buring R., Kiselev A. V. On the Kontsevich $\star$ -product associativity mechanism // PEPAN Letters. 2017. V. 14(2). P. 403–407; Buring R., Kiselev A. V. The expansion $\star \bmod \bar{o}(\hbar^4)$ and computer-assisted proof schemes in the Kontsevich deformation quantization // Experimental Math. doi:10.1080/10586458.2019.1680463 (Preprint arXiv:1702.00681 [math.CO])
- [5] Buring R., Kiselev A. V., Rutten N. J. Poisson brackets symmetry from the pentagon-wheel cocycle in the graph complex // PEPAN Letters 2018. V. 49(5). P. 924–928.
- [6] Buring R., Kiselev A. V., Rutten N. J. The heptagon-wheel cocycle in the Kontsevich graph complex // J. Nonlin. Math. Phys. 2017. V. 24, Suppl. 1. P. 157–173.
- [7] Dolgushev V. A., Rogers C. L., Willwacher T. H. Kontsevich’s graph complex, GRT, and the deformation complex of the sheaf of polyvector fields // Annals of Math. 2015. V. 182(3). P. 855–943.
- [8] Jost C. Globalizing L -infinity automorphisms of the Schouten algebra of polyvector fields // Differ. Geom. Appl. 2013. V. 31(2). P. 239–247.
- [9] Kiselev A. V. The calculus of multivectors on noncommutative jet spaces // J. Geom. Phys. 2018. V. 130. P. 130–167.
- [10] Kiselev A. V. Open problems in the Kontsevich graph construction of Poisson bracket symmetries // J. Phys.: Conf. Ser. 2019. V. 1416. Paper 012018 P. 1–8.
- [11] Kontsevich M. Formality conjecture // Proc. of Conf. “Deformation theory and symplectic geometry”. Ascona, June 17–21, 1996. Dordrecht: Kluwer Acad. Publ., 1997. P. 139–156; Kontsevich M. Derived Grothendieck–Teichmüller group and graph complexes [after T. Willwacher] // Séminaire Bourbaki (69ème année, 2016–2017). Janvier 2017. No. 1126. P. 183–212.
- [12] Laurent–Gengoux C., Pichereau A., Vanhaecke P. Poisson structures. Grundlehren der mathematischen Wissenschaften 347. Springer-Verlag Berlin Heidelberg, 2013.
- [13] Lichnerowicz A. Les variétés de Poisson et leurs algèbres de Lie associées // J. Differential Geom. 1977. V. 12(2). P. 253–300.
- [14] Pichereau A. Poisson (co)homology and isolated singularities // J. Algebra 2006. V. 299(2). P. 747–777.
- [15] Willwacher T. M. Kontsevich’s graph complex and the Grothendieck–Teichmüller Lie algebra // Invent. Math. 2015. V. 200(3). P. 671–760.