The physics and the mixed Hodge structure of Feynman integrals

Pierre Vanhove

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Abstract. This expository text is an invitation to the relation between quantum field theory Feynman integrals and periods. We first describe the relation between the Feynman parametrization of loop amplitudes and world-line methods, by explaining that the first Symanzik polynomial is the determinant of the period matrix of the graph, and the second Symanzik polynomial is expressed in terms of world-line Green's functions. We then review the relation between Feynman graphs and variations of mixed Hodge structures. Finally, we provide an algorithm for generating the Picard-Fuchs equation satisfied by the all equal mass banana graphs in a two-dimensional space-time to all loop orders.

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Amplitudes relations and monodromies

1. Unitarity methods

Constructions and computations of quantum field theory amplitudes have experienced tremendous progress, leading to powerful methods for evaluating loop amplitudes [**BDK96, Bern92, BDDK94, BCF04**]. These methods made computable many unknown amplitudes and provide an increasing knowledge of gauge theory and gravity amplitudes in various dimensions.

These methods are based on the unitarity properties of the scattering amplitudes in quantum field theory. A quantum field theory amplitude is a multivalued function presenting branch cuts associated to particle production.

For local and Lorentz invariant quantum field theories, the matrix of diffusion S is unitary $SS^\dagger = 1$. Therefore the scattering matrix T, defined as $S = 1 + iT$ satisfies the relation $T - T^\dagger = iTT^\dagger$. The perturbative expansion of the scattering matrix $T = \sum_{n\geq 0} g^n A_n$ leads to unitarity relation on the perturbative amplitudes A_n. This implies that the imaginary (absorptive) part of the amplitudes A_n is expressible as some phase integral of product of lowest order amplitudes through Cutkosky rules [**C60**], and dispersion relation are used to reconstruct the full amplitude. In general the evaluation of the dispersion relations is difficult.

Fortunately, at one-loop order, in four dimensions, we know a basis of scalar integral functions $\{I_r\}$ specified by boxes, triangles, bubbles, tadpoles and rational terms [**BDK96, OPP06, EZ07, EKMZ11**]

$$\mathfrak{A}_n^{1-\text{loop}} = \sum_r c_r \, I_r \tag{1.1}$$

where c_r are rational functions of the kinematics invariants.

An interesting aspect of this construction is that the scalar integral functions have distinctive analytic properties across their branch cuts. For instance the massless four-point amplitude can get a contribution from the massless box $I_4(s,t)$, the one-mass triangles $I_3^{1m}(s)$ and $I_3^{1m}(t)$, the massive massive bubbles $I_2(s)$ and $I_2(t)$. The finite part of these functions contain contributions with distinctive discontinuities that can be isolated by cuts

$$I_4(s,t) \sim \log(-s)\log(-t) \tag{1.2}$$

$$I_3^{1m}(s) \sim \log^2(-s) \tag{1.3}$$

$$I_2(s) \sim \log(-s)\,. \tag{1.4}$$

Higher-point one-loop amplitudes have dilogarithm functions entering the expression of the finite part, e.g. $\mathrm{Li}_2(1 - s_{12}s_{23}/(s_{34}s_{56}))$. Picking a particular kinematic region $s_{12} \to \infty$, this function reduces to its branch cut behaviour $\mathrm{Li}_2(1 - s_{12}s_{23}/(s_{34}s_{56})) \sim -\log(-s_{12})\log(-s_{23}) + \dots$ which can be isolated by the cut.

It is now enough to look at the discontinuities across the various branch cuts to extract the coefficients c_r in (1.1). The ambiguity has to be a rational function of the kinematic invariants. There are various methods to fix this ambiguity that are discussed for instance in [**BDK96**].

One of the advantages of having a basis of integral functions is that it permits us to state properties of the amplitudes without having to explicitly compute them, like the no-triangle property in $\mathcal{N} = 8$ supergravity [**BCFIJ07, BBV08a, BBV08b, ACK08**], or in multi-photon QED amplitudes at one-loop [**BBBV08**].

We hope that this approach can help to get a between control of the higher-loop amplitudes contributions in field theory. At higher loop order no basis is known for the amplitudes although it known that a basis must exist at each loop order [**SP10**].

Feynman integrals from multi-loop amplitudes in quantum field theory are multivalued functions. They have monodromy properties around the branch cuts in the complex energy plane, and satisfy differential equations. This is a strong motivation for looking at the relation between integrals from amplitudes and *periods* of multivalued functions. The relation between Feynman integrals and periods is described in 4.

2. Monodromy and tree-level amplitude relations

Before considering higher loop integrals we start discussing tree-level amplitudes. Tree-level amplitudes are not periods but they satisfy relations inherited from to the branch cuts of the integral definition of their string theory ancestor. This will serve as an illustration of how the monodromy properties can constraint the structure of quantum field theory amplitudes in Yang-Mills and gravity.

2.1. Gauge theory amplitudes. An n-point tree-level amplitude in (non-Abelian) gauge theory can be decomposed into color ordered gluon amplitudes

$$\mathfrak{A}_n^{\rm tree}(1,\dots,n)=g_{\rm YM}^{n-2}\sum_{\sigma\in\mathfrak{S}_n/\mathbb{Z}_n}{\rm tr}(t^{\sigma(1)}\cdots t^{\sigma(n)})\,A_n^{\rm tree}(\sigma(1,\dots,n))\,. \tag{2.1}$$

The color stripped amplitudes $A_n^{\rm tree}(\sigma(1,\dots,n))$ are gauge invariant quantities. We are making use of the short hand notation where the entry i is for the polarization ϵ_i and the momenta k_i, and $\mathfrak{S}_n/\mathbb{Z}_n$ denotes the group of permutations $\mathfrak{S}_n$ of n letters modulo cyclic permutations. We will make use of the notation $\sigma(a_1,\dots,a_n)$ for the action of the permutation σ on the a_i.

The color ordered amplitudes satisfy the following properties

- Flip Symmetry

$$A_n^{\rm tree}(1,\dots,n)=(-1)^n\,A_n^{\rm tree}(n,\dots,1) \tag{2.2}$$

- the photon decoupling identity. There is no coupling between the Abelian field (photon) and the non-Abelian field (the gluon), therefore for $t^1=\mathbb{I}$, the identity we have

$$\sum_{\sigma\in\mathfrak{S}_{n-1}}A_n^{\rm tree}(1,\sigma(2,\dots,n))=0 \tag{2.3}$$

These relations show that the color ordered amplitudes are not independent. The number of independent integrals is easily determined by representing the field theory tree amplitudes as the infinite tension limit, $\alpha'\to 0$, limit of the string amplitudes

$$A_n^{\rm tree}(\sigma(1,\dots,n))=\lim_{\alpha'\to 0}\mathcal{A}^{\rm tree}(\sigma(1,\dots,n)) \tag{2.4}$$

where $\mathcal{A}^{\rm tree}(\cdots)$ is the ordered string theory integral

$$\mathcal{A}^{\rm tree}(\sigma(1,\dots,n)):=\int_\Delta f(x_1,\dots,x_n)\prod_{1\le i<j\le n-1}(x_i-x_j)^{\alpha' k_i\cdot k_j}\prod_{i=2}^{n-2}dx_i\,. \tag{2.5}$$

In this integral we have made the following choice for three points along the real axis $x_1 = 0$, $x_{n-1} = 1$ and $x_n = +\infty$ and the domain of integration is defined by

$$\Delta := \{-\infty < x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n-1)} < +\infty\}\,. \tag{2.6}$$

The function $f(x_1, \ldots, x_n)$ depends only on the differences $x_i - x_j$ for $i \neq j$. This function has poles in some of the $x_i - x_j$ but does not have any branch cut.

Since for generic values of the external momenta the scalar products $\alpha' k_i \cdot k_j$ are real numbers, the factors $(x_i - x_j)^{\alpha' k_i \cdot k_j}$ in the integrand require a determination of the power x^α for $x < 0$

$$x^\alpha = |x|^\alpha \begin{cases} e^{i\pi\alpha} & \mathrm{Im}(x) \geq 0\,, \\ e^{-i\pi\alpha} & \mathrm{Im}(x) < 0\,. \end{cases} \tag{2.7}$$

Therefore the different orderings of the external legs, corresponding to different choices of the permutation σ in (2.5), are affected by choice of the branch cut. The different orderings are obtained by contour deformation of integrals [**KLT85, BBDV09, St09**]. This leads to a monodromy matrix that can simply expressed in terms of the *momentum kernel* in string theory [**BBDSV10**]

$$\mathcal{S}_{\alpha'}[i_1, \ldots, i_k | j_1, \ldots, j_k]_p := \prod_{t=1}^{k} \frac{1}{\pi\alpha'} \sin \alpha'\pi \big(p \cdot k_{i_t} + \sum_{q>t}^{k} \theta(t,q)\, k_{i_t} \cdot k_{i_q}\big)\,, \tag{2.8}$$

and its field theory limit when $\alpha' \to 0$ [**BBFS10, BBDFS10a, BBDFS10b**]

$$\mathcal{S}[i_1, \ldots, i_k | j_1, \ldots, j_k]_p = \prod_{t=1}^{k} \big(p \cdot k_{i_t} + \sum_{q>t}^{k} \theta(t,q)\, k_{i_t} \cdot k_{i_q}\big)\,, \tag{2.9}$$

where $\theta(i_t, i_q)$ equals 1 if the ordering of the legs i_t and i_q is opposite in the sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$, and 0 if the ordering is the same.

As a consequence of the proprieties of the string theory integral around the branch points one obtains that the color-ordered amplitudes satisfy the annihilation relations both in string theory and in the field theory limit

$$\sum_{\sigma \in \mathfrak{S}_{n-2}} \mathcal{S}[\sigma(2, \ldots, n-1) | \beta(2, \ldots, n-1)]_1\, A_n(n, \sigma(2, \ldots, n-1), 1) = 0\,, \tag{2.10}$$

for all permutations $\beta \in \mathfrak{S}_{n-2}$.

These relations are equivalent to the BCJ relations between tree-level amplitudes [**BCJ08**], and they imply that the all color-ordered amplitude can be expressed in a basis of $(n-3)!$ amplitudes [**BBDV09, St09**].

2.2. The gravity amplitudes. In the same way one can express the gravity amplitude by considering string theory amplitudes on the sphere with n marked points.

After fixing the three points $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$, the n-point closed string amplitude takes the general form

$$\mathcal{M}_n = \left(\frac{i}{2\pi\alpha'}\right)^{n-3} \int \prod_{1 \leq i < j \leq n-1} |z_j - z_i|^{2\alpha' k_i \cdot k_j}\, f(z_i)\, g(\bar{z}_i) \prod_{i=2}^{n-2} d^2 z_i\,, \tag{2.11}$$

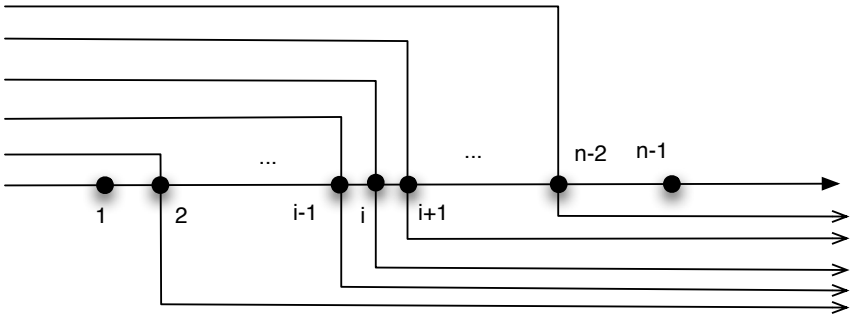


FIGURE 1. *The nested structure of the contours of integration for the variable v_i^- corresponding to the ordering $0 < v_2^+ < v_3^+ < \cdots < v_{n-2}^+ < 1$ of the v_+ variables.*

where $f(z_i)$ and $g(\bar{z}_i)$ arise from the operator product expansion of the vertex operators. They are functions without branch cuts of the differences $z_i - z_j$ and $\bar{z}_i - \bar{z}_j$ with possible poles in these variables. The precise form of these functions depends on the external states.

Changing variables to $z_i = v_i^1 + iv_i^2$, one can factorize the integral (we refer to [**BBDSV10**] for details)

$$\begin{aligned}\mathcal{M}_n = \left(\frac{-1}{4\pi\alpha'}\right)^{n-3} \int_{-\infty}^{+\infty} \prod_{i=2}^{n-2} dv_i^+ dv_i^- \, f(v_i^-)\, g(v_i^+) \\ \times (v_i^+)^{\alpha' k_1\cdot k_i} (v_i^-)^{\alpha' k_1\cdot k_i} (v_i^+ - 1)^{\alpha' k_{n-1}\cdot k_i} (v_i^- - 1)^{\alpha' k_{n-1}\cdot k_i} \\ \times \prod_{i<j\leq n-2} \left(v_i^+ - v_j^+\right)^{\alpha' k_i\cdot k_j} \left(v_i^- - v_j^-\right)^{\alpha' k_i\cdot k_j} .\end{aligned} \tag{2.12}$$

We now consider the deformations of the contours of integration for the v_i^- variables given in figure 1. Because the contours cannot cross each other we need to close them either to the right, turning around the branch cut at $z = 1$ by starting with the rightmost, or close the contours to the left, turning around the branch cut at $z = 0$, starting with the leftmost.

There is evidently an arbitrariness in the number of contours that are closed to the left or to the right. For a given $2 \leq j \leq n-2$, we can pull the contours for the set between 2 and $j-1$ to the left, and the set between j and $n-2$ to the right. The independence of the amplitude under this choice is a consequence of the monodromy relations in eq. (2.10).

To get the full closed string amplitude (2.11) we need to multiply the left-moving amplitude of the v^+ integrations with the right-moving contribution from the integration over v^- and then sum over all orderings to get

$$\begin{aligned}\mathcal{M}_n = (-i/4)^{n-3} \times \sum_{\sigma\in\mathfrak{S}_{n-3}} \sum_{\gamma\in\mathfrak{S}_{j-2}} \sum_{\beta\in\mathfrak{S}_{n-j-1}} \\ \mathcal{S}_{\alpha'}[\gamma\circ\sigma(2,\ldots,j-1)|\sigma(2,\ldots,j-1)]_{k_1} \mathcal{S}_{\alpha'}[\beta\circ\sigma(j,\ldots,n-2)|\sigma(j,\ldots,n-2)]_{k_{n-1}} \\ \times \mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)\, \widetilde{\mathcal{A}}_n(\gamma\circ\sigma(2,\ldots,j-1),1,n-1,\beta\circ\sigma(j,\ldots,n-2),n) .\end{aligned} \tag{2.13}$$

where the amplitudes $\mathcal{A}(\cdots)$ (respectively $\widetilde{\mathcal{A}}(\cdots)$) are obtained from integration over the variables v_i^+ (respectively v_i^-). This provides a general form of the closed/open string relation between external gauge bosons and gravitons at tree-level. When restricted to the case of graviton external states the field theory limit of this expression reduces to the form derived in [**BBDFS10a, BBDFS10b**].

The choice of contour deformation made by KLT in [**KLT85**] consists in closing half of the contours to the left and and the other half to the right. This leads to the smallest number of terms in the sum (2.13).

For $j = n-1$ the field theory gravity amplitude takes a form characteristic of the expression of gravity amplitudes as a sum of square of Yang-Mills amplitudes

$$M_n = (-1)^{n-3} \sum_{\sigma,\gamma\in\mathfrak{S}_{n-3}} \mathcal{S}[\gamma(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_{k_1} \times A_n(1,\sigma(2,\ldots,n-2),n-1,n)\widetilde{A}_n(n-1,n,\gamma(2,\ldots,n-2),1)\,. \tag{2.14}$$

The previous construction provided amplitudes relations between massless tree-level amplitudes in gauge and gravity amplitudes.

This construction does not make any explicit reference to a given space-time dimension, seeing a massive particle in, say in dimension $D = 4$, as the dimensional reduction of a massless particle in higher dimensions, it is immediate that the amplitude relations formulated with the momentum kernel are valid for massive external particles. This has been applied to amplitudes between massive matter field in pure gravity [**BBDV13**].

This construction made an important use of the multivalueness of the factors $\prod_{1\leq i<j\leq n}(x_i-x_j)^{a_{ij}}$ with $a_{ij} = \alpha\, k_i\cdot k_j \in \mathbb{R}$. Although the tree-level integrals are not periods, their infinity tension expansion for $\alpha' \to 0$ is expressible in terms of multiple zeta values that are periods [**Brown13, BSS13, St13, ST14**].

Feynman integrals and periods

3. Feynman integral

3.1. The Feynman parametrization. A connected Feynman graph Γ is determined by the number n of propagators (internal edges), the number l of loops, and the number v of vertices. The Euler characteristic of the graph relates these three numbers as $l = n-v+1$, therefore only the number of loops l and the number n of propagators are needed.

In a momentum representation an l-loop with n propagators Feynman graph reads[1]

$$I^D_\Gamma(p_i,m_i) := \frac{(\mu^2)^{\sum_{i=1}^n \nu_i - l\frac{D}{2}}}{\pi^{\frac{lD}{2}}} \frac{\prod_{i=1}^n \Gamma(\nu_i)}{\Gamma(\sum_{i=1}^n \nu_i - l\frac{D}{2})} \int_{(\mathbb{R}^{1,D-1})^l} \frac{\prod_{i=1}^l d^D\ell_i}{\prod_{i=1}^n (q_i^2-m_i^2+i\varepsilon)^{\nu_i}} \tag{3.1}$$

where μ^2 is a scale of dimension mass squared. Some of the vertices are connected to external momenta p_i with $i = 1,\ldots,v_e$ with $0 \leq v_e \leq v$. The internal masses are positive $m_i \geq 0$ with $1 \leq i \leq n$. Finally $+i\varepsilon$ with $\varepsilon > 0$ is the Feynman prescription

[1] In this text we will consider only graph without numerator factors. A similar discussion can be extended to this case but will not be considered here.

for the propagators for a space-time metric of signature $(+-\cdots-)$, and D is the space-time dimension, and we set $\nu := \sum_{i=1}^n \nu_i$.

Introducing the size l vector of loop momenta $L^\mu := (\ell_1^\mu, \ldots, \ell_l^\mu)^T$ corresponding to the minimal set of linearly independent momenta flowing along the graph. We introduce as well the size v_e vector of external momenta $P^\mu = (p_1^\mu, \ldots, p_{v_e}^\mu)^T$. Since we take the convention that all momenta are incoming momentum conservation implies that $\sum_{i=1}^{v_e} p_i = 0$.

Putting the momenta q_i flowing along the graph in a size n vector $q^\mu := (q_1^\mu, \ldots, q_n^\mu)^T$. Momentum conservation at each vertices of the graph gives the relation

$$q^\mu = \rho \cdot L^\mu + \sigma \cdot P^\mu \,. \tag{3.2}$$

The matrix ρ of size $n \times l$ has entries taking values in $\{-1, 0, 1\}$, the signs depend on an orientation of the propagators. (The orientation of the graph and the choice of basis for the loop momenta will be discussed further in section 3.3.) The matrix σ of size $n \times v_e$ has only entries taking values in $\{0, 1\}$ because have the convention that all external momenta are incoming.

We introduce the Schwinger proper-times α_i conjugated to each internal propagators

$$I_\Gamma^D(p_i, m_i) = \frac{(\mu^2)^{\nu - l\frac{D}{2}}}{\pi^{\frac{lD}{2}} \Gamma(\nu - l\frac{D}{2})} \int_{(\mathbb{R}^{1,D-1})^l} \int_{[0,+\infty[^n} e^{-\sum_{i=1}^n \alpha_i (q_i^2 - m_i^2 + i\varepsilon)} \prod_{i=1}^n \frac{d\alpha_i}{\alpha_i^{1-\nu_i}} \prod_{i=1}^l d^D \ell_i \,. \tag{3.3}$$

Setting $T = \sum_{i=1}^n \alpha_i$ and $\alpha_i = T\, x_i$ this integral becomes

$$I_\Gamma^D(p_i, m_i) = \frac{(\mu^2)^{\nu - l\frac{D}{2}}}{\pi^{\frac{lD}{2}} \Gamma(\nu - l\frac{D}{2})} \int_{(\mathbb{R}^{1,D-1})^l} \int_{[0,+\infty[^{n+1}} e^{-T\mathcal{Q}} \delta(\sum_{i=1}^n x_i - 1) \frac{dT}{T^{1-\nu}} \prod_{i=1}^n \frac{dx_i}{x_i^{1-\nu_i}} \prod_{i=1}^l d^D \ell_i \,, \tag{3.4}$$

where we have defined

$$\mathcal{Q} := \sum_{i=1}^n x_i (q_i^2 - m_i^2) \,. \tag{3.5}$$

Introducing the $n \times n$ diagonal matrix $X = \mathrm{diag}(x_1, \cdots, x_n)$, one rewrites this expression exhibiting the quadratic form in the loop momenta

$$\mathcal{Q} = (L^\mu + \Omega^{-1} Q^\mu)^T \cdot \Omega \cdot (L^\mu + \Omega^{-1} Q^\mu) - J - (Q^\mu)^T \cdot \Omega^{-1} \cdot Q^\mu \,, \tag{3.6}$$

where we have defined

$$\Omega := \rho^T X \rho, \quad Q^\mu := \rho^T X \sigma\, P^\mu, \quad J := (P^\mu)^T \sigma^T X \sigma P^\mu + \sum_{i=1}^n x_i (m_i^2 - i\varepsilon) \tag{3.7}$$

and we made use of the fact that the square $l \times l$ matrix Ω is symmetric and invertible. Performing the Gaussian integral over the loop momenta L^μ one gets

$$I_\Gamma^D(p_i, m_i) = \frac{(\mu^2)^{\nu - l\frac{D}{2}}}{\Gamma(\nu - l\frac{D}{2})} \int_{[0,+\infty[^{n+1}} e^{-T\mu^2 \mathcal{F}\mathcal{U}^{-1}} \frac{\delta(\sum_{i=1}^n x_i - 1)}{\mathcal{U}^{l\frac{D}{2}}} \prod_{i=1}^n \frac{dx_i}{x_i^{1-\nu_i}} \frac{dT}{T^{1-\nu+l\frac{D}{2}}} \,. \tag{3.8}$$

Introducing the notations for the first Symanzik polynomial

$$\mathcal{U} := \det(\Omega) \tag{3.9}$$

and using the adjugate matrix of $\mathrm{Adj}(\Omega) := \det\Omega\,\Omega^{-1}$, we define the second Symanzik polynomial

$$\mathcal{F} := \frac{-J\,\mathcal{U} + (Q^\mu)^T \cdot \mathrm{Adj}(\Omega) \cdot Q^\mu}{\mu^2}\,. \tag{3.10}$$

A modern approach to the derivation of these polynomials using graph theory is given in [**BW10**]. In section 3.3 we will give an interpretation of these quantities using the first quantized world-line formalism.

Performing the integration over T, one arrives at the expression for a Feynman graph given in quantum field theory textbooks like [**IZ80**]

$$I_\Gamma^D(p_i, m_i) = \int_{[0,+\infty[^n} \frac{\mathcal{U}^{\nu-(l+1)\frac{D}{2}}}{\mathcal{F}^{\nu-l\frac{D}{2}}}\,\delta(\sum_{i=1}^n x_i - 1)\prod_{i=1}^n x_i^{\nu_i-1}dx_i\,. \tag{3.11}$$

- Notice that $\mathcal{U}$ and $\mathcal{F}$ are independent of the dimension of space-time. The space-time dimension enters only in the powers of $\mathcal{U}$ and $\mathcal{F}$ in the expression for the Feynman graph in (3.11).
- The graph polynomial $\mathcal{U}$ is an homogeneous polynomial of degree l in the Feynman parameters x_i. $\mathcal{U}$ is linear in each of the x_i. This graph polynomial does not depend on the internal masses m_i or the external momenta p_i. In section 3.3, we argue that this polynomial is the determinant of the period matrix of the graph.
- The graph polynomial $\mathcal{F}$ is of degree $l+1$. This polynomial depends on the internal masses m_i and the kinematic invariants $p_i \cdot p_j$. If all internal masses are vanishing then $\mathcal{F}$ is linear in the Feynman parameters x_i as is $\mathcal{U}$.

Since the coordinate scaling $(x_1, \ldots, x_n) \to \lambda(x_1, \cdots, x_n)$ leaves invariant the integrand and the domain of integration, we can rewrite this integral as

$$I_\Gamma^D(p_i, m_i) = \int_\Delta \prod_{i=1}^n x_i^{\nu_i-1}\,\frac{\mathcal{U}^{\nu-(l+1)\frac{D}{2}}}{\mathcal{F}^{\nu-l\frac{D}{2}}}\,\omega \tag{3.12}$$

where ω is the differential $n-1$-form

$$\omega := \sum_{j=1}^n (-1)^{j-1}\,x_j\,dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \tag{3.13}$$

where $\widehat{dx_j}$ means that dx_j is omitting in this sum. The domain of integration $\mathcal{D}$ is defined as

$$\Delta := \{[x_1, \cdots, x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}, x_i \geq 0\}. \tag{3.14}$$

3.2. Ultraviolet and infrared divergences. The Feynman integrals in (3.1) and (3.11) evaluated in four dimensions $D = 4$ have in general ultraviolet and infrared divergences. One can work in a space-time dimension around four dimension by setting $D = 4 - 2\epsilon$ in the expression (3.11) and performing a Laurent series expansion around $\epsilon = 0$

$$I_\Gamma^{(4-2\epsilon)}(p_i, m_i) = \sum_{k \geq -2l} \epsilon^k \, I_\Gamma^{(k)}(p_i, m_i) + O(\epsilon) \, . \tag{3.15}$$

At one-loop around four dimensions the structure of the integrals is now very well understood [**BDK97, BCF04, OPP06, BDK96, Bri10, EKMZ11**]. General formulas for all one-loop amplitudes can be found in [**EZ07, QCD**], some higher-loop recent considerations can be found in [**P14**].

In this work we discuss properties of the Feynman integrals valid for any values of D but when we evaluate Feynman integrals in sections 8—10 we will work with both ultraviolet and infrared finite integrals. If all the internal masses are positive $m_i > 0$ for $1 \leq i \leq n$ then the integrals are free of infrared divergences. Working in $D = 2$ will make the integrals free of ultraviolet divergences. One can then relate the expansion around four dimensions to the one around two dimensions using the dimension shifting relations [**T96**].

3.3. The word-line formalism. The world-line formalism is a first quantized approach to amplitude computations. This formalism has the advantage of being close in spirit to the one followed by string theory perturbation.

Some of the rules for the world-line formalism can be deduced from a field theory limit infinite tension limit, $\alpha' \to 0$, of string theory. At one-loop order this leads to the so-called 'string based rules' used to compute amplitude in QCD [**BK87, BK90, BK91, BD91, Str92**] and in gravity [**BDS93, DN94**], as reviewed in [**Bern92, Sc01**]. One can motivate this construction by taking a field theory limit of string theory amplitudes as for instance in [**FMR99, dVMMLR96, MPRS13, T13**].

At one-loop order, this formalism has the advantage of making obvious some generic properties of the amplitudes like the no-triangle properties in $\mathcal{N} = 8$ supergravity [**BBV08a, BBV08b**] or for multi-photon amplitudes in QEDa at one-loop [**BBBV08**]. These rules have been extended to higher-loop orders in [**ScS94, RS96, RS97, SaS98**]. See for instance [**GKW99, GRV08**] for a treatment of the two-loop four-graviton $\mathcal{N} = 8$ supergravity amplitude in various dimensions. This formalism is compatible with the pure spinor formalism providing a first quantized approach for the super-particle [**Berk01, BB09**] that can be applied to amplitude computations in maximal supergravity [**AGV04, BG10, Bj10, CK12**].

In this section we will follow a more direct approach to describe the relation between Feynman graphs and the world-line approach. A more systematic derivation will be given elsewhere.

First consider an l-loop vacuum graph Γ_0 without external momenta with n_0 propagators (edges) and $v_0 = n_0 - l + 1$ vertices. One needs to assign a labeling and an orientation of the vacuum graph corresponding to a choice of l independent loop momenta circulating along the loop, this orientation conditions the signs in the incidence matrix in (3.2). Label the Schwinger proper-time of each propagator

by T_i with $i = 1, \ldots, n_0$. Therefore the expression for $\mathcal{Q}$ in (3.5) becomes

$$\mathcal{Q}_0 = (L^\mu)^T \cdot \Omega \cdot (L^\mu) - \sum_{i=1}^{n_0} T_i(m_i^2 - i\varepsilon)\,. \tag{3.16}$$

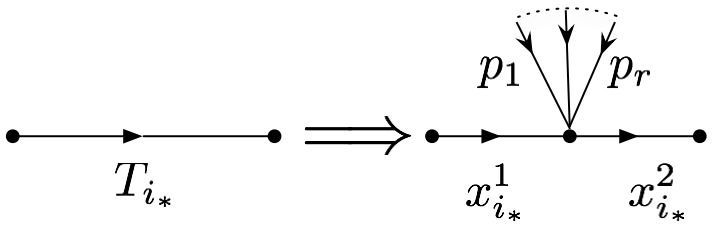
where $\Omega = \rho^T \mathrm{diag}(T_1, \ldots, T_{n_0})\rho$ is the *period matrix* associated with the graph. The presentation will follow the one given in [**GRV08**, section 2.1] for two-loop graphs. Let choose a basis of oriented closed loops C_i with $1 \leq i \leq l$ for the graph. In this context the loop number l is the first Betti number of the graph. And let ω_i be the elementary line element along the closed loop C_i. The entries of the matrix Ω constructed above are given by the oriented circulation of these line elements along each loop C_i

$$\Omega_{ij} = \oint_{C_i} \omega_j\,. \tag{3.17}$$

A direct construction of this matrix from graph theory is detailed in [**DS06**, section IV].

We now consider Feynman graphs with external momenta. One can construct such graphs by starting from a particular vacuum graph Γ_0 and adding to it external momenta.

- One can add external momenta to some vertices of the vacuum graph. This operation does not modify the numbers of vertices and propagators of the graph. This will affect external momentum dependence part in the definition of the existing momenta q_i in (3.16). This operation does not modify definition of the period matrix Ω.
- One can consider adding new vertices with incoming momenta. One vertex attached to external momenta, has to be added on a given internal edge (propagator) say i_*. Under this operator the number of vertices has increased by one unit as well as the number of propagators. The number of loops has not been modified. This operation splits the internal propagator i_* into two as depicted



  Under this operation the proper-time $T_{i_*} = x^1_{i_*} + x^2_{i_*}$ and the momentum flowing along the edge i_* is replaced by $q^1_{i_*} = q_{i_*}$ and $q^2_{i_*} = q^1_{i_*} + P$ where $P = \sum_{j=1}^r P_j$ is the sum of all incoming momenta on the added vertex. The insertion point of the new vertex is parametrized by $x^2_{i_*}$. The expression for $\mathcal{Q}$ in (3.5) then becomes

$$\mathcal{Q} = \sum_{\substack{i=1 \\ i\neq i_*}}^{n} T_i(q_i^2 - m_i^2 + i\varepsilon) + x^1_{i_*}((q^1_{i_*})^2 - m^2_{i_*} + i\varepsilon) + x^2_{i_*}((q^2_{i_*})^2 - m^2_{i_*} + i\varepsilon)\,. \tag{3.18}$$

Since $x^1_{i_*}((q^1_{i_*})^2 - m^2_{i_*} + i\varepsilon) + x^2_{i_*}((q^2_{i_*})^2 - m^2_{i_*} + i\varepsilon) = T_{i_*}((q_{i_*})^2 - m_{i_*} + i\varepsilon) + 2x^2_{i_*} q_{i_*} \cdot P$, we conclude that the matrix Ω entering the expressions for $\mathcal{Q}$ in (3.6) for a graph with external momenta is the same as the one of the associated vacuum graph.

We therefore conclude that in the representation of a Feynman graph Γ in (3.11) and (3.12) the first Symanzik polynomial $\mathcal{U}$ is the determinant of the period matrix of the vacuum graph Γ_0 associated to the graph Γ.

We now turn to the reinterpretation of the second Symanzik polynomial $\mathcal{F}$ in (3.10) using the world-line methods.

Define $\hat{\mathcal{F}} := \mathcal{F}/\mathcal{U}$ one can rewrite this expression in the following way

$$\hat{\mathcal{F}} = -\sum_{i=1}^{n} x_i(m_i^2 - i\varepsilon) + \sum_{1\leq r,s\leq m} p_r \cdot p_s \, G(x_r, x_s; \Omega)\,. \tag{3.19}$$

where we have introduced the *Green function*

$$G(x_r, x_s; \Omega) = -\frac{1}{2} d(x_r, x_s) + \frac{1}{2}\left(\int_{x_s}^{x_r} \omega\right) \cdot \Omega^{-1} \cdot \left(\int_{x_s}^{x_r} \omega\right)\,, \tag{3.20}$$

where $\omega = (\omega_1, \ldots, \omega_l)^T$ the size l vectors of elementary line elements along the loops, and $d(x_r, x_s)$ is the distance between the two vertices of coordinates x_r and x_s on the graph. This is the expression for the Green function between two points on the word-line graph constructed in [**DS06**, section IV]. In section 3.3.2 we provide a few examples.

One can therefore give an alternative form for the parametric representation of the Feynman graph Γ, by writing splitting the integration over the parameters into an integration over the proper-times T_i with $1 \leq i \leq n_0$ of the vacuum graph, and the insertion points x_i with $1 \leq i \leq m$ of the extra vertices carrying external momenta

$$I^D_\Gamma(p_i, m_i) = \int_{\mathcal{D}} \frac{1}{\hat{\mathcal{F}}^{3(l-1)+m-l\frac{D}{2}}} \prod_{i=1}^{m} dx_i \, \frac{\prod_{i=1}^{n_0} dT_i}{(\det \Omega)^{\frac{D}{2}}}\,. \tag{3.21}$$

For the case of φ^3 vertices, at the loop order l, the number of propagators of vacuum graph is $n_0 = 3(l-1)$ and this formulation leads to a treatment of field theory graphs in a string theory manner as in [**BBV08a, BBV08b, BBBV08, BG10, Bj10**].

3.3.1. *Examples of period matrices.* The construction applies to any kind of interaction since we never used the details of the valence of the vertices one needs to consider. We provide a few example based on φ^3 and φ^4 scalar theories.

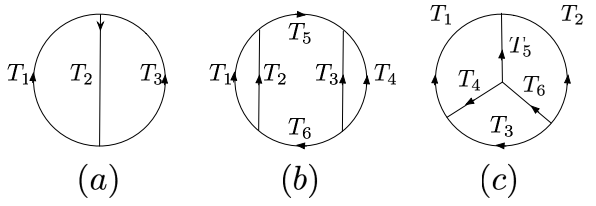


FIGURE 2. Examples of φ^3 vacuum graphs at (a) two-loop order, (b) and (c) at three-loop order.

For instance for the two-loop and three-loop graphs of figure 2 the period matrix are given by

$$\begin{aligned} \Omega_2 &= \begin{pmatrix} T_1+T_3 & T_3 \\ T_3 & T_2+T_3 \end{pmatrix} \text{ in figure } 2(a) \\ \Omega_3 &= \begin{pmatrix} T_1+T_2 & T_2 & 0 \\ T_2 & T_2+T_3+T_5+T_6 & T_3 \\ 0 & T_3 & T_3+T_4 \end{pmatrix} \text{ in figure } 2(b) \\ \Omega_3 &= \begin{pmatrix} T_1+T_4+T_5 & T_5 & T_4 \\ T_5 & T_2+T_5+T_6 & T_6 \\ T_4 & T_6 & T_3+T_4+T_6 \end{pmatrix} \text{ in figure } 2(c)\,. \end{aligned} \tag{3.22}$$

A list of period matrices for φ^3 vacuum graphs up to and including four loops can be found in [**BG10, Bj10**].

For the banana graphs with n propagators (and $n-1$ loops) in figure 3 the period matrix is given by

$$\Omega_{banana} = \begin{pmatrix} T_1+T_n & T_n & \cdots & & T_n \\ T_n & T_2+T_n & T_n & \cdots & T_n \\ \vdots & & & & \vdots \\ T_n & \cdots & & T_n & T_{n-1}+T_n \end{pmatrix}. \tag{3.23}$$

These graphs will be discussed in detail in section 8

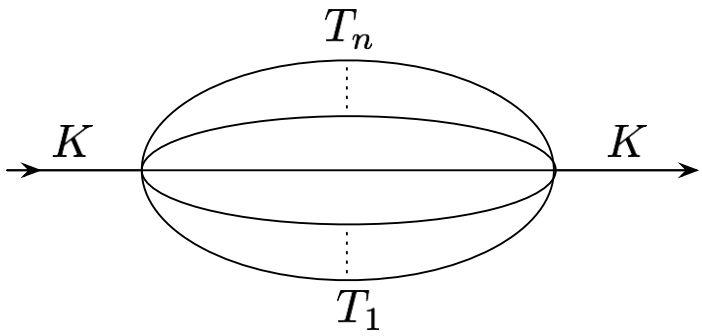


FIGURE 3. Graph for the banana graph with n propagators.

3.3.2. *Example of Green functions.* We provide a example of the construction of the second Symanzik polynomial $\hat{\mathcal{F}}$ using the Green function method.

For the one-loop graph of figure 4(a) the period matrix $\Omega = T$ is the length of the loop. The Green function between the external states with momenta p_r and p_s is given by

$$G^{1-\text{loop}}(x_r, x_s; L) = -\frac{1}{2}\,|x_s - x_r| + \frac{1}{2}\,\frac{(x_r - x_s)^2}{T}\,. \tag{3.24}$$

For massless external states $p_r^2 = 0$ for $1 \leq r \leq n$, the reduced second Symanzik polynomial $\hat{\mathcal{F}}^{1-loop}$ is given by

$$\hat{\mathcal{F}}^{1-\text{loop}} = \sum_{1 \leq r < s \leq n} p_r \cdot p_s\, G^{1-loop}(x_r, x_s; L)\,. \tag{3.25}$$

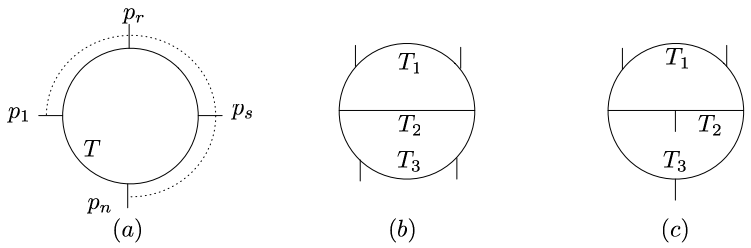


FIGURE 4. (a) One-loop n-point graph and (b)-(c) two-loop four-point graphs.

In the massless case the two-loop Green's function have been derived in [**GRV08**, section 2.1]. If the two external states with momenta p_r and p_s are on the same line, say the one of length T_1, then

$$G^{2-\text{loop}}(x_r,x_s;\Omega_2)=-\frac{1}{2}|x_s-x_r|+\frac{T_2+T_3}{2}\frac{(x_s-x_r)^2}{(T_1T_2+T_1T_3+T_2T_3)}\,, \tag{3.26}$$

where we used the two-loop period matrix Ω_2 in (3.22) such that $\det\Omega_2=T_1T_2+T_1T_3+T_2T_3$. If the external states with momenta p_r and p_s are on different lines, say p_r is on T_1 and p_s on T_2, then the Green's function is given by

$$G^{2-\text{loop}}(x_r,x_s;\Omega_2)=-\frac{1}{2}(x_r+x_s)+\frac{T_3(x_r+x_s)^2+T_2x_r^2+T_1x_s^2}{2(T_1T_2+T_1T_3+T_2T_3)}\,. \tag{3.27}$$

With these Green functions the reduced second Symanzik polynomial for the four-point two-loop graphs in figure 4(b)-(c) read

$$\hat{\mathcal{F}}^{2-loop}=\sum_{1\le r<s\le 4}p_r\cdot p_s\,G^{2-loop}(x_r,x_s;\Omega_2)\,. \tag{3.28}$$

4. Periods

In the survey [**KZ01**], Kontsevich and Zagier give the following definition of the ring $\mathcal{P}$ of periods: *a period is a complex number that can be expressed as an integral of an algebraic function over an algebraic domain.*

In more precise terms $z\in\mathcal{P}$ is a period if its real part $\Re\mathrm{e}(z)$ and imaginary part $\Im\mathrm{m}(z)$ are of the form

$$\int_\Delta \frac{f(x_1,\dots,x_n)}{g(x_1,\dots,x_n)}\prod_{i=1}^n dx_i\,, \tag{4.1}$$

where $f(x_1,\dots,x_n)$ and $g(x_1,\dots,x_n)$ belong to $\mathbb{Z}[x_1,\cdots,x_n]$ and the domain of integration Δ is a domain in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Since sums and products of periods remain periods, therefore the periods form a ring, and the periods form a sub $\bar{\mathbb{Q}}$-algebra of $\mathbb{C}$ (where $\bar{\mathbb{Q}}$ is the set of algebraic numbers).

Examples of periods represented by single integral

$$\sqrt{2}=\int_{2x^2\le 1}dx;\quad \log(2)=\int_{1\le x\le 2}\frac{dx}{x} \tag{4.2}$$

or by a double integral

$$\zeta(2) = \frac{\pi^2}{6} = \int_{0\leq t_1\leq t_2\leq 1} \frac{dt_2}{t_2}\,\frac{dt_1}{1-t_1}\,. \tag{4.3}$$

This example the value at 1 of the dilogarithm $\mathrm{Li}_2\,(1) = \zeta(2)$ where

$$\mathrm{Li}_2\,(x) = \sum_{n\geq 1} \frac{x^n}{n^2}; \qquad \text{for} \qquad 0 \leq x < 1\,. \tag{4.4}$$

In particular, it is familiar to the quantum field theory practitioner that the finite part of one-loop amplitudes in four dimensions is expressed in terms of dilogarithms.

Under change of variables and integration a period can take a form given in (4.1) or not. One example is π which can be represented by the following two-dimensional or one-dimensional integrals

$$\pi = \int_{x^2+y^2\leq 1} dxdy = 2\int_0^{+\infty} \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \tag{4.5}$$

or by the following contour integral

$$2i\pi = \oint \frac{dz}{z}\,. \tag{4.6}$$

At a first sight, the definition of a period given in (4.1) and the Feynman representation of the Feynman graph in (3.11) *look similar*. A relation between these two objects was remarked in the pioneer work of Broadhurst and Kreimer [**BrK95, BrK96**].

We actually need a more general definition of abstract periods given in [**KZ01**]. Let's consider $X(\mathbb{C})$ a smooth algebraic variety of dimension n over $\mathbb{Q}$. Consider $D \subset X$ a divisor with normal crossings, which means that locally this is a union of coordinate hyperplanes of dimension $n-1$. Let $\eta \in \Omega^n(X)$, and let $\Delta \in H_n(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$ a singular n-chain on $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$. To the quadruple (X, D, ω, Δ) we can associate a complex number called the *period of the quadruple*

$$P(X, D, \omega, \Delta) = \int_\Delta \eta\,. \tag{4.7}$$

In order that this definition is compatible with the example of periods given previously, in particular the behaviour under change of variables one needs to introduce the notation of equivalence classes of quadruples for periods leading to the same period (4.7). To this end one defines the space $\mathbf{P}$ of *effective periods* as the $\mathbb{Q}$-vector space of equivalence classes modulo (a) linearity in η and Δ, (b) under change variables, (c) and integration by part of Stokes formula. The map from $\mathbf{P}$ to the space of periods $\mathcal{P}$ is clearly surjective, and it is conjectured to be injective providing an isomorphism. We refer to the review article [**KZ01**] for more details.

5. Mixed Hodge structures for Feynman graph integrals

In this text we are focusing on ultraviolet and infrared finite Feynman graph integrals. A discussion of logarithmically divergent graphs can be found in [**BEK05**].

To a Feynman graph one can associate two graph hyper-surfaces. One defined from the determinant of the vacuum graph period matrix (the first Symanzik polynomial introduced in section 3.1)

$$X_\Gamma^0 := \{\det \Omega = \mathcal{U}(x_i) = 0 | x_i \in \mathbb{P}^{n-1}(\mathbb{R})\}\,, \tag{5.1}$$

and one graph hyper-surface X_Γ defined by the locus for the zeros of the second Symanzik polynomial $\mathcal{F}$ in (3.12)

$$X_\Gamma := \{\mathcal{F}(x_i) = 0 | x_i \in \mathbb{P}^{n-1}(\mathbb{R})\}\,. \tag{5.2}$$

The polar part X_η of the Feynman integral in (3.12) is the union of these two graph hyper-surfaces unless for $2\nu = (l+1)D$ when it is only given by X_Γ or $2\nu = lD$ when it is only given by X_Γ^0. Although the integrand η is a closed form such that $\eta \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_\eta)$, in general the domain Δ has a boundary and therefore its homology class is not in $H_n(\mathbb{P}^{n-1}\backslash X_\eta)$. This difficulty will be resolved by considering the relative cohomology.

In general the polar part X_η of the differential form entering the expression of the Feynman integral in (3.12), intersects the boundary of the domain of integration $\partial\Delta \cap X_\eta \neq \emptyset$. We need to consider a blow-up in $\mathbb{P}^{n-1}$ of linear space $f : \mathcal{P} \to \mathbb{P}^{n-1}$, such that all the vertices of Δ lie in $\mathcal{P}\backslash\mathcal{X}$ where $\mathcal{X}$ is the strict transform of X_η. Let $\mathcal{B}$ be the total inverse image of the coordinate simplex $\{x_1x_2\cdots x_n = 0 | [x_1,\ldots,x_n] \in \mathbb{P}^n\}$.

As been explained by Bloch, Esnault and Kreimer in [**BEK05**] all of this lead to the mixed Hodge structure associated to the Feynman graph

$$M(\Gamma) := H^{n-1}(\mathcal{P}\backslash\mathcal{X}, \mathcal{B}\backslash\mathcal{B}\cap\mathcal{X}; \mathbb{Q})\,. \tag{5.3}$$

In the second part of this text we will give a description of the motive for particular Feynman integral. A list of mixed Hodge structures associated with vacuum graphs can be found in [**Bloch08, Schn10**].

5.1. Example: The massive one-loop triangle. As an illustration we consider the example of the mixed Hodge structure for the massive one-loop triangle following [**BlochK10**, section 13].

The Feynman integral is given by

$$I_\triangleright(p_1,p_2,p_3) = \frac{\mu^2}{\pi^2}\int \frac{d^4\ell}{\ell^2\,(\ell+p_1)^2\,(\ell-p_3)^2}\,, \tag{5.4}$$

with $p_i^2 = m_i^2 \neq 0$ non vanishing external masses and the momentum conservation constraint $p_1+p_2+p_3 = 0$. This integral is finite being free of ultraviolet divergences in four dimensions, and of infrared divergences for non vanishing masses.

A direct application of section 3.1, for the case of a graph at $l = 1$ loop, with $n = 3$ edges in $D = 4$ dimensions, leads to the Schwinger representation

$$I_\triangleright(p_1,p_2,p_3) = \mu^2\int_{x_1,x_2,x_3\geq 0} \frac{x_1dx_2\wedge dx_3 - x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2}{(x_1+x_2+x_3)(x_1x_2m_3^2+x_1x_3m_2^2+x_2x_3m_1^2)}\,. \tag{5.5}$$

The graph hyper-surfaces are the line $X_\Gamma^0 = \{x_1+x_2+x_3 = 0 | x_i \in \mathbb{P}^2(\mathbb{R})\}$ and the conic $X_\triangleright := \{x_1x_2m_3^2 + x_1x_3m_2^2 + x_2x_3m_1^2 = 0 | x_i \in \mathbb{P}^2(\mathbb{R})\}$. The polar part of the integrand is $X_\eta = X_\Gamma^0 \cup X_\triangleright$. The domain of integration is the triangle $\Delta = \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. The intersection of the graph polar part and the domain of integration is given by the three points $X_\eta \cap \Delta = \{[1:0:0],[0:1:0],[0:0:1]\}$. Let $f : \mathcal{P} \to \mathbb{P}^2$ the blow-up of the three vertices $\{(x_1 = x_2 = 0),(x_1 = x_3 =$

$0), (x_2 = x_3 = 0)\}$. Let $E_i \subset \mathcal{P}$ the exceptional divisors such that E_i lies over the intersection $(x_j = x_k = 0)$, with (i, j, k) a permutation of $(1, 2, 3)$. Finally, let $F_i \subset \mathcal{P}$ be the strict transform of the locus $\{x_i = 0\}$. The blown up domain of integration is the hexagon $\mathcal{B} := f^*\Delta = \cup_{i=1}^3 E_i \cup_{i=1}^3 F_i$. If we denote by $\mathcal{X} = \mathcal{L} \cup \mathcal{C}$ the union of the strict transform $\mathcal{L}$ of the line $x_1 + x_2 + x_3 = 0$ and the strict transform $\mathcal{C}$ of the conic $X_\triangleright$. The mixed Hodge structure of the one-loop massive triangle graph in four dimension is given by (5.3).

6. Variation of mixed Hodge structures

A *pure Hodge structure of weight n* is an algebraic structure generalizing the Hodge theory for compact complex manifold. For a compact complex manifold $\mathcal{M}$ the de Rham cohomology groups $H^n(\mathcal{M}) := H^n(\mathcal{M}, \mathbb{R}) \otimes \mathbb{C}$ can be decomposed

$$H^n(\mathcal{M}) = \bigoplus_{p+q=n} H^{p,q}(\mathcal{M}), \qquad \text{with} \qquad \overline{H^{p,q}(\mathcal{M})} = H^{q,p}(\mathcal{M})\,. \tag{6.1}$$

The Dolbeaut cohomology groups $H^{p,q}(\mathcal{M})$ are defined as the $\bar\partial$-closed (p, q)-forms modulo $\bar\partial A^{p,q-1}(\mathcal{M})$ (see [**GH78**] for a more detailed exposition). By definition $H^n(\mathcal{M})$ is pure Hodge structure of weight n.

When one does not have a complex structure one defines a pure Hodge structure from a Hodge filtration. Let consider a finite dimensional $\mathbb{Q}$-vector space $H = H_\mathbb{Q}$. Suppose given a decreasing filtration $F^\bullet H_\mathbb{C}$ on $H_\mathbb{C} := H_\mathbb{Q} \otimes \mathbb{C}$,

$$H_\mathbb{C} \supseteq \cdots \supseteq F^{p-1}H_\mathbb{C} \supseteq F^p H_\mathbb{C} \supseteq F^{p+1}H_\mathbb{C} \supseteq \cdots \supseteq (0)\,. \tag{6.2}$$

One says that $F^\bullet H_\mathbb{C}$ defines a *pure Hodge structure of weight n*

$$H^n_\mathbb{C} := \bigoplus_{p+q=n} H^{p,q}, \qquad \text{where} \qquad H^{p,q} := F^p H_\mathbb{C} \cap \overline{F^q H_\mathbb{C}} \tag{6.3}$$

where $\overline{F^q H_\mathbb{C}}$ is the complex conjugate of $F^p H_\mathbb{C}$.

Pure Hodge structures are defined for smooth compact manifolds $\mathcal{M}$, but Feynman graph integrals involve non-compact or non-smooth varieties which require using the generalizations provided by the mixed Hodge structures introduced by Deligne [**D70**].

The only pure Hodge structure of dimension one is the *Tate Hodge structure* $\mathbb{Q}(n)$ with

$$F^p\mathbb{Q}(n)_\mathbb{C} = \begin{cases} 0 & \text{for } p > -n \\ \mathbb{Q}(n)_\mathbb{C} & \text{for } i \leq -n\,. \end{cases} \tag{6.4}$$

This means that $\mathbb{Q}(n)_\mathbb{C} = H^{-n,-n}(\mathbb{Q}(n)_\mathbb{C})$ and $\mathbb{Q}(n)$ has weight $-2n$. Notice that $\mathbb{Q}(n) \otimes \mathbb{Q}(m) = \mathbb{Q}(n + m)$, therefore $\mathbb{Q}(n) = \otimes^n \mathbb{Q}(1)$ for $n \in \mathbb{Z}$.

A *mixed Hodge structure* on H is a pair of (finite, separated, exhaustive) filtrations: (a) an increasing filtration $W_\bullet H_\mathbb{Q}$ called the *weight filtration*, (b) a decreasing filtration, the Hodge filtration $F^\bullet H_\mathbb{C}$ described earlier. The Hodge structure on H induces a filtration on the graded pieces for the weight filtration $gr^W_n H := W_n H / W_{n-1} H$. By definition for a mixed Hodge structure, the filtration $gr^W_n H$ should be a pure Hodge structure of weight n.

A mixed Hodge structure H is called *mixed Tate* if

$$gr_n^W H = \begin{cases} 0 & \text{for } n = 2m-1 \\ \bigoplus \mathbb{Q}(-m) & \text{for } n = 2m\,. \end{cases} \tag{6.5}$$

From mixed Hodge structures one can define a matrix of periods. For a mixed Tate Hodge structure the weight and the Hodge filtrations are opposite since $F^{p+1}H_{\mathbb{C}} \cap W_{2p}H_{\mathbb{C}} = (0)$, and $H_{\mathbb{C}} = \bigoplus_p F^p H_{\mathbb{C}} \cap W_{2p}H_{\mathbb{C}}$. We first make a choice of a basis $\{e_i^{p,p} \in F^p H_{\mathbb{C}} \cap W_{2p}H_{\mathbb{C}}\}$ of $H_{\mathbb{C}}$. Then expressing the basis elements $\{\varepsilon_i\}$ for $W_\bullet H$ in terms of the basis for $H_{\mathbb{C}}$ gives a *period matrix* with columns composed by the basis elements of $F^\bullet H$. With a proper choice of the basis for $W_\bullet H$ one can ensure that the period matrix is block lower triangular, with the block diagonal elements corresponding to $gr_n^W H$ given by $(2i\pi)^{-n}$. In the case of the polylogarithms such a period matrix is given in eq. (6.7).

Finally, we need to introduce the notation of *variation of Hodge structure* needed to take into account that Feynman integrals lead to families of Hodge structures parametrized by the variation of the kinematics invariants. These concepts have been introduced by Griffiths in [**G68**] and generalized to mixed Hodge modules over complex varieties by M. Saito in [**S89**]. We refer to these works for details about this, but a particular case of variation of mixed Hodge structure for polylogarithms is discussed in the next section.

A large class of amplitudes evaluate to (multiple) polylogarithms. In this case a study of the discontinuities of the amplitude can give access to interesting algebraic structures [**ABDG14**]. As well elliptic integrals arise from multiloop amplitudes [**CCLR98, MSWZ11, CHL12, ABW13, BV13, RT13**] . One example is the sunset Feynman integral studied in section 10.2. The value of the integral is obtained from a variation of mixed Hodge structure when the external momentum is varying [**BV13**].

6.1. Polylogarithms. A very clear motivic approach to polylogarithms is detailed in the article by Beilinson and Deligne [**DB94**]. We only refer to the main points needed for the present discussion, for details we refer to the articles [**H94, DB94**]. The iterated integral definition of the polylogarithms

$$\begin{aligned} \mathrm{Li}_1(z) &:= -\log(1-z) = \int_0^z \frac{dt}{1-t} \\ \mathrm{Li}_{k+1}(z) &:= \int_0^z \mathrm{Li}_k(z)\,\frac{dt}{t}, \qquad k \geq 1 \end{aligned} \tag{6.6}$$

imply that they provide multivalued function on $\mathbb{P}^1\backslash\{0,1,\infty\}$. These multivalued function have monodromy properties. To this end defined the lower triangular matrix of size $n \times n$ as

(6.7)

$$A(z) := \begin{pmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ -\mathrm{Li}_1(z) & 1 & 0 & \cdots & & \cdots & 0 \\ -\mathrm{Li}_2(z) & \log z & 1 & 0 & \cdots & \cdots & 0 \\ -\mathrm{Li}_3(z) & \frac{(\log z)^2}{2!} & \log z & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \end{pmatrix} \mathrm{diag}(1, 2i\pi, \ldots, (2i\pi)^n)\,.$$

so that $A_{1k}(z) = -\mathrm{Li}_k(z)$ for $1 \leq k \leq n$, and $A_{pq}(z) = (2i\pi)^{p-1}(\log z)^{q-p}/(q-p)!$ for $2 \leq p < q \leq n$.

For a fixed value of z this matrix is the period matrix associated with the mixed Hodge structure for the polylogarithms, the columns are the weight and the lines are the Hodge degree.

A determination of this matrix $A(z)$ depends on the path γ in $\mathbb{P}^1\backslash\{0,1,\infty\}$ and a point $z \in]0,1[$. For a counterclockwise path γ_0 around 0 or γ_1 around 1 the determination of $A(z)$ is changed as

$$A_{\gamma\gamma_i}(z) = A_\gamma(z)\, \exp(e_i); \qquad i = 0, 1 \tag{6.8}$$

where e_i are the nilpotent matrices

$$e_0 := \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & 0 \end{pmatrix}; \quad e_1 := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \cdots & \ddots & \cdots \end{pmatrix}. \tag{6.9}$$

The matrix $A(z)$ satisfies the differential equation

$$dA(z) = (e_0\, d\log(z) + e_1 d\log(z-1))\; A(z)\,. \tag{6.10}$$

This differential equation defined over $\mathbb{C}\backslash\{0,1\}$ defined the *nth polylogarithm local system*. This local system underlies a good variation of mixed Hodge structure whose weight graded quotients are canonically isomorphic to $\mathbb{Q}, \mathbb{Q}(1), \ldots, \mathbb{Q}(n)$ [**H94**, theorem 7.1].

One can define single-valued real analytic function on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, and continuous on $\mathbb{P}^1(\mathbb{C})$. The first important example is the *Bloch-Wigner dilogarithm* defined as [**Bloch00, Z03**]

$$D(z) := \Im\mathrm{m}\left(\mathrm{Li}_2(z) + \log|z|\log(1-z)\right). \tag{6.11}$$

The Bloch-Wigner dilogarithm function satisfies the following functional equations

$$\begin{aligned} D(z) &= -D(\bar{z}) = D(1-z^{-1}) = D((1-z)^{-1}) \\ &= -D(z^{-1}) = -D(1-z) = -D(-z(1-z)^{-1})\,. \end{aligned} \tag{6.12}$$

The differential of the Bloch-Wigner dilogarithm $D(z)$ is given by

$$dD(z) = \log|z| d\arg(1-z) - \log|1-z| d\arg(z)\,. \tag{6.13}$$

At higher-order there is no unique form for the real analytic version of the polylogarithm. A particularly nice version with respect to Hodge structure provided by Beilinson and Deligne in [**DB94**] is given by

$$\mathcal{L}_m(z) := \sum_{k=0}^{m-1} \frac{B_k}{k!} (\log(z\bar{z}))^k \times \begin{cases} \Re\mathrm{e}(\mathrm{Li}_{m-k}(z)) & \text{for } m = 1 \mod 2 \\ \Im\mathrm{m}(\mathrm{Li}_{m-k}(z)) & \text{for } m = 0 \mod 2\,. \end{cases} \tag{6.14}$$

where B_k are Bernoulli numbers $x/(e^x - 1) = \sum_{k\geq 0} B_k\, x^k/k!$.

A *dilogarithm Hodge structure*, relevant to one-loop amplitudes in four dimensions, has been defined in [**BlochK10**] as a mixed Tate Hodge structure such that for some integer n, $gr^W_{2p} H = (0)$ for $p \neq n, n+1, n+2$.

6.2. Polylogarithms and Feynman integrals. The construction of the mixed Hodge structure of the massive triangle in section 5.1 was shown in [**BEK05, BlochK10**] to correspond to the dilogarithm Hodge structure describe above.

Are all the Feynman integral expressible as polylogarithms or multiple polylogarithms in several variables? It is conjectured in [**Henn13**] that using integration by parts one could always express Feynman integrals as a combination of a finite set of master integrals satisfying the differential equation (6.10), therefore leading to multiple polylogarithm functions. Various high-loop graphs have been shown to evaluate to multiple polylogarithms [**HSS13, HS13, CHH14, ABDG14, P14**].

Counter-examples leading to elliptic functions are known in the massless case [**BS12, CHL12, NPSV13**] or the massive case by the sunset graph [**LR04, MSWZ11, ABW13, ABW14**] and three-loop banana graph with all equal internal masses [**BKV14**]. The banana graphs with all equal internal masses, discussed in section 8, lead to elliptic polylogarithms discussed below.

Different functions are expected from other classes of Feynman graphs. Determining and evaluating Feynman integrals are open and difficult questions, where one can hope that mixed Hodge structures or motivic methods be useful.

7. Elliptic polylogarithms

In this section we recall the main properties of the elliptic polylogarithms following [**Bloch00, Z90, GZ00, L97, BL94**].

Let $\mathcal{E}(\mathbb{C})$ be an elliptic curve over $\mathbb{C}$. The elliptic curve can be viewed either as the complex plane modded by a two-dimensional lattice $\mathcal{E}(\mathbb{C})\cong\mathbb{C}/(\mathbb{Z}\varpi_1+\mathbb{Z}\varpi_2)$. A point $z\in\mathbb{C}/(\mathbb{Z}\varpi_1+\mathbb{Z}\varpi_2)$ is associated to a point $P:=(\wp(z),\wp'(z))$ on $\mathcal{E}(\mathbb{C})$ where $\wp(z)=z^{-2}+\sum_{(m,n)\neq(0,0)}\left((z+m\varpi_1+n\varpi_2)^{-2}-(m\varpi_1+n\varpi_2)^{-2}\right)$ is the Weierstraß function and $q:=\exp(2i\pi\tau)$ with $\tau=\varpi_2/\varpi_1$ the period ratio in the upper-half plane $\mathbb{H}=\{\tau|\Re\mathrm{e}(\tau)\in\mathbb{R},\Im\mathrm{m}(\tau)>0\}$. Or we can see the elliptic curve as $\mathcal{E}(\mathbb{C})\cong\mathbb{C}^\times/q^{\mathbb{Z}}$. A point P on the elliptic curve is then mapped to $x:=e^{2i\pi z}$.

One defines an *elliptic polylogarithm* $\mathcal{L}^{\mathcal{E}}_n:\mathcal{E}(\mathbb{C})\to\mathbb{R}$ as the average of the real unvalued version of the polylogarithms

$$\mathcal{L}^{\mathcal{E}}_m(P):=\sum_{n\in\mathbb{Z}}\mathcal{L}_m(x\,q^n) \tag{7.1}$$

where $q:=\exp(2i\pi\tau)$ with $\tau\in\mathfrak{h}:=\{\tau|\Re\mathrm{e}(\tau)\in\mathbb{R},\Im\mathrm{m}(\tau)>0\}$. This series converges absolutely with exponential decay and is invariant under the transformation $x\mapsto qx$ and $x\mapsto q^{-1}x$.

If we have a collection of points P_r on the elliptic curve one can consider a linear combination of the elliptic polylogarithms. Such objects play an important role when computing regulators for elliptic curves, and in the so-call Beilinson conjecture relating the value of the regulator map to the value of L-function of the elliptic curve [**Beilinson85, Den97, Bloch00, Soule86, Bru07**].

Interestingly, as explained in [**BV13**], elliptic polylogarithms from Feynman graphs differ from (7.1). A simple physical reason is that the Feynman integral is a multivalued function therefore cannot be build from a real analytic version of the

polylogarithms. For the examples discussed in section 8 we will need the following sums of the elliptic dilogarithms

$$\sum_{r=1}^{n_r} c_r \sum_{n\geq 0} \mathrm{Li}_2\left(q^n z_r\right) \tag{7.2}$$

where z_r is a finite set of points on the elliptic curve and c_r are rational numbers. This expression is invariant under $z \mapsto qz$ and $z \mapsto q^{-1}z$ only for a *very special* choice of set of points depending on the (algebraic) geometry of the graph. A more precise definition of the quantity appearing from the two-loop sunset Feynman graph is given in equation (10.9).

7.1. Mahler measure. A *logarithmic Mahler measure* is defined by

$$\mu(F) := \oint_{|x_1|=\cdots=|x_n|=1} \log|F(x_1,\ldots,x_n)| \prod_{i=1}^{n} \frac{dx_i}{2i\pi x_i}\,, \tag{7.3}$$

and the *Mahler measure* is defined by $M(F) := \exp(\mu(F))$. In the definition $F(x_1,\ldots,x_n)$ is a Laurent polynomial in x_i.

Numerical experimentations by Boyd [**Bo98**] pointed out to a relation between the logarithmic Mahler measure for certain Laurent polynomials F and values of L-functions of the projective plane curve $C_F : F(x_1,\ldots,x_n) = 0$

$$\mu(F) = \mathbb{Q}^\times\, L'(Z_F, 0)\,. \tag{7.4}$$

In [**RV99**] (see as well [**BRV02, BRVD03**]) Rodrigez-Villegas showed that the logarithmic Mahler measure is given by evaluating the Bloch regulator leading to expressions given by the Bloch-Wigner dilogarithm. The relation in (7.4) is then a consequence of the conjectures by Bloch [**Bloch00**] and Beilinson [**Beilinson85**] relating regulators for elliptic curves to the values of L-functions (see [**Soule86, Bru07**] for some review on these conjectures).

Let consider the logarithmic Mahler measure defined using the second Symanzik polynomial $\mathcal{F}_2(x,y;t) = (1+x+y)(x+y+xy) - txy$ for the two-loop $n=3$ banana graph of figure 3

$$\mu_\ominus(t) = \frac{1}{(2i\pi)^2} \int_{|x|=|y|=1} \log(|\mathcal{F}_2(x,y;t)|) \frac{dxdy}{xy}\,. \tag{7.5}$$

The Mahler measure associated with this polynomial has been studied by Stienstra [**Stien05a, Stien05b**] and Lalin-Rogers in [**LR06**].

Consider the field $F = \mathbb{Q}(\mathcal{E}_\ominus)$ where $\mathcal{E}_\ominus = \{(x,y) \in \mathbb{P}^2 | \mathcal{F}_2(x,y;t) = 0\}$ is the sunset elliptic curve, and consider a Néron $\widehat{\mathcal{E}_\ominus}$ model of $\mathcal{E}_\ominus$ over $\mathbb{Z}$. The regulator map is an application from the higher regulator $K_2(\widehat{\mathcal{E}_\ominus})$ to $H^1(\mathcal{E}_\ominus, \mathbb{R})$ [**Bloch00**]. The regulator map is defined by

$$\begin{aligned} r : K_2(\mathcal{E}_\ominus) &\to H^1(\mathcal{E}_\ominus, \mathbb{R}) \\ \{x,y\} &\mapsto \{\gamma \to \int_\gamma \eta(x,y)\}\,, \end{aligned} \tag{7.6}$$

where

$$\eta(x,y) = \log|x| d\arg(y) - \log|y| d\arg x\,. \tag{7.7}$$

Notice that $\eta(x, 1-x) = dD(x)$ the differential of the Bloch-Wigner dilogarithm.

If x and y are non-constant function on $\mathcal{E}_\ominus$ with divisors $(x)=\sum_i x_i(a_i)$ and $(y)=\sum_i n_i(b_i)$ one associates the quantity $(x)\diamond(y)=\sum_{i,j} m_i n_i(a_i-b_j)$

A theorem by Beilinson states that if $\omega\in\Omega^1(\mathcal{E}_\ominus)$ then

$$\int_{\mathcal{E}_\ominus(\mathbb{C})}\omega\wedge\eta(x,y)=\varpi_r\, R_\tau((x)\diamond(y)) \tag{7.8}$$

where $R_\tau(z)$ is the Kronecker-Eisenstein series [**Weil76, Bloch00**] defined as

$$R_\tau(e^{2i\pi(a+b\tau)}):=\frac{\Im\mathrm{m}(\tau)^2}{\pi^2}\sum_{(p,q)\neq(0,0)}\frac{e^{2i\pi(aq-pb)}}{(p+q\tau)^2(p+q\bar\tau)}\,. \tag{7.9}$$

The logarithmic Mahler measure for the sunset graph is expressed as a sum of elliptic-dilogarithm evaluated at torsion points on the elliptic curve

$$\mu_\ominus(t)=-3\,\Im\mathrm{m}\left(R_\tau(\zeta_6)+R_\tau(\zeta_6^2)\right) \tag{7.10}$$

with $\zeta_6=\exp(i\pi/3)$ is a sixth root of unity and $\tau=\varpi_2/\varpi_1$ is the period ratio of the elliptic curve. This relation is true for t large enough so that the elliptic curve $\mathcal{E}_\ominus$ does not intersect the torus $\mathbb{T}^2=\{|x|=|y|=1\}$.

The Beilinson conjecture [**Beilinson85, Soule86, Bru07**] implies that the Mahler measure is rationally related to the value of the Hasse-Weil L-function for the sunset elliptic curve evaluated at $s=2$

$$\mu_\ominus(t)=\mathbb{Q}^\times\, L(\mathcal{E}_\ominus(t),2)\,. \tag{7.11}$$

which can be easily numerically checked using [**sage**].

Differentiating the Mahler measure with respect to t gives

$$g_1(t):=-t\frac{d\mu_\ominus(t)}{dt}=\frac{1}{(2i\pi)^2}\oint_{|x|=1}\oint_{|y|=1}\frac{t\,dxdy}{\mathcal{F}_2(x,y;t)}\,. \tag{7.12}$$

This quantity is actually a period of the elliptic curve [**Stien05a**] since we are integrating the two-form $\omega=\frac{t\,dxdy}{\mathcal{F}_2(x,y;t)}$ over a two-cycle given by the torus $\mathbb{T}^2=\{|x|=1,|y|=1\}$ (for t large enough so that the elliptic curve does not intersect the torus).

The banana integrals in two dimensions

In this section we will discuss the two-point $n-1$-loop all equal mass banana graphs in two dimensions.

We first provide an algorithm for determining the differential equation satisfied by these amplitudes to all loop order, and present the solution for the one-loop banana (bubble) graph and the two-loop banana (sunset) graph given in [**BV13**]. The solution to the three-loop banana graph will appear in [**BKV14**].

8. Schwinger representation

We look at the $n-1$-loop banana graph of figure 3 evaluated in $D=2$ euclidean dimensions

$$I_n^2(m_1,\dots,m_n;K)=\int_{\mathbb{R}^{2n}}\frac{\prod_{i=1}^n d^2\ell_i\,\delta^{(2)}(\sum_{i=1}^n\ell_i=K)}{\prod_{i=1}^n(\ell_i^2+m_i^2)}\,, \tag{8.1}$$

where ℓ_i for $1 \leq i \leq n$ are the momenta of each propagator. The steps described in section 3.1 lead to the following representation of the banana integrals in two dimensions

$$I_n^2 = \int_{x_i \geq 0} \frac{\delta(x_n = 1)}{\mathcal{F}_n} \prod_{i=1}^{n} dx_i \tag{8.2}$$

with

$$\mathcal{F}_n = (\sum_{i=1}^{n} x_i m_i^2)\mathcal{U}_n - K^2 \prod_{i=1}^{n} x_i \tag{8.3}$$

where $\mathcal{U}_n$ is the determinant of the period matrix Ω given in (3.23)

$$\mathcal{U}_n = \prod_{i=1}^{n} x_i \left(\sum_{i=1}^{n} \frac{1}{x_i} \right). \tag{8.4}$$

In order to determine the differential equation satisfied by the all equal mass banana graphs we provide an alternative expression for the banana integrals. For $K^2 < (\sum_{i=1}^{n} m_i)^2$, one can perform a series expansion

$$\begin{aligned} I_n^2 &= \int_{[0,+\infty[^{n-1}} \frac{\delta(x_n = 1)}{\sum_{i=1}^{n} m_i^2 x_i \sum_{i=1}^{n} x_i^{-1} - t} \prod_{i=1}^{n-1} \frac{dx_i}{x_i} \\ &= \sum_{k \geq 0} t^k \int_{[0,+\infty[^{n-1}} \frac{\delta(x_n = 1)}{(\sum_{i=1}^{n} m_i^2 x_i)^{k+1} (\sum_{i=1}^{n} x_i^{-1})^{k+1}} \prod_{i=1}^{n} \frac{dx_i}{x_i}. \end{aligned} \tag{8.5}$$

Exponentiating the denominators

$$I_n^2 = \sum_{k \geq 0} \frac{t^k}{k!^2} \int_{[0,+\infty[^{n+1}} \delta(x_n = 1)\, e^{-u(\sum_{i=1}^{n} m_i^2 x_i) - v(\sum_{i=1}^{n} x_i^{-1})} \frac{du dv}{u^{-k} v^{-k}} \prod_{i=1}^{n} \frac{dx_i}{x_i}. \tag{8.6}$$

Using the integral representation for the K_0 Bessel function

$$\int_0^{+\infty} e^{-u m^2 x - \frac{v}{x}} \frac{dx}{x} = 2K_0(2m\sqrt{uv}) \tag{8.7}$$

one gets

$$I_n^2 = 2^{n-1} \sum_{k \geq 0} \frac{t^k}{k!^2} \prod_{i=1}^{n-1} K_0(2m_i\sqrt{uv}) \int_{[0,+\infty[^2} e^{-u m_n^2 - v}\, u^k v^k du dv. \tag{8.8}$$

Now setting $uv = (x/2)^2$, the integral over v gives a K_0 Bessel function with the result

$$I_n^2 = 2^{n-1} \int_{[0,+\infty[} \sum_{k \geq 0} \left(\frac{\sqrt{t}x}{2} \right)^{2k} \frac{1}{k!^2} \prod_{i=1}^{n} K_0(m_i x) dx. \tag{8.9}$$

Using the series expansion of the I_0 Bessel function

$$I_0(x) = \sum_{k \geq 0} \left(\frac{x}{2} \right)^{2k} \frac{1}{k!^2} \tag{8.10}$$

we get the following representation for the banana graph (see [**BBBG08**] for a previous appearance of this formula at the special values $K^2 = m^2$ and all equal masses $m_i = m$)

$$I_n^2 = 2^{n-1} \int_0^{+\infty} x\, I_0(\sqrt{K^2}x) \prod_{i=1}^{n} K_0(m_i x)\, dx\,. \tag{8.11}$$

9. The differential equation for the banana graphs at all loop orders

We derive a differential equation for the $n-1$-loop all equal mass banana graphs

$$I_n^2(t) := 2^{n-1} \int_0^{\infty} x\, I_0(\sqrt{t}x)\, K_0(x)^n dx\,. \tag{9.1}$$

This will generalize to all loop order the differential equations given for the two loops case in [**LR04, MSWZ11, ABW13**] We first prove the existence of a differential equation for the integral. In [**BS07**] it is proven that the Bessel function $K_0(x)$ satisfies the differential equation

$$L_{n+1} K_0(x)^n = 0 \tag{9.2}$$

where L_{n+1} is a degree $n+1$ differential operator expressed as a polynomial in the differential operator $\theta_x = x\frac{d}{dx}$ of the form $L_{n+1} = \theta_x^{n+1} + \sum_{k=0}^{n} p_k(x^2)\theta_x^k$. This operator is obtained by the recursion given in [**BS07**]

$$L_1 = \theta_x \tag{9.3}$$

$$L_{k+1} = \theta_x L_k - x^2 k(n+1-k) L_{k-1}, \qquad 1 \le k \le n\,. \tag{9.4}$$

Setting $\theta_t := t\frac{d}{dt}$ we have the following relations

$$2\theta_t I_0(\sqrt{t}x) = \theta_x I_0(\sqrt{t}x); \tag{9.5}$$

$$\theta_t^2 I_0(\sqrt{t}x) = t\left(\frac{x}{2}\right)^2 I_0(\sqrt{t}x); \tag{9.6}$$

$$(\theta_t^3 - \theta_t^2) I_0(\sqrt{t}x) = \frac{tx^2}{8}\theta_x(I_0(\sqrt{t}x))\,, \tag{9.7}$$

and integrating by parts, one can convert the differential (9.2) into a differential equation for $\hat{L}_{n+1} I_n^2(t) = 0$ for the banana integral $I_n^2(t)$. Since

$$\int x^k f(x) \theta_x g(x)\, dx = -(k+1)\int x^k f(x) g(x) - \int x^k f(x) g(x)\, dx\,. \tag{9.8}$$

The polynomial coefficients $p_k(x)$ in (9.2) are polynomials in x^2 such that $p_k(0) = 0$ [**BS07**]. Therefore the differential operator $\hat{L}_{n+1} = \theta_t^2\, \tilde{L}_{n-1}$ where $\tilde{L}_{n-1}$ is a differential operator of at most degree $n-1$. We conclude that the integral satisfies the differential equation

$$\tilde{L}_{n-1} I_n^2(t) = S_n + \tilde{S}_n\, \log(t)\,. \tag{9.9}$$

where S_n and $\tilde{S}_n$ are constants. It is easy to check that $I_n^2(t)$ has a finite value of $t = 0$. Therefore $\tilde{S}_n = 0$ and the $n-1$-loop banana integral satisfy a differential equation of order $n-1$ with a constant inhomogeneous term.

The differential operator acting on the integral $I_n^2(t)$ is given by

$$\tilde{L}_{n-1} = \sum_{k=0}^{n-1} q_k(t)\, \frac{d^k}{dt^k} \tag{9.10}$$

with the coefficient $q_k(t)$ polynomials of degree $k+1$ for $0 \leq k \leq n-1$. The top degree and lowest-degree polynomial are given by

$$q_{n-1}(t) = t^{\lfloor \frac{n}{2} \rfloor + \eta(n)} \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (t - (n-2i)^2) \tag{9.11}$$

$$q_{n-2}(t) = \frac{n-1}{2} \frac{dq_{n-1}(t)}{dt} \tag{9.12}$$

$$q_0(t) = t - n\,. \tag{9.13}$$

with $\eta(n) = 0$ if $n \equiv 1 \mod 2$ and 1 if $n \equiv 0 \mod 2$. The inhomogeneous term S_n is a constant given by

$$S_n = \int_0^\infty 2^{n-1}\, x \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - \eta(n)} q_k(0) \left(\frac{x}{2}\right)^{2k} \right) K_0(x)^n dx \tag{9.14}$$

Numerical evaluations for various loop orders give that $S_n = -n!$.

In section 9.1 we provide `Maple` codes for generating the differential equations for the all equal mass banana integrals

$$\left(\sum_{k=0}^{n-1} q_k(t) \frac{d^k}{dt^k} \right) I_n^2(t) = -n!\,. \tag{9.15}$$

# loops= $n-1$	differential equation
$n=2$	$(t-2)\,f\,(t) + t(t-4)\,f^{(1)}(t) = -2!$
$n=3$	$(t-3)\,f\,(t) + \left(3\,t^2 - 20\,t + 9\right) f^{(1)}(t) + t(t-1)(t-9) f^{(2)}(t) = -3!$
$n=4$	$(t-4)\,f\,(t) + \left(7\,t^2 - 68\,t + 64\right) f^{(1)}(t) + \left(6\,t^3 - 90\,t^2 + 192\,t\right) f^{(2)}(t)$ $+t^2(t-4)(t-16) f^{(3)}(t) = -4!$
$n=5$	$(t-5)\,f\,(t) + (3t-5)(5t-57) f^{(1)}(t) + \left(25\,t^3 - 518\,t^2 + 1839\,t - 450\right) f^{(2)}(t)$ $+ \left(10\,t^4 - 280\,t^3 + 1554\,t^2 - 900\,t\right) f^{(3)}(t) + t^2(t-25)(t-1)(t-9) f^{(4)}(t) = -5!$
$n=6$	$(t-6)\,f\,(t) + \left(31\,t^2 - 516\,t + 1020\right) f^{(1)}(t) + \left(90\,t^3 - 2436\,t^2 + 12468\,t - 6912\right) f^{(2)}(t)$ $+ \left(65\,t^4 - 2408\,t^3 + 19836\,t^2 - 27648\,t\right) f^{(3)}(t)$ $+ \left(15\,t^5 - 700\,t^4 + 7840\,t^3 - 17280\,t^2\right) f^{(4)}(t) + t^3(t-36)(t-4)(t-16) f^{(5)}(t) = -6!$

TABLE 1. Examples of the differential equations satisfied by the all equal mass $n-1$-loop banana graph up to $n=6$ generated with the `Maple` code of section 9.1. We made use of the notation $f^{(n)}(t) := \frac{d^n f(t)}{dt^n}$.

9.1. Maple codes for the differential equations. A derivation of the differential equations for the banana graph can be obtained using *Maple* and the routines `compute_Q` and `rec_Q` from the paper [**BS07**] together with the package `gfun` [**SZ94**].

For $t < n^2$ the integral converges and we can perform the series expansion

$$I_n^2(t) = \sum_{k \geq 0} t^k\, J_n^k\,, \tag{9.16}$$

where we have introduced the Bessel moments

$$J_n^k = \frac{2^n}{\Gamma(k+1)^2} \int_0^{+\infty} \left(\frac{x}{2}\right)^{2k+1} K_0(x)^n \, dx \, . \tag{9.17}$$

Using the result of [**BS07**] on the recursion relations satisfied by these Bessel moment $c_{n,2k+1} = 2^{2k+1-n}\Gamma(k+1)^2 \, J_n^k$ we deduce that these moments satisfy a recursion relation for $k \geq 0$

$$(k+1)^{n-1} \, J_n^k + \sum_{1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor} P_{n,2\ell}(k) \, J_n^{k+\ell} = 0 \tag{9.18}$$

A first step is to construct the recursion relations satisfied by the coefficients J_n^k. For this we use the routines from [**BS07**]

```
compute_Q:=proc(n,theta,t)
local k, L;
    L[0]:=1; L[1]:=theta;
    for k to n do
        L[k+1]:=expand(series(
            t*diff(L[k],t)+L[k]*theta-k*(n-k+1)*t^2*L[k-1],
            theta,infinity))
    od;
    series(convert(L[n+1],polynom),t,infinity)
end:

rec_c:=proc(c::name,n::posint,k::name)
local Q,theta,t,j;
    Q:=compute_Q(n,theta,t);
    add(factor(subs(theta=-1-k-j,coeff(Q,t,j)))*c(n,k+j),j=0..n+1)=0
end:
```

The recursion relation (9.18), for the J_n^k in eq. (9.17), is then obtained by the routine

```
Brec:=proc(n::posint)  local e1,e2,Jnktmp,vtmp,itmp;
    Jnktmp:= (n, k) -> 2^(-n+2*k+1)*factorial(k)^2*J(k):
    e1 := subs([k = 2*K+1], rec_c(c, n, k)):
    e2 := subs([c(n, 2*K+1) = Jnktmp(n, K)], e1):
    for itmp from 1 to ceil(n/2) do
    e2:=subs([c(n,2*K+1+2*itmp)=Inktmp(n,K+itmp)],e2) od:
    vtmp:=seq(j(K+i),i=0..ceil(n/2)):
    collect(simplify((-1)^(n-1)*e2/(4^(K+1)*factorial(K+1)^2)),{vtmp},factor)
    end:
```

Then the differential equation is obtained from the previous recursion relation using the command `rectodiffeq` from the package `gfun` [**SZ94**]

```
with(gfun):
rectodiffeq({Brec(n),seq(J(k)=j(n,k),k=0..floor(n/2)-1+(n mod 2))},J(K),f(t));
```

9.2. The Picard-Fuchs equation of Feynman graphs. Feynman graph hyper-surfaces lead to Calabi-Yau geometries. It is therefore not surprising that the Picard-Fuchs operators acting on the Feynman integrals are similar to the one

arising in the context of open mirror symmetry discussed in [**MW09, JW13**] for instance. The inhomogeneous term is however different because Feynman integrals lead to different extensions of mixed Hodge structure than the one encountered in the mirror symmetry case.

For the banana graphs we give a formal solution to the Picard-Fuchs equations given in the previous section. Let $\{y_1(t), \ldots, y_{n-1}(t)\}$ solutions of the homogeneous Picard-Fuchs equation $\tilde{L}_n(t)y_i(t) = 0$ with $1 \leq i \leq n-1$ of equation (9.10), and the generalized Wronskian

$$W(x,t) = \det \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_{n-1}(x) \\ y_1'(x) & y_2'(x) & \cdots & y_{n-1}'(x) \\ \vdots & \ldots & \vdots & \\ y_1^{(n-3)}(x) & y_2^{(n-3)}(x) & \cdots & y_{n-1}^{(n-3)}(x) \\ y_1(t) & y_2(t) & \cdots & y_{n-1}(t) \end{pmatrix} \tag{9.19}$$

Clearly $\partial_t^i \, W(x,t)|_{x=t} = 0$ for $0 \leq i \leq n-2$, and $\partial_t^{n-1} \, W(x,t)|_{x=t} = \exp(-\int^t q_{n-2}(x)/q_{n-1}(x)dx) = W_0/q_{n-1}^{\frac{n-1}{2}}(t)$ is the Wronskian of the differential Picard-Fuchs operator $\tilde{L}_n(t)$ in (9.10). With a convenient choice of homogeneous solutions one can set the constant of integration $W_0 = 1$. A formal solution to the banana Picard-Fuchs reads

$$I_n^2(t) = \sum_{i=1}^{n-1} \alpha_i \, y_i(t) - n! \int_0^t W(x,t) \, q_{n-1}^{\frac{n-3}{2}}(x) \, dx \,, \tag{9.20}$$

where α_i are constant of integrations. This method has been used [**BV13, BKV14**] to solve the differential equation for $n = 2$ and $n = 3$. In the following we describe the solution of the lowest order banana graphs at one- and two-loop orders. For a detailed discussion of the three-loop banana graphs we refer to [**BKV14**].

10. Some explicit solutions for the all equal masses banana graphs

In [**B13**] Broadhurst provided a mixture of proofs and numerical evidences that up to and including four loops the special values $t = K^2/m^2 = 1$ for the all equal mass banana graphs are given by values of L-functions.

For generic values of $t = K^2/m^2 \in [0, (n+1)^2]$, the solution is expressible as an elliptic dilogarithm at $n = 2$ loops order [**BV13**] and elliptic trilogarithm at $n = 3$ loops order [**BKV14**]. The situation at higher-order is not completely clear.

In the following we present the one- and two-loop order solutions.

10.1. The massive one-loop bubble. In $D = 2$ dimensions the one-loop banana graph, is the massive bubble, which evaluates to

$$I_2^2(m_1, m_2, K^2) = \frac{\log(z^+) - \log(z^-)}{\sqrt{\Delta}} \tag{10.1}$$

where

$$\begin{aligned} z^{\pm} &= (K^2 - m_1^2 - m_2^2 \pm \sqrt{\Delta})/(2m_1^2) \\ \Delta_\circ &= (K^2)^2 + m_1^4 + m_2^4 - 2(K^2 m_1^2 + K^2 m_2^2 + m_1^2 m_2^2) \end{aligned} \tag{10.2}$$

where $\Delta_\circ$ is the discriminant of the equation

$$(m_1^2 x + m_2^2)(1 + x) - K^2 x = m_1^2 (x - z^+)(x - z^-) = 0 \tag{10.3}$$

In the single mass case $m_1 = m_2 = m_3 = 1$ the integral reads

$$I_2^2(t) = -4\,\frac{\log\left(\sqrt{t}+\sqrt{t-4}\right)+\log 2}{\sqrt{t(t-4)}}\,. \tag{10.4}$$

This expression satisfies the differential equation for $n = 2$ in table 1.

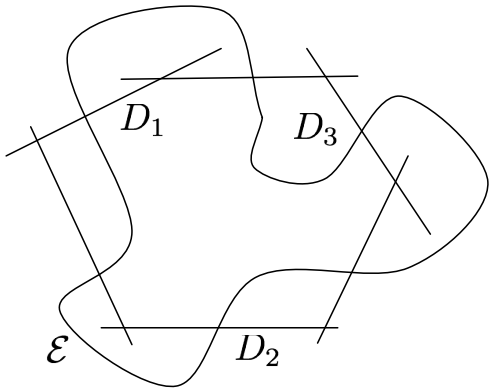


FIGURE 5. After blowup, the coordinate triangle becomes a hexagon in P with three new divisors D_i. The elliptic curve $X_\ominus = \{\mathcal{F}_2(x, y; t) = 0\}$ now meets each of the six divisors in one point.

10.2. The sunset integral. The domain of integration for the sunset is the triangle $\Delta = \{[x, y, z] \in \mathbb{P}^2 | x, y, z \geq 0\}$ and the second Symanzik polynomial $\mathcal{F}_2(x, y, z; t) = (x + y + z)(xy + xz + yz) - txyz$. The integral is given by

$$I_3^2(t) = \int_\Delta \frac{z dx \wedge dy + x dy \wedge dz - y dx \wedge dz}{\mathcal{F}_2(x, y, z; t)}\,. \tag{10.5}$$

This integral is very similar to the period integral in equation (7.12) for the elliptic curve $\mathcal{E}_\ominus := \{\mathcal{F}_2(x, y; t) = 0\}$. The only difference between these two integrals is the domain of integration. In the case of the period integral in (7.12) on integrates over a two-cycle and, for well chosen values of t, the elliptic curve has no intersection with the domain of integration, and therefore is a period of *a pure Hodge structure*. In the case of the Feynman integral the domain of integration has a boundary, so it is not a cycle, and for *all* values of t the elliptic curve intersects the domain of integration. This is precisely because the domain of integration of Feynman graph integral is given as in (3.14) that *Feynman integrals lead to period of mixed Hodge structures*.

As explained in section 5 one needs to blow-up the points where the elliptic curve $\mathcal{E}_\ominus := \{\mathcal{F}_2(x, y, z; t) = 0\}$ (the graph polar part) intersects the boundary of the domain of integration $\partial\Delta \cap \mathcal{E}_\ominus = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$. The blown-up domain is the hexagon $\mathfrak{h}$ in figure 5. The associated mixed Hodge structure is given by [**BV13**] for the relative cohomology $H^2(\mathcal{P} - \mathcal{E}_\ominus, \mathfrak{h} - \mathcal{E}_\ominus \cap \mathfrak{h})$

$$0 \to H^1(\mathfrak{h} - \mathcal{E}_\ominus \cap \mathfrak{h}) \to H^2(\mathcal{P} - E, \mathfrak{h} - \mathcal{E}_\ominus \cap \mathfrak{h}) \to H^2(\mathcal{P} - \mathcal{E}_\ominus, \mathbb{Q}) \to 0 \tag{10.6}$$

and for the domain of integration we have the dual sequence

$$0 \to H_2(P-E) \to H_2(P-\mathcal{E}_\ominus, \mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h}) \to H_1(\mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h}) \to 0 \tag{10.7}$$

The Feynman integral for the sunset graph coincides with $I_3^2(t) = \langle \omega, s(1)\rangle$ where ω in $F^1H^1(\mathcal{E}_\ominus, \mathbb{C})$ is an element in the smallest Hodge filtration piece $F^2H^1(\mathcal{E}_\ominus, \mathbb{C})(-1)$, and $s(1)$ is a section in $H^1(\mathcal{E}_\ominus, \mathbb{Q}(2))$ [**BV13**].

The integral is expressed as the following combination of elliptic dilogarithms

$$-\frac{I_3^2(t)}{6} = -i\frac{\pi}{6}\,\varpi_r(t))(1-2\tau) + \frac{\varpi_r(t)}{\pi}\,E_\ominus(\tau)\,, \tag{10.8}$$

where the Hauptmodul $t = \frac{\pi}{\sqrt{3}}\,\eta(\tau)^6\eta(2\tau)^{-3}\eta(3\tau)^{-2}\eta(6\tau)$, the real period $\varpi_r(t) = \frac{\pi}{\sqrt{3}}\,\eta(\tau)^6\eta(2\tau)^{-3}\eta(3\tau)^{-2}\eta(6\tau)$ and τ is the period ratio for the elliptic curve $\mathcal{E}_\ominus$. Using $q := \exp(2i\pi\tau)$ the elliptic dilogarithm is given by

$$\begin{aligned} E_\ominus(\tau) &= -\frac{1}{2i}\sum_{n\geq 0}\left(\mathrm{Li}_2\left(q^n\zeta_6^5\right) + \mathrm{Li}_2\left(q^n\zeta_6^4\right) - \mathrm{Li}_2\left(q^n\zeta_6^2\right) - \mathrm{Li}_2\left(q^n\zeta_6\right)\right) \\ &+ \frac{1}{4i}\left(\mathrm{Li}_2\left(\zeta_6^5\right) + \mathrm{Li}_2\left(\zeta_6^4\right) - \mathrm{Li}_2\left(\zeta_6^2\right) - \mathrm{Li}_2\left(\zeta_6\right)\right)\,. \end{aligned} \tag{10.9}$$

which we can write as well as q-expansion

$$E_\ominus(\tau) = \frac{1}{2}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{k-1}}{k^2}\,\frac{\sin(\frac{n\pi}{3}) + \sin(\frac{2n\pi}{3})}{1-q^k}\,. \tag{10.10}$$

As we mentioned earlier this integral is not given by an elliptic dilogarithm obtained by evaluating the real analytic function $D(z)$ to the contrary to the Mahler measure described in section 7.1.

The amplitude is closely related to the *regulator* in arithmetic algebraic geometry [**Beilinson85, Bloch00, Soule86, Bru07**]. Let $conj : M_\mathbb{C} \to M_\mathbb{C}$ be the real involution which is the identity on $M_\mathbb{R}$ and satisfies $conj(c\,m) = \bar{c}\,m$ for $c \in \mathbb{C}$ and $m \in M_\mathbb{R}$. With notation as above, the extension class $s(1) - s_F \in H^1(\mathcal{E}_\ominus, \mathbb{C})$ is well-defined up to an element in $H^1(\mathcal{E}_\ominus, \mathbb{Q}(2))$ (i.e. the choice of $s(1)$). Since $conj$ is the identity on $H^1(\mathcal{E}_\ominus, \mathbb{Q}(2))$, the projection onto the minus eigenspace $(s(1) - s_F)^{conj=-1}$ is canonically defined. The regulator is then

$$\langle \omega, (s(1) - s_F)^{conj=-1}\rangle \in \mathbb{C}\,. \tag{10.11}$$

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Institut des Hautes Etudes Scientifiques, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France

Institut de Physique Théorique,, CEA, IPhT, F-91191 Gif-sur-Yvette, France, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France

E-mail address: pierre.vanhove@cea.fr