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A matrix model for the topological string I
Deriving the matrix model

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Abstract

We construct a matrix model that reproduces the topological string partition function on arbitrary toric Calabi-Yau 3-folds. This demonstrates, in accord with the BKMP "remodeling the B-model" conjecture, that Gromov-Witten invariants of any toric Calabi-Yau 3- fold can be computed in terms of the spectral invariants of a spectral curve. Moreover, it proves that the generating function of Gromov-Witten invariants is a Tau-function for an integrable hierarchy. In a follow-up paper, we will explicitly construct the spectral curve of our matrix model and argue that it equals the mirror curve of the toric Calabi-Yau.

Contents

1 Introduction

In the topological string A-model, the object of study is the moduli space of maps from a Riemann surface Σ_g of genus g to a given Calabi-Yau target space $\mathfrak{X}$. Its partition function is the generating function of Gromov-Witten invariants of $\mathfrak{X}$, which roughly speaking count these maps.

In recent years, deep connections have been unrooted between the topological string on various geometries and random matrix models. A classic result in the field is that intersection numbers, which are related to the Gromov-Witten theory of a point, are computed by the Kontsevich matrix integral [1], see also [2]. In the Dijkgraaf-Vafa conjecture [3]

such a connection is obtained between the topological B-model on certain non-compact Calabi-Yau manifolds and a 1-matrix model.

In the 20 years that have passed since topological string theory was formulated [4, 5], various techniques have been developed for computing the corresponding partition function. The topological vertex method [6] solves this problem completely for toric Calabi-Yau 3-folds at large radius, furnishing the answer as a combinatorial sum over partitions. On geometries with unit first Betti number (the conifold and $\mathcal{O}(-2) \to \mathbb{CP}^1 \times \mathbb{C}$), this formalism yields the partition function as a sum over a single partition with Plancherel measure. In [7], such a sum was rewritten as a 1-matrix integral. More complicated examples, such as the topological string on geometries underlying Seiberg-Witten $SU(n)$ theory, can be written as sums over multiple partitions [8, 9, 10]. 1-matrix integrals that reproduce the corresponding partition functions were formulated in [11]. Multi-matrix integrals have arisen in rewriting the framed vertex as a chain of matrices integral [12]. Its Hurwitz-numbers limit (infinite framing of the framed vertex geometry) was shown to be reproduced by a 1-matrix model with an external field in [13, 14].

Here, generalizing the method of [7], we are able to formulate a matrix model which reproduces the topological string partition function on a certain fiducial geometry, which we introduce in the next section. Flop transitions and limits in the Kähler cone relate the fiducial geometry to an arbitrary toric Calabi-Yau. As we can follow the effect of both of these operations on the topological string partition function, our matrix model provides a description for the topological string on an arbitrary toric Calabi-Yau manifold.

By providing a matrix model realization, we are able to transcribe deep structural insights into matrix models to the topological string setting. E.g., our matrix model involves a chain of matrices, and chain of matrices integrals are always Tau functions for an integrable system. By providing a matrix model realization, we hence prove integrability of the generating function of Gromov-Witten invariants.

Moreover, matrix models satisfy loop equations, which are known to be equivalent to W-algebra constraints. A general formal solution to these equations was found in [15], centered around the introduction of an auxiliary Riemann surface, referred to as the spectral curve of the system. The partition and correlation functions of the matrix model are identified with so-called symplectic invariants of this curve. The BKMP conjecture [16] identifies the spectral invariants of the mirror curve to a toric Calabi-Yau manifold with the topological string partition function with the Calabi-Yau manifold as target space. In a forthcoming publication [17], we will compute the spectral curve of our matrix model explicitly, thus establishing the validity of this conjecture.

The outline of this paper is as follows. In section 2, after a very brief review of toric geometry basics, we introduce the fiducial geometry and the notation that we will use in discussing it throughout the paper. We also review the transformation properties of the topological string partition function under flop transitions, which will relate the fiducial to an arbitrary toric geometry, in this section. We recall the topological vertex formalism and its application to geometries on a strip [18] in section 3. Section 4 contains our main

result: we introduce a chain of matrices matrix model and demonstrate that it reproduces the topological string partition function on the fiducial geometry. We discuss implications of this result in section 5, and end with conclusions in section 6.

2 The fiducial geometry and flop transitions

Toric geometries present a rich class of very computable examples for many questions in algebraic geometry. The topological vertex formalism provides an algorithm for computing the generating function for Gromov-Witten invariants on toric 3 dimensional Calabi-Yau manifolds. These are necessarily non-compact and have rigid complex structure.

The geometry of toric manifolds of complex dimension d can be encoded in terms of a d dimensional fan Σ, consisting of cones of dimensions 0 to d. We denote the set of all n dimensional cones as $\Sigma(n)$. Each such n-cone represents the closure of a $(C^*)^{d-n}$ orbit. In particular, 1-cones correspond to hypersurfaces, and for $d = 3$, our case of interest, 2-cones correspond to curves.

The fan for the class of geometries we are interested in is constructed by triangulating a finite connected region of the $\mathbb{Z}^2$ lattice containing the origin, embedding this lattice in $\mathbb{Z}^3$ within the (x, y) plane at $z = 1$, and defining the cones of the fan via half-lines emanating at the origin and passing through the vertices of this triangulation.[1]

We can associate a dual diagram to such toric fans, a so-called web diagram, spanned by lines orthogonal to the projection of 2-cones onto the $\mathbb{Z}^2$ lattice. In the web diagram, the relation between the dimension of the components of the diagram and the submanifold of the toric geometry they represent coincide: 3-cones (points) correspond to vertices, and 2-cones (curves) to lines, see figure 1.

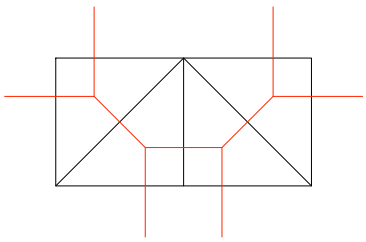

Figure 1: Example of a box triangulation, corresponding to a 3 dimensional toric fan. The diagram in red is the dual web diagram. Vertices of the triangulations (faces of the web diagram) correspond to 1-cones, edges correspond to 2-cones, and faces (vertices of the dual) correspond to 3-cones.

[1]The canonical class of a toric manifold is given by the sum over all torically invariant divisors. The construction sketched above guarantees that this sum is principal, hence the canonical class trivial: the monomial associated to the 1-cone $(0, 0, 1)$ generates the class in question. See e.g. [19].

2.1 The fiducial geometry

The geometry $\mathfrak{X}_0$ we will take as the starting point of our considerations is depicted in figure 2.

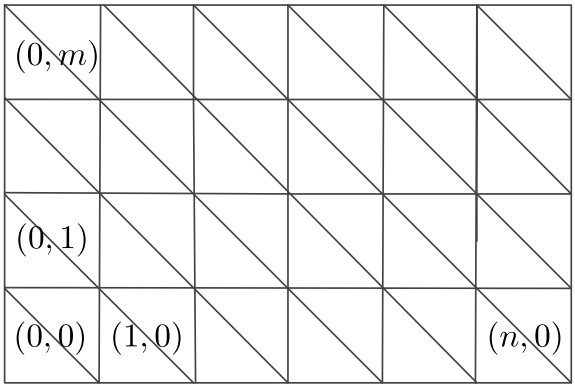

Figure 2: Fiducial geometry $\mathfrak{X}_0$ with boxes numbered.

Since the torically invariant curves play a central role in our considerations, we introduce a labeling scheme for these in figure 3: (i, j) enumerates the boxes as in figure (2), and we will explain the a-parameters further below. In the following, we will, when convenient,

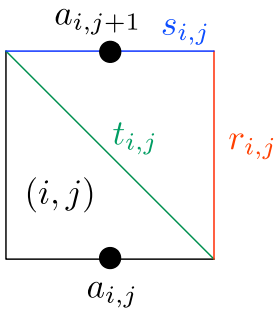

Figure 3: Labeling curve classes, and introducing a-parameters.

use the same notation for a torically invariant curve Σ, its homology class $[\Sigma] \in H_2(\mathfrak{X}_0, \mathbb{Z})$, and its volume or associated Kähler parameter $\int_\Sigma J$, given a Kähler form J on $\mathfrak{X}_0$. The classes of the curves $r_{i,j}, s_{i,j}, t_{i,j}$ are not independent. To determine the relations among these, we follow [20, page 39, 40]. Consider the integer lattice Λ spanned by formal generators e_ρ, $\rho \in \Sigma(1)$ 1-cones of the toric fan,

$$\Lambda = \{ \sum_{\rho \in \Sigma(1)} \lambda_\rho e_\rho | \lambda_\rho \in \mathbb{Z} \} .$$

Each torically invariant curve, corresponding to a 2-cone of the fan, maps to a relation between 1-cones, and thus to an element of the lattice Λ, as follows: a 2-cone σ is spanned by two integral generators v_1 and v_2, and it is contained in precisely two 3-cones, which are each spanned by v_1, v_2 and one additional generator v_3, v_4 respectively. These vectors satisfy the relation $\sum_{i=1}^{4} \lambda_i v_i = 0$, where the λ_i can be chosen as relatively prime integers, and as v_3 and v_4 lie on opposite sides of σ, we can assume that $\lambda_3, \lambda_4 > 0$. [20] shows that on a smooth variety, the sublattice Λ_h generated by the elements $\sum_{i=1}^{4} \lambda_i e_i$ of Λ is isomorphic to $H_2(\mathfrak{X}_0, \mathbb{Z})$. We will also call this isomorphism λ,

$$\lambda : H_2(\mathfrak{X}_0, \mathbb{Z}) \to \Lambda_h \, .$$

Figure 4 exemplifies this map. It allows us to easily work out the relation between the

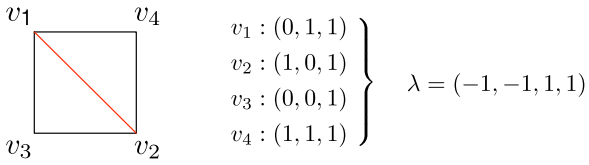

Figure 4: The 2-cone σ corresponds to the relation $\vec{\lambda}$ among 1-cones.

various curve classes. Consider figure 5. The images of the curve classes depicted there

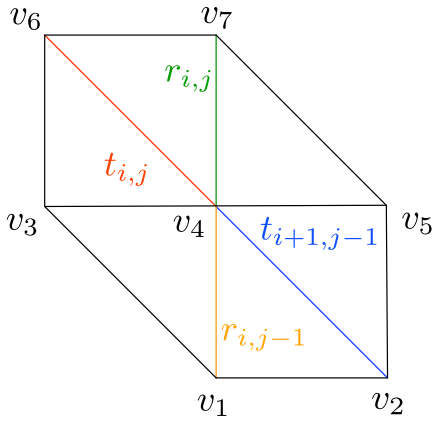

Figure 5: Determining the relation between curve classes.

under λ are,

$$\begin{aligned} \lambda(r_{i,j}) &= e_5 + e_6 - e_4 - e_7 \\ \lambda(r_{i,j-1}) &= e_2 + e_3 - e_1 - e_4 \\ \lambda(t_{i,j}) &= e_3 + e_7 - e_4 - e_6 \\ \lambda(t_{i+1,j-1}) &= e_1 + e_5 - e_2 - e_4 \end{aligned}$$

We read off the relation

$$t_{i,j} + r_{i,j} = t_{i+1,j-1} + r_{i,j-1} \,. \tag{2.1}$$

By symmetry, we also have

$$t_{i,j} + s_{i,j-1} = t_{i+1,j-1} + s_{i+1,j-1} \,.$$

A moment's thought convinces us that this constitutes a complete basis for the space of relations. We can solve these in terms of the classes of the curves $r_i, s_i, t_{i,j}$, $i, j = 0, 1, \ldots$ depicted in figure 6, which hence generate $H_2(\mathfrak{X}_0, \mathbb{Z})$. The explicit relations are

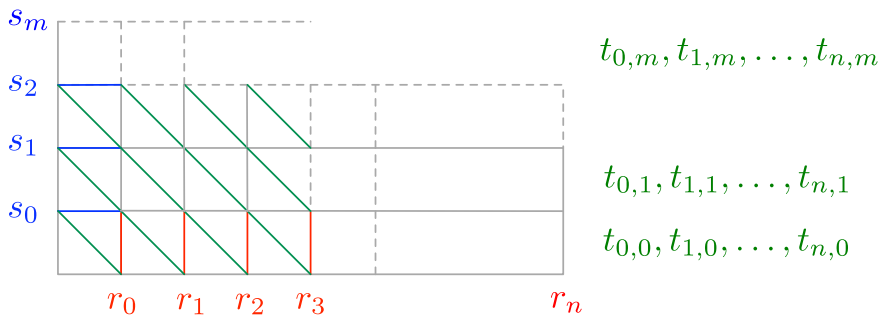

Figure 6: Fiducial geometry with choice of basis of $H_2(\mathfrak{X}_0, \mathbb{Z})$.

$$\begin{aligned} r_{i,j} &= r_i + \sum_{k=1}^{j} (t_{i+1,k-1} - t_{i,k}) \\ s_{i,j} &= s_j + \sum_{k=1}^{i} (t_{k-1,j+1} - t_{k,j}) \end{aligned}$$

Our computation for the partition function on $\mathfrak{X}_0$ will proceed by first considering the horizontal strips in the toric fan describing the geometry, as depicted in figure 2, individually, and then applying a gluing algorithm to obtain the final result.

For each strip, we find it convenient to write the curve class $w_{IJ} \in H_2(\mathfrak{X}_0, \mathbb{Z})$ of the curve extending between two 3-cones which we label by I and J (recall that 3-cones correspond to vertices in the dual web diagram), with J to the right of I, as the difference between two parameters a_I and a_J associated to each 3-cone,

$$w_{IJ} = a_I - a_J \,. \tag{2.2}$$

We call these parameters, somewhat prosaically, a-parameters. It is possible to label the curve classes in this way due to their additivity along a strip. In terms of the notation introduced in figure 3, we obtain

$$t_{i,j} = a_{i,j} - a_{i,j+1} \quad , \qquad r_{i,j} = a_{i,j+1} - a_{i+1,j} \,.$$

By invoking the relation (2.1), we easily verify that upon gluing two strips, the curve class of a curve extending between two 3-cones I and J on the lower strip is equal to the class of the curve between the 3-cones I' and J' on the upper strip, where the cones I and I' are glued together, as are the cones J and J',

$$w_{IJ} = w_{I'J'} \tag{2.3}$$

This allows us to identify the parameters $a_I = a_{I'}$ and $a_J = a_{J'}$ associated to 3-cones glued together across strips.

Note that the basic curve classes s_i are not captured by the parameters $a_{i,j}$.

2.2 Flop invariance of toric Gromov-Witten invariants

Under the proper identification of curve classes, Gromov-Witten invariants (at least on toric manifolds) are invariant under flops. Assume $\mathfrak{X}$ and $\mathfrak{X}^+$ are related via a flop transition, $\phi : \mathfrak{X} \to \mathfrak{X}^+$. In a neighborhood of the flopped $(-1,-1)$ curve, the respective toric diagrams are depicted in figure 7.

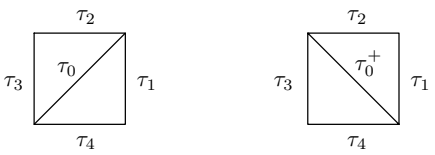

Figure 7: $\mathfrak{X}$ and $\mathfrak{X}^+$ in the vicinity of the (-1,-1) curve.

The 1-cones of $\Sigma_{\mathfrak{X}}$, corresponding to the toric invariant divisors of $\mathfrak{X}$, are not affected by the flop, hence can be canonically identified with those of $\mathfrak{X}^+$. The 2-cones τ_i in these diagrams correspond to toric invariant 2-cycles C_i, C_i^+ in the geometry. The curve classes of $\mathfrak{X}$ push forward to classes in $\mathfrak{X}^+$ via

$$\phi_*([C_0]) = -[C_0^+]\,, \quad \phi_*([C_i]) = [C_i^+] + [C_0^+]\,. \tag{2.4}$$

All other curve classes of $\mathfrak{X}$ are mapped to their canonical counterparts in $\mathfrak{X}^+$. Under appropriate analytic continuation and up to a phase factor (hence the $\propto$ in the following formula), the following identity then holds [21, 18, 22]

$$Z_{GW}(\mathfrak{X}, Q_0, Q_1, \ldots, Q_4, \vec{Q}) \propto Z_{GW}(\mathfrak{X}^+, 1/Q_0, Q_0Q_1, \ldots, Q_0Q_4, \vec{Q})\,. \tag{2.5}$$

i.e.

$$GW_g(\mathfrak{X}, Q_0, Q_1, \ldots, Q_4, \vec{Q}) = GW_g(\mathfrak{X}^+, 1/Q_0, Q_0Q_1, \ldots, Q_0Q_4, \vec{Q})\,.$$

Any toric Calabi-Yau manifold $\mathfrak{X}$ with Kähler moduli $\vec{Q}$ can be obtained from a sufficiently large fiducial geometry $(\mathfrak{X}_0, \vec{Q}_0)$ upon performing a series of flop transitions and taking unwanted Kähler moduli of $\mathfrak{X}_0$ to ∞. As an example, we show how to obtain the $\mathbb{P}^2$ geometry from the fiducial geometry with 2×2 boxes in figure 8.

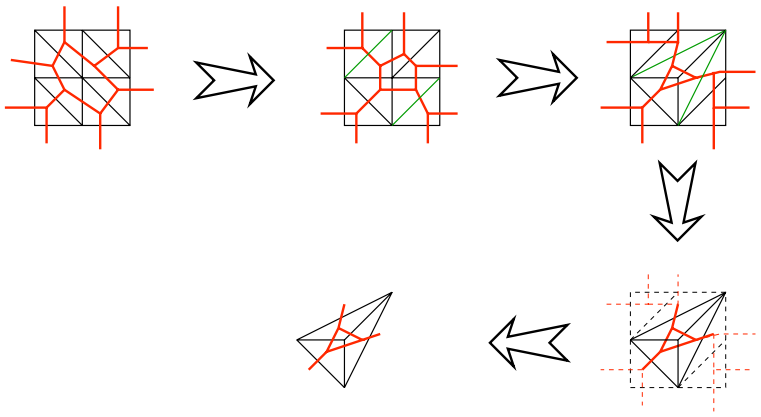

Figure 8: We obtain local $\mathbb{P}^2$ from the fiducial geometry with 2×2 boxes by performing five flops and then sending the Kähler parameters of the unwanted edges to ∞.

3 The partition function via the topological vertex

3.1 Gromov-Witten invariants

Gromov-Witten invariants $\mathcal{N}_{g,D}(\mathfrak{X})$ roughly speaking count the number of maps from a Riemann surface of genus g into the target space $\mathfrak{X}$, with image in a given homology class $D = (D_1, \ldots, D_k) \in H_2(\mathfrak{X}, \mathbb{Z})$. They can be assembled into a generating series

$$GW_g(\mathfrak{X}, Q) = \sum_D \mathcal{N}_{g,D}(\mathfrak{X})\, Q^D.$$

Each $GW_g(\mathfrak{X}, Q)$ is a formal series in powers $Q^D = \prod_i Q_i^{D_i}$ of the parameters $Q = (Q_1, Q_2, \ldots, Q_k)$, the exponentials of the Kähler parameters.

We can introduce a generating function for Gromov-Witten invariants of all genera by introducing a formal parameter g_s (the string coupling constant) and writing

$$GW(\mathfrak{X}, Q, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2}\; GW_g(\mathfrak{X}, Q)\,.$$

It is in fact more convenient to introduce disconnected Gromov-Witten invariants $\mathcal{N}^*_{\chi,D}(\mathfrak{X})$, for possibly disconnected surfaces, of total Euler characteristics χ, and to define

$$Z_{GW}(\mathfrak{X}, Q, g_s) = e^{GW(\mathfrak{X},Q,g_s)} = \sum_D Q^D \sum_\chi g_s^{-\chi}\, \mathcal{N}^*_{\chi,D}(\mathfrak{X}).$$

For toric Calabi-Yau manifolds, an explicit algorithm was presented in [6] for computing Z_{GW} via the so-called topological vertex formalism, proved in [23, 24].

3.2 The topological vertex

In the topological vertex formalism, each vertex of the web diagram contributes a factor $C_q(\alpha, \beta, \gamma)$ to the generating function of GW-invariants, where the α, β, γ are Young tableaux associated to each leg of the vertex, and $C_q(\alpha, \beta, \gamma)$ is a formal power series in the variable q, where

$$q = e^{-g_s}.$$

Topological vertices are glued along edges (with possible framing factors, see [6]) carrying the same Young tableaux α by performing a sum over α, weighted by $Q^{|\alpha|}$, with Q encoding the curve class of this connecting line,

$$Z_{\text{vertex}}(\mathfrak{X}, Q, q) = \sum_{\text{Young tableaux } \alpha_e} \prod_{\text{edges } e} Q_e^{|\alpha_e|} \prod_{\text{vertices } v=(e_1,e_2,e_3)} C_q(\alpha_{e_1}, \alpha_{e_2}, \alpha_{e_3})\,.$$

Note than in practical computations, the sum over representations can ordinarily not be performed analytically. A cutoff on the sum corresponds to a cutoff on the degree of the maps being counted.

The equality

$$Z_{GW}(\mathfrak{X}, Q, g_s) = Z_{\text{vertex}}(\mathfrak{X}, Q, q)$$

holds at the level of formal power series in the Q's, i.e. in the large radius expansion. It was proved in [24] that the log of the right hand side indeed has a power series expansion in powers of g_s.

3.3 Notations for partitions and q-numbers

Before going further in the description of the topological vertex formula, we pause to fix some notations and introduce special functions that we will need in the following.

3.3.1 Representations and partitions

Representations of the symmetric group are labelled by Young tableaux, or Ferrer diagrams.

For a representation γ, we introduce the following notation:

- γ_i: number of boxes in the i-th row of the Young tableau associated to the representation γ, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d \geq 0$.
- The weight $|\gamma| = \sum_i \gamma_i$: the total number of boxes in the corresponding Young tableau.
- The length $l(\gamma)$: the number of non-vanishing rows in the Young tableau, i.e. $\gamma_i = 0$ iff $i > l(\gamma)$.
- The Casimir $\kappa(\gamma) = \sum_i \gamma_i(\gamma_i - 2i + 1)$.

- γ^T denotes the conjugate representation, which is obtained by exchanging the rows and columns of the associated Young tableau. We have $|\gamma^T| = |\gamma|$, $l(\gamma^T) = \gamma_1$, and $\kappa(\gamma^T) = -\kappa(\gamma)$.

Moreover, an integer $d > 0$ will denote a cut-off on the length of representations summed over,

$$l(\gamma) \leq d.$$

Most expressions we are going to write will in fact be independent of d, and we shall argue in [17], following the same logic as in [7] based on the arctic circle property [25], that our results depend on d only non-perturbatively.

Also, to each representation γ, we shall associate a parameter a as introduced in (2.2).

Instead of dealing with a partition γ, characterized by the condition $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d \geq 0$, it will prove convenient to define the quantities

$$h_i(\gamma) = \gamma_i - i + d + a\,, \tag{3.1}$$

which satisfy instead

$$h_1 > h_2 > h_3 > \cdots > h_d \geq a\,.$$

The relation between γ and $h(\gamma)$, for the off-set $a = 0$, is depicted in figure 9.

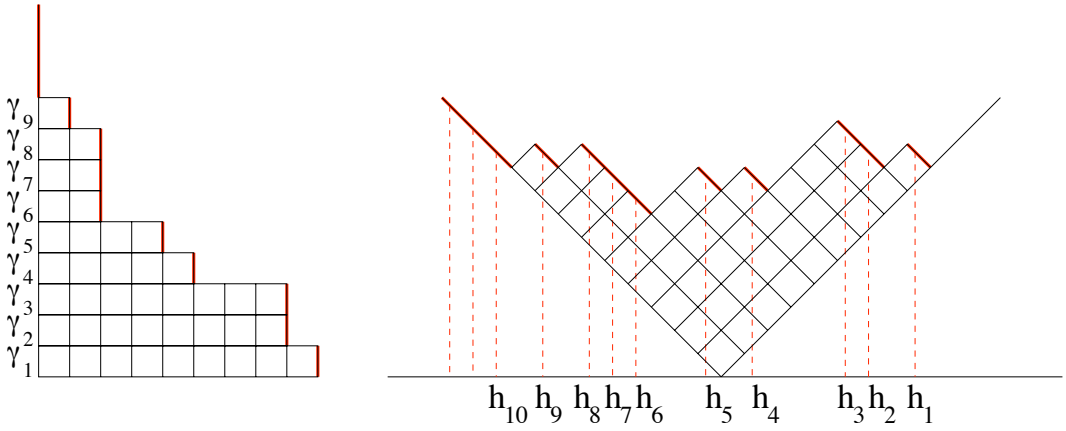

Figure 9: Relation between a partition γ and $h(\gamma)$.

We finally introduce the functions

$$x_i(\gamma) = q^{h_i(\gamma)}\,.$$

In terms of the $h_i(\gamma)$, we have

$$\kappa(\gamma) = \sum_i h_i^2 - (2d + 2a - 1)\sum_i h_i + d\,C_{d,a}\,,$$

where $C_{d,a} = \frac{1}{3}(d-1)(2d-1) + a(a+2d-1)$.

3.3.2 q-numbers

We choose a string coupling constant g_s such that the quantum parameter $q = e^{-g_s}$ satisfies $|q| < 1$. A q-number $[x]$ is defined as

$$[x] = q^{-\frac{x}{2}} - q^{\frac{x}{2}} = 2\sinh\frac{x\,g_s}{2}\,. \tag{3.2}$$

q-numbers are a natural deformation away from the integers; in the limit $q \to 1$, $\frac{1}{g_s}[x] \to x$.

We also define the q-product

$$g(x) = \prod_{n=1}^{\infty}(1 - \frac{1}{x}\,q^n)\,.$$

The function $g(x)$ is related to the quantum Pochhammer symbol, $g(x) = [q/x;q]_\infty$, and to the q-deformed gamma function via $\Gamma_q(x) = (1-q)^{1-x}\,g(1)/g(q^{1-x})$. $g(x)$ satisfies the functional relation

$$g(qx) = (1 - \frac{1}{x})\;g(x)\,.$$

For Γ_q, this implies $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\;\Gamma_q(x)$, the quantum deformation of the functional equation $\Gamma(x+1) = x\Gamma(x)$ of the gamma function, which is recovered in the classical limit $q \to 1$. The central property of $g(x)$ for our purposes is that it vanishes on integer powers of q,

$$g(q^n) = 0 \quad \text{if } n \in \mathbb{N}^*.$$

Moreover, it has the following small $\ln q$ behavior,

$$\ln g(x) = \frac{1}{\ln q}\sum_{n=0}^{\infty}\frac{(-1)^n\,B_n}{n!}\,(\ln q)^n\;\mathrm{Li}_{2-\mathrm{n}}(1/\mathrm{x})\,,$$

where $\mathrm{Li}_{\mathrm{n}}(\mathrm{x}) = \sum_{\mathrm{k}=1}^{\infty}\frac{\mathrm{x}^{\mathrm{k}}}{\mathrm{k}^{\mathrm{n}}}$ is the polylogarithm, and B_n are the Bernouilli numbers

$$B_0 = 1\;,\quad B_1 = -\frac{1}{2}\;,\quad B_2 = \frac{1}{6}\;,\quad \ldots$$

$B_{2k+1} = 0$ if $k \geq 1$ (see the appendix).

We shall also need the following function $f(x)$,

$$\begin{aligned}\frac{1}{f(x)} &= \frac{g(x)\,g(q/x)}{g(1)^2\,\sqrt{x}}\;e^{\frac{(\ln x)^2}{2\ln q}}\;e^{\frac{-i\pi\ln x}{\ln q}}\\ &= \frac{-\ln q}{\theta'(\frac{1}{2}-\frac{i\pi}{\ln q},-\frac{2i\pi}{\ln q})}\;\theta\left(\frac{\ln x}{\ln q}+\frac{1}{2}-\frac{i\pi}{\ln q},\frac{-2i\pi}{\ln q}\right),\end{aligned}$$

where θ is the Riemann theta-function for the torus of modulus $-2i\pi/\ln q$. This relationship is the quantum deformation of the classical gamma function identity

$$e^{-i\pi x}/\Gamma(1-x)\Gamma(x) = \sin(\pi x)/\pi\,.$$

3.4 The partition function via the vertex

We begin by considering a single horizontal strip of the fiducial geometry, as depicted in figure 10.

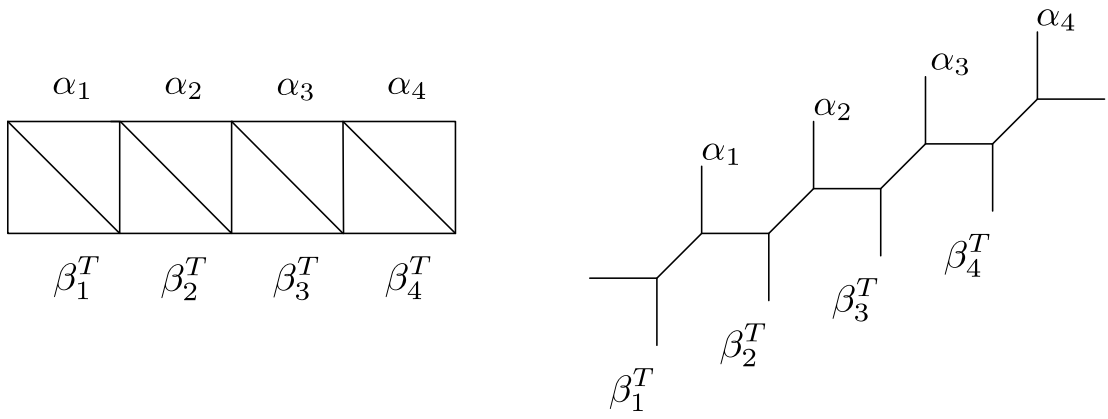

Figure 10: A horizontal strip of the fiducial geometry and its corresponding web diagram.

Of the three legs of the vertex, two point in the direction of the strip and connect the vertex to its neighbors. One leg points out of the strip, either above or below. This leg carries a free representation, α_i or β_i^T in the notation of figure 10. The partition function will hence depend on representations, one per vertex (i.e. face of the triangulation).

A note on notation: since each 3-cone carries a representation (which up to the final paragraph of this subsection is held fixed) and an a-parameter (see figure 3), we will identify the a-parameters by the corresponding representations when convenient.

Using the topological vertex, it was shown in [18] that the A-model topological string partition function of the strip is given by a product of terms, with the individual factors depending on the external representations and all possible pairings of these. Applied to the fiducial strip, the results there specialize to

$$Z_{\text{strip}}(\alpha_0;\beta^T) \;=\; \prod_{i=0}^{n} \frac{[\alpha_i][\beta_i^T]}{[\beta_i,\alpha_i^T]_{Q_{\beta_i,\alpha_i}}} \; \frac{\prod_{i<j}[\alpha_i,\alpha_j^T]_{Q_{\alpha_i,\alpha_j}} \prod_{i<j}[\beta_i,\beta_j^T]_{Q_{\beta_i,\beta_j}}}{\prod_{i<j}[\alpha_i,\beta_j^T]_{Q_{\alpha_i,\beta_j}}[\beta_i,\alpha_j^T]_{Q_{\beta_i,\alpha_j}}} \,. \tag{3.3}$$

We explain each factor in turn.

• Each vertex $\gamma = \alpha_i$ or $\gamma = \beta_i^T$ contributes a representation dependent factor to the partition function, which we have denoted by $[\gamma]$. It is the $n \to \infty$ limit of the Schur

polynomial evaluated for $x_i = q^{\frac{1}{2}-i}$, $i = 1, \ldots, n$, given explicitly by

$$\begin{aligned}
[\gamma] &= (-1)^d q^{\frac{1}{4}\kappa(\gamma)} \prod_{1\le i<j\le d} \frac{[\gamma_i - \gamma_j + j - i]}{[j-i]} \prod_{i=1}^{d} \prod_{j=1}^{\gamma_i} \frac{1}{[d+j-i]} \\
&= \prod_{1\le i<j\le d} (q^{h_j} - q^{h_i}) \prod_{i=1}^{d} \left(\frac{g(q^{a_\gamma - h_i})}{g(1)} \; q^{\frac{1}{2}h_i^2 - (a_\gamma + d - 1)h_i + \frac{a_\gamma(a_\gamma+d-1)}{2} + \frac{(d-1)(2d-1)}{12}} \right) \\
&= \Delta(X(\gamma)) \; e^{-\frac{1}{g_s}\mathrm{tr}\, U(X(\gamma), a_\gamma)} \; e^{-\frac{1}{g_s}\mathrm{tr}\, U_1(X(\gamma), a_\gamma)}.
\end{aligned}$$

We recall that $h_i(\gamma) = \gamma_i - i + d + a_\gamma$, and we have defined $x_i = q^{h_i}$ and $X(\gamma)$ is the diagonal matrix $X(\gamma) = \mathrm{diag}(q^{h_1}, q^{h_2}, \ldots, q^{h_d})$. Furthermore, $\Delta(X)$ denotes the Vandermonde determinant of the matrix X

$$\Delta(X) = \prod_{1\le i<j\le d} (x_j - x_i)\,,$$

and we have written

$$U(X, a) = -g_s \ln \left(\frac{g(\frac{q^a}{X})}{g(1)} \right),$$

$$U_1(X, a) = \frac{(\ln X)^2}{2} - (a + d - 1) \ln X \; \ln q + C(a, d)\,,$$

where $C(a, d) = \frac{a(a+d-1)}{2} + \frac{(d-1)(2d-1)}{12}$.
We have

$$[\gamma] = q^{\frac{\kappa(\gamma)}{2}} [\gamma^T]\,, \quad \kappa(\gamma^T) = -\kappa(\gamma)\,,$$

and thus

$$[\gamma^T] = \Delta(X(\gamma)) \; e^{-\frac{1}{g_s}\mathrm{tr}\, U(X(\gamma), a_\gamma)} \; e^{-\frac{1}{g_s}\mathrm{tr}\, \tilde{U}_1(X(\gamma), a_\gamma)}\,,$$

where

$$\tilde{U}_1(X, a) = \frac{1}{2} \ln X \; \ln q + \tilde{C}(a, d).$$

and $\tilde{C}_{a,d}$ is another constant which depends only on a and d and which will play no role for our purposes.

• In addition, each pair of representations contributes a factor, reflecting the contribution of the curve extended between the respective vertices. In the nomenclature of [18], the representations α_i are all of same type, and of opposite type relative to the β_i. If we take $i < j$, representations of same type (corresponding to (-2,0) curves) contribute a factor of

$$[\alpha_i, \alpha_j^T] \quad \text{or} \quad [\beta_i^T, \beta_j]\,,$$

whereas representations of different type (corresponding to (-1,-1) curves) contribute a factor of

$$\frac{1}{[\alpha_i, \beta_j]} \quad \text{or} \quad \frac{1}{[\beta_i^T, \alpha_j^T]}\,.$$

The pairing is given by [26, 9, 27, 18]

$$
\begin{aligned}
[\gamma, \delta^T] &= Q_{\gamma,\delta}^{-\frac{|\gamma|+|\delta|}{2}} q^{-\frac{\kappa(\gamma)-\kappa(\delta)}{4}} \prod_{i=1}^{d}\prod_{j=1}^{d} \frac{[h_i(\gamma)-h_j(\delta)]}{[a_\gamma - a_\delta + j - i]} \\
&\quad \times \prod_{i=1}^{d}\prod_{j=1}^{\gamma_i} \frac{1}{[a_\gamma - a_\delta + j - i + d]} \prod_{i=1}^{d}\prod_{j=1}^{\delta_i} \frac{1}{[a_\gamma - a_\delta - j + i - d]} \prod_{k=0}^{\infty} g(Q_{\gamma,\delta}^{-1} q^{-k}) \\
&= (-1)^{\frac{d(d-1)}{2}} \prod_{i=1}^{d} \frac{q^{\frac{1}{2}(h_i(\delta)^2 - h_i(\delta)(2a_\gamma + 2d - 1) - a_\delta^2 + 2a_\gamma a_\delta + (d-2i)a_\gamma + (2i-d-1)a_\delta)}}{[a_\gamma - a_\delta]^d} \prod_{i=1}^{d} (-1)^{\delta_i} \\
&\quad \prod_{i,j=1}^{d} (q^{h_j(\delta)} - q^{h_i(\gamma)}) \prod_{i=1}^{d} \frac{g(q^{a_\gamma - h_i(\delta)})}{g(q^{a_\gamma - a_\delta})} \frac{g(q^{a_\delta - h_i(\gamma)})}{g(q^{a_\delta - a_\gamma})} \\
&\propto \Delta(X(\gamma), X(\delta))\; e^{-\frac{1}{g_s}\mathrm{Tr}\, U(X(\gamma), a_\delta)} e^{-\frac{1}{g_s}\mathrm{Tr}\, U(X(\delta), a_\gamma)}\; e^{-\frac{1}{g_s}\left(\mathrm{Tr}\, U_2(X(\gamma), a_\delta) + \mathrm{Tr}\, \tilde{U}_2(X(\delta), a_\gamma)\right)},
\end{aligned}
\tag{3.4}
$$

where the square brackets on the RHS denote q-numbers as defined in (3.2), the symbol $\Delta(X(\gamma), X(\delta))$ signifies

$$
\Delta(X(\gamma), X(\delta)) = \prod_{i,j} (X_i(\delta) - X_j(\gamma)) = \prod_{i,j} (q^{h_i(\delta)} - q^{h_j(\gamma)}), \tag{3.5}
$$

and

$$
U_2(X, a) = 0,
$$

$$
\tilde{U}_2(X, a) = \frac{(\ln X)^2}{2} - (a + d - \frac{1}{2})\, \ln X\, \ln q + i\pi \ln X.
$$

The parameter $Q_{\gamma,\delta}$ reflects, given a choice of Kähler class J of the metric on $\mathfrak{X}_0$, the curve class of the curve $\mathcal{C}$ extended between the vertices labeled by γ and δ via

$$
w_{\gamma,\delta} = \int_{\mathcal{C}} J\ , \qquad Q_{\gamma,\delta} = q^{w_{\gamma,\delta}}\ ,
$$

and by the definition of the a-parameters

$$
w_{\gamma,\delta} = a_\gamma - a_\delta.
$$

Substituting these expressions into (3.3), we obtain

$$
\begin{aligned}
&Z_{\mathrm{strip}}(\alpha_0, \ldots, \alpha_n; \beta_0^T, \ldots, \beta_n^T) = \\
&\quad = \frac{\prod_i \Delta(X(\alpha_i)) \prod_{i<j} \Delta(X(\alpha_i), X(\alpha_j))\ \prod_i \Delta(X(\beta_i)) \prod_{i<j} \Delta(X(\beta_i), X(\beta_j))}{\prod_{i,j} \Delta(X(\alpha_i), X(\beta_j))}
\end{aligned}
$$

$$\times \prod_i e^{-\frac{1}{g_s}\,\mathrm{tr}\,(V_{\vec{a}}(X(\alpha_i))-V_{\vec{b}}(X(\alpha_i)))} \prod_i e^{-\frac{1}{g_s}\,\mathrm{tr}\,V_i(X(\alpha_i))}$$
$$\times \prod_i e^{\frac{1}{g_s}\,\mathrm{tr}\,(V_{\vec{a}}(X(\beta_i))-V_{\vec{b}}(X(\beta_i)))} \prod_i e^{-\frac{1}{g_s}\,\mathrm{tr}\,\tilde{V}_i(X(\beta_i))}\,, \tag{3.6}$$

where we have denoted by $\vec{a} = (a_0, a_2, \ldots, a_n)$ (resp. $\vec{b} = (b_0, b_2, \ldots, b_n)$) the a-parameters of representations on the upper side (resp. lower side) of the strip, and defined

$$V_{\vec{a}}(X) = -g_s \sum_{j=0}^{n} \ln\left(g(q^{a_j}/X)\right) \tag{3.7}$$

and

$$V_i(X) = \ln X\ \ln q\ \left(\frac{1}{2} - \sum_{j\le i}(a_j - b_j)\right) + i\pi \ln X$$
$$\tilde{V}_i(X) = \ln X\ \ln q\ \left(\frac{1}{2} - \sum_{j<i}(b_j - a_j)\right)\,.$$

3.5 Gluing strips

To obtain the partition function for the full multistrip fiducial geometry $\mathfrak{X}_0$, we must glue these strips along the curves labelled $s_{i,j}$ in figure 3.
Denoting the representations $\alpha_{j,i}$ on line i collectively by

$$\vec{\alpha}_i = (\alpha_{0,i}, \alpha_{1,i}, \ldots, \alpha_{n,i})\,,$$

this yields

$$Z_{\mathrm{vertex}}(\mathfrak{X}_0) = Z_{(n,m)}(\vec{\alpha}_{m+1}, \vec{\alpha}_0^T) = \sum_{\alpha_{j,i},\, j=0,\ldots,n;\, i=1,\ldots,m} \prod_{i=1}^{m+1} Z_{\mathrm{strip}}(\vec{\alpha}_i, \vec{\alpha}_{i-1}^T) \prod_{j=0}^{n}\prod_{i=1}^{m} q^{s_{j,i}\,|\alpha_{j,i}|}\,. \tag{3.8}$$

Our goal now is to find a matrix integral which evaluates to this sum.

4 The matrix model

4.1 Definition

Consider the fiducial geometry $\mathfrak{X}_0$ of size $(n+1)\times(m+1)$, with Kähler parameters $t_{i,j} = a_{i,j} - a_{i,j+1}$, $r_{i,j} = a_{i,j+1} - a_{i+1,j}$, and $s_{i,j}$, as depicted in figures 3 and 6. We write

$$\vec{a}_i = (a_{0,i}, a_{1,i}, \ldots, a_{n,i}).$$

Assume that the external representations are fixed to $\vec{\alpha}_{m+1} = (\alpha_{0,m+1}, \alpha_{1,m+1}, \ldots, \alpha_{n,m+1})$ on the upper line, and $\vec{\alpha}_0 = (\alpha_{0,0}, \alpha_{1,0}, \ldots, \alpha_{n,0})$ on the lower line (for most applications, one prefers to choose these to be trivial).

We now define the following matrix integral $\mathcal{Z}_{\rm MM}$ ($_{\rm MM}$ for Matrix Model),

$$\begin{aligned}\mathcal{Z}_{\rm MM}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) &= \Delta(X(\vec{\alpha}_{m+1}))\ \Delta(X(\vec{\alpha}_0))\ \prod_{i=0}^{m+1} \int_{H_N(\Gamma_i)} dM_i\ \prod_{i=1}^{m+1} \int_{H_N(\mathbb{R}_+)} dR_i \\ & \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(M_i) - V_{\vec{a}_{i-1}}(M_i)\right]}\ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(M_{i-1}) - V_{\vec{a}_i}(M_{i-1})\right]} \\ & \prod_{i=1}^{m+1} e^{\frac{1}{g_s}{\rm tr}\,(M_i - M_{i-1})R_i}\ \prod_{i=1}^{m} e^{(S_i + \frac{i\pi}{g_s})\,{\rm tr}\ \ln M_i} \\ & e^{{\rm tr}\,\ln f_0(M_0)}\ e^{{\rm tr}\,\ln f_{m+1}(M_{m+1})}\ \prod_{i=1}^{m} e^{{\rm tr}\,\ln f_i(M_i)}\,. \end{aligned} \tag{4.1}$$

All matrices are taken of size

$$N = (n+1)\,d\,,$$

where d is the cut-off discussed in section (3.3.1). We have introduced the notation

$$X(\vec{\alpha}_{m+1}) = {\rm diag}(X(\vec{\alpha}_{m+1})_i)_{i=1,\ldots,N}\ , \qquad X(\vec{\alpha}_{m+1})_{jd+k} = q^{h_k(\alpha_{j,m+1})},$$

$$X(\vec{\alpha}_0) = {\rm diag}(X(\vec{\alpha}_0)_i)_{i=1,\ldots,N}\ , \qquad X(\vec{\alpha}_0)_{jd+k} = q^{h_k(\alpha_{j,0})},$$

for $k = 1, \ldots, d$, $j = 0, \ldots, n$. $\Delta(X) = \prod_{i<j}(X_i - X_j)$ is the Vandermonde determinant. $V_{\vec{a}_i}(x)$ was introduced in (3.7). For $i = 1, \ldots, m$, we have defined

$$f_i(x) = \prod_{j=0}^{n} \frac{g(1)^2\ e^{(\frac{1}{2} + \frac{i\pi}{\ln q})\,\ln(xq^{1-a_{j,i}})}\ e^{\frac{(\ln(xq^{1-a_{j,i}}))^2}{2g_s}}}{g(x\,q^{1-a_{j,i}})\,g(q^{a_{j,i}}/x)}\,.$$

The denominator of these functions induces simple poles at $x = q^{a_{j,i}+l}$ for $j = 0, \ldots, n$ and $l \in \mathbb{Z}$. The numerator is chosen such that they satisfy the relation $f_i(qx) = f_i(x)$. This enforces a simple l-dependence of the residues taken at $x = q^{a_{j,i}+l}$, given by a prefactor q^l – a fact which will be important in the following. These residues are in fact given by

$$\operatorname*{Res}_{q^{a_{j,i}+l}} f_i(x) = q^{a_{j,i}+l}\ \hat{f}_{j,i} = -\,q^{a_{j,i}+l}\ \prod_{k\neq j} \frac{g(1)^2\ e^{(\frac{1}{2} + \frac{i\pi}{\ln q})\,(1+a_{j,i}-a_{k,i})\ln q}\ e^{\frac{(\ln(q^{1+a_{j,i}-a_{k,i}}))^2}{2g_s}}}{g(q^{a_{j,i}-a_{k,i}})\,(1-q^{a_{k,i}-a_{j,i}})g(q^{a_{k,i}-a_{j,i}})}\,, \tag{4.2}$$

where $\hat{f}_{j,i}$ is independent of the integer l.

The parameters S_i are defined by

$$S_i = s_{0,i-1} + t_{0,i-1} = s_{j,i-1} - \sum_{k<j} t_{k,i} + \sum_{k\leq j} t_{k,i-1}\,. \tag{4.3}$$

The final equality holds for arbitrary j, and can be verified upon invoking (2.3) repeatedly.

For $i = 0$ and $i = m + 1$, we define

$$f_0(x) = \frac{1}{\prod_{j=0}^{n} \prod_{i=1}^{d} (x - q^{h_i(\alpha_{j,0})})} \,,$$

$$f_{m+1}(x) = \frac{1}{\prod_{j=0}^{n} \prod_{i=1}^{d} (x - q^{h_i(\alpha_{j,m+1})})} \,.$$

Notice that if the representations $\vec{\alpha}_0$ or $\vec{\alpha}_{m+1}$ are trivial, i.e. $h_i(\alpha_{j,0}) = d - i + a_{j,0}$ or $h_i(\alpha_{j,m+1}) = d - i + a_{j,m+1}$, we have

$$f_0(x) = \prod_{j=0}^{n} \frac{g(x\, q^{1-a_{j,0}-d})}{x^d\, g(x\, q^{1-a_{j,0}})} \,, \qquad f_{m+1}(x) = \prod_{j=0}^{n} \frac{g(x\, q^{1-a_{j,m+1}-d})}{x^d\, g(x\, q^{1-a_{j,m+1}})}$$

respectively. The functions f_0 and f_{m+1} have simple poles at $x = q^{h_l(\alpha_{j,0})}$ (resp. $x = q^{h_l(\alpha_{j,m+1})}$) for $l = 1, \ldots, d$, with residue

$$\hat{f}_{j,0;l} = \operatorname*{Res}_{q^{h_l(\alpha_{j,0})}} f_0(x) = \frac{1}{\prod_{j' \neq j} \prod_{i=1}^{d} (q^{h_l(\alpha_{j,0})} - q^{h_i(\alpha_{j',0})})} \, \frac{1}{\prod_{i \neq l} (q^{h_l(\alpha_{j,0})} - q^{h_i(\alpha_{j,0})})} \,,$$

$$\hat{f}_{j,m+1;l} = \operatorname*{Res}_{q^{h_l(\alpha_{j,m+1})}} f_{m+1}(x) = \frac{1}{\prod_{j' \neq j} \prod_{i=1}^{d} (q^{h_l(\alpha_{j,m+1})} - q^{h_i(\alpha_{j',m+1})})} \, \frac{1}{\prod_{i \neq l} (q^{h_l(\alpha_{j,m+1})} - q^{h_i(\alpha_{j,m+1})})} \,.$$

The l dependence here is more intricate than above, but this will not play any role since the partitions $\alpha_{j,0}$ and $\alpha_{j,m+1}$ are kept fixed, and not summed upon.

The integration domains for the matrices R_i are $H_N(\mathbb{R}_+^N)$, i.e. the set of hermitian matrices having only positive eigenvalues. For the matrices $M_i, i = 1, \ldots, m$, the integration domains are $H_N(\Gamma_i)$, where

$$\Gamma_i = \prod_{j=0}^{n} (\gamma_{j,i})^d \,.$$

$\gamma_{j,i}$ is defined as a contour which encloses all points of the form $q^{a_{j,i}+\mathbb{N}}$, and does not intersect any contours $\gamma_{k,l}$, $(j,i) \neq (k,l)$. For this to be possible, we must require that the differences $a_{j,i} - a_{j',i'}$ be non-integer. The normalized logarithms of two such contours are depicted in figure 11.

We have defined

$$H_N(\Gamma_i) = \{ M = U\, \Lambda\, U^\dagger \,, \quad U \in U(N) \,, \;\; \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \;\in \Gamma_i \} \,,$$

i.e. $H_N(\Gamma_i)$ is the set of normal matrices with eigenvalues on Γ_i. By definition, the measure on $H_N(\Gamma_i)$ is (see [28])

$$dM = \frac{1}{N!} \; \Delta(\Lambda)^2 \; dU \, d\Lambda \,, \tag{4.4}$$

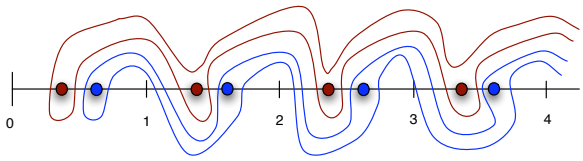

Figure 11: Two contours surrounding points $a+\mathbb{N}$ and $b+\mathbb{N}$, such that $a-b\notin\mathbb{Z}$.

where dU is the Haar measure on $U(N)$, (normalized not to 1, but to a value depending only on N, such that the Itzykson-Zuber integral evaluates as given in (4.6) with prefactor 1), and $d\Lambda$ is the product of the measures for each eigenvalue along its integration path.

The integration domains for the matrices M_0, M_{m+1} are $H_N(\Gamma_0)$, $H_N(\Gamma_{m+1})$ respectively, where

$$\Gamma_0=(\sum_{j=0}^{n}\gamma_{j,0})^N\,,\qquad\qquad \Gamma_{m+1}=(\sum_{j=0}^{n}\gamma_{j,m+1})^N\,. \tag{4.5}$$

The goal of the rest of this section is to prove that the matrix integral (4.1) reproduces the topological string partition function for target space the fiducial geometry $\mathfrak{X}_0$.

4.2 Diagonalization

Let us first diagonalize all matrices. We write

$$M_i=U_i\,X_i\,U_i^\dagger\,,$$

$$R_i=\tilde U_i\,Y_i\,\tilde U_i^\dagger,$$

where U_i and $\tilde U_i$ are unitary matrices.

By the definition (4.4), the measures dM_i and dR_i are given by

$$dM_i=\frac{1}{N!}\,\Delta(X_i)^2\,dU_i\,dX_i\quad,\quad dR_i=\frac{1}{N!}\,\Delta(Y_i)^2\,d\tilde U_i\,dY_i\,.$$

The matrix integral thus becomes

$$\begin{aligned}\mathcal{Z}_{\rm MM}(Q,g_s,\vec\alpha_{m+1},\vec\alpha_0^T) \;=\; & \frac{\Delta(X(\vec\alpha_{m+1}))\;\Delta(X(\vec\alpha_0))}{(N!)^{2m+3}}\;\prod_{i=0}^{m+1}\int_{\Gamma_i}dX_i\,\Delta(X_i)^2\;\prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N}dY_i\,\Delta(Y_i)^2\\ & \prod_{i=0}^{m+1}dU_i\;\prod_{i=1}^{m+1}d\tilde U_i\\ & \prod_{i=1}^{m}e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec a_i}(X_i)-V_{\vec a_{i-1}}(X_i)\right]}\;\prod_{i=1}^{m}e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec a_{i-1}}(X_{i-1})-V_{\vec a_i}(X_{i-1})\right]}\end{aligned}$$

$$\prod_{i=1}^{m+1} e^{\frac{1}{g_s}\mathrm{tr}\, X_i U_i^\dagger \tilde{U}_i Y_i \tilde{U}_i^\dagger U_i}\; e^{\frac{-1}{g_s}\mathrm{tr}\, X_{i-1} U_{i-1}^\dagger \tilde{U}_i Y_i \tilde{U}_i^\dagger U_{i-1}}\; \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,\mathrm{tr}\,\ln X_i}$$
$$e^{\mathrm{tr}\,\ln f_0(X_0)}\; e^{\mathrm{tr}\,\ln f_{m+1}(X_{m+1})}\; \prod_{i=1}^{m} e^{\mathrm{tr}\,\ln f_i(X_i)}\,.$$

Next, we introduce the matrices $\hat{U}_i$, $\check{U}_i$, for $i = 1, \ldots, m+1$, via

$$\hat{U}_i = U_i^\dagger \tilde{U}_i \quad , \quad \check{U}_i = \tilde{U}_i^\dagger U_{i-1}\,.$$

We can express $U_0, \ldots, U_{m+1}$, and $\tilde{U}_1, \ldots, \tilde{U}_{m+1}$, in terms of these matrices and U_{m+1},

$$U_i = U_{m+1}\,\hat{U}_{m+1}\,\check{U}_{m+1}\,\hat{U}_m\,\check{U}_m \ldots \hat{U}_{i+1}\,\check{U}_{i+1}\,,$$
$$\tilde{U}_i = U_{m+1}\,\hat{U}_{m+1}\,\check{U}_{m+1}\,\hat{U}_m\,\check{U}_m \ldots \hat{U}_{i+1}\,\check{U}_{i+1}\,\hat{U}_i\,.$$

With this change of variables, we arrive at

$$\begin{aligned}
\mathcal{Z}_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \;=\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\;\Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}} \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\,\Delta(X_i)^2 \prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i\,\Delta(Y_i)^2 \\
& \int dU_{m+1} \prod_{i=1}^{m+1} d\hat{U}_i \prod_{i=1}^{m+1} d\check{U}_i \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_i}(X_i) - V_{\vec{a}_{i-1}}(X_i)\right]} \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,\mathrm{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1}) - V_{\vec{a}_i}(X_{i-1})\right]} \\
& \prod_{i=1}^{m+1} e^{\frac{1}{g_s}\mathrm{tr}\, X_i \hat{U}_i Y_i \hat{U}_i^\dagger}\; e^{\frac{-1}{g_s}\mathrm{tr}\, X_{i-1} \check{U}_i^\dagger Y_i \check{U}_i}\; \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,\mathrm{tr}\,\ln X_i} \\
& e^{\mathrm{tr}\,\ln f_0(X_0)}\; e^{\mathrm{tr}\,\ln f_{m+1}(X_{m+1})}\; \prod_{i=1}^{m} e^{\mathrm{tr}\,\ln f_i(X_i)}\,.
\end{aligned}$$

Notice that the integral over U_{m+1} decouples, and $\int dU_{m+1} = \mathrm{Vol}(U(N))$.

4.3 Itzykson-Zuber integral and Cauchy determinants

The $\hat{U}_i$ and $\check{U}_i$ appear in the form of Itzykson-Zuber integrals [29],

$$I(X,Y) = \int dU\, e^{\mathrm{tr}\, XUYU^\dagger} = \frac{\det_{p,q}(e^{x_p\, y_q})}{\Delta(X)\,\Delta(Y)}\,, \tag{4.6}$$

where x_p and y_q are the eigenvalues of X and Y. We thus have

$$\mathcal{Z}_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \;\propto\; \frac{\Delta(X(\vec{\alpha}_{m+1}))\;\Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}} \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\,\Delta(X_i)^2 \prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i\,\Delta(Y_i)^2$$

$$
\begin{aligned}
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]} \ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]} \\
& \prod_{i=1}^{m+1} I(\frac{1}{g_s}X_i, Y_i)\ I(-\frac{1}{g_s}X_{i-1}, Y_i)\ \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\operatorname{tr}\ \ln X_i} \\
& e^{\operatorname{tr}\ln f_0(X_0)}\ e^{\operatorname{tr}\ln f_{m+1}(X_{m+1})}\ \prod_{i=1}^{m} e^{\operatorname{tr}\ln f_i(X_i)} \\
\propto\ & \frac{\Delta(X(\vec{\alpha}_{m+1}))\ \Delta(X(\vec{\alpha}_0))}{(N!)^{2m+3}}\ \prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\ \prod_{i=1}^{m+1}\int_{\mathbb{R}_+^N} dY_i \\
& \Delta(X_0)\,\Delta(X_{m+1})\ \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]} \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]}\ \ \prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\operatorname{tr}\ \ln X_i} \\
& \prod_{i=1}^{m+1} \det_{p,q}\left(e^{\frac{1}{g_s}(X_i)_p\,(Y_i)_q}\right)\ \det_{p,q}\left(e^{\frac{-1}{g_s}(X_{i-1})_p\,(Y_i)_q}\right) \\
& e^{\operatorname{tr}\ln f_0(X_0)}\ e^{\operatorname{tr}\ln f_{m+1}(X_{m+1})}\ \prod_{i=1}^{m} e^{\operatorname{tr}\ln f_i(X_i)}\,,
\end{aligned}
$$

where we have dropped an overall sign, powers of g_s, and the group volume $\mathrm{Vol}(U(N))$ which are constant prefactors of no interest to us.

Next, we perform the integrals over Y_i along $\mathbb{R}_+^N$.

$$
\begin{aligned}
& \int_{\mathbb{R}_+^N} dY \det_{p,q}\left(e^{\frac{1}{g_s}(X_i)_p\,(Y)_q}\right)\ \det_{p,q}\left(e^{\frac{-1}{g_s}(X_{i-1})_p\,(Y)_q}\right) \\
= & \sum_{\sigma}\sum_{\tilde{\sigma}}(-1)^{\sigma}(-1)^{\tilde{\sigma}}\ \prod_{p=1}^{N}\int_0^{\infty} dy_p\, e^{\frac{y_p}{g_s}((X_i)_{\sigma(p)}-(X_{i-1})_{\tilde{\sigma}(p)})} \\
= & \sum_{\sigma}\sum_{\tilde{\sigma}}(-1)^{\sigma}(-1)^{\tilde{\sigma}}\ \prod_{p=1}^{N}\frac{g_s}{(X_{i-1})_{\tilde{\sigma}(p)}-(X_i)_{\sigma(p)}} \\
= & N!\, g_s^N\ \det_{p,q}\left(\frac{1}{(X_{i-1})_p-(X_i)_q}\right).
\end{aligned}
$$

Note that the integral is only convergent for $(X_i)_{\sigma(p)}-(X_{i-1})_{\tilde{\sigma}(p)}<0$. For X_i that violate this inequality, we will define the integral via its analytic continuation given in the third line.

An application of the Cauchy determinant formula,

$$
\det\left(\frac{1}{x_i+y_j}\right)_{1\le i<j\le n} = \frac{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{n}(x_i+y_j)}\,,
$$

yields

$$\int_{\mathbb{R}_+^N} dY \det_{p,q}(e^{\frac{1}{g_s}(X_i)_p\,(Y)_q}) \;\; \det_{p,q}(e^{\frac{-1}{g_s}(X_{i-1})_p\,(Y)_q}) = (-1)^{\binom{N}{2}} N!\, g_s^N \;\frac{\Delta(X_i)\,\Delta(X_{i-1})}{\Delta(X_{i-1},X_i)}\,,$$

where the notation $\Delta(X_{i-1},X_i)$ was introduced in (3.5). Evaluating the Y_i integrals thus, and continuing to drop overall signs and powers of g_s, our matrix integral becomes

$$\begin{aligned}\mathcal{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \;\; \propto \;\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\;\Delta(X(\vec{\alpha}_0))}{(N!)^{m+3}} \;\prod_{i=0}^{m+1}\int_{\Gamma_i} dX_i\;\Delta(X_i)^2 \\ & \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]} \;\prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]} \\ & \prod_{i=1}^{m+1}\frac{1}{\Delta(X_{i-1},X_i)} \;\;\prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\;\ln X_i} \\ & e^{{\rm tr}\,\ln f_0(X_0)}\;\; e^{{\rm tr}\,\ln f_{m+1}(X_{m+1})}\;\prod_{i=1}^{m} e^{{\rm tr}\,\ln f_i(X_i)}\,.\end{aligned}$$

4.4 Recovering the sum over partitions

Following the steps introduced in [11] in reverse, we next decompose the diagonal matrix X_i into blocks,

$$X_i = {\rm diag}\,(X_{0,i}, X_{1,i}, \ldots, X_{n,i})\,,$$

where each matrix $X_{j,i}$ is a $d\times d$ diagonal matrix whose eigenvalues are integrated on the contours $\gamma_{j,i}$ surrounding points of the form $q^{a_{j,i}+\mathbb{N}}$. We arrive at

$$\begin{aligned}\mathcal{Z}_{\rm MM}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) \;\; \propto \;\; & \frac{\Delta(X(\vec{\alpha}_{m+1}))\;\Delta(X(\vec{\alpha}_0))}{(N!)^{m+3}} \;\prod_{i=0}^{m+1}\prod_{j=0}^{n}\int_{(\gamma_{j,i})^d} dX_{j,i} \\ & \Delta(X_0)\,\Delta(X_{m+1})\;\prod_{i=1}^{m+1}\frac{\Delta(X_{i-1})\Delta(X_i)}{\Delta(X_{i-1},X_i)} \\ & \prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_i}(X_i)-V_{\vec{a}_{i-1}}(X_i)\right]} \;\prod_{i=1}^{m} e^{\frac{-1}{g_s}\,{\rm tr}\left[V_{\vec{a}_{i-1}}(X_{i-1})-V_{\vec{a}_i}(X_{i-1})\right]} \\ & e^{{\rm tr}\,\ln f_0(X_0)}\;\; e^{{\rm tr}\,\ln f_{m+1}(X_{m+1})}\;\prod_{i=1}^{m} e^{{\rm tr}\,\ln f_i(X_i)}\;\;\prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\;\ln X_i}\,,\end{aligned}$$

with

$$\frac{\Delta(X_{i-1})\Delta(X_i)}{\Delta(X_{i-1},X_i)} = \frac{\prod_j \Delta(X_{j,i-1})\prod_j \Delta(X_{j,i})\;\prod_{j<l}\Delta(X_{j,i-1},X_{l,i-1})\;\prod_{j<l}\Delta(X_{j,i},X_{l,i})}{\prod_{j,l}\Delta(X_{j,i-1},X_{l,i})}\,.$$

Our next step is to evaluate the $dX_{j,i}$ integrals via Cauchy's residue theorem. The poles of the integrands lie at the poles of f_i, and the zeros of $\Delta(X_{i-1},X_i)$. However, we have

been careful to define our contours $\gamma_{j,i}$ in a way that only the poles of f_i contribute. These lie at the points $q^{a_{j,i}+\mathbb{N}}$. Hence, the integrals evaluate to a sum of residues over the points

$$(X_{j,i})_l = q^{a_{j,i}+(h_{j,i})_l}\,,$$

where each $(h_{j,i})_l$ is a positive integer.

Since the integrand contains a Vandermonde of the eigenvalues of $X_{j,i}$, the residues vanish whenever two eigenvalues are at the same pole of f_i, i.e. if two $(h_{j,i})_l$ coincide. Moreover, since the integrand is symmetric in the eigenvalues, upon multiplication by $N!$, we can assume that the $(h_{j,i})_l$ are ordered,

$$(h_{j,i})_1 > (h_{j,i})_2 > (h_{j,i})_3 > \cdots > (h_{j,i})_d \geq 0.$$

The $(h_{j,i})_l$ hence encode a partition $\alpha_{j,i}$ via $(h_{j,i})_l = (\alpha_{j,i})_l - i + d$, and we have reduced our integrals to a sum over partitions. In terms of the function $h_l(\alpha)$ introduced in (3.1),

$$(X_{j,i})_l = q^{h_l(\alpha_{j,i})}\,, \quad h_l(\alpha_{j,i}) = (h_{j,i})_l + a_{j,i}\,,$$

$$h_1(\alpha_{j,i}) > h_2(\alpha_{j,i}) > \cdots > h_d(\alpha_{j,i}) \geq a_{j,i}.$$

Notice that unlike f_i, $i = 1, \ldots, m$, f_0 and f_{m+1} only have a finite number of $N = (n+1)d$ poles. Since the $(h_{j,0})_l$, $(h_{j,m+1})_l$ respectively can be chosen pairwise distinct and ordered, f_0 and f_{m+1} act as delta functions in the integrals over the $N \times N$ matrices X_0 and X_{m+1}, and fix these to the prescribed values $X(\vec{\alpha}_0)$ and $X(\vec{\alpha}_{m+1})$ respectively.

Performing the integrals hence yields

$$\begin{aligned}\mathcal{Z}_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \;\propto\; & \Delta(X(\vec{\alpha}_{m+1}))^2 \Delta(X(\vec{\alpha}_0))^2 \\ & \sum_{\{\alpha_{j,i} | j=0,\ldots,n;\, i=1,\ldots,m+1\}} \prod_{i=1}^{m+1} \frac{\Delta(X(\vec{\alpha}_{i-1}))\Delta(X(\vec{\alpha}_i))}{\Delta(X(\vec{\alpha}_{i-1}), X(\vec{\alpha}_i))} \\ & \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{\alpha}_i}(X(\vec{\alpha}_i)) - V_{\vec{\alpha}_{i-1}}(X(\vec{\alpha}_i))\right]} \prod_{i=1}^{m} e^{\frac{-1}{g_s}\operatorname{tr}\left[V_{\vec{\alpha}_{i-1}}(X(\vec{\alpha}_{i-1})) - V_{\vec{\alpha}_i}(X(\vec{\alpha}_{i-1}))\right]} \\ & \prod_{i=1}^{m} e^{(S_i + \frac{i\pi}{g_s})\operatorname{tr}\,\ln X(\vec{\alpha}_i)} \prod_{i=0}^{m+1} \prod_{j=0}^{n} \prod_{l=1}^{d} \left(\operatorname*{Res}_{q^{h_l(\alpha_{j,i})}} f_i \right).\end{aligned}$$

Notice that

$$\prod_j \prod_l \operatorname*{Res}_{q^{h_l(\alpha_{j,0})}} f_0 = \frac{1}{\Delta(X(\vec{\alpha}_0))^2}\,,$$

$$\prod_j \prod_l \operatorname*{Res}_{q^{h_l(\alpha_{j,m+1})}} f_{m+1} = \frac{1}{\Delta(X(\vec{\alpha}_{m+1}))^2}\,.$$

Furthermore,

$$\operatorname*{Res}_{q^{h_l(\alpha_{j,i})}} f_i = q^{h_l(\alpha_{j,i})}\, \hat{f}_{j,i}$$

where $\hat{f}_{j,i}$ computed in (4.2) is independent of $h_l(\alpha_{j,i})$. We thus have

$$e^{S_i \,\mathrm{tr}\,\ln X(\vec{\alpha}_i)} \prod_{j=0}^{n} \prod_{l=1}^{d} \left(\operatorname*{Res}_{q^{h_l(\alpha_{j,i})}} f_i \right) = e^{(S_i+1)\mathrm{tr}\,\ln X(\vec{\alpha}_i)} \prod_{j=0}^{n} (\hat{f}_{j,i})^d \,.$$

Upon substituting the expression (4.3) for S_i, we finally arrive at

$$\begin{aligned}
&\mathcal{Z}_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \\
\propto\ & \prod_{i=1}^{m} \prod_{j=0}^{n} (\hat{f}_{j,i})^d \sum_{\{\alpha_{j,i} | j=0,\ldots,n;\, i=1,\ldots,m+1\}} \\
& \prod_{i=1}^{m+1} \frac{\prod_j \Delta(X(\alpha_{j,i-1})) \prod_j \Delta(X(\alpha_{j,i})) \prod_{j<l} \Delta(X(\alpha_{j,i-1}), X(\alpha_{l,i-1})) \prod_{j<l} \Delta(X(\alpha_{j,i}), X(\alpha_{l,i}))}{\prod_{j,l} \Delta(X(\alpha_{j,i-1}), X(\alpha_{l,i}))} \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s} \mathrm{tr}\left[V_{\vec{a}_i}(X(\vec{\alpha}_i)) - V_{\vec{a}_{i-1}}(X(\vec{\alpha}_i))\right]} \prod_{i=1}^{m} \prod_{k=0}^{n} e^{(\frac{1}{2} - \sum_{j \le k}(a_{j,i} - a_{j,i-1}) - \frac{i\pi}{g_s})\mathrm{tr}\,\ln X(\alpha_{k,i})} \\
& \prod_{i=1}^{m} e^{\frac{-1}{g_s} \mathrm{tr}\left[V_{\vec{a}_{i-1}}(X(\vec{\alpha}_{i-1})) - V_{\vec{a}_i}(X(\vec{\alpha}_{i-1}))\right]} \prod_{i=1}^{m} \prod_{k=0}^{n} e^{(\frac{1}{2} - \sum_{j<k}(a_{j,i} - a_{j,i+1}))\mathrm{tr}\,\ln X(\alpha_{k,i})} \\
& \prod_{i=1}^{m} \prod_{j=0}^{n} e^{s_{j,i} \mathrm{tr}\,\ln X(\alpha_{j,i})} \,.
\end{aligned} \tag{4.7}$$

Comparing to (3.6) and (3.8), we conclude

$$\mathcal{Z}_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \propto \sum_{\alpha_{j,i}, j=0,\ldots,n;\, i=1,\ldots,m} \prod_{i=1}^{m+1} Z_{\mathrm{strip}}(\vec{\alpha}_i, \vec{\alpha}_{i-1}^T) \prod_{j=0}^{n} \prod_{i=1}^{m} q^{s_{j,i} |\alpha_{j,i}|} \,,$$

i.e.

$$\mathcal{Z}_{\mathrm{MM}}(Q, g_s, \vec{\alpha}_{m+1}, \vec{\alpha}_0^T) \propto Z_{\mathrm{vertex}}(\mathfrak{X}_0) = e^{\sum_g g_s^{2g-2} GW_g(\mathfrak{X}_0)} \,.$$

Up to a trivial proportionality constant, we have thus succeeded in rewriting the topological string partition function on the fiducial geometry $\mathfrak{X}_0$ as a *chain of matrices* matrix integral. By our reasoning in section 2.2, this result extends immediately to arbitrary toric Calabi-Yau 3-folds as follows. We have argued that any such 3-fold can be obtained from a sufficiently large choice of fiducial geometry via flops and limits. The respective partition functions are related via (2.5). Upon the appropriate variable identification, we hence arrive at a matrix model representation of the topological string on an arbitrary toric Calabi-Yau 3-fold.

5 Implications of our result

We have rewritten the topological string partition function as a matrix integral. This allows us to bring the rich theory underlying the structure of matrix models to bear on the study of topological string.

The type of matrix integral we have found to underlie the topological string on toric Calabi-Yau 3-folds is a so-called chain of matrices. This class of models has been studied extensively [30, 28], and many structural results pertaining to it are known.

5.1 Loop equations and Virasoro constraints

The loop equations of matrix models provide a set of relations among correlation functions. They are Schwinger-Dyson equations; they follow from the invariance of the matrix integral under a change of integration variables, or by an integration by parts argument.

Loop equations for a general chain of matrices have been much studied in the literature, in particular in [31, 32, 33, 34]. They can be viewed as W-algebra constraints (a generalization of Virasoro constraints) [35]. Having expressed the topological string partition function as a matrix integral, we can hence conclude that Gromov-Witten invariants satisfy W-algebra constraints.

Moreover, a general formal solution of loop equations was found in [34], and expressed in terms of so-called symplectic invariants F_g of a spectral curve. The spectral curve for a matrix integral is related to the expectation value of the resolvent of the first matrix in the chain,

$$W(x) = \left\langle \mathrm{tr}\, \frac{1}{x - M_0} \right\rangle^{(0)} .$$

The superscript $^{(0)}$ indicates that the expectation value is evaluated to planar order in a Feynman graph expansion.

The symplectic invariants $F_g(\mathcal{C})$ of an arbitrary spectral curve $\mathcal{C}$ were defined in [36]. [34] proved that for any chain of matrices integral Z, one has

$$\ln Z = \sum_g F_g(\mathcal{C})$$

with $\mathcal{C}$ the spectral curve associated to the matrix integral.

Calculating the spectral curve of a chain of matrices matrix model with complicated potentials poses some technical challenge. We will present the spectral curve for our matrix model (4.1) in a forthcoming publication [17].

5.2 Mirror symmetry and the BKMP conjecture

The mirror $\hat{\mathfrak{X}}$ of a toric Calabi-Yau 3-fold $\mathfrak{X}$ is a conic bundle over $\mathbb{C}^* \times \mathbb{C}^*$. The fiber is singular over a curve, which we will refer to as the mirror curve $\mathcal{S}_{\hat{\mathfrak{X}}}$ of $\hat{\mathfrak{X}}$. It is a plane curve described by an equation

$$\mathcal{S}_{\hat{\mathfrak{X}}} \qquad : \qquad H(e^x, e^y) = 0\,,$$

where H is a polynomial whose coefficients follow from the toric data of $\mathfrak{X}$ and the Kähler parameters of the geometry.

Mirror symmetry is the statement that the topological A-model partition function with target space $\mathfrak{X}$ is equal to the topological B-model partition function with target space $\hat{\mathfrak{X}}$.

Bouchard, Klemm, Mariño and Pasquetti (BKMP) have conjectured in [16] that

$$GW_g(\mathfrak{X}) \stackrel{?}{=} F_g(\mathcal{S}_{\hat{\mathfrak{X}}})\,.$$

Here, the F_g's are the symplectic invariants introduced in [36]. The main interest of this conjecture is that it provides a systematic method for computing the topological string partition function, genus by genus, away from the large radius limit, and without having to solve differential equations.

This conjecture was motivated by the fact that symplectic invariants have many intriguing properties reminiscent of the topological string free energies. They are invariant under transformations $\mathcal{S} \to \tilde{\mathcal{S}}$ which conserve the symplectic form $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$, whence their name [36]. They satisfy holomorphic anomaly equations [37], they have an integrable structure similar to Givental's formulae [38, 39, 40, 41, 42], they satisfy some special geometry relations, WDVV relations [43], and they give the Witten-Kontsevich theory as a special case [36, 44].

BKMP succesfully checked their claim for various examples to low genus.

The conjecture was proved for arbitrary genus in [7] for $\mathfrak{X}$ a Hirzebruch rank 2 bundle over $\mathbb{P}^1$ (this includes the conifold). Marshakov and Nekrasov [10] proved $F_0 = GW_0$ for the family of $SU(n)$ Seiberg-Witten models. Klemm and Sułkowski [11], generalizing [7] to Nekrasov's sums over partitions for $SU(n)$ Seiberg-Witten gauge theories, proved the relation for F_0, building on work in [45]. In fact, it seems easy to extend their computation to arbitrary genus F_g. In [46], Sułkowski provided a matrix model realization of $SU(n)$ gauge theory with a massive adjoint hypermultiplet, using again a generalization of [7] for more general sums over partitions. Bouchard and Mariño [47] noticed that an infinite framing limit of the BKMP conjecture for the framed vertex $\mathfrak{X} = \mathbb{C}^3$ implies another conjecture for the computation of Hurwitz numbers, namely that the Hurwitz numbers of genus g are the symplectic invariants of genus g for the Lambert spectral curve $e^x = y\,e^{-y}$. That conjecture was proved recently by another generalization of [7] using a matrix model for summing over partitions [13], and also by a direct cut and join combinatorial method [48]. The BKMP conjecture was also proved for the framed vertex $\mathfrak{X} = \mathbb{C}^3$ in [49, 50], using the ELSV formula and a cut and join combinatorial approach.

Since we have demonstrated that the topological string partition function is reproduced by a matrix model, we can conclude that the Gromov-Witten invariants coincide with the symplectic invariants

$$\sum_g g_s^{2g-2}\, GW_g = \sum_G F_g(\mathcal{C})\,,$$

with $\mathcal{C}$ the spectral curve of our matrix model. We will compute $\mathcal{C}$ explicitly in a forthcoming work [17], and demonstrate that it indeed coincides, up to symplectic transformations, with the mirror curve $\mathcal{S}_{\hat{\mathfrak{X}}}$.

5.3 Simplifying the matrix model

The matrix models associated to the conifold or to geometries underlying Seiberg-Witten theory have a remarkable property: the spectral curve is the same (perturbatively and up to symplectic transformations) as the one of a simpler matrix model withe all g-functions replaced by only the leading term in their small $\ln q$ expansion. We will demonstrate in a forthcoming work [17] that this property also holds for our matrix integral (4.1). We can hence simplify the potentials of our matrix model, arriving at

$$\begin{aligned}\mathcal{Z}_{\rm simp}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) &= \Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))\prod_{i=0}^{m+1}\int_{H_{\bar{n}_i}(\Gamma_i)}dM_i\prod_{i=1}^{m+1}\int_{H_{\bar{n}}(\mathbb{R}_+)}dR_i \\ &\prod_{i=1}^{m} e^{\frac{1}{g_s}\,{\rm tr}\,\sum_{j=0}^{n}(Li_2(q^{a_{j,i}}/M_i)-Li_2(q^{a_{j,i-1}}/M_i))} \\ &\prod_{i=0}^{m-1} e^{\frac{1}{g_s}\,{\rm tr}\,\sum_{j=0}^{n}(Li_2(q^{a_{j,i}}/M_i)-Li_2(q^{a_{j,i+1}}/M_i))} \\ &\prod_{i=1}^{m+1} e^{\frac{1}{g_s}{\rm tr}\,(M_i-M_{i-1})R_i}\;\;\prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\;\ln M_i}\,,\end{aligned}$$

where the matrix M_i is of size $\bar{n}_i=\sum_j \bar{n}_{j,i}$.

Classical limit

In the classical limit, the dilogarithm Li_2 becomes the function $x\ln x$, and we have

$$\begin{aligned}\mathcal{Z}_{\rm eff.\,cl}(Q,g_s,\vec{\alpha}_{m+1},\vec{\alpha}_0^T) &= \Delta(X(\vec{\alpha}_{m+1}))\,\Delta(X(\vec{\alpha}_0))\prod_{i=0}^{m+1}\int_{H_{\bar{n}_i}(\Gamma_i)}dM_i\prod_{i=1}^{m+1}\int_{H_{\bar{n}}(\mathbb{R}_+)}dR_i \\ &\prod_{i=1}^{m} e^{\frac{1}{g_s}\,{\rm tr}\,\sum_{j=0}^{n}(M_i-a_{j,i})\,\ln(a_{j,i}-M_i)-(M_i-a_{j,i-1})\,\ln(a_{j,i-1}-M_i)} \\ &\prod_{i=0}^{m-1} e^{\frac{1}{g_s}\,{\rm tr}\,\sum_{j=0}^{n}(M_i-a_{j,i})\,\ln(a_{j,i}-M_i)-(M_i-a_{j,i+1})\,\ln(a_{j,i+1}-M_i)} \\ &\prod_{i=1}^{m+1} e^{\frac{1}{g_s}{\rm tr}\,(M_i-M_{i-1})R_i}\;\;\prod_{i=1}^{m} e^{(S_i+\frac{i\pi}{g_s})\,{\rm tr}\;\ln M_i}\,.\end{aligned}$$

This model shares features with the Eguchi-Yang matrix model [51], see also [10].

6 Conclusion

We have rewritten the topological vertex formula for the partition function of the topological A-model as a matrix integral.

Having expressed the topological string in terms of a matrix model, we can bring the immense matrix model toolkit which has been developed since the introduction of random matrices by Wigner in 1951 to bear on questions concerning the topological string and Gromov-Witten invariants. We already started down this path in section 5 above. Going further, we can apply the method of bi-orthogonal polynomials [28] to our matrix model to unearth the integrable system structure (Miwa-Jimbo [52, 53]) underlying the topological string, at least in the case of toric targets, together with its Lax pair, its Hirota equations (which arise as orthogonality relations), etc. In a related vein, free fermions [54, 55] arise in the theory of matrix models when invoking determinantal formulae to express the matrix model measure [56]. It will be very interesting to explore how this is related to the occurrence of free fermions in topological string theory, as studied in [57, 58, 59, 60]. More generally, one should study what can be learned about the non-perturbative topological string from its perturbative reformulation as a matrix model, as in the works [61, 62, 63, 64]. A recurrent such question, which could be addressed in the matrix model framework (in fact, it was already latently present in the calculations in this work), is that of the quantization of Kähler parameters.

On a different note, notice that the matrix model derived in this article, with a potential which is a sum of logs of q-deformed Γ functions, looks very similar to the matrix model counting plane partitions introduced in [65]. This is a hint that it could be possible to recover the topological vertex formula, corresponding to the topological string with target $\mathbb{C}^3$ and appropriate boundary conditions, directly from the matrix model approach. Either along these lines or the lines pursued in this paper, it would be interesting to derive a matrix model related to the Nekrasov deformation [8, 66] of the topological string.

A completely open question is whether the close relation between topological strings and matrix models persists beyond toric target spaces, and more ambitiously yet, whether there exists a general notion of geometry underlying matrix models.

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A q-product

The g-function, which plays a central role in the definition of our matrix model, is defined as an infinite product,

$$g(x) = \prod_{n=1}^{\infty}(1 - \frac{1}{x}\, q^n)\,.$$

It is the quantum Pochhammer symbol $g(x) = [q/x;q]_\infty$, and it is related to the q-deformed gamma function via $\Gamma_q(x) = (1-q)^{1-x}\, g(1)/g(q^{1-x})$.

The RHS is convergent for $|q| < 1$ and arbitrary complex $x \neq 0$. $g(x)$ satisfies the functional equation

$$g(qx) = (1 - \frac{1}{x})\, g(x)\,.$$

For $n \in \mathbb{N}$, we have

$$g(q^n) = 0$$

and

$$g'(q^n) = (-1)^{n-1} g(1)\, q^{-\frac{n(n+1)}{4}} \prod_{m=1}^{n-1}[m] = g(1)\, q^{-\frac{n(n+1)}{2}}\ [n-1]! = (-1)^{n-1}\, q^{-\frac{n(n+1)}{2}}\ \frac{g(1)^2}{g(q^{1-n})}\,.$$

Via the triple product representation of the theta function,

$$\theta(z;\tau) = \prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau})(1 + e^{(2m-1)\pi i\tau + 2\pi i z})(1 + e^{(2m-1)\pi i\tau - 2\pi i z})\,,$$

we obtain the identity

$$\theta\left(\frac{1}{2} + \frac{1}{4\pi i}\ln\frac{q}{x^2}\,;\ \frac{\ln q}{2\pi i}\right) = g(x) g(\frac{q}{x}) g(1)\,.$$

We have

$$\frac{g(x)\, g(q/x)}{g(1)^2\ \sqrt{x}}\ e^{\frac{(\ln x)^2}{2\ln q}}\ e^{\frac{-i\pi\ln x}{\ln q}} = \frac{-\ln q}{\theta'(\frac{1}{2} - \frac{i\pi}{\ln q}, -\frac{2i\pi}{\ln q})}\ \theta\left(\frac{\ln x}{\ln q} + \frac{1}{2} - \frac{i\pi}{\ln q}, \frac{-2i\pi}{\ln q}\right)$$

where θ is the Riemann theta-function for the torus of modulus $-2i\pi/\ln q$.

At small $\ln q$, the following expansion is valid,

$$\ln g(x) = \frac{1}{\ln q}\sum_{n=0}^{\infty}\frac{(-1)^n\, B_n}{n!}\,(\ln q)^n\ \mathrm{Li}_{2-n}(1/x)\,,$$

where we have used the definition of the Bernoulli numbers B_n as the coefficients in the expansion of $t/(e^t - 1)$,

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\,.$$

$\mathrm{Li_n}$ is the polylogarithm function, defined as

$$\mathrm{Li_n(x)} = \sum_{\mathrm{k}=1}^{\infty} \frac{\mathrm{x^k}}{\mathrm{k^n}}\,.$$

This is a generalization of the logarithm function, recovered at $n = 1$,

$$\mathrm{Li_1(x)} = -\ln(1-\mathrm{x})\,.$$

It satisfies the functional relation

$$\mathrm{Li_n'(x)} = \frac{1}{\mathrm{x}}\mathrm{Li_{n-1}(x)}\,. \tag{A.1}$$

Note in particular that this implies that $\mathrm{Li_n}$ is an algebraic function of x for $n \leq 0$. E.g.,

$$\mathrm{Li_0(x)} = \frac{\mathrm{x}}{1-\mathrm{x}}\,.$$

We also define the function

$$\psi_q(x) = x\,\frac{g'(x)}{g(x)}\,.$$

Using the functional equation (A.1) of the polylogarithm, we find its small $\ln(q)$ expansion

$$\begin{aligned}\psi_q(x) &= -\frac{1}{\ln q}\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}(\ln q)^n\ \mathrm{Li_{1-n}(1/x)} \\ &= \frac{1}{\ln q}\Big[\ln(1-\frac{1}{x}) - \frac{\ln q}{2(x-1)} - \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\ (\ln q)^{2n}\,\mathrm{Li_{1-2n}(x)}\Big]\,.\end{aligned}$$

For the second equality, we have used $B_0 = 1, B_1 = -\frac{1}{2}$, and $B_{2n+1} = 0$ for $n > 1$.

We have near $x \to \infty$

$$\psi_q(x) \sim \frac{q}{1-q}\ \frac{1}{x} + O(x^{-2})$$

and near $x \to 0$:

$$\psi(x) \sim \frac{1}{2} + \frac{i\pi + \ln x}{g_s} + O(x).$$

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