VANISHING THEOREM FOR TAME HARMONIC BUNDLES VIA L^2-COHOMOLOGY

YA DENG AND FENG HAO

Abstract. Using L^2-methods, we prove a vanishing theorem for tame harmonic bundles over quasi-Kähler manifolds in a very general setting. As a special case, we give a completely new proof of the Kodaira type vanishing theorems for Higgs bundles due to Arapura and for parabolic Higgs bundles by Arapura, Li and the second named author. To prove our vanishing theorem, we construct a fine resolution of the Dolbeault complex for tame harmonic bundles via the complex of sheaves of L^2-forms, and we establish the Hörmander L^2-estimate and solve $(\bar{\partial}_E + \theta)$-equations for the Higgs bundle (E, θ).

Contents

0. Introduction

0.1. **Main result.** Let (X, ω) be a compact Kähler manifold and let D be a simple normal crossing divisor on D. Let (E, θ, h) be a tame harmonic bundle over $X - D$ (see § 1.1 for precise definition), and let $^\diamond E$ be the subsheaf of $\iota_* E$ consisting of sections whose norms

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with respect to h have sub-polynomial growth, where $\iota : X - D \hookrightarrow X$ is the inclusion. By Simpson-Mochizuki, ${}^\diamond E$ is a locally free coherent sheaf, and (E, θ) extends to a logarithmic Higgs bundle

$$\theta : {}^\diamond E \to {}^\diamond E \otimes \Omega^1_X(\log D)$$

such that

$$\theta \wedge \theta = 0.$$

We refer to § 3.2 for more details.

In this paper, we prove the following vanishing theorem.

Theorem A (=Theorem 3.18)**.** *Let (X, ω) be a compact Kähler manifold of dimension n, and let D be a simple normal crossing divisor on X. Let (E, θ) be a tame harmonic bundle on $X - D$, and let $({}^\diamond E, \theta)$ be the extension of (E, θ) on X as introduced above. Let L be a line bundle on X equipped with a smooth metric h_L so that its curvature $\sqrt{-1}R(h_L) \geq 0$ and has at least $n - k$ positive eigenvalues. Let P be a nef line bundle on X. Then for the following (Dolbeault) complex of sheaves*

$$\mathrm{Dol}({}^\diamond E, \theta) := {}^\diamond E \xrightarrow{\wedge\theta} {}^\diamond E \otimes \Omega^1_X(\log D) \xrightarrow{\wedge\theta} \cdots \xrightarrow{\wedge\theta} {}^\diamond E \otimes \Omega^n_X(\log D) \tag{0.1.1}$$

the hypercohomology

$$\mathbb{H}^i\Big(X, \mathrm{Dol}({}^\diamond E, \theta) \otimes L \otimes P\Big) = 0$$

for any $i > n + k$.

Theorem A seems new even if the tame harmonic bundle (E, θ, h) comes from a complex variation of polarized Hodge structures. It indeed interpolates the Kodaira-Akizuki-Nakano type vanishing theorems for parabolic Higgs bundles [AHL19, Corollary 7.3] by Arapura, Li and the second named author (in the case that L is ample, see Corollary 3.20), and the log Girbau vanishing theorem by Huang-Liu-Wan-Yang [HLWY16, Corollary 1.2] (in the case that $(E, \theta) = (\mathscr{O}_{X-D}, 0)$, see Remark 3.19). We stress here that our proof of Theorem A is essentially self-contained (in particular we do not apply the deep Simpson-Mochizuki correspondence) and is purely in characteristic 0 (since we are working on Kähler manifolds), while [AHL19] applies heavy machinery (*e.g.* the complicated construction of parabolic Higgs bundles over quasi-projective manifold, moduli spaces of parabolic Higgs bundles by Yokogawa and the Biswas correspondence) to reduce the problem to the celebrated vanishing theorem by Arapura [Ara19] whose proof is in characteristic p (see § 0.3 for more details). The main technique in the proof of Theorem A is a new application of L^2-methods to tame harmonic bundles, and we hope that it can bring some new input in the study of L^2-cohomology for Higgs bundles. Let us also mention a few byproducts of our proof: we construct explicitly a complex of sheaves of L^2-forms which is quasi-isomorphic to the Dolbeault complex (0.1.1) (see Theorem 3.17) in a similar manner (but using different metric) as [Zuc79] in which Zucker did this for variation of polarized Hodge structures over a quasi-projective curve; we also establish the Hörmander L^2-estimate and solvability criteria for $(\bar\partial_E + \theta)$-equations for general Higgs bundles (E, θ) (see Theorem 2.6 and Corollary 2.7).

If we apply the Simpson-Mochizuki correspondence [Sim90, Moc09] for parabolic Higgs bundles on projective manifolds to Theorem A, we can obtain the following vanishing theorem for parabolic Higgs bundles.

Corollary B (=Corollary 3.20)**.** *Let X be a complex projective manifold of dimension n, and let D be simple normal crossing divisor on X. Let $(E, {}_{\boldsymbol{a}}E, \theta)$ be poly-stable parabolic Higgs bundle on (X, D) with trivial parabolic degrees which is locally abelian. Let L be a line bundle on X equipped with a smooth metric h_L so that its curvature $\sqrt{-1}R(h_L) \geq 0$ and has at least $n - k$ positive eigenvalues. Let P be a nef line bundle on X. Then for the weight 0 filtration ${}^\diamond E$ of $(E, {}_{\boldsymbol{a}}E, \theta)$, one has*

$$\mathbb{H}^i\Big(X, \mathrm{Dol}({}^\diamond E, \theta) \otimes L \otimes P\Big) = 0$$

for any $i > n + k$.

For the notions in Corollary B we refer to §§ 3.1 and 3.7 for more details.

0.2. **Idea of the proof.** Let us briefly explain the main idea of our proof of Theorem A. We first construct a complex of L^2 fine sheaves which is quasi-isomorphic to the Dolbeault complex

$$\mathrm{Dol}(E,\theta) := {}^\diamond E \xrightarrow{\theta} {}^\diamond E \otimes \Omega_X(\log D) \xrightarrow{\theta} \cdots \xrightarrow{\theta} {}^\diamond E \otimes \Omega_X^n(\log D) \tag{0.2.1}$$

For a given a Kähler metric on $X - D$, and a smooth metric g for E over $X - D$, we let $\mathfrak{L}^m_{(2)}(X,E)_{g,\omega}$ be the sheaf on E of germs of E-valued m-forms σ with measurable coefficients so that $|\sigma|^2_{g,\omega}$ is locally integrable and $(\bar\partial + \theta)(\sigma)$ exists weakly as a locally L^2 E-valued $(m+1)$-forms. Here the L^2 norms $|\sigma|^2_{g,\omega}$ are induced by ω on differential forms and by g on elements in E. Since $(\bar\partial + \theta)^2 = 0$, it thus gives rise to a complex of fine sheaves

$$\mathfrak{L}^0_{(2)}(X,E)_{g,\omega} \overset{\bar\partial+\theta}{\to} \cdots \overset{\bar\partial+\theta}{\to} \mathfrak{L}^{2n}_{(2)}(X,E)_{g,\omega} \tag{0.2.2}$$

As the harmonic metric h is a *canonical* metric on the E, it is quite natural to make the choice that g is the harmonic metric h and ω is a Poincaré-type metric ω_P over $X - D$ as [Zuc79, CKS87, KK87]. However, even for the case when (E,θ) comes from a variation of polarized Hodge structures over $X - D$, it turns out to be a quite difficult problem that $(\mathfrak{L}^\bullet_{(2)}(X,E)_{h,\omega_P}, \bar\partial + \theta)$ is quasi isomorphic to $\mathrm{Dol}(E,\theta)$, and one essentially cannot avoid the delicate norm estimate for Hodge metrics near D in [Sch73, Kas85, CKS86] (see *e.g.* [Zuc79, JYZ07]). In this paper, we make a slight perturbation $h(\boldsymbol{a}, N)$ of the harmonic metric h (see Lemma 3.10 for more details) as [Moc02, §4.5.3] so that $h(\boldsymbol{a}, N)$ will degenerate mildly, albeit the norm of harmonic metric h for ${}^\diamond E$ is of sub polynomial growth. This construction indeed brings us several advantages (among others): we can prove that $(\mathfrak{L}^\bullet_{(2)}(X,E)_{h(\boldsymbol{a},N),\omega_P}, \bar\partial + \theta)$ is indeed quasi-isomorphic to $\mathrm{Dol}(E,\theta)$, and the negative contribution of the curvature $(E,\theta,h(\boldsymbol{a},N))$ is small enough which can be absorbed completely by the curvature $\sqrt{-1}R(h_L)$ of any (partially) positive metrized line bundle (L, h_L).

Once this fine resolution of $\mathrm{Dol}(E,\theta)$ is established, to prove Theorem A (we assume now $P = \mathscr{O}_X$ for simplicity), the hypercohomology of $\mathrm{Dol}(E,\theta) \otimes L$ is isomorphic to the cohomology of the complex of global sections of (0.2.2)

$$(L^\bullet_{(2)}(X - D, E \otimes L|_{X-D})_{h(\boldsymbol{a},N)\cdot h_L,\omega_P}, D''), \tag{0.2.3}$$

where $D'' := \bar\partial_{E\otimes L} + \theta \otimes \mathbb{1}_L$ satisfying $D''^2 = 0$. We then reduce the proof of Theorem A to the vanishing of L^2-cohomology of (0.2.3) for $i > \dim X + k$. To prove this, we first generalize the L^2-estimate by Hörmander, Andreotti-Vesentini, Skoda, Demailly and others to Higgs bundles. Roughly speaking, we prove that under certain curvature conditions for Higgs bundles (E,θ), we can solve the D''-equation as the $\bar\partial$-equation in a similar way (see Theorem 2.6 and Corollary 2.7). We then choose the perturbation $h(\boldsymbol{a}, N)$ of h carefully so that such required curvature condition can be fulfilled and it enables us to prove the vanishing result for the L^2-cohomology of (0.2.3). This idea of solving D''-equation for Higgs bundles using L^2-method seems a new ingredient as we are aware of.

0.3. **Previous results.** For X a complex projective manifold with a simple normal crossing divisor D, Arapura [Ara19] gives a vanishing theorem for semistable Higgs bundles (E,θ) over $X - D$ with trivial parabolic structure, trivial Chern classes and nilpotent Higgs field θ. In the spirit of the algebraic proof of the Kodaira vanishing theorem by Deligne-Illusie [DI87], the proof of Arapura's vanishing theorem is reduced to the mod p-setting and boils down to a periodic sequence of Higgs bundles $(E_i,\theta_i) = B^i(E,\theta)$ through an operator B raised from the absolute Frobenius morphism, which is due to Lan-Sheng-Yang-Zuo

[LSZ19, LSYZ13] and Langer [Lan15]. The dimension of the cohomology $\mathbb{H}^i(X, \mathrm{Dol}(E_i, \theta_i) \otimes L^{p^i})$ is non-decreasing for $\{(E_i, \theta_i)\}$, then Arapura's vanishing theorem follows from Serre's vanishing theorem. With his vanishing theorem, Arapura reproves the Saito's vanishing theorem (see, e.g. Popa [Pop16]) for polarized variations of Hodge structures with unipotent monodromy on the complement of a normal crossing divisor on any complex projective manifold. In the following up article [AHL19], Arapura's vanishing theorem for Higgs bundles is generalized to parabolic Higgs bundles coming from tame harmonic bundles over $X - D$, especially the nilpotency condition for Higgs field θ is get rid of, which is inevitable due to technical reasons in [Ara19]. Also, the parabolic structures in the generalized vanishing theorem are allowed to be nontrivial and the jumping numbers of the parabolic Higgs bundles are real. Two main steps of the proof in [AHL19] reduce the generalized vanishing theorem to Arapura's vanishing theorem [Ara19]. After perturbing the jump numbers to rational numbers and using Biswas's correspondence [Bis97], one reduces the proof to the case in which the Higgs bundles have trivial parabolic structures. Another reduction step is using the $\mathbb{C}^*$-action on the moduli space of parabolic Higgs bundles, and the properness of the Hitchin fibration due to Yokogawa [Yok93] to reduce the proof to the case with parabolic Higgs bundle admitting nipotent Higgs field.

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Notations and conventions

- A couple (E, h) is a *Hermitian vector bundle* on a complex manifold X if E a holomorphic vector bundle on X equipped with a smooth hermitian metric h. $\bar{\partial}_E$ denotes the complex structure of E, and we sometimes simply write $\bar{\partial}$ if no confusion arises.
- Two hermitian metrics h and $\tilde{h}$ of a holomorphic vector bundle on X if *mutually bounded* if $C^{-1}h \leq \tilde{h} \leq Ch$ for some constant $C > 0$, and we shall denote by $h \sim h'$.
- For a hermitian vector bundle (E, h) on a complex manifold, $R(h)$ denotes its Chern curvature.
- For a Higgs bundle (E, h) with a smooth metric h on a complex manifold, $F(h) := R(h) + [\theta, \theta_h^*]$, where θ_h^* is the adjoint of θ with respect to h.
- Δ denotes the unit disk in $\mathbb{C}$.
- The complex manifold X in this paper are always assumed to be connected and of dimension n.
- Throughout the paper we always work over the complex number field $\mathbb{C}$.

1. Technical preliminary

1.1. **Higgs bundle and tame harmonic bundle.**

In this section we recall the definition of Higgs bundles and tame harmonic bundles. We refer the readers to [Sim88, Sim90, Sim92, Moc02, Moc07] for further details.

Definition 1.1. Let X be a complex manifold. A *Higgs bundle* on X is a pair (E, θ) where E is holomorphic vector bundle with $\bar{\partial}_E$ its complex structure, and $\theta : E \to E \otimes \Omega_X^1$ is a holomorphic one form with value in $\mathrm{End}(E)$, say *Higgs field*, satisfying $\theta \wedge \theta = 0$.

Let (E,θ) be a Higgs bundle over a complex manifold X. Write $D'' := \bar\partial_E+\theta$. Then $D''^2 = 0$. Suppose h is a smooth hermitian metric of E. Denote by $\partial_h + \bar\partial_E$ the Chern connection with respect to h, and θ_h^* be the adjoint of θ with respect to h. Write $D'_h := \partial_h + \theta_h^*$. The metric h is *harmonic* if the operator $D_h := D'_h + D''$ is integrable, that is, if $D_h^2 = 0$.

Definition 1.2 (Harmonic bundle)**.** A harmonic bundle on a complex manifold X is a Higgs bundle (E,θ) endowed with a harmonic metric h.

Let X be an n-dimensional complex manifold, and let D be a simple normal crossing divisor.

Definition 1.3. (Admissible coordinate) Let p be a point of X, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ be components of D containing p. An *admissible coordinate* around p is the tuple $(U;z_1,\ldots,z_n;\varphi)$ (or simply $(U;z_1,\ldots,z_n)$ if no confusion arises) where

- U is an open subset of X containing p.
- there is a holomorphic isomorphism $\varphi : U \to \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1,\ldots,\ell$.

We shall write $U^* := U-D$, $U(r) := \{z \in U \mid |z_i| < r, \forall i = 1,\ldots,n\}$ and $U^*(r) := U(r)\cap U^*$.

For any harmonic bundle (E,θ,h), let p be any point of X, and $(U;z_1,\ldots,z_n)$ be an admissible coordinate around p. On U, we have the description:

$$\theta = \sum_{j=1}^{\ell} f_j d\log z_j + \sum_{k=\ell+1}^{n} g_k dz_k \tag{1.1.1}$$

Definition 1.4 (Tameness)**.** Let t be a formal variable. We have the polynomials $\det(f_j-t)$, and $\det(g_k-t)$, whose coefficients are holomorphic functions defined over U^*. When the functions can be extended to the holomorphic functions over U, the harmonic bundle is called *tame* at p. A harmonic bundle is *tame* if it is tame at each point.

Recall that the Poincaré metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ is described as

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j\wedge d\bar z_j}{|z_j|^2(\log|z_j|^2)^2} + \sum_{k=\ell+1}^{n} \frac{\sqrt{-1}dz_k\wedge d\bar z_k}{(1-|z_k|^2)^2}.$$

Note that

$$\omega_P = -\sqrt{-1}\partial\bar\partial\log\Big(\prod_{j=1}^{\ell}(-\log|z_j|^2)\cdot\prod_{k=\ell+1}^{n}(1-|z_k|^2)\Big).$$

For the tame harmonic bundle, we have the following crucial norm estimate for Higgs field θ due to Simpson [Sim90] for the one dimensional case and Mochizuki [Moc07, Lemma 8.3] in general.

Theorem 1.5. *Let (E,θ,h) be a tame harmonic bundle. Let f_j, g_k be the matrix-valued holomorphic functions as in Definition 1.3. Then there exists a positive constant $C>0$ satisfying that*

$$\begin{aligned} |f_j|_h &\le C, && \textit{for}\quad j = 1,\ldots,\ell;\\ |g_k|_h &\le C, && \textit{for}\quad k = \ell+1, n. \end{aligned}$$

In other words, the norm

$$|\theta|_{h,\omega_P} \le C\sum_{j=1}^{\ell}(-\log|z_j|^2)$$

holds over some $U^*(r)$ for some constant $C>0$ and $1<r\ll 1$. □

1.2. **Curvature property of Higgs bundles.** Suppose now (E,θ) is a Higgs bundle of rank r equipped with a metric h over a Kähler manifold (X,ω) of dimension n .

We make the following assumption for (E,θ,h) throughout this section .

Assumption 1.6. $\bar{\partial}_E\theta^*_h=0$.

Let us denote by $D_h:=D'_h+D''$ a connection and $F(h):=D_h^2$. Assumption 1.6 is equivalent to that $\partial_h\theta=0$. Hence

$$F(h)=D_h^2=[D'_h,D'']=R(h)+[\theta,\theta^*]\in A^{1,1}(X,\mathrm{End}(E)), \tag{1.2.1}$$

where $R(h):=(\partial_h+\bar{\partial}_E)^2$. Moreover, one can easily see that $(\sqrt{-1}F(h))^*=\sqrt{-1}F(h)$. In other words, $\sqrt{-1}F(h)$ is a $(1,1)$-form with $\mathrm{Herm}(E)$-value, where $\mathrm{Herm}(E)$ is the hermitian endomorphism of (E,h).

By Simpson [Sim88], one has the following Kähler identity:

$$\sqrt{-1}[\Lambda_\omega,D'']=(D'_h)^* \tag{1.2.2}$$

$$\sqrt{-1}[\Lambda_\omega,D'_h]=-(D'')^* \tag{1.2.3}$$

where $(D'_h)^*$ and $(D'')^*$ are the formally adjoint operators of D'_h and D'' with respect to h and ω, and Λ_ω is the adjoint operator of $\wedge\omega$ with respect to Hodge inner product on differential forms. Define the Laplacians

$$\begin{aligned}\Delta' &= D'_hD'^*_h+(D'_h)^*D'_h\\ \Delta'' &= D''(D'')^*+(D'')^*D''\end{aligned}$$

A computation can easily derive the following equality.

Lemma 1.7 (Bochner-Kodaira-Nakano identity for Higgs bundles)**.**

$$\Delta''=\Delta'+[\sqrt{-1}F(h),\Lambda_\omega] \tag{1.2.4}$$

Proof. By (1.2.3), one has

$$\Delta''=D''(D'')^*+(D'')^*D''=-\sqrt{-1}[D'',[\Lambda_\omega,D'_h]].$$

By the Jacobi identity, one has

$$\begin{aligned}\Delta'' &= \sqrt{-1}[D'_h,[\Lambda_\omega,D'']]-\sqrt{-1}[\Lambda_\omega,[D'_h,D'']]\\ &\overset{(1.2.2)}{=} [D'_h,(D'_h)^*]+[\sqrt{-1}[D'_h,D''],\Lambda_\omega]\\ &\overset{(1.2.1)}{=} \Delta'+[\sqrt{-1}F(h),\Lambda_\omega],\end{aligned}$$

which is the desired equality. □

1.3. **Notions of positivity for Higgs bundles.** Let (E,θ) be a Higgs bundle endowed with a smooth metric h, which satisfies assumption 1.6. For any $x\in X$, let $e_1,\ldots,e_r$ be a frame of E at x, and let $e^1,\ldots,e^r$ be its dual in E^*. Let $z_1,\ldots,z_n$ be a local coordinate centered at x. We write

$$F(h)=R(h)+[\theta,\theta^*_h]=R^\beta_{j\bar{k}\alpha}dz_j\wedge d\bar{z}_k\otimes e^\alpha\otimes e_\beta$$

Set $R_{j\bar{k}\alpha\bar{\beta}}:=h_{\gamma\bar{\beta}}R^\gamma_{j\bar{k}\alpha}$, where $h_{\gamma\bar{\beta}}=h(e_\gamma,e_\beta)$. $F(h)$ is called *Nakano semi-positive* at x if

$$\sum_{j,k,\alpha,\beta}R_{j\bar{k}\alpha\bar{\beta}}u^{j\alpha}\overline{u^{k\beta}}\geq 0$$

for any $u=\sum_{j,\alpha}u^{j\alpha}\frac{\partial}{\partial z_j}\otimes e_\alpha\in(T^{1,0}_X\otimes E)_x$. $F(h)$ is called *Griffiths semi-positive* at x if

$$\sum_{j,k,\alpha,\beta}R_{j\bar{k}\alpha\bar{\beta}}\xi^j\zeta^\alpha\overline{\xi^k\zeta^\beta}\geq 0$$

for any $\xi = \sum_j \xi^j \frac{\partial}{\partial z_j} \in T^{1,0}_{X,x}$ and any $\zeta = \sum_\alpha \zeta^\alpha e_\alpha \in E_x$. (E,θ,h) is called Nakano (resp. Griffiths) semipositive if $F(h)$ is Nakano (resp. Griffiths) semi-positive at every $x \in X$. When $\theta = 0$, this reduces to the same positivity concepts in [Dem12, Chapter VII, §6] for vector bundles.

We write

$$F(h) \geq_{\mathrm{Nak}} \lambda(\omega \otimes \mathbb{1}_E) \quad \text{for } \lambda \in \mathbb{R}$$

if

$$\sum_{j,k,\alpha,\beta} (R_{j\bar{k}\alpha\bar{\beta}} - \lambda\omega_{j\bar{k}} h_{\alpha\bar{\beta}})(x) u^{j\alpha}\overline{u^{k\beta}} \geq 0$$

for any $x \in X$ and any $u = \sum_{j,\alpha} u^{j\alpha} \frac{\partial}{\partial z_j} \otimes e_\alpha \in (T^{1,0}_X \otimes E)_x$. We denote by

$$F(h) \geq_{\mathrm{Gri}} \lambda(\omega \otimes \mathbb{1}_E)$$

if

$$\sum_{j,k,\alpha,\beta} (R_{j\bar{k}\alpha\bar{\beta}} - \lambda\omega_{j\bar{k}} h_{\alpha\bar{\beta}})(x) \xi^j \zeta^\alpha \overline{\xi^k \zeta^\beta} \geq 0$$

for any $x \in X$, any $\xi = \sum_j \xi^j \frac{\partial}{\partial z_j} \in T^{1,0}_{X,x}$ and any $\zeta = \sum_\alpha \zeta^\alpha e_\alpha \in E_x$. A Higgs bundle is Griffiths semi-positive (resp. semi-negative) if it is Nakano positive (resp. semi-negative).

Lemma 1.8. *Let (E,h) be a hermitian vector bundle on a Kähler manifold (X,ω). If there is a positive constant C so that $|R(h)(x)|_{h,\omega} \leq C$ for any $x \in X$, then*

$$C\omega \otimes \mathbb{1}_E \geq_{Nak} R(h) \geq_{Nak} -C\omega \otimes \mathbb{1}_E$$

Proof. For any $x \in X$, let $z_1, \ldots, z_n$ be a local coordinate centered at x so that

$$\omega_x = \sqrt{-1}\sum_{\ell=1}^n dz_\ell \wedge d\bar{z}_\ell$$

Let $e_1, \ldots, e_r$ be a local holomorphic frame of E which is orthonormal at x. Write

$$R(h) = R^\beta_{j\bar{k}\alpha} dz_j \wedge d\bar{z}_k \otimes e^\alpha \otimes e_\beta.$$

Then $R_{j\bar{k}\alpha\bar{\beta}}(x) = R^\beta_{j\bar{k}\alpha}(x)$, and we have

$$\sum_{j,k,\alpha,\beta} |R_{j\bar{k}\alpha\bar{\beta}}(x)|^2 = |R(h)(x)|^2_{h,\omega} \leq C^2.$$

Hence for any $u = \sum_{j,\alpha} u^{j\alpha} \frac{\partial}{\partial z_j} \otimes e_\alpha \in (T^{1,0}_X \otimes E)_x$, one has

$$\begin{aligned}
|\sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\bar{\beta}}(x) u^{j\alpha}\overline{u^{k\beta}}|^2 &\leq \sum_{j,\alpha} |\sum_{k,\beta} R_{j\bar{k}\alpha\bar{\beta}}(x) u^{j\alpha}\overline{u^{k\beta}}|^2 \\
&\leq \sum_{j,\alpha} (\sum_{k,\beta} |R_{j\bar{k}\alpha\bar{\beta}}(x) u^{j\alpha}|^2) \cdot (\sum_{k,\beta} |\overline{u^{k\beta}}|^2) \\
&= |u|^2_{h,\omega} \cdot \sum_{k,\beta} (\sum_{j,\alpha} |R_{j\bar{k}\alpha\bar{\beta}}(x) u^{j\alpha}|^2) \\
&\leq |u|^2_{h,\omega} \cdot \sum_{k,\beta} (\sum_{j,\alpha} |R_{j\bar{k}\alpha\bar{\beta}}(x)|^2)(\sum_{j,\alpha} |u^{j\alpha}|^2) \\
&\leq |u|^4_{h,\omega} \cdot \sum_{j,k,\alpha,\beta} |R_{j\bar{k}\alpha\bar{\beta}}(x)|^2 \leq |u|^4_{h,\omega} \cdot |R(h)|^2_{h,\omega}.
\end{aligned}$$

Hence one has

$$-C|u|^2_{h,\omega} \leq \sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\bar{\beta}}(x) u^{j\alpha}\overline{u^{k\beta}} \leq C|u|^2_{h,\omega}$$

The lemma is proved. □

The following easy fact will be useful in this paper.

Lemma 1.9. *Let (E_1, h_1) and (E_2, h_2) are two hermitian vector bundles over a Kähler manifold (X, ω) such that $|R(h_1)(x)|_{h_1,\omega} \leq C_1$ and $|R(h_2)(x)|_{h_2,\omega} \leq C_2$ for all $x \in X$. Then for the hermitian vector bundle $(E_1 \otimes E_2, h_1h_2)$, one has*

$$|R(h_1h_2)(x)|_{h_1h_2,\omega} \leq \sqrt{2r_2C_1^2 + 2r_1C_2^2}$$

for all $x \in X$. Here $r_i := \mathrm{rank} E_i$.

2. L^2-METHOD FOR HIGGS BUNDLES

2.1. **A quick tour for the simplest case.** In this subsection, we assume that (E, θ, h) is a harmonic bundle over a projective manifold X. We will show how to apply Bochner technique to give a simple and quick proof of Theorem A in the case L is ample. The main goal of this subsection is to show the general strategy and we will discuss how to generalize these ideas to prove Theorem A.

For a Higgs bundle (E, θ) over a projective manifold X of dimension n, one has the following *holomorphic Dolbeault complex*

$$\mathrm{Dol}(E, \theta) := E \xrightarrow{\wedge\theta} E \otimes \Omega_X^1 \xrightarrow{\wedge\theta} \cdots \xrightarrow{\wedge\theta} E \otimes \Omega_X^n \tag{2.1.1}$$

By Simpson [Sim92], the complex of $\mathscr{C}^\infty$ sections of E

$$\mathscr{A}^0(E) \xrightarrow{D''} \mathscr{A}^1(E) \xrightarrow{D''} \cdots \xrightarrow{D''} \mathscr{A}^{2n}(E) \tag{2.1.2}$$

gives a fine resolution of the above holomorphic Dolbeault complex. Indeed, it can be proven easily from the Dolbeault lemma. Here $\mathscr{A}^m(E)$ is the sheaf of germs of smooth m-forms with value in E. Hence the cohomology of complex of its global sections $\big(A^\bullet(E), D''\big)$ computes the hypercohomology $\mathbb{H}^\bullet\big(X, \mathrm{Dol}(E, \theta)\big)$.

Suppose now $(\tilde{E}, \tilde{\theta})$ is a stable Higgs bundle with vanishing Chern classes. By the Simpson correspondence, there is a unique (up to a constant rescaling) hermitian metric $\tilde{h}$ over $\tilde{E}$ so that the curvature $R(\tilde{E}, \tilde{h}) = 0$. Assume that L is an ample line bundle on X equiped with a smooth metric h_L so that its curvature tensor $\sqrt{-1}R(L, h_L)$ is a Kähler form ω.

Let us define a new Higgs bundle $(E, \theta) := (\tilde{E} \otimes L, \tilde{\theta} \otimes \mathbb{1})$. We introduce a hermitian metric h on E defined by $h := \tilde{h} \otimes h_L$. One can easily check that (E, θ, h) satisfies Assumption 1.6 and the curvature

$$\sqrt{-1}F(E, h) := \sqrt{-1}R(E, h) + \sqrt{-1}[\theta, \theta^*] = \sqrt{-1}R(L, h_L) \otimes \mathbb{1}_E = \omega \otimes \mathbb{1}_E. \tag{2.1.3}$$

By the Hodge theory, for each $i \in \mathbb{Z}_{\geqslant 0}$, we know that the space of harmonic forms

$$\mathscr{H}^i := \{\alpha \in A^i(E) \mid \Delta''\alpha = 0\}$$

is isomorphic to the cohomology $H^i\big(A^\bullet(E), D''\big) \simeq \mathbb{H}^i\big(X, \mathrm{Dol}(E, \theta)\big)$.

Theorem 2.1 (Theorem A in the case that $D = \varnothing$ and L is ample)**.** *With the notations in this subsection, $\mathbb{H}^i\big(X, \mathrm{Dol}(\tilde{E}, \tilde{\theta}) \otimes L\big) = \mathbb{H}^i\big(X, \mathrm{Dol}(E, \theta)\big) = 0$ for $i > n$.*

Proof. Note that $\mathrm{Dol}(E, \theta) = \mathrm{Dol}(\tilde{E}, \tilde{\theta}) \otimes L$. It suffices to prove that $\mathscr{H}^i = 0$ for $i > n$. We will prove by contradiction. Let us take the Kähler form $\omega := \sqrt{-1}R(L, h_L)$. Assume that there exists a non-zero $\alpha \in \mathscr{H}^i$. Then by Lemma 1.7, one has

$$0 = \Delta''\alpha = \Delta'\alpha + [\sqrt{-1}F(E, h), \Lambda_\omega]\alpha \tag{2.1.4}$$

An integration by parts yields

$$\langle \Delta'\alpha, \alpha\rangle_{h,\omega} = \|D'_h\alpha\|^2_{h,\omega} + \|(D'_h)^*\alpha\|^2_{h,\omega} \geqslant 0.$$

Hence

$$\begin{aligned} 0 &\geqslant \int_X \langle [\sqrt{-1}F(E,h), \Lambda_\omega]\alpha, \alpha\rangle_{h,\omega} d\mathrm{Vol}_\omega \\ &\overset{(2.1.3)}{=} \int_X \langle [\omega \otimes \mathbb{1}, \Lambda_\omega]\alpha, \alpha\rangle_{h,\omega} d\mathrm{Vol}_\omega \\ &= \int_X (i-n)|\alpha|_{h,\omega} d\mathrm{Vol}_\omega > 0 \end{aligned}$$

for $i > n$. Here $d\mathrm{Vol}_\omega := \frac{\omega^n}{n!}$ denotes the volume form of (X, ω). Hence the contradiction. □

Hence the above proof inspires us that, to prove Theorem A in full generality, we shall find a 'proper' complex of fine sheaves which is quasi-isomorphic to $\mathrm{Dol}(E, \theta)$, so that its cohomology of global sections can be computed explicitly. Inspired by the work [Zuc79, DPS01, HLWY16], we will consider the L^2-complex as the candidate for this complex of fine sheaves. However, instead of solving $\bar{\partial}$-equation for vector bundles to prove the vanishing theorem, we shall consider L^2-estimate and solvability criteria of $(\bar{\partial}_E + \theta)$-equations for Higgs bundles (E, θ). This is the main content of next subsection.

2.2. **Hörmander L^2-estimate for Higgs bundles.** Solvability criteria for $\bar{\partial}$-equations on complex manifolds are often described as cohomology vanishing theorems. It is essentially based on the abstract theory of functional analysis. Since the Kähler identities (1.2.2) and (1.2.3) hold for Higgs bundles, it inspires us that the following principle should hold.

Principle. *The package of L^2-estimate by Hörmander, Andreotti-Venssetti, Bombieri, Skoda, Demailly et. al. should hold without modification for Higgs bundles, provided that the D'' is used in place of $\bar{\partial}$ and that m-forms are used instead of (p, q)-forms.*

In this subsection we prove that for Higgs bundles over a complete Kähler manifold under certain curvature condition, one can solve the D''-equation in the same vein as [Dem12, Chapter VIII, Theorem 4.5]. We follow the standard method of L^2 estimate as that in [Dem12, Chapter VIII], and we provide full details for completeness sake.

Let us denote by $A^m(X, E)$ (resp. $A^{p,q}(X, E)$) the set of smooth E-valued m-forms (resp. (p, q)-forms) on X, and denote by $A^m_0(X, E)$ (resp. $A^{p,q}_0(X, E)$) the set of smooth E-valued m-forms (resp. (p, q)-forms) on X with compact support over the Kähler manifold (X, ω). The pointwise length of $u \in A^m(X, E)$ with respect to the fiber metric induced by h and ω, is denoted by $|u|_{h,\omega}$. The pointwise inner product of u and v is denoted by $\langle u, v\rangle_{h,\omega}$, or simply by $\langle u, v\rangle$. Then the L^2-norm of u denoted by $\|u\|_{h,\omega}$, or simply by $\|u\|$, is defined as the square root of the integral

$$\|u\|^2 := \int_X |u|^2_{h,\omega} d\mathrm{Vol}_\omega$$

where $d\mathrm{Vol}_\omega := \frac{\omega^n}{n!}$, which is finite if $u \in A^m_0(X, E)$. The inner product of u and v associated to this norm is defined by

$$\langle\!\langle u, v\rangle\!\rangle_{h,\omega} := \int_X \langle u, v\rangle_{h,\omega} d\mathrm{Vol}_\omega$$

which is simply denoted by $\langle\!\langle u, v\rangle\!\rangle$. Note that the Hodge decomposition $A^m_0(X, E) = \oplus_{p+q=m} A^{p,q}_0(X, E)$ is orthogonal with respect to this inner product $\langle\!\langle \bullet, \bullet\rangle\!\rangle$.

We shall denote by $L^m_{(2),\mathrm{loc}}(X,E)$ (resp. $L^{p,q}_{(2),\mathrm{loc}}(X,E)$) E-valued m-forms (resp. (p,q)-forms) with locally integrable coefficients. One has a natural decomposition

$$L^m_{(2),\mathrm{loc}}(X,E) = \oplus_{p+q=m} L^{p,q}_{(2),\mathrm{loc}}(X,E)$$

Moreover, the operators D'' (and D'_h, $\bar{\partial}_E$ respectively) act on $L^m_{(2),\mathrm{loc}}(X,E)$ in the sense of distribution, or precisely speaking, *E-valued currents.* Note that those objects are all defined without the choice of the metrics ω and h. A section $s \in L^m_{(2),\mathrm{loc}}(X,E)$ is said to be in the domain of definition of D'', denoted by $\mathrm{Dom}_{\mathrm{loc}}D''$, if $D''s \in L^{m+1}_{(2),\mathrm{loc}}(X,E)$.

Let $L^m_{(2)}(X,E)_{h,\omega}$ (resp. $L^{p,q}_{(2)}(X,E)_{h,\omega}$) be the completion of the pre-Hilbert space $A^m_0(X,E)$ (resp. $A^{p,q}_0(X,E)$) with respect to the above inner product $\langle\!\langle \bullet,\bullet\rangle\!\rangle$. We simply write $L^m_{(2)}(X,E)$ (resp. $L^{p,q}_{(2)}(X,E)$) if no confusion happens. By the Lebesgue's theory of integration, $L^m_{(2)}(X,E)$ (resp. $L^{p,q}_{(2)}(X,E)$) is a subset of $L^m_{(2),\mathrm{loc}}(X,E)$ (resp. $L^{p,q}_{(2),\mathrm{loc}}(X,E)$). The natural decomposition

$$L^m_{(2)}(X,E) = \oplus_{p+q=m} L^{p,q}_{(2)}(X,E)$$

is orthogonal with respect to the inner product $\langle\!\langle \bullet,\bullet\rangle\!\rangle$.

Hence D'' (and D'_h, $\bar{\partial}_E$ respectively) act on them respectively, and these operators are unbounded, densely defined linear operators

$$L^m_{(2)}(X,E) \to L^{m+1}_{(2)}(X,E).$$

The domain of definition of D'' denoted by $\mathrm{Dom}D''$ are defined by

$$\{u \in L^m_{(2)}(X,E) \mid D''u \in L^{m+1}_{(2)}(X,E)\},$$

for which one has $\mathrm{Dom}D'' \subset \mathrm{Dom}_{\mathrm{loc}}D''$. Note that $\mathrm{Dom}D''$ depends on the choice of the metric ω and h, up to mutual boundedness. Namely, if $\tilde{\omega} \sim \omega$ and $\tilde{h} \sim h$, $\mathrm{Dom}D''$ remains the same in terms of the new metrics $\tilde{\omega}$ and $\tilde{h}$.

By the argument in [Dem12, Chapter VIII, Theorem 1.1], this extended operator D'' (the so-called *weak extension* in the literature) is closed, namely its graph is closed. $\mathrm{Dom}D'_h$ is defined in exactly same manner.

The following result in [Dem12, Chapter VIII, Theorem 3.2.(a)] is crucial in applying the L^2-estimate. Roughly speaking, it gives a condition when the weak extension of D'' is the strong one, in terms of the graph norm, and it enables us to apply the integration by parts for L^2-sections as in Lemma 2.4.

Theorem 2.2. *Let (X,ω) be a complete Kähler manifold and $(E,\bar{\partial}_E,\theta,h)$ is a Higgs bundle on X satisfying Assumption 1.6. Then $A^m_0(X,E)$ is dense in $\mathrm{Dom}D''$, $\mathrm{Dom}D''^*$ and $\mathrm{Dom}D''\cap\mathrm{Dom}D''^*$ respectively for the graph norm*

$$u \mapsto \|u\|+\|D''u\|, \quad u \mapsto \|u\|+\|(D'')^*u\|, \quad u \mapsto \|u\|+\|D''u\|+\|D''^*u\|.$$

We recall the following theorem of functional analysis by Von Neumann and Hömander, which is crucial in obtaining the L^2-estimate for Higgs bundles.

Lemma 2.3. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ be complex Hilbert spaces, and $T:\mathscr{H}_1\to\mathscr{H}_2$ and $S:\mathscr{H}_2\to\mathscr{H}_3$ are closed and densely defined linear operators satisfying $\mathrm{Dom}S \supset \mathrm{Im}\,T$. Then*

(i) $\mathscr{H}_2 = \ker S \oplus \overline{\mathrm{Im}\,S^*}$.

(ii) *Let $v \in \mathscr{H}_2$, Then $v \in \mathrm{Im}\,T$ if and only if there exists a nonnegative number C such that*

$$|\langle\!\langle u,v\rangle\!\rangle_2| \le C\|T^*u\|_1 \tag{2.2.1}$$

holds for any $u \in \mathrm{Dom}T^*$.

Note that $y \in \mathrm{Dom}T^*$ if the linear form

$$\mathrm{Dom}T \ni x \mapsto \langle\!\langle Tx, y\rangle\!\rangle_2$$

is bounded in $\mathscr{H}_1$-norm. Since $\mathrm{Dom}T$ is dense, there exists for every y in $\mathrm{Dom}T^*$ a unique element T^*y $\mathscr{H}_1$ such that $\langle\!\langle x, T^*y\rangle\!\rangle_1 = \langle\!\langle Tx, y\rangle\!\rangle_2$ for all $x \in \mathrm{Dom}T$.

Note that $A_m := [\sqrt{-1}F(h), \Lambda_\omega]$ acts on $\wedge^m T_X^* \otimes E$ as a *hermitian operator.* As A_m is smooth, for any $u \in L^m_{(2),\mathrm{loc}}(X, E)$, $A_m(u) \in L^m_{(2),\mathrm{loc}}(X, E)$. If A_m is semi-positively definite, $A_m^{\frac{1}{2}}$ is also a densely defined hermitian operator from $L^m_{(2)}(X, E)$ to itself. The following result is exactly the same vein as the Kodaira-Nakano inequality (see [Dem82, lemme 4.4])

Lemma 2.4. *Let (X, ω) be a complete Kähler manifold and $(E, \bar\partial_E, \theta, h)$ is a Higgs bundle on X satisfying Assumption 1.6. Assume that A_m is semi-positively definite. Then for every $u \in \mathrm{Dom}D'' \cap \mathrm{Dom}D''^*$, one has*

$$\|D''u\|^2 + \|D''^*u\|^2 \geq \langle\!\langle A_m u, u\rangle\!\rangle := \int_X \langle A_m u, u\rangle_{h,\omega} dVol_\omega \tag{2.2.2}$$

Proof. Since (X, ω) is complete, by the proof of [Dem12, Chapter VIII, Theorem 3.2.(a)], there exists an exhaustive sequence $\{K_\nu\}_{\nu\in\mathbb{N}}$ of compact subsets of X and functions ρ_ν such that $\rho_\nu = 1$ in a neighborhood of K_ν, $\mathrm{Supp}(\rho_\nu) \subset K_{\nu+1}$, $0 \leq \rho_\nu \leq 1$, and $|d\rho_\nu|_\omega \leq 2^{-\nu}$. One can show that $\rho_\nu u \to u$ in the graph norm $u \mapsto \|u\| + \|D''u\| + \|D''^*u\|$. Since A_m is supposed to be semi-positively definite, hence

$$\lim_{\nu\to+\infty} \int_X \langle A_m(\rho_\nu u), \rho_\nu u\rangle_{h,\omega} d\mathrm{Vol}_\omega = \int_X \langle A_m(u), u\rangle_{h,\omega} d\mathrm{Vol}_\omega,$$

which might be $+\infty$ in general. Hence it suffices to prove (2.2.2) under the assumption that u has compact support.

By the convolution arguments in [Dem12, Chapter VIII, Theorem 3.2.(a)], there exists $u_\ell \in A_0^m(X, E)$ so that u_ℓ tends to u as $\ell \to \infty$ with respect to the graph norm $\|u\| + \|D''u\| + \|D''^*u\|$, and there is a uniform compact set K so that $\mathrm{Supp}(u_\ell) \subset K$ for all ℓ. By Lemma 1.7, one has

$$\langle\!\langle \Delta'' u_\ell, u_\ell\rangle\!\rangle = \langle\!\langle \Delta' u_\ell, u_\ell\rangle\!\rangle + \langle\!\langle A_m u_\ell, u_\ell\rangle\!\rangle$$

As u_ℓ has compact support, one applies integration by parts to obtain

$$\langle\!\langle \Delta'' u_\ell, u_\ell\rangle\!\rangle = \|D'' u_\ell\|^2 + \|D''^* u_\ell\|^2$$

and

$$\langle\!\langle \Delta' u_\ell, u_\ell\rangle\!\rangle = \|D'_h u_\ell\|^2 + \|D'^* u_\ell\|^2 \geq 0$$

which gives rise to

$$\|D'' u_\ell\|^2 + \|D''^* u_\ell\|^2 \geq \langle\!\langle A_m u_\ell, u_\ell\rangle\!\rangle$$

(2.2.2) follows from the above inequality when ℓ tends to infinity. The lemma is proved. □

Remark 2.5. Suppose that A_m is a semi-positively definite hermitian operator on $\wedge^m T_X^* \otimes E$. For some $v \in L^m_{(2)}(X, E)$, assume that for almost all $x \in X$, there exists $\alpha(x) \in [0, +\infty[$ so that

$$|\langle v, f\rangle_{h,\omega}|^2 \leq \alpha(x)\langle v, A_m(x)v\rangle_{h,\omega}$$

for any $f \in A_0^m(X, E)_x$. If the operator $A_m(x)$ is invertible, the minimum of $\alpha(x)$ is $\langle A_m(x)^{-1}u, u\rangle_{h,\omega}$. Hence we shall always formally write it in this way even when $A_m(x)$ is no longer invertible, following [Dem12, Chapter VIII, §4].

Now we are able to state our main result on L^2-estimate for Higgs bundles.

Theorem 2.6 (Solving D''-equation for Higgs bundle). *Let (X,ω) be a complete Kähler manifold, and $(E,\bar\partial_E,\theta,h)$ be a Higgs bundle on X satisfying Assumption 1.6. Assume that A_m is semi-positively definite on $\wedge^m T_X^*\otimes E$ at every $x\in X$. Then for any $v\in L^m_{(2)}(X,E)$ such that $D''v=0$ and*

$$\int_X \langle A_m^{-1}v,v\rangle d\,Vol_\omega<+\infty,$$

there exists $u\in L^{m-1}_{(2)}(X,E)$ so that $D''u=v$ and

$$\|u\|^2\leqslant \int_X \langle A_m^{-1}v,v\rangle d\,Vol_\omega.$$

Proof. We will apply Lemma 2.3.(ii) to prove this theorem. We have the following comparison

$$\mathscr{H}_1=L^{m-1}_{(2)}(X,E)\xrightarrow{T=D''}\mathscr{H}_2=L^m_{(2)}(X,E)\xrightarrow{S=D''}\mathscr{H}_3=L^{m+1}_{(2)}(X,E),$$

which satisfies the conditions in Lemma 2.3.(ii).

For any $f\in \mathrm{Dom}S\cap\mathrm{Dom}T^*$, one has

$$\begin{aligned}|\langle\!\langle f,v\rangle\!\rangle|^2=|\int_X\langle f,v\rangle d\mathrm{Vol}_\omega|^2&\leq|\int_X\langle A_m^{-1}v,v\rangle^{\frac12}\cdot\langle A_mf,f\rangle^{\frac12}d\mathrm{Vol}_\omega|^2\\&\leq\int_X\langle A_m^{-1}v,v\rangle d\mathrm{Vol}_\omega\cdot\int_X\langle A_mf,f\rangle d\mathrm{Vol}_\omega\end{aligned}$$

by Cauchy-Schwarz inequality. By (2.2.2) one has

$$|\langle\!\langle f,v\rangle\!\rangle|^2\leq C(\|Sf\|^2+\|T^*f\|^2),\tag{2.2.3}$$

where $C:=\int_X\langle A_m^{-1}v,v\rangle d\mathrm{Vol}_\omega>0$.

Note that $T^*\circ S^*=0$ by $S\circ T=0$. By Lemma 2.3.(i), for any $f\in\mathrm{Dom}T^*$, there is an orthogonal decomposition $f=f_1+f_2$, where $f_1\in\ker S$ and $f_2\in(\ker S)^\perp=\overline{\mathrm{Im}\,S^*}\subset\ker T^*$. Since $v\in\ker S$, by (2.2.3) we then have

$$|\langle\!\langle f,v\rangle\!\rangle|^2=|\langle\!\langle f_1,v\rangle\!\rangle|^2\leqslant C(\|Sf_1\|^2+\|T^*f_1\|^2)=C\|T^*f_1\|^2=C\|T^*f\|^2.$$

By Lemma 2.3.(ii), we conclude that there is $u\in L^{m-1}_{(2)}(X,E)$ so that $Tu=v$ with $\|u\|_2\leq C$. The theorem is proved. □

A direct consequence is the following result which can be seen as a Higgs bundle version of Girbau vanishing theorem (see [Dem12, Chapter VII, Theorem 4.2]) in the log setting [HLWY16, Theorem 4.1].

Corollary 2.7. *Let (X,ω) be a complete Kähler manifold, and $(\tilde E,\tilde\theta,\tilde h)$ be any harmonic bundle on X. Let L be a line bundle on X equipped with a metric h_L. Assume that for some $m>0$, one has*

$$\langle[\sqrt{-1}R(h_L),\Lambda_\omega]f,f\rangle_\omega\geq\varepsilon|f|^2_\omega\tag{2.2.4}$$

for any $f\in\Lambda^{p,q}T^*_{X,x}$, any x and any $p+q=m$. Set $(E,\theta,h):=(\tilde E\otimes L,\tilde\theta\otimes\mathbb{1}_L,\tilde h h_L)$. Then for any $v\in L^m_{(2)}(X,E)$ such that $D''v=0$, there exists $u\in L^{m-1}_{(2)}(X,E)$ so that $D''u=v$ and

$$\|u\|^2\leqslant\frac{\|v\|^2}{\varepsilon}.$$

Proof. Note that

$$\begin{aligned}\sqrt{-1}F(h) &= \sqrt{-1}\Big(R(h) + [\theta, \theta_h^*]\Big)\\ &= \sqrt{-1}R(\tilde{h}) \otimes \mathbb{1}_L + \sqrt{-1}R(h_L) \otimes \mathbb{1}_E + [\tilde{\theta} \otimes \mathbb{1}_L, \tilde{\theta}_{\tilde{h}}^* \otimes \mathbb{1}_L]\\ &= \sqrt{-1}F(\tilde{h}) \otimes \mathbb{1}_L + \sqrt{-1}R(h_L) \otimes \mathbb{1}_E\\ &= \sqrt{-1}R(h_L) \otimes \mathbb{1}_E, \end{aligned} \tag{2.2.5}$$

where the last equality follows from that $F(\tilde{h}) = 0$ since $(\tilde{E}, \tilde{\theta}, \tilde{h})$ is a harmonic bundle. In this case, it is easy to see that for any $f \in (\Lambda^m T_X^* \otimes E)_x$, decomposing $f = \sum_{p+q=m} f^{p,q}$ with $f^{p,q}$ its (p, q)-component, one has

$$\langle A_m f, f\rangle_{h,\omega} = \sum_{p+q=m} \langle [\sqrt{-1}R(h_L), \Lambda_\omega] \otimes \mathbb{1}_E(f^{p,q}), f^{p,q}\rangle_{h_L,\omega} \geq \sum_{p+q=m} \varepsilon |f^{p,q}|^2_{h,\omega} = \varepsilon |f|^2_{h,\omega}.$$

Hence $\langle A_m^{-1} f, f\rangle_{h,\omega} \leq \varepsilon^{-1}|f|^2_{h,\omega}$. Applying Theorem 2.6, we conclude that there is $u \in L^{m-1}_{(2)}(X, E)$ so that $D''u = v$ and

$$\|u\|^2 \leqslant \int_X \langle A_m^{-1} v, v\rangle_{h,\omega} d\mathrm{Vol}_\omega \leq \frac{\|v\|^2}{\varepsilon}.$$

□

3. Vanishing theorem for tame harmonic bundles

3.1. **Parabolic Higgs bundle.** In this section, we recall the notions of parabolic Higgs bundles. For more details refer to [AHL19, section 1, 3, 4, 5] and [MY92, section 1]. Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a reduced simple normal crossing divisor and $U = X - D$ be the complement of D.

Definition 3.1. A parabolic sheaf $(E, {}_{\boldsymbol{a}}E, \theta)$ on (X, D) is a torsion free O_U-module E, together with an $\mathbb{R}^l$-indexed filtration ${}_{\boldsymbol{a}}E$ (*parabolic structure*) by coherent subsheaves such that

1). $\boldsymbol{a} \in \mathbb{R}^l$ and ${}_{\boldsymbol{a}}E|_U = E$.
2). For $1 \leq i \leq l$, ${}_{\boldsymbol{a}+1_i}E = {}_{\boldsymbol{a}}E(-D_i)$, where $\mathbf{1}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in i-th component.
3). ${}_{\boldsymbol{a}-\boldsymbol{\epsilon}}E = {}_{\boldsymbol{a}} E$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.
4). The set of *weights* $\boldsymbol{a}$ such that ${}_{\boldsymbol{a}}E/{}_{\boldsymbol{a}+\boldsymbol{\epsilon}}E \neq 0$ is discrete in $\mathbb{R}^l$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.

A weight is normalized if it lies in $[0, 1)^l$. Denote ${}_{\mathbf{0}}E$ by ${}^{\diamond}E$, where $\mathbf{0} = (0, \ldots, 0)$. Note that the parabolic structure of $(E, {}_{\boldsymbol{a}}E, \theta)$ is uniquely determined by the filtration for weights lying in $[0, 1)^l$. A *parabolic bundle* on (X, D) consists of a vector bundle E on X with a parabolic structure, such that as a filtered bundle.

Definition 3.2. A *parabolic Higgs bundle* on (X, D) is a parabolic bundle $(E, {}_{\boldsymbol{a}}E, \theta)$ together with $\mathscr{O}_X$ linear map

$$\theta : {}^{\diamond}E \to \Omega^1_X(\log D) \otimes {}^{\diamond}E$$

such that

$$\theta \wedge \theta = 0$$

and

$$\theta({}_{\boldsymbol{a}}E) \subseteq \Omega^1_X(\log D) \otimes {}_{\boldsymbol{a}}E,$$

for $\boldsymbol{a} \in [0, 1)^l$.

A natural class of parabolic Higgs bundles comes from prolongations of tame harmonic bundle, which is discussed in the following section.

3.2. **Prolongation by an increased order.** By a celebrated theorem of Simpson and Mochizuki, there is a natural parabolic Higgs bundle induced by tame harmonic bundle (E,θ,h).

We recall some notions in [Moc07, §2.2.1]. Let (X,D) be the pair in subsection 3.1. Let E be holomorphic vector bundle with a $\mathscr{C}^\infty$ hermitian metric h over $X-D$.

Let U be an open subset of X, which is admissible with respect to D. For any section $\sigma\in\Gamma(U-D,E|_{U-D})$, let $|\sigma|_h$ denote the norm function of σ with respect to the metric h. We denote $|\sigma|_h\in O(\prod_{i=1}^{\ell}|z_i|^{-b_i})$ if there exists a positive number C such that $|\sigma|_h\leq C\cdot\prod_{i=1}^{\ell}|z_i|^{-b_i}$. For any $\boldsymbol{b}\in\mathbb{R}^\ell$, say $-\mathrm{ord}(\sigma)\leq\boldsymbol{b}$ means the following:

$$|\sigma|_h=O(\prod_{i=1}^{\ell}|z_i|^{-b_i-\varepsilon})$$

for any real number $\varepsilon>0$. For any $\boldsymbol{b}$, the sheaf ${}_{\boldsymbol{b}}E$ is defined as follows:

$$\Gamma(U-D,{}_{\boldsymbol{b}}E):=\{\sigma\in\Gamma(U-D,E|_{U-D}\mid-\mathrm{ord}(\sigma)\leq\boldsymbol{b}\}. \tag{3.2.1}$$

The sheaf ${}_{\boldsymbol{b}}E$ is called the prolongment of E by an increasing order $\boldsymbol{b}$. In particular,we use the notation ${}^\diamond E$ in the case $\boldsymbol{b}=(0,\ldots,0)$.

According to Simpson [Sim90, Theorem 2] and Mochizuki [Moc09, Propostion 2.53], the above prolongation gives a parabolic Higgs bundles, especially θ preserves the filtration.

Theorem 3.3 (Simpson, Mochizuki)**.** *Let (X,D) be a complex manifold X with a simple normal crossing divisor D. If (E,θ,h) is a tame harmonic bundle on $X-D$, then the corresponding filtration ${}_{\boldsymbol{b}}E$ according to the increasing order in the prolongment of E defines a parabolic bundle $(E,{}_{\boldsymbol{b}}E,\theta)$ on (X,D).* □

Definition 3.4 (Acceptable bundle)**.** Let $(E,\bar\partial_E,h)$ be a hermitian vector bundle over $X-D$. We say that $(E,\bar\partial_E,h)$ is acceptable at $p\in D$, if the following holds: there is an admissible coordinate $(U;z_1,\ldots,z_n)$ around p, so that the norm $|R(E,h)|_{h\otimes\omega_P}\leq C$ for $C>0$. When $(E,\bar\partial_E,h)$ is acceptable at any point p of D, it is called acceptable.

3.3. **Modification of the metric.** This subsection is mainly inspired by [Moc02, §4.5.3]. Let us consider the case $X=\Delta^n$, and $D=\sum_{i=1}^{\ell}D_i$ with $D_i=(z_i=0)$. Let $(E,\bar\partial_E,h)$ be an acceptable bundle over $X-D$. For any $\boldsymbol{a}\in\mathbb{R}^{\ell}_{\geq0}$ and $N\in\mathbb{Z}$, we define

$$\chi(\boldsymbol{a},N):=-\sum_{j=1}^{\ell}a_j\log|z_j|^2-N\Big(\sum_{j=1}^{\ell}\log(-\log|z_j|^2)+\sum_{k=\ell+1}^{n}\log(1-|z_k|^2)\Big). \tag{3.3.1}$$

Set $h(\boldsymbol{a},N):=h\cdot e^{-\chi(\boldsymbol{a},N)}$. Then

$$R(h(\boldsymbol{a},N))=R(h)+\sqrt{-1}\partial\bar\partial\chi(\boldsymbol{a},N)=R(h)+N\omega_P.$$

Note that $\Omega_{X^*}=\oplus_{i=1}^{n}L_i$ where L_i is the trivial line bundle defined by $L_i:=p_i^*\Omega_{\Delta^*}$ for $i=1,\ldots,\ell$ and $L_k=p_k^*\Omega_\Delta$ for $k=\ell+1,\ldots,n$ where p_i is the projection of $(\Delta^*)^\ell\times\Delta^{n-\ell}$ to its i-th factor. For any $p=1,\ldots,n$, set h_p to be the hermitian metric on $\Omega^p_{X^*}$ induced by ω_P. Then there is a positive constant $C(p,\ell)>0$ depending only on p and ℓ so that $|R(h_p)|_{h_p,\omega_P}\leq C(p,\ell)$. Set $C_0:=\sup_{p=0,\ldots,n;\ell=1,\ldots,n}C(p,\ell)$.

Proposition 3.5. *Let $(E,\bar\partial_E,h)$ be an acceptable bundle over $X-D$, where X is a compact complex manifold and D is a simple normal crossing divisor. Then there is a constant $N_0>0$ so that, for any $p\in D$, one has an admissible coordinate $(U;z_1,\ldots,z_n)$ around p (which can be made arbitrary small), and for vector bundles $\mathscr{E}_p:=T^p_{U^*}\otimes E$ and $\mathscr{F}_p:=\Omega^p_{U^*}\otimes E$, which are all equipped with the $\mathscr{C}^\infty$-metric $h_{\mathscr{E}_p}$ and $h_{\mathscr{F}_p}$ induced by $h(\boldsymbol{a},N)$ and ω_P, one has the following estimate*

$$\sqrt{-1}R(h_{\mathscr{E}_p})\geqslant_{Nak}\omega_P\otimes\mathbb{1}_{\mathscr{E}_p};\quad\sqrt{-1}R(h_{\mathscr{F}_p})\leqslant_{Gri}2N\omega_P\otimes\mathbb{1}_{\mathscr{F}_p} \tag{3.3.2}$$

over U^* for any $N\geqslant N_0$. Such N_0 does not depend on the choice of $\boldsymbol{a}$.

Proof. As (E,h) is assumed to be acceptable, for any $x \in D$, one can find an admissible coordinate $(U; z_1, \ldots, z_n; \varphi)$ around x so that $|R(h)|_{h,\omega_P} \leq C$. By the above argument, one has $|R(h_p)|_{h_p,\omega_P} \leq C_0$. By Lemma 1.9, we conclude that there is a constant $C_1 > 0$ which depends only on C so that

$$|R(h_p^{-1}h)|_{h_p^{-1}h,\omega_P} \leq C_1, \quad |R(h_p h)|_{h_p h,\omega_P} \leq C_1$$

for any $p = 0, \ldots, n$, where $h_p^{-1}h$ is the metric for $\mathscr{E}_p$ and $h_p h$ is the metric for $\mathscr{F}_p$. By Lemma 1.8, one has

$$\sqrt{-1}R(h_p^{-1}h) \geq_{\rm Nak} -C_1\omega_P \otimes \mathbb{1}_{\mathscr{E}_p}, \quad \sqrt{-1}R(h_p h) \leq_{\rm Nak} C_1\omega_P \otimes \mathbb{1}_{\mathscr{F}_p}.$$

As $h_{\mathscr{E}_p} = h_p^{-1}h(\boldsymbol{a}, N)$ and $h_{\mathscr{F}_p} = h_p h(\boldsymbol{a}, N)$, we then have

$$\sqrt{-1}R(h_{\mathscr{E}_p}) \geq_{\rm Nak} (N - C_1)\omega_P \otimes \mathbb{1}_{\mathscr{E}_p}, \quad \sqrt{-1}R(h_{\mathscr{F}_p}) \leq_{\rm Nak} (N + C_1)\omega_P \otimes \mathbb{1}_{\mathscr{F}_p}.$$

If we take $N_x = C_1 + 1$, then the desired estimate (3.3.2) follows for any $N \geq N_x$.

Now we will prove that for points near x, the above estimate N_x holds uniformly. As C_1 depends only on C, one has to prove that there is a constant C so that for any point z near x, there is an admissible coordinate with respect to z so that $|R(h)|_{h,\omega_P} \leq C$.

Claim 3.6. *Let $\phi : \Delta \to \Delta^*$ defined by $\phi(t) = \frac{t}{4} + \frac{1}{2}$. Then*

$$\phi^* \frac{\sqrt{-1}dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2} = \frac{\sqrt{-1}dt \wedge d\bar{t}}{|\phi(t)|^2(\log|\phi(t)|^2)^2} \leq C_2\sqrt{-1}dt \wedge d\bar{t} \leq C_2\frac{\sqrt{-1}dt \wedge d\bar{t}}{(1-|t|^2)^2},$$

where $C_2 = 16(\log\frac{9}{16})^{-2}$.

For any $z \in U$, we first assume that $z_1 = \cdots = z_\ell = 0$, namely the components of D passing to z are the same as x. Take isomorphisms of unit disk $\{\phi_j \in \operatorname{Aut}(\Delta)\}_{j=\ell+1,\ldots,n}$ so that $\phi_j(z_j) = x_j$. Note that $x_1 = \cdots = x_\ell = 0$. Hence $(\mathbb{1}_\Delta, \ldots, \mathbb{1}_\Delta, \phi_{\ell+1}, \ldots, \phi_n) \circ \varphi : U \to \Delta^n$ gives rise to the admissible coordinate for z, and the Poincaré metric ω_P is invariant under this transformation. Hence one can take $N_z = N_x$.

Now we assume that $z_1 = \cdots = z_m = 0$, and that any of $\{z_{m+1}, \ldots, z_\ell\}$ is not equal to zero. We first take automorphisms $\{\eta_i\}_{i=m+1,\ldots,\ell} \subset \operatorname{Aut}(\Delta^*)$ so that $\eta_i(\frac{1}{2}) = z_i$. Set $\phi_i = \eta_i \circ \phi : \Delta \to \Delta^*$ for $i = m+1, \ldots, \ell$. Take isomorphisms of unit disk $\{\phi_j \in \operatorname{Aut}(\Delta)\}_{j=\ell+1,\ldots,n}$ so that $\phi_j(z_j) = x_j$. Then $\varphi^{-1} \circ (\mathbb{1}_\Delta, \ldots, \mathbb{1}_\Delta, \phi_{m+1}, \ldots, \phi_n) : \Delta^n \to X$ will gives rise to the desired admissible coordinate for such z. By the above claim, one has $|R(h)|_{h,\omega_P} \leq C_2 C$. Hence the above estimate N_x can be made uniformly in U. As X and D is compact, one can cover D by finite such open sets, and the desired N_0 in the theorem can be achieve.

We now show that these admissible coordinates can be made arbitrarily small. For $0 < \varepsilon < 1$, set

$$\begin{aligned} \phi_\varepsilon : \Delta^n &\to \Delta^n \\ (z_1, \ldots, z_n) &\to (\varepsilon z_1, \ldots, \varepsilon z_n). \end{aligned}$$

For any admissible coordinate $(U; z_1, \ldots, z_n; \varphi)$ around x so that $|R(h)|_{h,\omega_P} \leq C$, one can introduce a new one $(U(\varepsilon); w_1, \ldots, w_n; \varphi_\varepsilon)$ around x with

$$\begin{aligned} \varphi_\varepsilon : U(\varepsilon) &\xrightarrow{\sim} \Delta^n \\ x &\to \varphi \circ \phi_\varepsilon(x). \end{aligned}$$

When $\varepsilon \ll 1$, this admissible coordinate will be arbitrarily small. Note that $\phi_\varepsilon^*\omega_P \leq \omega_P$. Hence in $(U(\varepsilon); w_1, \ldots, w_n; \varphi_\varepsilon)$, one still have $|R(h)|_{h,\omega_P} \leq C$. The constant N_x is thus unchanged. The proposition is proved. □

This result will be important for us to construct a fine resolution of parabolic Higgs bundles in § 3.5.

3.4. **From L^2-integrability to $\mathscr{C}^0$-estimate.** Note that in order to show the quasi-isomorphism between some complex of sheaves of L^2-forms and (0.1.1), one has to deduce some norm estimate of sections from the L^2-integrability condition. In the case that (E,θ) is a line bundle without Higgs field, this is not difficult and has been carried out in [DPS01, §2.4.2] and [HLWY16, Theorem 3.1]. This subsection is devoted to show this using *mean value inequality* following [Moc09, Lemma 7.12].

We first recall the following well-known lemma and we provide the proof for completeness sake.

Lemma 3.7. *Assume that $R(h)$ is Griffiths negative. Then for any holomorphic section $s \in H^0(X,E)$, one has*

$$\sqrt{-1}\partial\overline{\partial}\log|s|_h \geqslant 0.$$

Proof. Outside the zero locus $(s=0)$, one has

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\log|s|_h^2 &= \sqrt{-1}\frac{\{D's,D's\}_h}{|s|_h^2} - \sqrt{-1}\frac{\{D's,s\}_h\wedge\{s,D's\}_h}{|s|_h^4} - \frac{\{\sqrt{-1}R(h)s,s\}_h}{|s|_h^2}\\ &\geq -\frac{\{\sqrt{-1}R(h)s,s\}_h}{|s|_h^2} \geq 0\end{aligned}$$

where the first inequality is due to Cauchy-Schwarz inequality and the second one follows from the assumption that $R(h)$ is Griffiths negative. As $\log|s|_h^2$ is locally bounded from above, it is thus a global plurisubharmonic function on X. □

Proposition 3.8. *With the same setting as Proposition 3.5, for any $p \in D$, we take an admissible coordinate $(U;z_1,\ldots,z_n)$ around p and pick $N \geq N_0$ as in Proposition 3.5. Then for any section $s \in H^0(U^*,\Omega^p_{U^*}\otimes E|_{X^*})$, when $0<r\ll 1$, one has*

$$|s|_{h,\omega_P}(z) \leq C\|s\|_{h(\boldsymbol{a},N),\omega_P}\cdot\Big(\prod_{i=1}^{\ell}|z_i|^{-a_i-\delta}\Big) \tag{3.4.1}$$

for any $\delta>0$ and any $z \in U^*(r)$.

Proof. By Proposition 3.5, for the hermitian vector bundle $(\Omega^p_{U^*}\otimes E,(h_ph(\boldsymbol{a},-N))$ one thus has

$$R(h_ph(\boldsymbol{a},-N)) = R(h_ph(\boldsymbol{a},N)) - 2N\omega_P\otimes\mathbb{1}_{\Omega^p_{U^*}\otimes E} \leqslant_{\rm Gri} 0$$

over U^* for $N \geqslant N_0$. For any section $s \in H^0(U^*,\Omega^p_{U^*}\otimes E)$, by Lemma 3.7 one has

$$\sqrt{-1}\partial\overline{\partial}\log|s(z)|^2_{h(\boldsymbol{a},-N),\omega_P} \leq 0,$$

where we omit h_p for simplicity. For any $z \in U^*(r)$ where $0 < r \ll 1$, one has $\log |s(z)|^2_{h(\boldsymbol{a},-N),\omega_P} < 0$, and

$$\begin{aligned}
\log |s(z)|^2_{h(\boldsymbol{a},-N),\omega_P} &\leqslant \frac{4^n}{\pi^n \prod_{i=1}^{\ell} |z_i|^2} \int_{\Omega_z} \log |s(w)|^2_{h(\boldsymbol{a},-N),\omega_P} d\mathrm{vol}_g \\
&\leqslant \log \Big(\frac{4^n}{\pi^n \prod_{i=1}^{\ell} |z_i|^2} \cdot \int_{\Omega_z} |s(w)|^2_{h(\boldsymbol{a},-N),\omega_P} d\mathrm{vol}_g \Big) \\
&\leqslant \log \Big(C \int_{\Omega_z} \frac{1}{\prod_{i=1}^{\ell} |w_i|^2} |s(w)|^2_{h(\boldsymbol{a},-N),\omega_P} d\mathrm{vol}_g \Big) \\
&\leqslant \log C_1 + \log \int_{\Omega_z} |s(w)|^2_{h(\boldsymbol{a},-N),\omega_P} \cdot |\prod_{i=1}^{\ell} (\log |w_i|^2)^2| \prod_{j=\ell+1}^{n} (1-|w_j|^2)^2 d\mathrm{vol}_{\omega_P} \\
&\leqslant \log C_1 + \log \int_{\Omega_z} |s(w)|^2_{h(\boldsymbol{a},N),\omega_P} d\mathrm{vol}_{\omega_P} \\
&\leqslant \log C_1 + \log \|s\|^2_{h(\boldsymbol{a},N),\omega_P}
\end{aligned}$$

where $\Omega_z := \{w \in U^* \mid |w_i - z_i| \leq \frac{|z_i|}{2} \text{ for } i \leq \ell; |w_i - z_i| \leq \frac{1}{2} \text{ for } i > \ell\}$ and g is the Euclidean metric. The first inequality is due to mean value inequality, and the second one is Jensen inequality. Hence

$$\begin{aligned}
|s(z)|_{h,\omega_P} &= |s(z)|_{h(\boldsymbol{a},-N),\omega_P} \cdot (-\prod_{i=1}^{\ell} \log |z_i|^2)^{\frac{N}{2}} \cdot (\prod_{i=1}^{\ell} |z_i|^{-a_i}) \\
&\leq e^{\frac{C_1}{2}} \|s\|_{h(\boldsymbol{a},N),\omega_P} \cdot (-\prod_{i=1}^{\ell} \log |z_i|^2)^{\frac{N}{2}} \cdot (\prod_{i=1}^{\ell} |z_i|^{-a_i}) \\
&\leqslant C_\delta \|s\|_{h(\boldsymbol{a},N),\omega_P} \cdot (\prod_{i=1}^{\ell} |z_i|^{-a_i-\delta})
\end{aligned}$$

for any $\delta > 0$ and some positive constant C_δ depending on δ. □

3.5. **A fine resolution for Dolbeault complex of Higgs bundles.** Let (E, θ, h) be a tame harmonic bundle on $X - D$, where (X, ω) is a compact Kähler manifold and $D = \sum_{i=1}^{\ell} D_i$ is a simple normal crossing divisor on D.

Let L be a line bundle on X equipped with a smooth metric h_L so that $\sqrt{-1}R(h_L) \geq 0$ and has at least $n-k$ positive eigenvalues[1]. Let P be a nef line bundle on X. Let σ_i be the section $H^0(X, \mathscr{O}_X(D_i))$ defining D_i, and we pick any smooth metric h_i for the line bundle $\mathscr{O}_X(D_i)$ so that $|\sigma_i|_{h_i}(z) < 1$ for any $z \in X$. Write $\sigma_D := \prod_{i=1}^{\ell} \sigma_i \in H^0(X, \mathscr{O}_X(D))$ and $h_D := \prod_{i=1}^{\ell} h_i$ the smooth metric for $\mathscr{O}_X(D)$. Pick a positive constant N greater than N_0, which is the constant in Proposition 3.5 so that (3.3.2) holds.

Given a smooth metric h_P on P, note that for $\mathscr{L} := L \otimes P|_{X^*}$ equiped with the metric

$$h_{\mathscr{L}} := h_L h_P \prod_{i=1}^{\ell} |\sigma_i|^{2a_i}_{h_i} \cdot (-\prod_{i=1}^{\ell} \log |\sigma_i|^2_{h_i})^N, \tag{3.5.1}$$

[1] Such a metrized line bundle (L, h_L) is called *k-positive* in [SS85].

its curvature

$$\sqrt{-1}R(h_{\mathscr{L}}) = \sqrt{-1}R(h_L) + \sqrt{-1}R(h_P) + \sum_{i=1}^{\ell} 2\sqrt{-1}a_iR(h_i) + \sqrt{-1}N\sum_{i=1}^{\ell}\frac{\partial\log|\sigma_i|^2_{h_i}\wedge\bar\partial\log|\sigma_i|^2_{h_i}}{(\log|\sigma_i|^2_{h_i})^2} - N\sum_{i=1}^{\ell}\frac{\sqrt{-1}R(h_i)}{(\log|\sigma_i|^2_{h_i})^2} \tag{3.5.2}$$

Here $R(h_i)$ is the curvature of $\left(\mathscr{O}_X(D_i), h_i\right)$.

Let $0 \leq \gamma_1(x) \leq \cdots \leq \gamma_n(x)$ be eigenvalues of $\sqrt{-1}R(h_L)$ with respect to ω. Set $\varepsilon_0 := \inf_X \gamma_{k+1}(x)$ which is strictly positive by our assumption on $\sqrt{-1}R(h_L)$.

Note that for the prolongation ${}^{\diamond}E$ on X of (E,θ,h) in § 3.2, by the semi-continuity of parabolic Higgs bundles, there is a $\boldsymbol{b} = (b_1,\ldots,b_\ell) \in \mathbb{R}^\ell_{<0}$, so that one has ${}_{\boldsymbol{b}}E = {}^{\diamond}E$.

Lemma 3.9. *We can rescale h_i by timing a positive small constant, take proper metric h_P for P and pick $\boldsymbol{a} \in \mathbb{R}^\ell_{>0}$ and $\delta > 0$ small enough so that*

(1) $a_i + b_i + \delta < 0$ for $i = 1,\ldots,\ell$.
(2) $a_i > \delta$ for $i = 1,\ldots,\ell$
(3) One has

$$\sqrt{-1}R(h_{\mathscr{L}}) \geq \sqrt{-1}R(h_L) - \varepsilon_1\omega \geq -\varepsilon_1\omega. \tag{3.5.3}$$

for $\varepsilon_1 = \frac{\varepsilon_0}{100n^2}$.
(4) The metric

$$\omega_{\boldsymbol{a},N} := \varepsilon_2\omega + \sqrt{-1}R(h_{\mathscr{L}}) \tag{3.5.4}$$

is a Kähler metric when restricted on $X^* = X - D$ for $\varepsilon_2 = \frac{\varepsilon_0}{10n}$.

Proof. Note that (1), (2) are easy to made, and (4) is a consequence of (3). Let us explain how to achieve (3). The possible negative contribution for $\sqrt{-1}R(h_{\mathscr{L}})$ only can come from $\sqrt{-1}R(h_P) + \sum_{i=1}^{\ell} 2\sqrt{-1}a_iR(h_i) - N\sum_{i=1}^{\ell}\frac{\sqrt{-1}R(h_i)}{(\log|\sigma_i|^2_{h_i})^2}$. As P is nef, one can take h_P so that $\sqrt{-1}R(h_P) \geq -\frac{1}{2}\varepsilon_1\omega$. As N is fixed, we can replace h_i by $c\cdot h_i$ for $c \to 0^+$ and let a_i's small enough, so that $\sum_{i=1}^{\ell} 2\sqrt{-1}a_iR(h_i) - N\sum_{i=1}^{\ell}\frac{\sqrt{-1}R(h_i)}{(\log|\sigma_i|^2_{h_i})^2} \geq -\frac{1}{2}\varepsilon_1\omega$. □

We know that $\omega_{\boldsymbol{a},N}$ is a *complete* Kähler metric. Indeed, write $h_i \overset{\text{loc}}{=} e^{-\varphi_i}$ in terms of the trivialization $D_i \cap U = (z_i = 0)$ of any admissible coordinate $(U; z_1,\ldots,z_n)$, one has

$$\omega_{\boldsymbol{a},N} = \Big(\varepsilon_2\omega + \sum_{i=1}^{\ell} 2\sqrt{-1}a_iR(h_i) + \sqrt{-1}R(h_P)\Big) + N\sum_{i=1}^{\ell}\frac{1}{(\log|z|_i^2+\varphi_i)^2}\Big(\frac{dz_i}{z_i}+\partial\varphi_i\Big)\wedge\Big(\frac{d\bar z_i}{\bar z_i}+\bar\partial\varphi_i\Big) - N\sum_{i=1}^{\ell}\frac{\sqrt{-1}\partial\bar\partial\varphi_i}{\log|z|_i^2+\varphi_i}$$

From this local expression one can also see that $\omega_{\boldsymbol{a},N} \sim \omega_P$ on any $U^*(r)$ for $0 < r < 1$. We also can show the following

Lemma 3.10. *For the smooth metric $h_{\boldsymbol{a},N} := h\cdot\prod_{i=1}^{\ell}|\sigma_i|^{2a_i}_{h_i}\cdot(-\prod_{i=1}^{\ell}\log|\sigma_i|^2_{h_i})^N$ of E, it is mutually bounded with $h(\boldsymbol{a},N)$ defined in § 3.3 on any $U^*(r)$ for $0 < r < 1$.*

Let us prove that such construction satisfies the positivity condition in Corollary 2.7.

Proposition 3.11. *With the above notations, for any $p + q > n + k$, one has*

$$\langle[\sqrt{-1}R(h_{\mathscr{L}}), \Lambda_{\omega_{\boldsymbol{a},N}}]f, f\rangle_{\omega_{\boldsymbol{a},N}} \geq \frac{\varepsilon}{2}|f|^2_{\omega_{\boldsymbol{a},N}} \tag{3.5.5}$$

for any $f \in \Lambda^{p,q}T^*_{X^*,x}$ and any $x \in X^*$.

Proof. For any point $x \in X$, one can choose local coordinate $(z_1, \ldots, z_n)$ around x so that at x, $\omega = \sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ and $\sqrt{-1}R(h_{\mathscr{L}}) = \sqrt{-1}\sum_{i=1}^n \tilde{\gamma}_i dz_i \wedge d\bar{z}_i$, where $\tilde{\gamma}_1 \leq \cdots \leq \tilde{\gamma}_n$ are eigenvalues of $\sqrt{-1}R(h_{\mathscr{L}})$ with respect to ω. By (3.5.3) one has $\tilde{\gamma}_i \geq \gamma_i - \varepsilon_1$. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be eigenvalues of $\sqrt{-1}R(h_{\mathscr{L}})$ with respect to $\omega_{\boldsymbol{a},N}$. Then $\lambda_i = \frac{\tilde{\gamma}_i}{\varepsilon_2 + \tilde{\gamma}_i}$, and thus at each point $x \in X^*$, one has

- $-\frac{\varepsilon_1}{\varepsilon_2 - \varepsilon_1} \leq \lambda_i \leq 1$ for $i = 1, \ldots, n$.
- $\lambda_i \geq 1 - \frac{\varepsilon_2}{\varepsilon_0 - \varepsilon_1}$ for for $i = k+1, \ldots, n$.

We can assume that $p \geq q$. Then

$$\begin{aligned}\langle[\sqrt{-1}R(h_{\mathscr{L}}), \Lambda_{\omega_{\boldsymbol{a},N}}]f, f\rangle_{\omega_{\boldsymbol{a},N}} &\geq (\sum_{i=1}^p \lambda_i + \sum_{j=1}^q \lambda_j - \lambda_1 - \cdots - \lambda_n)|f|^2_{\omega_{\boldsymbol{a},N}}\\ &\geq \Big((p-k)(1 - \frac{\varepsilon_2}{\varepsilon_0 - \varepsilon_1}) - \frac{k\varepsilon_1}{\varepsilon_2 - \varepsilon_1} - (n-q)\Big)|f|^2_{\omega_{\boldsymbol{a},N}}\\ &\geq \Big(1 - n(\frac{\varepsilon_2}{\varepsilon_0 - \varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2 - \varepsilon_1})\Big)|f|^2_{\omega_{\boldsymbol{a},N}} \geq \frac{1}{2}|f|^2_{\omega_{\boldsymbol{a},N}}.\end{aligned}$$

□

Remark 3.12. Let us mention that Lemma 3.9 and proposition 3.11 are indeed inspired by the proof of *Girbau vanishing theorem* in [Dem12, Chapter VII, Theorem 4.2] and its logarithmic generalization in [HLWY16, Theorem 4.1].

We equip E with the metric $h_{\boldsymbol{a},N}$ and X^* with the complete Kähler metric $\omega_{\boldsymbol{a},N}$ having the same growth as ω_P near D. Let $\mathfrak{L}^m_{(2)}(E)_{h_{\boldsymbol{a},N},\omega_{\boldsymbol{a},N}}$ be the sheaf on X (rather than on X^*!) of germs of locally L_2, E-valued m-forms, for which $D''(u)$ exists weakly as locally L^2-forms. Namely, for any open set $U \subset X$, we define

$$\mathfrak{L}^m_{(2)}(E)(U) := \{u \in L^m_{(2)}(U - D, E) \mid D''u \in L^{m+1}_{(2)}(U - D, E)\}. \tag{3.5.6}$$

Here we write $\mathfrak{L}^m_{(2)}(E)$ instead of $\mathfrak{L}^m_{(2)}(E)_{h_{\boldsymbol{a},N},\omega_{\boldsymbol{a},N}}$ for short.

We also define $\mathfrak{L}^{p,q}_{(2)}(E)$ to be be the sheaf on X of germs of locally L_2, E-valued (p,q)-forms, for which both $\bar{\partial}_E(u)$ exist weakly as locally L^2-forms. Namely, for any open set $U \subset X$, one has

$$\mathfrak{L}^{p,q}_{(2)}(E)(U) := \{u \in L^{p,q}_{(2)}(U - D, E) \mid \bar{\partial}_E u \in L^{p,q+1}_{(2)}(U - D, E)\} \tag{3.5.7}$$

Note that for any admissible coordinate $(U; z_1, \ldots, z_n)$, as $\omega_{\boldsymbol{a},N} \sim \omega_P$ and $h_{\boldsymbol{a},N} \sim h(\boldsymbol{a}, N)$ on any $U^*(r)$ for $0 < r < 1$, one has

$$\mathfrak{L}^m_{(2)}(E)(U^*(r)) = L^m_{(2)}(U^*(r), E)_{h(\boldsymbol{a},N),\omega_P}, \quad \mathfrak{L}^{p,q}_{(2)}(E)(U^*(r)) = L^{p,q}_{(2)}(U^*(r), E)_{h(\boldsymbol{a},N),\omega_P}.$$

The following lemma will be crucial for us.

Lemma 3.13. $\mathfrak{L}^m_{(2)}(E) = \oplus_{p+q=m}\mathfrak{L}^{p+q}_{(2)}(E)$.

Proof. For any $x \in D$ and any admissible coordinate $(U; z_1, \ldots, z_n)$, we have

$$|\theta|_{h(\boldsymbol{a},N),\omega_P} = |\theta|_{h,\omega_P} \cdot e^{-\chi(\boldsymbol{a},N)}$$

where $\chi(\boldsymbol{a}, N)$ is defined in (3.3.1). By Theorem 1.5, one has

$$|\theta|_{h(\boldsymbol{a},N),\omega_P} \leq C$$

for any C. Hence θ is a bounded linear operator

$$L^{p,q}_{(2)}(U - D, E) \to L^{p+1,q}_{(2)}(U - D, E).$$

The theorem follows from that $D'' = \bar{\partial}_E + \theta$. □

Proposition 3.14. *Let $\mathscr{A}^m({}^\diamond E)$ be the sheaf on X of germs of smooth m-forms with values in $\oplus_{p+q=m}{}^\diamond E\otimes\Omega_X^{p,q}(\log D)$. One has the inclusion*

$$\mathscr{A}^m({}^\diamond E)\subset\mathfrak{L}^m_{(2)}(E)$$

which is densely defined.

Proof. For any $x\in D$, we pick an admissible coordinate $(U;z_1,\ldots,z_n)$ with $D\cap U=(z_1\ldots z_\ell=0)$ and a holomorphic frame $v_1,\ldots,v_r\in\Gamma(U,{}^\diamond E)$. By the definition of ${}^\diamond E$ and our choice of $\boldsymbol{a}$ in (2), one has $|v_\alpha|_h\leq C\prod_{i=1}^{\ell}|z_i|^{-a_i+2\delta}$ for each $\alpha=1,\ldots,r$.

Write $w_i=\log z_i$ for $i=1,\ldots,\ell$ and $w_j=z_j$ for $j=\ell+1,n$. For the basis $\{dw_I\wedge d\bar{w}_J\}_{|I|+|J|=m}$ of $\oplus_{p+q=m}\Omega_X^{p,q}(\log D)$, on $U^*(r)$ with $0<r<1$, one has

$$|dw_I\wedge d\bar{w}_J|_{\omega_P}\leq C\prod_{i\in(I\cup J)\cap\{1,\ldots,\ell\}}(-\log|z_i|^2)$$

which have at most logarithmic growth. For any smooth section $s\in\mathscr{A}^m({}^\diamond E)(U)$ with any $0<r<1$, we can write $s=\sum_{|I|+|J|=m,\alpha}f^\alpha_{I,J}dw_I\wedge d\bar{w}_J\otimes v_\alpha$ with $\{f^\alpha_{I,J}\}$ smooth functions on U. Hence one has

$$|s|_{h,\omega_P}\leq C'\prod_{i=1}^{\ell}|z_i|^{-a_i+\delta}$$

on any $U^*(r)$ with $0<r<1$. Therefore,

$$|s|_{h(\boldsymbol{a},N),\omega_P}\leq C'\prod_{i=1}^{\ell}|z_i|^{-a_i+\delta}\cdot e^{-\chi(\boldsymbol{a},N)}\leq C''$$

where $\chi(\boldsymbol{a},N)$ is defined in (3.3.1), and we use the fact that there is a constant $C(N,\delta)$ depending on positive constants N and δ so that

$$\log(-|z_i|^2)^N\leq C(N,\delta)|z_i|^{-\delta}$$

when z_i tends to 0. Hence

$$\int_{U^*(r)}|s|^2_{h(\boldsymbol{a},N),\omega_P}d\mathrm{Vol}_{\omega_P}<+\infty,$$

and since $h(\boldsymbol{a},N)\sim h_{\boldsymbol{a},N}$ and $\omega_P\sim\omega_{\boldsymbol{a},N}$ on any $U^*(r)$ with $0<r<1$, we conclude that

$$\int_{U^*(r)}|s|^2_{h_{\boldsymbol{a},N},\omega_{\boldsymbol{a},N}}d\mathrm{Vol}_{\omega_{\boldsymbol{a},N}}<+\infty.$$

Note that $\theta:E\to E\otimes\Omega^1_{X^*}$ extends to $\theta:{}^\diamond E\to{}^\diamond E\otimes\Omega^1_X(\log D)$, and $\bar\partial_E$ for E also extends to the complex structure $\bar\partial_{{}^\diamond E}$ of ${}^\diamond E$, one thus can define $D''=\bar\partial_{{}^\diamond E}+\theta:\mathscr{A}^m({}^\diamond E)\to\mathscr{A}^{m+1}({}^\diamond E)$ which extends the original $\bar\partial_E+\theta$ over X^*. Hence

$$s\in\mathfrak{L}^m_{(2)}(E)(U^*(r)),$$

which proves the theorem. □

Recall that one has $D''^2=0$. Let $(\mathfrak{L}^\bullet_{(2)}(E),D'')$ be a complex of fine sheaves over X defined by

$$\mathfrak{L}^0_{(2)}(E)\xrightarrow{D''}\mathfrak{L}^1_{(2)}(E)\xrightarrow{D''}\cdots\xrightarrow{D''}\mathfrak{L}^m_{(2)}(E).\tag{3.5.8}$$

By Proposition 3.14, there is a natural inclusion
(3.5.9)

$$\begin{tikzcd} {}^{\diamond}E \arrow[r, "\theta"] \arrow[d] & {}^{\diamond}E\otimes\Omega_X^1(\log D) \arrow[r, "\theta"] \arrow[d] & \cdots \arrow[r, "\theta"] \arrow[d] & {}^{\diamond}E\otimes\Omega_X^n(\log D) \arrow[d] & & \\ \mathfrak{L}^0_{(2)}(E) \arrow[r, "D''"] & \mathfrak{L}^1_{(2)}(E) \arrow[r, "D''"] & \cdots \arrow[r, "D''"] & \mathfrak{L}^n_{(2)}(E) \arrow[r, "D''"] & \cdots \arrow[r, "D''"] & \mathfrak{L}^{2n}_{(2)}(E) \end{tikzcd}$$

and we are going to show that this morphism between two complexes are quasi-isomorphism.

We now recall a celebrated theorem (in a weaker form) by Demailly [Dem82, Théorème 4.1], which enables us to solve the $\bar{\partial}$-equation on weakly pseudo-convex Kähler manifold (might not be complete). When the metric is complete, it is due to Andreotti-Vesentini [AV65].

Theorem 3.15 (Demailly). *Let (X,ω) be a Kähler manifold (ω might not be complete), where X possesses a complete Kähler metric (*e.g. *X is weakly pseudo-convex). Let E be a vector bundle on X equipped with a smooth hermitian metric h so that*

$$\sqrt{-1}R_h(E)\geq_{Nak}\varepsilon\omega\otimes\mathbb{1}_E,$$

where $\varepsilon>0$ is a positive constant. Assume that $g\in L^{n,q}_{(2)}(X,E)$ so that $\bar{\partial}g=0$. Then there exists $f\in L^{n,q-1}_{(2)}(X,E)$ so that $\bar{\partial}f=g$ and

$$\|f\|^2_{h,\omega}\leqslant\varepsilon^{-1}\|g\|^2_{h,\omega}.$$

This theorem by Demailly is used to solve the $\bar{\partial}$-equation locally.

Proposition 3.16. *For any $x\in X$, there is an open set $U\subset X$ (can be made arbitrary small) containing x so that for any $g\in\mathfrak{L}^{p,q}_{(2)}(E)(U)$ with $q\geq 1$ and $\bar{\partial}_E(g)=0$, there exists a section $f\in\mathfrak{L}^{p,q-1}_{(2)}(E)(U)$ so that $\bar{\partial}_E f=g$.*

Proof. If $x\notin D$, then we can take an open set $U\subset X-D$ containing x which is biholomorphic to a polydisk, and the theorem follows from the usual L^2-Dolbeault lemma. Assume $x\in D$. Let $(\tilde{U};z_1,\ldots,z_n)$ be an admissible coordinate around x. By Proposition 3.5, $\mathscr{E}_p:=T^p_{\tilde{U}^*}\otimes E$ equipped with the $\mathscr{C}^\infty$-metric $h_{\mathscr{E}_p}=h_p^{-1}h(\boldsymbol{a},N)$ induced by $h(\boldsymbol{a},N)$ and ω_P, satisfying

$$\sqrt{-1}R(h_{\mathscr{E}_p})\geq_{\mathrm{Nak}}\omega_P\otimes\mathbb{1}_{\mathscr{E}_p}$$

for any $p=0,\ldots,n$. Note that $\omega_P|_{\tilde{U}^*(\frac{1}{2})}\sim\omega_{\boldsymbol{a},N}|_{\tilde{U}^*(\frac{1}{2})}$ and $h(\boldsymbol{a},N)|_{\tilde{U}^*(\frac{1}{2})}\sim h_{\boldsymbol{a},N}|_{\tilde{U}^*(\frac{1}{2})}$. Hence one has

$$L^{n,q}_{(2)}(\tilde{U}^*(\frac{1}{2}),\mathscr{E}_{n-p})_{h_{\mathscr{E}_{n-p}},\omega_P}=L^{p,q}_{(2)}(\tilde{U}^*(\frac{1}{2}),E)_{h_{\boldsymbol{a},N},\omega_{\boldsymbol{a},N}}\tag{3.5.10}$$

for any $p=0,\ldots,n$. For any $g\in L^{n,q}_{(2)}(\tilde{U}^*(\frac{1}{2}),\mathscr{E}_{n-p})_{h_{\mathscr{E}_{n-p}},\omega_P}$ with $\bar{\partial}(g)=0$, if $q\geq 1$, by Theorem 3.15, there is $f\in L^{n,q-1}_{(2)}(\tilde{U}^*(\frac{1}{2}),\mathscr{E}_{n-p})_{h_{\mathscr{E}_{n-p}},\omega_P}$ so that $\bar{\partial}f=g$. The proposition then follows from (3.5.10), and $\tilde{U}^*(\frac{1}{2})$ is the desired open set U in the proposition. □

Now we are ready to prove that the L^2-complex is the desired fine resolution for our Higgs bundle.

Theorem 3.17. *The morphism between two complexes in (3.5.9) is quasi-isomorphism.*

Proof. Pick any $m\in\{0,\ldots,n\}$. We are going to show that $\iota:\ker\theta/\operatorname{Im}\theta\to\ker D''/\operatorname{Im}D''$ at ${}^{\diamond}E\otimes\Omega_X^m(\log D)$ is an isomorphism. For any $x\in D$, we pick an open set $U\ni x$ as in Proposition 3.16 and set $U^*=U-D$. Indeed, $U^*=\tilde{U}^*(\frac{1}{2})$ where $(\tilde{U};z_1,\ldots,z_n)$ is an admissible coordinate around x and thus $h_{\boldsymbol{a},N}\sim h(\boldsymbol{a},N)$ and $\omega_{\boldsymbol{a},N}\sim\omega_P$ on U^*. Pick any $g \in \mathfrak{L}^m_{(2)}(E)(U) = L^m_{(2)}(U^*, E)_{h(\boldsymbol{a},N),\omega_P}$ so that $D''g = 0$. By Lemma 3.13, we can write $g = \sum_{p+q=m} g_{p,q}$ where $g_{p,q} \in \mathfrak{L}^{p,q}_{(2)}(E)(U)$, and let q_0 be the largest integer for q so that $g_{p,q} \neq 0$. If $q_0 = 0$, then $g = g_{m,0}$ and one has $\bar{\partial}_E g_{m,0} = 0$ and $\theta g_{m,0} = 0$. By the elliptic regularity of $\bar{\partial}$ one concludes that $g \in \Gamma(U^*, \Omega^m_{U^*} \otimes E|_{U^*})$. By Proposition 3.8, one has

$$|g|_{h,\omega_P} \leq C \cdot (\prod_{i=1}^{\ell} |z_i|^{-a_i-\delta})$$

If we write $g = \sum_{|I|=m} dw_I \otimes e_I$, where $(w_1, \ldots, w_n) = (\log z_1, \ldots, \log z_\ell, z_{\ell+1}, \ldots, z_n)$ and $e_I \in \Gamma(U^*, E|_{U^*})$. Then

$$|g|_{h,\omega_P} = \sum_{|I|=m} |e_I|_h |dw_I|_{\omega_P} \geq C' \sum_{|I|=m} |e_I|_h$$

Hence $|e_I|_h \leq C'' \cdot (\prod_{i=1}^{\ell} |z_i|^{-a_i-\delta}) \leq C''' \cdot (\prod_{i=1}^{\ell} |z_i|^{b_i})$ by our choice of $\boldsymbol{a}$ in (1). As ${}_{\boldsymbol{b}}E$ is defined via the increasing order of sections of E in (3.2.1), we conclude that $e_I \in \Gamma(U, {}_{\boldsymbol{b}}E|_U)$. As ${}_{\boldsymbol{b}}E = {}^\diamond E$ by our choice of $\boldsymbol{b}$ at the beginning of this subsection, one has $e_I \in \Gamma(U, {}^\diamond E)$. Hence $g \in \Gamma(U, \Omega^m_X(\log D) \otimes {}^\diamond E|_U)$, which means that ι is surjective.

Now we assume that $q_0 > 0$ and p_0 is the largest integer for p so that $g_{p,q} \neq 0$. By Lemma 3.13, we can decompose $D''g$ into bidegrees, so that

$$\begin{cases} \bar{\partial}_E g_{m-q_0,q_0} = 0 \\ \theta g_{m-q_0,q_0} + \bar{\partial}_E g_{m-q_0+1,q_0-1} = 0 \\ \vdots \\ \theta g_{p_0-1,m-p_0+1} + \bar{\partial}_E g_{p_0,m-p_0} = 0 \\ \theta g_{p_0,m-p_0} = 0 \end{cases}$$

for which, the operators act in the sense of distribution. Hence $g_{m-q_0,q_0} \in \mathfrak{L}^{m-q_0,q_0}_{(2)}(E)(U)$ with $\bar{\partial}_E g_{m-q_0,q_0} = 0$. Applying Proposition 3.16, there is a section $f_{m-q_0,q_0-1} \in \mathfrak{L}^{m-q_0,q_0-1}_{(2)}(E)(U)$ so that $\bar{\partial}_E f_{m-q_0,q_0-1} = -g_{m-q_0,q_0}$. By Lemma 3.13, $D'' f_{m-q_0,q_0-1} \in \mathfrak{L}^m_{(2)}(E)(U)$, and we define $g' := D'' f_{m-q_0,q_0-1} + g \in \mathfrak{L}^m_{(2)}(E)(U)$. One thus has $D''g' = 0$. Write $g' = \sum_{p+q=m} g'_{p,q}$ where $g'_{p,q} \in \mathfrak{L}^{p,q}_{(2)}(E)(U)$. Note that

$$\begin{cases} g'_{m-q_0,q_0} = \bar{\partial}_E f_{m-q_0,q_0-1} - g_{m-q_0,q_0} = 0 \\ g'_{m-q_0+1,q_0-1} = \theta f_{m-q_0,q_0-1} + g_{m-q_0+1,q_0-1} \\ g'_{m-q_0+2,q_0-2} = g_{m-q_0+2,q_0-2} \\ \vdots \\ g'_{p_0,m-p_0} = g_{p_0,m-p_0} \end{cases}$$

One can perform the same manner inductively to find $f \in \mathfrak{L}^{m-1}_{(2)}(E)(U)$ so that $g_0 = g + D''f \in \mathfrak{L}^{m,0}_{(2)}(E)(U)$ so that $D''g_0 = 0$. By the above argument, we know that $g_0 \in \Gamma(U, \Omega^m_X(\log D) \otimes {}^\diamond E|_U)$, which shows the surjectivity of ι.

Now we prove the injectivity of ι. Let $g \in \Gamma(U, \Omega^m_X(\log D) \otimes {}^\diamond E|_U) \subset \mathfrak{L}^m_{(2)}(E)(U)$ so that $g = D''f$. Write $f = \sum_{p+q=m} f_{p,q}$ where $f_{p,q} \in \mathfrak{L}^{p,q}_{(2)}(E)(U)$. Then $D''(f_{m,0} + f_{m-1,1}) = g$ thanks to the bidegree condition. Hence $\bar{\partial}_E f_{m-1,1} = 0$. Applying Proposition 3.16, there is a section $h_{m-1,0} \in \mathfrak{L}^{m-1,0}_{(2)}(E)(U)$ so that $\bar{\partial}_E h_{m-1,0} = -f_{m-1,1}$. Then $g = D''(f_{m,0} + f_{m-1,1} + D'' h_{m-1,0}) = D''(f_{m,0} + \theta h_{m-1,0}) = \theta(f_{m,0} + \theta h_{m-1,0}) = \theta(f_{m,0})$. The injectivity is thus proved.

When $m > n$, the exactness of D'' can be proven in the same way. Let $g \in \mathfrak{L}^m_{(2)}(E)(U)$ so that $D''g = 0$. Applying Proposition 3.16 once again, we can find $f \in \mathfrak{L}^{m-1}_{(2)}(E)(U)$ so that

$D''f+g\in\mathfrak{L}^{n,m-n}_{(2)}(E)(U)$. As $\theta(D''f+g)=0$, this implies that $\bar\partial_E(D''f+g)=0$, and by Proposition 3.16 again one can find $h\in\mathfrak{L}^{n,m-n-1}_{(2)}(E)(U)$ so that $D''h=\bar\partial_E h=D''f+g$. This shows the exactness of D'' when $m>n$. We complete the proof of the theorem. □

3.6. **Proof of the main theorem.** In this subsection, we will prove the following vanishing theorem for tame harmonic bundle.

Theorem 3.18. *Let (X,ω) be a compact Kähler manifold of dimension n and let D be a simple normal crossing divisor on X. Let $(E,{}_{\boldsymbol{a}}E,\theta)$ be the parabolic Higgs bundle on X induced by a tame harmonic bundle (E,θ,h) on $X^*=X-D$. Let L be a line bundle on X equipped with a smooth metric h_L so that $\sqrt{-1}R(h_L)\geq 0$ and has at least $n-k$ positive eigenvalues. Let P be a nef line bundle on X. Then*

$$\mathbb{H}^m\Big(X,({}^\diamond E\otimes\Omega^\bullet_X(\log D),\theta)\otimes L\otimes P\Big)=0$$

for any $m>n+k$.

Proof. We will use the notations in § 3.5. Recall that $(X^*,\omega_{\boldsymbol{a},N})$ is a complete Kähler manifold. Write $\mathscr{L}:=L\otimes P|_{X^*}$ and we equip it with the metric $g=h_Lh_P$ where h_P is properly chosen as Lemma 3.9. Then g is the restriction to X^* of a smooth metric on X. We introduce a new Higgs bundle $(\tilde E,\tilde\theta,\tilde h):=(E\otimes\mathscr{L},\theta\otimes\mathbb{1}_{\mathscr{L}},h(\boldsymbol{a},N)\cdot g)$. We still use the notation $D'':=\bar\partial_{\tilde E}+\tilde\theta$ abusively, and D''^* denotes its adjoint with respect to $\tilde h$. We will apply Corollary 2.7 to solve D''-equation for this new Higgs bundle.

Note that $h(\boldsymbol{a},N)g=hh_{\mathscr{L}}$ by (3.5.1) and Lemma 3.10. By proposition 3.11, the metrized line bundle $(\mathscr{L},h_{\mathscr{L}})$ satisfies the condition in Corollary 2.7 when $m>n+k$. Hence by Corollary 2.7 for any section $g=L^m_{(2)}(X^*,\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}}$, if $D''g=0$ and $m>n+k$, there exists $f\in L^{m-1}_{(2)}(X^*,\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}}$ so that

$$D''f=g.$$

Let $\mathfrak{L}^m_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}}$ be the sheaf on X (rather than on X^*!) of germs of locally L_2, $\tilde E$-valued m-forms, for which both $D''(u)$ (as a distribution) exist weakly as locally L^2-forms. Namely, for any open set $U\subset X$, one has

(3.6.1) $$\mathfrak{L}^m_{(2)}(\tilde E)(U):=\{u\in L^m_{(2)}(U-D,\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}}\mid D''u\in L^{m+1}_{(2)}(U-D,E)_{\tilde h,\omega_{\boldsymbol{a},N}}\}$$

Then the above argument proves that the cohomology H^i of the complex of global sections of the sheaves $(\mathfrak{L}^\bullet_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}},D'')$ vanishes for $m>n+k$.

As g is smooth over the whole X, the metric $\tilde h\sim h(\boldsymbol{a},N)$ near D (fix any trivialization of $L\otimes P$). Hence the natural inclusion

(3.6.2)

$$\begin{tikzcd}
{}^\diamond E\otimes L\otimes P \arrow[r, "\tilde\theta"] \arrow[d] & {}^\diamond E\otimes L\otimes P\otimes\Omega_X(\log D) \arrow[r, "\tilde\theta"] \arrow[d] & \cdots \arrow[r, "\tilde\theta"] \arrow[d] & {}^\diamond E\otimes L\otimes P\otimes\Omega^n_X(\log D) \arrow[d] & & \\
\mathfrak{L}^0_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}} \arrow[r, "D''"] & \mathfrak{L}^1_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}} \arrow[r, "D''"] & \cdots \arrow[r, "D''"] & \mathfrak{L}^n_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}} \arrow[r, "D''"] & \cdots \arrow[r, "D''"] & \mathfrak{L}^{2n}_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}}
\end{tikzcd}$$

is thus also a quasi-isomorphism by Theorem 3.17.

As the complex $(\mathfrak{L}^\bullet_{(2)}(\tilde E)_{\tilde h,\omega_{\boldsymbol{a},N}},D'')$ is a fine sheaf, its cohomology computes the hypercomology of the complex $({}^\diamond E\otimes L\otimes\Omega^\bullet_X(\log D),\tilde\theta)$. We thus conclude that $\mathbb{H}^m(X,({}^\diamond E\otimes L\otimes\Omega^\bullet_X(\log D),\tilde\theta))=0$ for $m>n+k$. The theorem is proved. □

Remark 3.19. Let us show how to derive the log Girbau vanishing theorem in [HLWY16, Corollary 1.2] from Theorem A. With the same setting as Theorem A, let $(E,\theta,h):=(\mathscr{O}_{X-D},0,h)$

where h is the canonical metric on the trivial line bundle $\mathscr{O}_{X-D}$. According to the prolongation of (E,θ,h) defined in § 3.2, one has $({}^\diamond\!E,\theta)=(\mathscr{O}_X,0)$. Hence the Dolbeault complex in (0.1.1)

$$\mathrm{Dol}({}^\diamond\!E,\theta)=\mathscr{O}_X\xrightarrow{0}\Omega^1_X(\log D)\xrightarrow{0}\cdots\xrightarrow{0}\Omega^n_X(\log D)$$

which is a direct sum of sheaves of logarithmic p-forms shifting p places to the right:

$$\mathrm{Dol}({}^\diamond\!E,\theta)=\oplus_{p=0}^n\Omega^p_X(\log D)[p],$$

where $\Omega^p_X(\log D)[p]$ is the obtained by shifting the *single degree complex* $\Omega^p_X(\log D)$ in degree p. Hence if $m>n+k$, by Theorem 3.18 one has

$$\begin{aligned}0=\mathbb{H}^m\Big(X,\mathrm{Dol}({}^\diamond\!E,\theta)\otimes L\otimes P\Big)&=\oplus_{p=0}^nH^m(X,\Omega^p_X(\log D)\otimes P\otimes N[p])\\&=\oplus_{p=0}^nH^{m-p}(X,\Omega^p_X(\log D)\otimes P\otimes N).\end{aligned}$$

We thus conclude that

$$H^q(X,\Omega^p_X(\log D)\otimes P\otimes N)$$

if $p+q>n+k$. This is the log Girbau vanishing theorem by Huang-Liu-Wan-Yang.

3.7. **Vanishing theorem for parabolic Higgs bundles.** Let X be a complex projective manifold and let D be simple normal crossing divisor on X. For a parabolic Higgs bundle $(E,{}_{\boldsymbol{a}}E,\theta)$ on (X,D), its parabolic Chern classes, denoted by para-$c_i(E)$, is the usual Chern class of ${}^\diamond\!E$ with a modification along the boundary divisor D (see, e.g., [AHL19, §3] for more details). With a polarization, i.e., an ample line bundle H on X, the parabolic degree para-deg(E) of $(E,{}_{\boldsymbol{a}}E,\theta)$ is defined to be para-$c_1(E)\cdot H^{n-1}$. We say $(E,{}_{\boldsymbol{a}}E,\theta)$ *slope stable* if for any coherent torsion free subsheaf V of ${}^\diamond\!E$, with $0<\mathrm{rank}V<\mathrm{rank}{}^\diamond\!E=\mathrm{rank}E$ and $\theta(V)\subseteq V\otimes\Omega^1_X(\log D)$, the condition

$$\frac{\text{para-deg}(V)}{\mathrm{rank}(V)}<\frac{\text{para-deg}(E)}{\mathrm{rank}(E)}$$

is satisfied, where V carries the induced the parabolic structure from $(E,{}_{\boldsymbol{a}}E,\theta)$, i.e. ${}_{\boldsymbol{a}}V:=V\cap{}_{\boldsymbol{a}}E$. A parabolic Higgs bundle $(E,{}_{\boldsymbol{a}}E,\theta)$ is *poly-stable* if it is a direct sum of slope stable parabolic Higgs bundles. By [IS07], $(E,{}_{\boldsymbol{a}}E,\theta)$ is called *locally abelian* if in a Zariski neighborhood of any point $x\in X$ there is an isomorphism between the underlying parabolic vector bundle $(E,{}_{\boldsymbol{a}}E)$ and a direct sum of parabolic line bundles.

By the celebrated Simpson-Mochizuki correspondence [Moc09, Theorem 9.4], a parabolic Higgs bundle $(E,{}_{\boldsymbol{a}}E,\theta)$ on (X,D) is poly-stable with trivial parabolic degrees and locally abelian if and only if it is induced by a tame harmonic bundle over $X-D$ defined in § 3.2. Based on this deep theorem, our theorem can thus be restated as follows.

Corollary 3.20. *Let $(E,{}_{\boldsymbol{a}}E,\theta)$ be poly-stable parabolic Higgs bundle on (X,D) with trivial parabolic degrees which is locally abelian. Let L be a line bundle on X equipped with a smooth metric h_L so that its curvature $\sqrt{-1}R(h_L)\geq 0$ and has at least $n-k$ positive eigenvalues. Let P be a nef line bundle on X. Then for the weight 0 filtration ${}^\diamond\!E$ of $(E,{}_{\boldsymbol{a}}E,\theta)$, one has*

$$\mathbb{H}^m\Big(X,({}^\diamond\!E\otimes\Omega^\bullet_X(\log D),\theta)\otimes L\otimes P\Big)=0$$

for any $m>\dim X+k$.

This above corollary generalizes [AHL19, Corollary 7.3] in which they further assume that L is ample.

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Institut des Hautes Études Scientifiques, Université Paris-Saclay, 35 route de Chartres, 91440, Bures-sur-Yvette, France
E-mail address: deng@ihes.fr
URL: https://www.ihes.fr/~deng

Department of Mathematics, KU LEUVEN, Celestijnenlaan 200B, bus 2400 B-3001 Leuven, Belgium
E-mail address: feng.hao@kuleuven.be
URL: https://www.kuleuven.be/wieiswie/nl/person/00133186